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UNIQUENESS OF POSITIVE SOLUTIONS OF QUASILINEAR DIFFERENTIAL EQUATIONS

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Dedicated to Professor Takaŝi Kusano on his 60th birthday

(Submitted by: L.A. Peletier)

Abstract. In this paper we are concerned with the uniqueness of positive solutions of boundary value problems for quasilinear differential equations of the type $(|u'|^{m-2}u')' + p(t)f(u) = 0$, $m > 1$. The key ingredient of the method is the generalized Prüfer transformation. These problems arise, for example, in the study of the m-Laplace equation in annular regions.

1. Introduction. In this paper we are concerned with the uniqueness of positive solutions of the equation

$$(|u'|^{m-2}u')' + p(t)f(u) = 0, \qquad a < t < b, \quad m > 1, \tag{1.1}$$

subject to one of the following sets of boundary conditions:

$$u(a) = u'(b) = 0, \tag{1.2a}$$
$$u'(a) = u(b) = 0, \tag{1.2b}$$
$$u(a) = u(b) = 0. \tag{1.2c}$$

In equation (1.1), we assume that p satisfies

$$p \in C[a,b] \quad \text{and} \quad p(t) > 0 \quad \text{for} \quad t \in (a,b)$$

and that f satisfies

$$f \in C[0,\infty), \quad f(u) > 0 \quad \text{for} \quad u \in (0,\infty), \quad \text{and} \quad \frac{f(u)}{u^{m-1}} \quad \text{is decreasing in} \quad u \in (0,\infty). \tag{1.3}$$

The case, where $f(u) = u^\gamma$ with $0 < \gamma < m-1$, is an important special case.

By a solution of (1.1) we mean a function $u \in C^1(a,b)$ which has the property $|u'|^{m-2}u' \in C^1(a,b)$ and satisfies the equation in (a,b).

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The problems under consideration arise in studies of radially symmetric solutions of the m-Laplace equation

$$\nabla \cdot (|\nabla u|^{m-2}\nabla u) + q(|x|)f(u) = 0, \quad R_0 < |x| < R_1, \quad x \in \mathbb{R}^N, \quad N \geq 2, \tag{1.4}$$

with one of the following sets of boundary conditions:

$$u = 0 \quad \text{on} \quad |x| = R_0 \quad \text{and} \quad \frac{\partial u}{\partial r} = 0 \quad \text{on} \quad |x| = R_1, \tag{1.5a}$$

$$\frac{\partial u}{\partial r} = 0 \quad \text{on} \quad |x| = R_0 \quad \text{and} \quad u = 0 \quad \text{on} \quad |x| = R_1, \tag{1.5b}$$

$$u = 0 \quad \text{on} \quad |x| = R_0 \quad \text{and} \quad u = 0 \quad \text{on} \quad |x| = R_1. \tag{1.5c}$$

Here, $r = |x|$ and $\partial/\partial r$ denotes differentiation in the radial direction. A radially symmetric solution $u = u(r)$, $r = |x|$, of (1.4) satisfies the ordinary differential equation

$$(|u'|^{m-2}u')' + \frac{N-1}{r}|u'|^{m-2}u' + q(r)f(u) = 0, \qquad R_0 < r < R_1. \tag{1.6}$$

With the change of variables, $t = r^{\frac{m-N}{m-1}}$ (for $m \neq N$) or $t = \log r$ (for $m = N$), (1.6) becomes an equation of the form (1.1), where

$$a = \min\{R_0^{\frac{m-N}{m-1}},\ R_1^{\frac{m-N}{m-1}}\}, \quad b = \max\{R_0^{\frac{m-N}{m-1}},\ R_1^{\frac{m-N}{m-1}}\}, \quad \text{and}$$

$$p(t) = \left|\frac{m-1}{m-N}\right|^m t^{\frac{m(N-1)}{m-N}} q(t^{\frac{m-1}{m-N}}) \quad \text{for} \quad m \neq N,$$

$$a = \log R_0,\ b = \log R_1, \quad \text{and} \quad p(t) = e^{Nt}q(e^t) \quad \text{for} \quad m = N,$$

and boundary conditions (1.5a) – (1.5c) become (1.2a) – (1.2c), respectively.

The uniqueness problem concerning (1.1), for the case $m = 2$, has been studied by several authors (see, for example, [3, 6, 7]). However, it seems that very little is known for the case $m \neq 2$. In this paper, we establish uniqueness results regarding the case $m > 1$, assuming existence of the solution. In Section 2, we introduce the generalized Prüfer transformation for (1.1), and in Section 3, we give comparison results which play an important role in proving uniqueness theorems. In Section 4, we show the uniqueness theorems for (1.1) subject to (1.2a), (1.2b), and (1.2c), respectively.

Concerning the existence of solutions of boundary value problems for the equation of the type (1.1), we refer to the papers by del Pino, Elgueta and Manasevich [1] and Kaper, Knaap and Kwong [4].

2. Generalized Prüfer transformation. In this section we introduce the generalized Prüfer transformation for (1.1). First, following the idea of Elbert [2] we define the function $S = S(t)$ as the solution of differential equation

$$(|S'|^{m-2}S')' + (m-1)|S|^{m-2}S = 0, \qquad t \in \mathbb{R}, \tag{2.1}$$

with the initial condition

$$S(0) = 0 \quad \text{and} \quad S'(0) = 1. \tag{2.2}$$

We note that S exists and is unique on the whole interval $\mathbb{R}$; see Kitano and Kusano [5]. It is easy to see that if $S' \neq 0$ on a subinterval I of $\mathbb{R}$, then $S \in C^2(I)$ and S satisfies

$$|S'|^{m-2}S'' + |S|^{m-2}S = 0 \qquad \text{in } \ I. \tag{2.3}$$

The following result is needed in this paper.

Lemma 1. *The function S satisfies the following properties*:

$$S(t) > 0 \ \ \textit{and} \ \ S'(t) > 0 \ \ \textit{for} \ \ 0 < t < \frac{1}{2}\hat{\pi}, \quad S'(\tfrac{1}{2}\hat{\pi}) = 0, \quad \textit{and}$$

$$(S'(t))^m + (S(t))^m = 1 \quad \textit{for} \ \ 0 \le t \le \frac{1}{2}\hat{\pi}, \tag{2.4}$$

where

$$\hat{\pi} = \big(\frac{2\pi}{m}\big)/\sin(\frac{\pi}{m}). \tag{2.5}$$

The above result can be found in [2], although we give a proof of the assertion for the sake of completeness.

Proof. Let

$$\tau = \inf\{t > 0 : S'(t) \le 0\}.$$

Then, by virtue of (2.2), we see that $\tau \in (0, \infty]$ and that $S'(t) > 0$ and $S(t) > 0$ for $0 < t < \tau$. Define the function T by

$$T(t) = \frac{S(t)}{S'(t)}, \quad 0 \le t < \tau.$$

We find that $T(t) > 0$ for $0 < t < \tau$, $T(0) = 0$ and $\lim_{t\to\tau} T(t) = \infty$. By a simple computation, we obtain

$$\frac{T'(t)}{1 + (T(t))^m} = 1, \quad 0 < t < \tau.$$

An integration of the above on $[0, t]$, $0 < t < \tau$, yields

$$\int_0^{T(t)} \frac{d\sigma}{1 + \sigma^m} = t.$$

Letting $t \to \tau$, we have

$$\tau = \int_0^{\infty} \frac{d\sigma}{1 + \sigma^m} = \frac{1}{2}\hat{\pi},$$

where $\hat{\pi}$ is given in (2.5). Therefore, $S'(t) > 0$ for $0 < t < \frac{1}{2}\hat{\pi}$ and $S'(\frac{1}{2}\hat{\pi}) = 0$. By using (2.3) we see that

$$(|S(t)|^m + |S'(t)|^m)' = mS'(t)\{|S(t)|^{m-2}S(t) + |S'(t)|^{m-2}S''(t)\} = 0, \quad 0 < t < \frac{1}{2}\hat{\pi}.$$

By virtue of (2.2), this implies that (2.4) holds. This completes the proof. □

We define the generalized polar coordinates of solutions of (1.1) with the aid of the function S. Let $c \in (a, b]$ and let u be a positive solution of (1.1) satisfying $u(a) = u'(c) = 0$. We define continuous functions $\rho(t)$ and $\phi(t)$ on $[a, c]$ by

$$u(t) = \rho(t)S(\phi(t)), \tag{2.6}$$

$$u'(t) = \rho(t)S'(\phi(t)), \tag{2.7}$$

$$\phi(a) = 0. \tag{2.8}$$

By virtue of (2.4), ρ is written in the form

$$\rho(t) = ((u'(t))^m + (u(t))^m)^{\frac{1}{m}}.$$

Since $\rho(t) > 0$, $\phi(t)$ is well defined and uniquely determined as $0 < \phi < \frac{1}{2}\hat{\pi}$ in (a, c) and $\phi(c) = \frac{1}{2}\hat{\pi}$. We easily see that u is twice continuously differentiable in (a, c), because $u' \neq 0$ in (a, c). Then, ρ and ϕ are continuously differentiable in (a, c). Differentiating (2.6) with respect to t, we obtain

$$S(\phi)\rho' + \rho S'(\phi)\phi' = \rho S'(\phi). \tag{2.9}$$

Substituting (2.6) and (2.7) into (1.1), yields

$$(\rho^{m-1}(S'(\phi))^{m-1})' + p(t)\rho^{m-1}(S(\phi))^{m-1}\left(\frac{f(u)}{u^{m-1}}\right) = 0.$$

By using (2.3), we obtain

$$(S'(\phi))^{m-1}\rho' - \rho(S(\phi))^{m-1}\phi' = -\frac{p(t)}{m-1}\left(\frac{f(u)}{u^{m-1}}\right)\rho(S(\phi))^{m-1}. \tag{2.10}$$

Multiplying (2.9) and (2.10) by $(S'(\phi))^{m-1}$ and $-S(\phi)$, respectively, and adding them, we have

$$\phi'(t) = (S'(\phi(t)))^m + \frac{p(t)}{m-1}\left(\frac{f(u)}{u^{m-1}}\right)(S(\phi))^m, \quad a < t < c. \tag{2.11}$$

By virtue of (2.6), the identity (2.11) is equivalent to

$$\phi'(t) = (S'(\phi(t)))^m + \frac{p(t)}{m-1}\left(\frac{f(u)}{\rho^{m-1}}\right)S(\phi), \quad a < t < c.$$

Then we observe that $\phi'(a) = \lim_{t\to a}\phi'(t) = 1$.

Here we prepare the following lemma.

Lemma 2. *Let* $c \in (a, b]$ *and let* u *be a positive solution of* (1.1) *such that* $u(a) = u'(c) = 0$. *Define* ρ *and* ϕ *by* (2.6)–(2.8). (*Then,* ϕ *satisfies* $\phi'(a) = 1$ *and* (2.11).) *Assume that* ψ *satisfies* $\psi(a) = 0, \psi'(a) = 1$, *and*

$$\psi'(t) < (S'(\psi))^m + \frac{p(t)}{m-1}\left(\frac{f(u)}{u^{m-1}}\right)(S(\psi))^m, \quad a < t < c. \tag{2.12}$$

Then

$$\phi(t) > \psi(t), \quad a < t \le c. \tag{2.13}$$

Proof. The proof is by contradiction. Suppose (2.13) does not hold. Then, there exists $t_0 \in (a, c]$ such that $\phi(t_0) \le \psi(t_0)$. First we show that

$$\phi(t) \le \psi(t), \quad a \le t \le t_0. \tag{2.14}$$

Suppose to the contrary that $\phi(t_1) > \psi(t_1)$ for some $t_1 \in (a, t_0)$. Then, there exists $t_2 \in (t_1, t_0]$ satisfying $\phi(t) > \psi(t)$ for $t_1 < t < t_2$ and $\phi(t_2) = \psi(t_2)$. This implies $\phi'(t_2) \le \psi'(t_2)$. On the other hand, from (2.11) and (2.12), we have $\psi'(t_2) < \phi'(t_2)$. We have a contradiction. Thus, (2.14) has to hold. Since S' is decreasing in $(0, \frac{1}{2}\hat{\pi})$, from (2.11) and (2.12), we have

$$\psi'(t) - \phi'(t) < \frac{p(t)}{m-1}\left(\frac{f(u)}{u^{m-1}}\right)[(S(\psi))^m - (S(\phi))^m], \quad a < t < t_0.$$

According to the mean value theorem,

$$(S(\psi))^m - (S(\phi))^m \le m(S(\theta))^{m-1}S'(\theta)[\psi(t) - \phi(t)]$$

for some $\theta(t) \in [\phi(t), \psi(t)]$. Since S is increasing in $(0, \frac{1}{2}\hat{\pi})$ and $S' \le 1$, we have the following differential inequality

$$[\psi(t) - \phi(t)]' < \frac{mp(t)}{m-1}\left(\frac{f(u)}{\rho^{m-1}}\right)\frac{(S(\psi))^{m-1}}{(S(\phi))^{m-1}}[\psi(t) - \phi(t)], \quad a < t < t_0. \tag{2.15}$$

Since $\lim_{t\to a}(S(\psi(t)))^{m-1}/(S(\phi(t)))^{m-1}$ exists and is finite, the function appearing in the right-hand of (2.15) is continuous on $[a, t_0]$. Solving the differential inequality (2.15), we have $\psi(t) - \phi(t) < 0$, $a < t < t_0$, which contradicts (2.14). Therefore, (2.13) holds. This completes the proof of Lemma 2.

3. Comparison results.

In this section we establish comparison results which are crucial for the proofs of uniqueness theorems in Section 4.

Proposition 1. *Let* $c \in (a, b]$ *and let* u_1 *be a positive solution of* (1.1) *satisfying* $u_1(a) = u_1'(c) = 0$. *Assume that* u_2 *is a solution of* (1.1) *such that* $u_2(a) = 0$ *and* $u_2'(a) > u_1'(a)$. *Then*

$$u_2(t) > u_1(t) \quad \text{for} \ \ a < t \le c \ \ \text{and} \ \ u_2'(t) > 0 \ \ \text{for} \ \ a \le t \le c. \tag{3.1}$$

By using a reflective argument, we have the following proposition.

Proposition 2. *Let $c \in [a,b)$ and let u_1 be a positive solution of* (1.1) *satisfying $u_1'(c) = u_1(b) = 0$. Assume that u_2 is a solution of* (1.1) *such that $u_2(b) = 0$ and $u_2'(b) < u_1'(b)$. Then*

$$u_2(t) > u_1(t) \quad \text{for} \quad c \le t < b \quad \text{and} \quad u_2'(t) < 0 \quad \text{for} \quad c \le t \le b.$$

We need the next lemma for the proof of Proposition 1.

Lemma 3. *Let u_1 and u_2 be solutions of* (1.1) *appearing in Proposition* 1*. Assume that*

$$u_2(t) > u_1(t), \qquad a < t \le \xi, \tag{3.2}$$

for some $\xi \in (a,c]$. Then

$$u_2'(t) > u_1'(t), \qquad a \le t \le \xi. \tag{3.3}$$

Proof. The proof is by contradiction. Suppose (3.3) does not hold. Then there exists $\eta \in (a, \xi]$ such that

$$u_2'(t) > u_1'(t), \quad a \le t < \eta \quad \text{and} \quad u_2'(\eta) = u_1'(\eta). \tag{3.4}$$

As in Section 2, we introduce the generalized polar coordinates ρ_i and ϕ_i $(i = 1,2)$ by

$$\begin{aligned} u_i(t) &= \rho_i(t) S(\phi_i(t)), \\ u_i'(t) &= \rho_i(t) S'(\phi_i(t)), \\ \phi_i(a) &= 0. \end{aligned} \tag{3.5}$$

Then ϕ_i $(i = 1,2)$ fulfills $\phi_i'(a) = 1$ and

$$\phi_i'(t) = (S'(\phi_i))^m + \frac{p(t)}{m-1}\Big(\frac{f(u_i)}{u_i^{m-1}}\Big)(S(\phi_i))^m, \quad a < t < \eta.$$

From (1.3) and (3.2) we have

$$\frac{f(u_1(t))}{(u_1(t))^{m-1}} > \frac{f(u_2(t))}{(u_2(t))^{m-1}}, \quad a < t \le \eta.$$

Then ϕ_2 satisfies

$$\phi_2'(t) < (S'(\phi_2))^m + \frac{p(t)}{m-1}\Big(\frac{f(u_1)}{u_1^{m-1}}\Big)(S(\phi_2))^m, \quad a < t < \eta.$$

Applying Lemma 2 with $\phi = \phi_1$ and $\psi = \phi_2$, we obtain $\phi_1(t) > \phi_2(t)$ for $a < t \le \eta$. By virtue of $0 < \phi_2(\eta) < \phi_1(\eta) \le \frac{1}{2}\hat{\pi}$, we see that

$$S(\phi_1(\eta)) > S(\phi_2(\eta)) \quad \text{and} \quad S'(\phi_2(\eta)) > S'(\phi_1(\eta)) \ge 0. \tag{3.6}$$

From (3.2), (3.5), and (3.6), we have $\rho_1(\eta) < \rho_2(\eta)$, which implies that

$$u_1'(\eta) = \rho_1(\eta)S'(\phi_1(\eta)) < \rho_2(\eta)S'(\phi_2(\eta)) = u_2'(\eta).$$

This contradicts the assumption (3.4). Hence (3.3) has to hold. This completes the proof of Lemma 3.

Proof of Proposition 1. First we show the left part of (3.1). By assumptions, we have $u_1(t) < u_2(t)$ in a right-neighborhood of $t = a$. Suppose that

$$u_1(t) < u_2(t), \quad a < t < \zeta \quad \text{and} \quad u_1(\zeta) = u_2(\zeta) \tag{3.7}$$

for some $\zeta \in (a, c]$. By Lemma 3, we have $u_1'(\xi) < u_2'(\xi)$ for any $\xi \in (a, \zeta)$. It follows that

$$u_1(\zeta) = \int_a^\zeta u_1'(\xi)d\xi < \int_a^\zeta u_2'(\xi)d\xi = u_2(\zeta),$$

which contradicts (3.7). Thus, the left part of (3.1) holds. Using again Lemma 3 with $\xi = c$, we obtain $u_2'(t) > u_1'(t) \geq 0$ for $a \leq t \leq c$, which implies the right part of (3.1). This completes the proof of Proposition 1.

4. Uniqueness theorems. First we prepare the following lemma.

Lemma 4. *Let u be a positive solution of* (1.1) *in* (a, b) *satisfying* $u(a) = 0$ (*resp.* $u(b) = 0$). *Then,* $u'(a) > 0$ (*resp.* $u'(b) < 0$).

Proof. Since $u(t) > 0$ in (a, b), we have $u'(a) \geq 0$. Suppose that $u'(a) = 0$. Since $|u'|^{m-2}u'$ is decreasing by (1.1), we see that $u'(t) \leq 0$ for all $a \leq t \leq b$. Then, we have $u(t) \leq 0$ for $a \leq t \leq b$, which contradicts the assumption. Thus we have $u'(a) > 0$. For the case $u(b) = 0$, we can prove it analogously. □

We impose the following hypotheses:

- (Ha) For any $\lambda > 0$, a solution u of (1.1) satisfying the initial condition $u(a) = 0$ and $u'(a) = \lambda$ is determined uniquely as long as $u'(t) > 0$;
- (Hb) For any $\lambda < 0$, a solution u of (1.1) satisfying the initial condition $u(b) = 0$ and $u'(b) = \lambda$ is determined uniquely as long as $u'(t) < 0$.

The results concerning the uniqueness of solutions of the initial value problem for the case $f(u) = u^\gamma$ with $\gamma > 0$ may be found in a recent paper of Kitano and Kusano [5]. In the present paper we impose one of the following conditions (F1) and (F2) on f:

- (F1) f is locally Lipschitz continuous on $[0, \infty)$;
- (F2) $f(u)$ is nondecreasing in $u \in (0, \infty)$.

We obtain the following lemma, which is proved in the Appendix.

Lemma 5. *Assume that either* (F1) *or* (F2) *is satisfied. Then, hypotheses* (Ha) *and* (Hb) *are satisfied.*

First we consider the problem (1.1)-(1.2a).

Theorem 1. *Under the hypothesis* (Ha)*, boundary value problem* (1.1)–(1.2a) *has at most one positive solution.*

Proof. Suppose (1.1) – (1.2a) has more than one solution. Let u and v be two distinct positive solutions. By virtue of Lemma 4 and (Ha), we see that $u'(a) \neq v'(a)$. We may assume without loss of generality that $v'(a) > u'(a)$. Applying Proposition 1 with $c = b$, $u_1 = u$, and $u_2 = v$, we have $v'(b) > 0$. This contradicts the assumption that v satisfies (1.2a). Thus the proof of Theorem 1 is complete. □

By using Proposition 2, we can prove the following theorem in a similar fashion as in Theorem 1.

Theorem 2. *Under the hypothesis* (Hb)*, boundary value problem* (1.1)–(1.2b) *has at most one positive solution.*

Next we consider the problem (1.1)–(1.2c).

Theorem 3. *Under the hypotheses* (Ha) *and* (Hb), *boundary value problem* (1.1)–(1.2c) *has at most one positive solution.*

Proof. Suppose that (1.1) – (1.2c) has more than one solution. Let u and v be two distinct positive solutions. By virtue of Lemma 4, (Ha) and (Hb), either $u'(a) \neq v'(a)$ or $u'(b) \neq v'(b)$ holds. We may assume that $u'(b) \neq v'(b)$ and $u'(b) > v'(b)$. Define c_u and c_v so that $u'(c_u) = v'(c_v) = 0$. Since $|u'|^{m-2}u'$ and $|v'|^{m-2}v'$ are decreasing in (a, b), c_u and c_v are determined uniquely. Applying Proposition 2 with $c = c_u$, $u_1 = u$, and $u_2 = v$, we have

$$v(t) > u(t), \quad c_u \leq t < b \quad \text{and} \quad v'(t) < 0, \quad c_u \leq t \leq b. \tag{4.1}$$

Then, we see that $c_v < c_u$ and

$$u(c_v) < v(c_v). \tag{4.2}$$

For the case $u'(a) < v'(a)$, applying Proposition 1 with $c = c_u$, $u_1 = u$ and $u_2 = v$, we have $v'(c_u) > 0$. This contradicts the right part of (4.1). For the case $u'(a) > v'(a)$, applying Proposition 1 with $c = c_v$, $u_2 = u$ and $u_1 = v$, we have $u(c_v) > v(c_v)$. This contradicts (4.2). Finally, for the case $u'(a) = v'(a)$, we have $c_u = c_v$ by (Ha). This contradicts $c_u > c_v$. Therefore, the proof of Theorem 3 is complete. □

By virtue of Lemma 5, we obtain the following corollary.

Corollary 1. *Assume that either* (F1) *or* (F2) *is satisfied. Then each of the problems* (1.1)–(1.2a), (1.1)–(1.2b)*, and* (1.1)–(1.2c) *has at most one positive solution.*

Appendix.

Proof of Lemma 5. We only consider the hypothesis (Ha). The hypothesis (Hb) is treated similarly. We break the proof into the cases (F1) and (F2).

(i) *The case where* (F1) *holds.* The scalar equation (1.1) can be written as the system

$$v' = |w|^{\frac{1}{m-1}-1}w,$$
$$w' = -p(t)f(v),$$

where $u = v$ and $w = |u'|^{m-2}u'$. Since $|w|^{\frac{1}{m-1}-1}w$ is locally Lipschitz continuous on $(-\infty,\infty)\backslash\{0\}$, a solution (v,w) satisfying $v(a) = 0$, $w(a) > 0$ is determined uniquely as long as $w > 0$. Therefore, we observe that hypothesis (Ha) is satisfied under condition (F1).

(ii) *The case where* (F2) *holds.* First we show that f is locally Lipschitz continuous on $(0,\infty)$. Let $u > v > 0$. From (1.3) we have $v^{m-1}f(u) \le u^{m-1}f(v)$. It follows that

$$(f(u) - f(v))u^{m-1} \le f(u)(u^{m-1} - v^{m-1}).$$

Then we obtain

$$0 \le f(u) - f(v) \le \frac{f(u)}{u^{m-1}}(u^{m-1} - v^{m-1}) \le (m-1)\frac{f(u)}{u^{m-1}}w^{m-2}(u-v), \tag{A.1}$$

where $w = u$ if $m \ge 2$ and $w = v$ if $1 < m < 2$. Let $u_2 > u_1 > 0$. For any $u, v \in (u_1,u_2)$, we have, from (A.1),

$$|f(u) - f(v)| \le (m-1)\delta\frac{f(u_1)}{u_1}|u-v|, \tag{A.2}$$

where $\delta = (u_2/u_1)^{m-2}$ if $m \ge 2$ and $\delta = 1$ if $1 < m < 2$. This implies that f is locally Lipschitz continuous on $(0,\infty)$.

By the argument of (i), it is clear that the uniqueness of the local solution of (1.1) satisfying $u(t_0) > 0$ and $u'(t_0) \ne 0$, $t_0 \in (a,b)$ holds. Thus, under the condition (F2), the question concerns the uniqueness in a right-neighborhood of a.

Let u_1 and u_2 be local solutions of (1.1) satisfying $u(a) = 0$ and $u'(a) = \lambda > 0$. There exists a number $t_1 \in (a,b]$ such that

$$\frac{\lambda}{2} \le u_i'(t) \le \lambda \quad \text{and} \quad \frac{\lambda}{2}(t-a) \le u_i(t) \le \lambda(t-a), \quad a \le t \le t_1, \quad i = 1,2. \tag{A.3}$$

We note that u_1 and u_2 satisfy

$$u_i'(t) = \left(\lambda^{m-1} - \int_a^t p(s)f(u_i(s))ds\right)^{\frac{1}{m-1}}, \quad a \le t \le t_1,\ i = 1,2, \tag{A.4}$$

and

$$u_i(t) = \int_a^t \left(\lambda^{m-1} - \int_a^s p(r)f(u_i(r))dr\right)^{\frac{1}{m-1}} ds, \quad a \le t \le t_1,\ i = 1,2.$$

From (A.3) and (A.4) we see that

$$\frac{1}{2}\lambda \le \left(\lambda^{m-1} - \int_a^t p(s)f(u_i(s))ds\right)^{\frac{1}{m-1}} \le \lambda, \quad a \le t \le t_1,\ i = 1,2.$$

By using the mean value theorem, we have

$$|u_1(t) - u_2(t)| \le \frac{c_1}{m-1}\int_a^t (t-s)p(s)|f(u_1(s)) - f(u_2(s))|ds, \quad a \le t \le t_1, \tag{A.5}$$

where $c_1 = (\lambda/2)^{\frac{2-m}{m-1}}$ if $m \ge 2$ and $c_1 = \lambda^{\frac{2-m}{m-1}}$ if $1 < m < 2$. By virtue of (A.2) we have

$$|f(u_1(s)) - f(u_2(s))| \le (m-1)c_2\frac{f(\frac{\lambda}{2}(s-a))}{s-a}|u_1(s) - u_2(s)|, \quad a < t \le t_1,$$

where $c_2 = 2^{m-1}/\lambda$ if $m \ge 2$ and $c_2 = 2/\lambda$ if $1 < m < 2$. Then it follows that

$$|u_1(t) - u_2(t)| \le c_1c_2(t-a)\int_a^t p(s)f(\frac{\lambda}{2}(s-a))\frac{|u_1(s) - u_2(s)|}{s-a}ds, \quad a < t \le t_1,$$

or equivalently,

$$\frac{|u_1(t)-u_2(t)|}{t-a} \le c_1c_2\int_a^t p(s)f(\frac{\lambda}{2}(s-a))\frac{|u_1(s)-u_2(s)|}{s-a}ds, \quad a<t\le t_1.$$

Notice here that the function $v(t) = |u_1(t) - u_2(t)|/(t-a)$ has the finite limit 0 as $t \to a+0$, and hence $v(t)$ can be regarded as a continuous function on the closed interval $[a, t_1]$. Then by Gronwall's inequality we see that $v(t) = 0$ for $a \le t \le t_1$, which implies that $u_1(t) \equiv u_2(t)$ for $a \le t \le t_1$. Thus, the uniqueness in a right-neighborhood of a has been proved. This completes the proof for the case (F2).

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