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# STABILITY OF MULTIDIMENSIONAL TRAVELING WAVES FOR A BENJAMIN-BONA-MAHONY TYPE EQUATION

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**Abstract.** In this paper we show that for a convenient choice of the nonlinear map $a(u)$ the equation $u_t+\operatorname{div}(a(u))-\Delta u_t=0$ has traveling waves solution $\phi_c(x-\vec{c}t)$, where $\vec{c}=(c,\dots,c)\in\mathbb{R}^n$. For $c$ varying in a suitable interval we show that these traveling waves are stable.

**1. Introduction.** In this paper we study the existence and stability of traveling waves for the equation

$$u_t+\operatorname{div}(a(u))-\Delta u_t=0 \qquad \text{in} \quad \mathbb{R}^n\times\mathbb{R}, \tag{1.1}$$

where $\Delta$ is the usual Laplacian operator in $\mathbb{R}^n$, $a:\mathbb{R}\to\mathbb{R}^n$ is a smooth map and $\operatorname{div}(a(u))=\sum_{j=1}^n\frac{\partial}{\partial x_j}\,(a_j(u))$. Equation (1.1) is a straightforward extension of the generalized Benjamin-Bona-Mahony (GBBM) equation

$$u_t+u_x+u^pu_x-u_{xxt}=0 \qquad \text{in} \quad \mathbb{R}\times\mathbb{R} \tag{1.2}$$

to higher dimensions and it also can be interpreted as a regularization of a single conservation law in several space dimensions. Equation (1.2) with $p=1$ models long waves with small amplitude and it has been proposed as an alternative model for the Korteweg-de Vries (KdV) equation $u_t+u_x+uu_x+u_{xxx}=0$.

A traveling wave (also called solitary wave) of equation (1.1) is a classical solution of (1.1) of the form $u(x,t)=\phi(x-\vec{c}t)$, where $\vec{c}$ is a constant vector of $\mathbb{R}^n$ and $\phi:\mathbb{R}^n\to\mathbb{R}$ is a function satisfying $\phi(y)\to 0$ as $|y|\to+\infty$.

For example, the equation (1.2) has a one-parameter family of traveling waves given explicitly by $u_c(x,t)=\phi_c(x-ct)$, where $c>1$ and

$$\phi_c(y)=\Big\{\frac{1}{2}(p+1)(p+2)(c-1)\operatorname{sech}^2\Big[\frac{p}{2}\Big(\frac{c-1}{c}\Big)^{\frac{1}{2}}y\Big]\Big\}^{\frac{1}{p}}.$$

Results concerning stability of traveling waves for the BBM and KdV equations were given first by Bona [4] and Benjamin [3]. They proved that the traveling waves for these equations are stable for all values of $c>1$. However, the same is not true for the GBBM and GKdV (generalized KdV) equations. In these cases the parameter $c$ (and hence

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$\phi_c$) must satisfy certain conditions (called *stability conditions*). For the existence and stability of traveling waves for GBBM, GKdV and other nonlinear dispersive evolution equations the reader is referred to [1], [5], [13] and [14].

In particular, for the equation (1.2), Weinstein in [14] proved that the traveling waves of (1.2) are stable for all $c > 1$ if $1 \leq p \leq 4$, while for $p > 4$ there is a $c^* = c^*(p) > 1$ for which the traveling waves are stable for all $c > c^*$. At the same time he conjectured that for $p > 4$ and $1 < c < c^*$ the corresponding traveling waves would be unstable. This conjecture and the above result were proved by Souganidis and Strauss [13] by using a different approach. Weinstein also suggested in [14] that under suitable conditions on $p$ and $n$, analogous results could be expected for equation (1.1).

In this paper we show that for a convenient choice of the map $a(s) = (a_1(s), \ldots, a_n(s))$ equation (1.1) has a one-parameter family of multidimensional traveling waves and we present a result on stability which is analogous to those of Weinstein [14] and Souganidis and Strauss [13] for the case $n = 1$. More precisely, we show that if

$$a_j(s) = a_0(s) \equiv c_0\, s + \frac{s^{p+1}}{p+1}, \qquad \forall j = 1, \ldots, n, \tag{1.3}$$

where $c_0 \geq 0$ and $p$ is a positive integer satisfying

$$p \leq \frac{2}{n-2} \quad \text{if} \quad n > 2 \qquad \text{and} \qquad p < +\infty \quad \text{if} \quad n \leq 2, \tag{1.4}$$

then, equation (1.1) has a family of traveling waves

$$u_c(x,t) = \phi_c(x - \vec{c}t), \quad \text{where} \quad \vec{c} = (c, \ldots, c) \in \mathbb{R}^n \quad \text{and} \quad c > c_0, \tag{1.5}$$

and the following holds:

(1) $\phi_c(x - \vec{c}t)$ is stable for all $c > 0$ if $c_0 = 0$;

(2) For $c_0 > 0$, $\phi_c(x - \vec{c}t)$ is stable for all $c > c_0$ if $p \leq \frac{4}{n}$, while for $p > \frac{4}{n}$ there is a $c^* = c^*(p) > c_0$ for which $\phi_c(x - \vec{c}t)$ is stable for all $c > c^*$.

Results (1) and (2) are consequences of

(3) If $c > c_0$ satisfies $q'(c) > 0$ then the corresponding traveling wave $\phi_c(x - \vec{c}t)$ is stable.

In (3), $q'(c)$ denotes the derivative with respect to $c$ of the function $q(c) = Q(\phi_c)$, where $Q$ is the functional $Q(w) = \frac{1}{2}\|w\|_1^2$ (which is conserved for solutions of (1.1)). The *stability condition* $q'(c) > 0$ appears in [14] in the one-dimensional case.

Results (1), (2) and (3) above improve results obtained by the author in [11]. To prove them we partially follow the approach of [14] (see also [15]). Several arguments used in the proof of (3) are natural adaptations of the corresponding ones in the one-dimensional case. The main difficulty is to prove that the kernel of the *linearized operator* $\mathcal{L}_c$ which arises in the stability analysis of the traveling waves is spanned by the functions $\frac{\partial \phi_c}{\partial x_j}$, $j = 1, \ldots, n$. This fact was partially proved by Weinstein in [16], but his proof can be extended for all dimensions by using ideas developed recently by Kwong [9].

We organize this paper as follows. In Section 2, we will show how traveling waves for equation (1.1) can be obtained under restrictions (1.3), (1.4) and (1.5). We also recall some facts concerning the existence of solutions of the initial value problem for equation (1.1) which will be used in this paper. Section 3 will be devoted to the study of the spectrum of $\mathcal{L}_c$. The above mentioned property of the kernel of $\mathcal{L}_c$ will be proved. In Section 4, we will prove the stability result (3) and hence (1) and (2) by assuming a crucial step, that is, Lemma 4.5 postponed to Section 5. Finally, in Section 5 we will use the spectral properties of $\mathcal{L}_c$ established in Section 3 to prove Lemma 4.5.

The following notations will be used throughout this paper. The norm of the space $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, will be denoted by $|\ |_p$. For $p = 2$ we will write $(\ ,\ )$ for the inner product of $L^2(\mathbb{R}^n)$ and $\|\ \|$ for its corresponding norm. $\|\ \|_m$ and $(\ ,\ )_m$ will denote the norm and the inner product of the real Sobolev space $H^m(\mathbb{R}^n)$, respectively. We will denote by $C^k(\mathbb{R}^n)$ the space of all continuous functions $u : \mathbb{R}^n \to \mathbb{R}$ whose partial derivatives up to order $k$ are continuous. $C^k(I; X)$ will denote the space of continuous maps of the real interval $I$ with values in the Banach space $X$ whose derivatives up to order $k$ are continuous. We will write $C^k(I; \mathbb{R}) = C^k(I)$; $\Delta$ and $\nabla$ are the usual Laplacian operator and the gradient in $\mathbb{R}^n$, respectively.

**2. The evolution equation and existence of solutions.** In this section we will discuss the existence of traveling waves for equation (1.1) and we will recall some facts concerning the initial value problem for (1.1). Let us initially consider the equation

$$u_t + \operatorname{div}(a(u)) - \Delta u_t = 0 \qquad \text{in} \quad \mathbb{R}^n \times \mathbb{R}, \tag{2.1}$$

where $a : \mathbb{R} \to \mathbb{R}^n$ is a $C^1$ map, $a(s) = (a_1(s), \dots , a_n(s))$, and

$$\operatorname{div}(a(u)) = \sum_{i=1}^{n} \frac{\partial}{\partial x_j}(a_j(u))$$

and try to find solutions of the form

$$u(x, t) = \phi(x - \vec{c}t), \tag{2.2}$$

where $\vec{c} = (c_1, \dots , c_n) \in \mathbb{R}^n$ and $\phi : \mathbb{R}^n \to \mathbb{R}$ is smooth and satisfies $\phi(y) \to 0$ as $|y| \to +\infty$.

Substitution of (2.2) into (2.1) yields

$$\sum_{j=1}^{n} \Big\{ -c_j \frac{\partial \phi}{\partial x_j}(x - \vec{c}t) + c_j \frac{\partial}{\partial x_j} \Delta\phi(x - \vec{c}t) + \frac{\partial}{\partial x_j}\big[a_j(\phi(x - \vec{c}t))\big] \Big\} = 0. \tag{2.3}$$

If we choose the map $a(s)$ as in (1.3), (1.4) and $\vec{c} = (c, \dots , c) \in \mathbb{R}^n$, with $c > c_0$ ($c_0 \ge 0$ fixed), then (2.3) implies

$$\sum_{j=1}^{n} \frac{\partial}{\partial x_j} \Big( -c\phi(x - \vec{c}t) + c\Delta\phi(x - \vec{c}t) + c_0\phi(x - \vec{c}t) + \frac{1}{p+1}\phi^{p+1}(x - \vec{c}t) \Big) = 0.$$

Thus, in order that (2.2) with the above choice of $\vec{c}$ defines a traveling wave of (2.1) it is sufficient that $\phi$ solves the elliptic problem

$$\begin{cases} c\Delta\phi - c\phi + c_0\phi + \frac{\phi^{p+1}}{p+1} = 0, & \text{in} \quad \mathbb{R}^n \\ \phi \in C^3(\mathbb{R}^n), \;\; \phi \neq 0 \quad \text{and} & \phi(y) \to 0, \quad \text{as} \quad |y| \to +\infty. \end{cases} \tag{2.4}$$

To solve (2.4) we initially consider the problem

$$\begin{cases} -\Delta\psi + \psi - \frac{\psi^{p+1}}{p+1} = 0, & \text{in} \quad \mathbb{R}^n \\ \psi \neq 0, \qquad \psi \in H^1(\mathbb{R}^n). \end{cases} \tag{2.5}$$

The existence of a positive radially symmetric solution of (2.5) can be obtained from the minimization problem

$$\begin{cases} \psi_0 \neq 0, \qquad \psi_0 \in H^1(\mathbb{R}^n) \\ J(\psi_0) = \inf\limits_{\substack{\psi \in H^1(\mathbb{R}^n) \\ \psi \neq 0}} J(\psi), \end{cases} \tag{2.6}$$

where $J$ is the functional defined on $H^1(\mathbb{R}^n)\backslash\{0\}$ by

$$J(\psi) = \frac{\frac{1}{2}\int_{\mathbb{R}^n} |\nabla\psi|^2 dx + \frac{1}{2}\int_{\mathbb{R}^n} \psi^2 dx}{\left(\int_{\mathbb{R}^n} |\psi|^{p+2} dx\right)^{\frac{2}{p+2}}}. \tag{2.7}$$

In (2.7) we assume that $p$ satisfies (1.4). Problem (2.6) can be solved by using the concentration-compactness method due to P.L. Lions [10]. In fact we have the following results concerning the existence of solutions of (2.6) and (2.5).

**Theorem 2.1.** *The minimization problem* (2.6) *has a solution* $\psi_0 \in H^1(\mathbb{R}^n)$ *with the following properties:*

(i) $\psi_0 > 0$*;*
(ii) $\psi_0$ *is radially symmetric, that is,* $\psi_0(x) = \psi_0(|x|)$*;*
(iii) $\psi_0$ *is strictly decreasing with* $r = |x|$*.*

*Moreover, if*

$$\gamma_0 = \frac{\int_{\mathbb{R}^n} |\nabla\psi_0|^2 dx + \int_{\mathbb{R}^n} \psi_0^2 dx}{\int_{\mathbb{R}^n} \psi_0^{p+2} dx}, \qquad \text{then}$$

(iv) $\psi = \gamma_0^{\frac{1}{p}}(p+1)^{\frac{1}{p}}\psi_0$ *is a solution of* (2.5).

**Proof.** See Theorems 3.1, 3.2 and 7.1 in Weinstein [14]. □

By regularity and decaying properties of solutions of (2.5) it follows that $\psi \in C^3(\mathbb{R}^n)$ and there exists positive constants $K$ and $\delta$ such that

$$|D^\alpha\psi(x)| \leq Ke^{-\delta|x|}, \qquad \forall x \in \mathbb{R}^n, \;\; \forall|\alpha| \leq 2. \tag{2.8}$$

We now define, for $c > c_0$,

$$\phi_c(x) = (c - c_0)^{\frac{1}{p}}\psi\Big(\big(1 - \frac{c_0}{c}\big)^{\frac{1}{2}}x\Big), \quad \forall x \in \mathbb{R}^n. \tag{2.9}$$

The functions $\phi_c$ are solutions of (2.4). In fact we have

**Theorem 2.2.** *The function $\phi_c$ defined in* (2.9) *is a solution of* (2.4)*. Moreover, if $\Lambda(c) = \phi_c$ for all $c > c_0$, then $\Lambda \in C^1\big((c_0, +\infty);\ H^1(\mathbb{R}^n)\big) \cap C^0\big((c_0, +\infty);\ H^2(\mathbb{R}^n)\big)$.*

**Proof.** By (2.8) it follows that $\phi_c \in H^2(\mathbb{R}^n)$ and $\phi_c(x) \to 0$ as $|x| \to +\infty$. A simple calculation using (2.8) and Theorem 2.1 shows that $\phi_c$ solves the differential equation in (2.4). Now, for $c > c_0$ we define $w_c(x) = \mu'(c)\psi(\nu(c)x) + \mu(c)\nu'(x)x \cdot \nabla\psi(\nu(c)x)$, for all $x \in \mathbb{R}^n$, where $\mu(c) = (c - c_0)^{\frac{1}{p}}$ and $\nu(c) = (1 - \frac{c_0}{c})^{\frac{1}{2}}$. Clearly $w_c \in H^1(\mathbb{R}^n)$. Fixing $c > c_0$, using properties of the functions $\nu(s)$, $\mu(s)$, $s \to \psi(\nu(s)x)$, $s \to \frac{\partial\psi}{\partial x_j}\ (\nu(s)x)$, (2.8) and the Lebesgue dominated convergence theorem we deduce that

$$\Big\|\frac{1}{h}(\Lambda(c+h) - \Lambda(c)) - w_c\Big\|^2 \to 0 \quad \text{and}$$
$$\Big\|\frac{1}{h}\Big(\frac{\partial}{\partial x_j}\Lambda(c+h) - \frac{\partial}{\partial x_j}\Lambda(c)\Big) - \frac{\partial w_c}{\partial x_j}\Big\|^2 \to 0, \tag{2.10}$$

for all $j = 1, \dots, n$, as $h \to 0$. From (2.10) we deduce that the map $\Lambda : (c_0, +\infty) \to H^1(\mathbb{R}^n)$ is differentiable with derivative $\Lambda'(c) = w_c$. Moreover, analogous arguments show that the map $c \to \Lambda'(c)$ is continuous and that $\Lambda \in C^0((c_0, +\infty);\ H^2(\mathbb{R}^n))$. □

Now, for $\phi_c$ as in (2.9) we define the linear operator on $L^2(\mathbb{R}^n)$, called the *linearized operator* associated to equation (2.4),

$$D(\mathcal{L}_c) = H^2(\mathbb{R}^n), \qquad \mathcal{L}_c w = -\Delta w + \Big(1 - \frac{c_0}{c}\Big)\, w - \frac{1}{c}\, \phi_c^p w. \tag{2.11}$$

The following property will be used in the next section.

**Lemma 2.3.**

$$(\mathcal{L}_c h, h) \geq -\frac{p}{c(p+1)}\, \frac{(\phi_c^{p+2}, h)}{|\phi_c|_{p+2}^{p+2}} \qquad \text{for all} \quad h \in D(\mathcal{L}_c).$$

**Proof.** Since $J$ attains its minimum value at $\psi_0$ by Theorem 2.1, then $\langle J''(\psi_0) \cdot h, h\rangle \geq 0$ for all $h \in H^1(\mathbb{R}^n)$. Thus, applying Theorem 2.1 (iv) we obtain after some calculations

$$\int_{\mathbb{R}^n} |\nabla h|^2 dx + \int_{\mathbb{R}^n} h^2 dx - 4\Big(\int_{\mathbb{R}^n} \nabla\psi \cdot \nabla h\, dx + \int_{\mathbb{R}^n} \psi h\, dx\Big) \frac{\int_{\mathbb{R}^n} \psi^{p+1} h\, dx}{\int_{\mathbb{R}^n} \psi^{p+2} dx}$$
$$- \int_{\mathbb{R}^n} \psi^p h^2 dx + \frac{p+4}{p+2}\, \frac{\Big(\int_{\mathbb{R}^n} \psi^{p+1} h\, dx\Big)^2}{\int_{\mathbb{R}^n} \psi^{p+2} dx} \geq 0, \qquad \forall h \in H^1(\mathbb{R}^n). \tag{2.12}$$

Observing that $\psi(x) = (c - c_0)^{-\frac{1}{p}} \phi_c(\frac{x}{\nu(c)})$, where $\nu(c) = (1 - \frac{c_0}{c})^{\frac{1}{2}}$, we can replace $h(x)$ by $h(\frac{x}{\nu(c)})$ in (2.12) and make the change of variable $y = \frac{x}{\nu(c)}$ to obtain

$$\int_{\mathbb{R}^n} |\nabla h|^2 dy + \Big(1 - \frac{c_0}{c}\Big) \int_{\mathbb{R}^n} h^2 dy - \frac{1}{c} \int_{\mathbb{R}^n} \phi_c^p h^2 dy + \frac{p}{c(p+1)} \frac{\int_{\mathbb{R}^n} \phi_c^{p+2} h\, dy}{\int_{\mathbb{R}^n} \phi_c^{p+2} dy}$$
$$- \frac{4 \int_{\mathbb{R}^n} \phi_c^{p+1} h\, dy}{\int_{\mathbb{R}^n} \phi_c^{p+2} dy} \Big(\int_{\mathbb{R}^n} \Big( - \Delta\phi_c + \Big(1 - \frac{c_0}{c}\Big)\phi_c - \frac{1}{c} \frac{\phi_c^{p+1}}{p+1}\Big) h\, dy\Big) \geq 0,$$

for all $h \in H^1(\mathbb{R}^n)$, which implies the conclusion of Lemma 2.3. $\square$

Next, we will consider the initial value problem

$$\begin{cases} u_t + \operatorname{div}(a(u)) - \Delta u_t = 0, & x \in \mathbb{R}^n, \ \ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{2.13}$$

where $a(s) = (a_1(s), \ldots, a_n(s))$ is as in (1.3), (1.4). We also define the functionals on $H^1(\mathbb{R}^n)$,

$$\begin{aligned} Q(w) &= \frac{1}{2} \int_{\mathbb{R}^n} |\nabla w|^2 dx + \frac{1}{2} \int_{\mathbb{R}^n} w^2 dx, \\ E(w) &= \frac{c_0}{2} \int_{\mathbb{R}^n} |\nabla w|^2 dx - \frac{1}{(p+1)(p+2)} \int_{\mathbb{R}^n} w^{p+2} dx. \end{aligned} \tag{2.14}$$

In (2.14), $p$ satisfies (1.4). Note that $\operatorname{div}(a(u)) = \sum_{j=1}^{n} (c_0 \frac{\partial u}{\partial x_j} + u^p \frac{\partial u}{\partial x_j})$. We recall the following.

**Theorem 2.4.** *Suppose that $a(\cdot)$ satisfies* (1.3) *and* (1.4). *Then,*

(i) *for any $u_0 \in H^m(\mathbb{R}^n)$, $m = 1, 2$, the problem* (2.13) *has a unique solution $u \in C^1([0, +\infty);\ H^m(\mathbb{R}^n))$;*
(ii) *for each $T > 0$, the map $u_0 \to u(\cdot, t)$ from $H^1(\mathbb{R}^n)$ to $C^0([0,T]; H^1(\mathbb{R}^n))$ is continuous; and*
(iii) $E(u(\cdot,t)) = E(u_0)$, $\ Q(u(\cdot,t)) = Q(u_0)$ *for all $t \geq 0$.*

Theorem 2.4 (i)–(ii) can be deduced from the results established in [2] (see also [7]) and Theorem 2.4 (iii) can be proved by direct calculation.

**3. Spectral properties of the linearized operator.** We recall that the linearized operator is the (unbounded) operator $\mathcal{L}_c$ defined in (2.11). Our aim here is to establish some spectral properties of $\mathcal{L}_c$ which will be used in section 5. We will denote by Ker $\mathcal{L}_c$ the kernel of $\mathcal{L}_c$ and by $[\frac{\partial \phi_c}{\partial x_1}, \ldots, \frac{\partial \phi_c}{\partial x_n}]$ the subspace of $L^2(\mathbb{R}^n)$ spanned by $\frac{\partial \phi_c}{\partial x_1}, \ldots, \frac{\partial \phi_c}{\partial x_n}$.

**Lemma 3.1.**

(i) *There exists $\beta > 0$ such that $(\sigma(\mathcal{L}_c) \cap (0, +\infty)) \subset [\beta, +\infty)$;*
(ii) $\alpha = \inf \sigma(\mathcal{L}_c)$ *is a simple eigenvalue of $\mathcal{L}_c$ with a strictly positive eigenfunction;*
(iii) $\operatorname{Ker} \mathcal{L}_c \supset [\frac{\partial \phi_c}{\partial x_1}, \ldots, \frac{\partial \phi_c}{\partial x_n}]$. (3.1)

**Proof.** (i) follows from the fact that the essential spectrum of $\mathcal{L}_c$ is equal to $[1 - \frac{c_0}{c}, +\infty)$. (ii) follows from Theorem XIII.44 of [12] by observing that $e^{-t\mathcal{L}_c}$ is ergodic for some $t > 0$. To obtain (iii) we find the derivative of equation (2.4) with respect to $x_j$, $j = 1, \cdots, n$. $\square$

For the stability of traveling waves we need to guarantee that $\alpha$ is the unique negative eigenvalue of $\mathcal{L}_c$ and that the opposite inclusion in (3.1) is also valid. We will prove these facts in the next proposition, where $Y_{kj}(w)$, $k = 0, 1, \cdots; j = 1, \cdots, \ell(k,n)$ will denote the orthonormal basis of spherical harmonics of $L^2(S^{n-1})$.

**Proposition 3.2.**

(i) *Ker* $\mathcal{L}_c = [\frac{\partial \phi_c}{\partial x_1}, \cdots, \frac{\partial \phi_c}{\partial x_n}]$. (3.2)

(ii) $\alpha = \inf\ \sigma(\mathcal{L}_c)$ *is the unique negative eigenvalue of* $\mathcal{L}_c$.

**Proof.** (i) For $n = 1$ see [16]. Let us consider $n \geq 2$. Following [16] we take $w \in H^2(\mathbb{R}^n)$ such that $\mathcal{L}_c w = 0$ and expand it as $w = \sum_{k,j} f_{kj}(r) Y_{kj}(w)$, where $f_{kj} \in L^2(0, +\infty;\ r^{n-1}dr)$. Since the potential $V(x) = V(|x|) = 1 - \frac{c_0}{c} - \frac{1}{c}\ \phi_c^p(r)$ of $\mathcal{L}_c = -\Delta + V(x)$ is radial we conclude that $f_{kj}$ must satisfy

$$\Big( - \frac{d^2}{dr^2} - \frac{n-1}{r}\ \frac{d}{dr} + V(r) + \frac{\lambda_k}{r^2} \Big)\ f_{kj}(r) = 0, \tag{3.3}$$

where $\lambda_k = k(n-2+k)$. By the regularity properties of the solutions of $\mathcal{L}_c w = 0$ it follows that $f_{kj} \in C^2((0, +\infty))$. Thus, to prove (i) it is sufficient to show the following statement:

Equation (3.3) does not have nontrivial classical solutions in

$$L^2(0, +\infty;\ r^{n-1}dr) \qquad \text{for} \qquad k = 0, 2, 3, \dots . \tag{3.4}$$

The cases $k = 2, 3, \cdots$ were eliminated in [16]. Let us consider $k = 0$. In this case we put $f_{01}(r) = f(r)$ and rewrite (3.3) as

$$f''(r) + \frac{1}{r}\ f'(r) + \Big[\frac{1}{c}\ \phi_c^p(r) - \Big(1 - \frac{c_0}{c}\Big)\Big]\ f(r) = 0, \qquad 0 < r < +\infty. \tag{3.5}$$

By (2.4) we also have

$$\phi_c''(r) + \frac{1}{r}\ \phi_c'(r) + \Big[\frac{1}{c}\ \frac{\phi_c^p(r)}{p+1} - \Big(1 - \frac{c_0}{c}\Big)\Big]\ \phi_c(r) = 0, \qquad 0 < r < +\infty, \tag{3.6}$$

where $\phi_c(r) > 0$, $\phi_c'(r) < 0$, for all $0 < r < +\infty$ and $\phi_c(r) \to 0$ as $r \to +\infty$.

Suppose that $f \neq 0$. As in [9] there exists a unique $\xi \in (0, +\infty)$ satisfying $\phi_c(\xi) = [(p+1)(c - c_0)]^{\frac{1}{p}}$ and by the comparison of the equations (3.5), (3.6) we conclude that $f(r)$ has at least one zero in $(0, \xi)$. Let $\theta_c(r) = -\frac{r\phi_c'(r)}{\phi_c(r)}$, for all $0 \leq r < +\infty$. If we show that $\theta_c(r)$ is increasing on $[0, \xi]$ then Lemmas 24 and 25 of [9] will imply that $f(r)$ has a unique zero in $(0, \xi)$ and that $\lim_{r\to+\infty} |f(r)| = +\infty$. This contradicts the fact that $f \in L^2(0, +\infty; r\,dr)$. Thus (3.4) will follow. The increase of $\theta_c(r)$ on $[0, \xi]$ is immediate if $n = 2$ because in this case $(-r\phi_c'(r))' = -r\phi_c''(r) - \phi_c'(r) = \frac{r}{c}\left(\frac{\phi_c^{p+1}(r)}{p+1} - (c - c_0)\phi_c(r)\right) \geq 0$ for all $0 \leq r \leq \xi$ and $\phi_c(r)$ is strictly increasing on $[0, +\infty)$. For $n > 2$, by Lemma 3.1 of [6] (see condition [F5] on page 1555), we only have to show that there exists an $\overline{s} > 0$ satisfying

$$\Big(\frac{1}{c}\ \frac{4 + (2-n)p}{(p+1)(p+2)}\ s^{p+2} - \frac{2}{c}(c - c_0)s^2\Big)(s - \overline{s}) \geq 0 \qquad \text{for all} \quad s \geq 0.$$

This is our case because the continuous function

$$g(s) = \frac{1}{c}\,\frac{4+(2-n)p}{(p+1)(p+2)}\,s^{p+2} - \frac{2}{c}(c-c_0)s^2$$

has a unique positive zero $\overline{s} = (\frac{2(p+1)(p+2)(c-c_0)}{4+(2-n)p})^{\frac{1}{p}}$ and $\lim_{s\to+\infty} g(s) = +\infty$. This proves (i).

To prove (ii) we first observe that $\alpha < 0$ because when $n = 1$, $\frac{\partial \phi_c}{\partial x_1}$ changes sign at $x = 0$ and for $n \geq 2$ zero is not a simple eigenvalue of $\mathcal{L}_c$. Let $\lambda_2$ denote the second eigenvalue of $\mathcal{L}_c$. By Min-Max Principle and Lemma 2.3 we have

$$\lambda_2 = \sup_{v\in L^2(\mathbb{R}^n)} \inf_{\substack{w\in H^2(\mathbb{R}^n)\cap[v]^\perp\\ \|w\|=1}} (\mathcal{L}_c w, w) \geq \inf_{w\in H^2(\mathbb{R}^n)\cap[\phi_c^{p+1}]^\perp} (\mathcal{L}_c w, w) \;\geq\; 0.$$

This shows that $\alpha$ is the unique eigenvalue of $\mathcal{L}_c$.

**4. Stability of traveling waves.** In this section the stability results already mentioned in the introduction will be proved. By stability we mean

**Definition 4.1.** A traveling wave $\phi_c(x-\vec{c}t)$ of equation (2.1) is called *stable* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $u_0 \in H^1(\mathbb{R}^n)$ satisfies $\|u_0 - \phi_c\|_1 < \delta$ then the solution $u(\cdot,t)$ of (2.13) corresponding to $u_0$ satisfies

$$\inf_{y\in\mathbb{R}^n} \|u(\cdot,t) - \phi_c(\cdot - y)\|_1 < \varepsilon, \qquad \text{for all} \quad t \geq 0.$$

We will consider the function $q(c) = Q(\phi_c)$, for all $c > c_0$, where $Q$ is the functional defined in (2.14) and $\phi_c$ is as in (2.9). Our results are the following.

**Theorem 4.2.** *Let $a(\cdot)$ and $\vec{c}$ be chosen as in* (1.3), (1.4) *and* (1.5). *If $q'(c) > 0$ then the corresponding traveling wave $\phi_c(x-\vec{c}t)$ of* (2.1) *is stable.*

**Theorem 4.3.** *Let $a(\cdot)$ and $\vec{c}$ be chosen as in* (1.3), (1.4) *and* (1.5). *Then*

(i) *$\phi_c(x-\vec{c}t)$ is stable for all $c > 0$ if $c_0 = 0$;*

(ii) *For $c_0 > 0$ the traveling wave $\phi_c(x-\vec{c}t)$ is stable for all $c > c_0$ if $p \leq \frac{4}{n}$, while for $p > \frac{4}{n}$ there exists $c^* = c^*(p) > c_0$ for which $\phi_c(x-\vec{c}t)$ is stable for all $c > c^*$.*

Theorem 4.3 is an immediate consequence of Theorem 4.2. In fact, by the definition of $Q$ and (2.9), $q'(c) > 0$ is equivalent to

$$\frac{2}{p} > c_0\left(\frac{n-2}{2c} + \frac{1}{k(c-c_0)+c}\right),$$

where $k = \frac{\int_{\mathbb{R}^n} |\nabla\psi|^2 dx}{\int_{\mathbb{R}^n} \psi^2 dx}$, from which we deduce (i) and (ii) of Theorem 4.3.

To prove Theorem 4.2 we will use Lemma 4.4 and Lemma 4.5 below. The proof of Lemma 4.5 will be postponed to Section 5. In what follows $\phi_c$ is as in (2.9) and $u(\cdot,t)$ will denote the solution of (2.13) with initial data $u_0 \in H^1(\mathbb{R}^n)$.

**Lemma 4.4.** *If for some $t \geq 0$ there exists $z_0 = z_0(t) \in \mathbb{R}^n$ such that $\|u(\cdot + z_0, t) - \phi_c\|_1 \leq \|\phi_c\|_1$, then, there exists $z(t) \in \mathbb{R}^n$ satisfying*

$$\|u(\cdot + z(t), t) - \phi_c\|_1^2 = \inf_{y \in \mathbb{R}^n} \|u(\cdot + y, t) - \phi_c\|_1^2. \tag{4.1}$$

**Proof.** It follows from the fact that the function

$$\rho(y) = \|u(\cdot + y, t) - \phi_c\|_1^2 = \|u(\cdot, t) - \phi_c(\cdot - y)\|_1^2$$

is continuous on $\mathbb{R}^n$ and $\lim_{|y| \to +\infty} \rho(y) = \|u(\cdot, t)\|_1^2 + \|\phi_c\|_1^2$. □

In the next lemma $h = h(\cdot, t) = u(\cdot + z(t), t) - \phi_c$ and $F_c(\cdot, \cdot)$ is the bilinear form on $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ defined by

$$F_c(v, w) = \int_{\mathbb{R}^n} \nabla v \cdot \nabla w \, dx + \Big(1 - \frac{c_0}{c}\Big) \int_{\mathbb{R}^n} vw \, dx - \frac{1}{c} \int_{\mathbb{R}^n} \phi_c^p vw \, dx. \tag{4.2}$$

**Lemma 4.5.** *Suppose $q'(c) > 0$ and $q(c) = Q(u_0)$. Then there exists positive constants $A_1$, $A_2$, $A_3$ depending only on $c$ such that*

$$\frac{c}{2} F_c(h, h) \geq A_1 \|h\|_1^2 - A_2 \|h\|_1^3 - A_3 \|h\|_1^4. \tag{4.3}$$

**Proof of Theorem 4.2.** Let $c > c_0$ satisfy $q'(c) > 0$ and let $\varepsilon > 0$ given. First we will show that there exists $\delta > 0$ for which if

$$u_0 \in H^1(\mathbb{R}^n), \quad \|u_0 - \phi_c\|_1 < \delta \quad \text{and} \quad q(c) = Q(u_0), \tag{4.4}$$

then the corresponding solution $u(x, t)$ of (2.13) satisfies

$$\inf_{y \in \mathbb{R}^n} \|u(\cdot, t) - \phi_c(\cdot - y)\|_1 < \varepsilon, \qquad \forall\, 0 \leq t < +\infty. \tag{4.5}$$

We define $\varepsilon_1 = \min\{1, \varepsilon, \|\phi_c\|_1\}$ and $\delta_1 = \frac{1}{2}\,\varepsilon_1$. Suppose that (4.4) is satisfied for $\delta = \delta_1$. Then, there exists $T > 0$ such that $\|u(\cdot, t) - \phi_c\|_1 < \|\phi_c\|_1$, for all $0 \leq t < T$, and hence by Lemma 4.4 there exists $z(t) \in \mathbb{R}^n$ satisfying

$$\|u(\cdot + z(t), t) - \phi_c\|_1 = \inf_{y \in \mathbb{R}^n} \|u(\cdot + y, t) - \phi_c\|_1, \qquad \forall\, 0 \leq t < T. \tag{4.6}$$

Let $T_{\max} = \sup\{T > 0 \text{ such that (4.6) holds}\}$ and let

$$h(\cdot, t) = u(\cdot + z(t), t) - \phi_c, \qquad \forall\, 0 \leq t < T_{\max}.$$

We also consider the invariant functional (*Lyapunov functional*) of (2.1),

$$L(w) = E(w) + (c - c_0)\, Q(w), \qquad w \in H^1(\mathbb{R}^n).$$

Since $\phi_c$ solves (2.4) a simple calculation using Taylor's theorem gives us

$$\begin{aligned} L(u_0) - L(\phi_c) &= L(u(\cdot,t)) - L(\phi_c) = L(\phi_c + h(\cdot,t)) - L(\phi_c) \\ &= \frac{c}{2}\, F_c(h(\cdot,t), h(\cdot,t)) + R_c(h(\cdot,t)), \quad \forall\, 0 \le t < T_{\max}, \end{aligned} \tag{4.7}$$

where $F_c(\cdot,\cdot)$ was defined in (4.2) and $R_c(h) = -\frac{1}{6}\, p \int_{\mathbb{R}^n} (\phi_c + \theta h)^{p-1} h^3 dx$, $0 < \theta < 1$ a constant independent of $h$.

Using (2.9) and the Sobolev imbeddings $H^1(\mathbb{R}^n) \hookrightarrow L^3(\mathbb{R}^n)$, $H^1(\mathbb{R}^n) \hookrightarrow L^{p+2}(\mathbb{R}^n)$ for $p$ as in (1.4) we obtain

$$L(u_0) - L(\phi_c) \le \frac{c}{2}\, \|h(\cdot,t)\|_1^2 + A_4 \|h(\cdot,t)\|_1^3 + A_5 \|h(\cdot,t)\|_1^{p+2},$$

for all $0 \le t < T_{\max}$, where

$$A_4 = \frac{2^{p-1}}{6}\, K_1 p (c - c_0)^{1-\frac{1}{p}} |\psi|_\infty^{p-1}, \quad A_5 = \frac{2^{p-1}}{6}\, K_2 p$$

and $K_1$, $K_2$ are positive constants depending only on $p$ and $n$.

Thus, using (4.6) we conclude that

$$L(u_0) - L(\phi_c) \le \frac{c}{2}\, \|u_0 - \phi_c\|_1^2 + A_4 \|u_0 - \phi_c\|_1^3 + A_5 \|u_0 - \phi_c\|_1^{p+2} \le A_6 \|u_0 - \phi_c\|_1, \tag{4.8}$$

where $A_6 = 3 \max\{\frac{c}{2}, A_4, A_5\}$.

On the other hand, by Lemma 4.5 and (4.7),

$$L(u_0) - L(\phi_c) \ge \gamma_c(\|h(\cdot,t)\|_1), \qquad \forall\, 0 \le t < T_{\max}, \tag{4.9}$$

where $\gamma_c(s) = A_1 s^2 - A_7 s^3 - A_3 s^4 - A_5 s^{p+2}$ and $A_7 = A_2 + A_4$.

We now select $0 < \delta_2 < \varepsilon_1$ for which $\gamma_c(s)$ is strictly increasing and positive on $(0, \delta_2)$ and define $\delta = \min\{\delta_1, \delta_2, \frac{\gamma_c(\frac{\delta_2}{2})}{A_6}\}$. Thus, if (4.4) is satisfied for this $\delta > 0$ then from (4.8) and (4.9) we deduce that $\gamma_c(\|h(\cdot,t)\|_1) \le \gamma_c\left(\frac{\delta_2}{2}\right)$, for all $0 \le t < T_{\max}$. By continuity of $\|h(\cdot,t)\|_1$ and the definition of $\delta_2$ we must have $\|h(\cdot,t)\|_1 < \frac{\delta_1}{2} < \varepsilon$ and consequently (4.5) holds for all $0 \le t < T_{\max}$.

The continuity of $\xi(t) = \inf_{y \in \mathbb{R}^n} \|u(\cdot + y, t) - \phi_c\|_1$ on $[0, +\infty)$ and Lemma 4.4 imply that $T_{\max} = +\infty$.

Now let us consider general initial data $u_0 \in H^1(\mathbb{R}^n)$. As before $c > c_0$ is fixed such that $q'(c) > 0$. We begin selecting $\mu = \mu(c) > 0$ for which $q(s)$ is strictly increasing on $I_c = [c - \mu, c + \mu]$ and $\eta = \min_{s \in I_c} \|\phi_s\|_1 > 0$. By the continuity of $Q$ on $H^1(\mathbb{R}^n)$ there exists $\delta_1 > 0$ such that if $\|u_0 - \phi_c\|_1 < \delta_1$ then $Q(u_0) \in q(I_c)$. Suppose that $c' \in I_c$ is such that $q(c') = Q(u_0)$. For such a $c'$ we have

$$\begin{aligned} \|u_0 - \phi_c\|_1 &\ge \big|\, \|u_0\|_1 - \|\phi_c\|_1 \,\big| = 2^{\frac{1}{2}} |q(c')^{\frac{1}{2}} - q(c)^{\frac{1}{2}}| \\ &\ge 2^{\frac{1}{2}} M_1 |c' - c| \ge 2^{\frac{1}{2}} M_1 M_2^{-1} \|\phi_{c'} - \phi_c\|_1 \end{aligned} \tag{4.10}$$

and

$$\|u_0 - \phi_{c'}\|_1 \le \|u_0 - \phi_c\|_1 + \|\phi_c - \phi_{c'}\|_1 \le M\|u_0 - \phi_c\|_1, \tag{4.11}$$

where

$$M_1 = \min_{s\in I_c} q'(s)^{\frac{1}{2}}, \qquad M_2 = \max_{s\in I_c} \|\phi_s\|_1, \qquad M = 1 + 2^{-\frac{1}{2}} M_1^{-1} M_2,$$

are positive constants.

We also consider the function $\gamma(s) = \widetilde{A}_1 s^2 - \widetilde{A}_7 s^3 - \widetilde{A}_3 s^4 - A_5 s^{p+2}$ on $[0, +\infty)$, where $\widetilde{A}_1 = \inf_{s\in I_c} A_1(s)$, $\widetilde{A}_7 = \max_{s\in I_c} A_7(s)$, $\widetilde{A}_3 = \max_{s\in I_c} A_3(s)$, and $A_1$, $A_3$, $A_5$ and $A_7$ are positive constants as before. By Lemma 5.4 and Lemma 4.5, $\widetilde{A}_1$, $\widetilde{A}_3$, $\widetilde{A}_7$ are positive constants depending only on $c$. Given $\varepsilon > 0$ we choose $\delta_2 > 0$, $\delta_2 < \varepsilon_1 = \min\{1, \varepsilon, \eta\}$ such that $\gamma(s)$ is strictly increasing and positive on $(0, \delta_2)$ and define $\delta = \min\{\delta_1, \frac{\delta_2}{2M}, \frac{\gamma(\frac{\delta_2}{2})}{\widetilde{A}_6}\}$, where $\widetilde{A}_6 = 3\max\{\frac{c+\mu}{2}, \widetilde{A}_4, A_5\}$ and $\widetilde{A}_4 = \max_{s\in I_c} A_4(s)$. Note that $\delta > 0$ depends only on $c$. Thus, if $u_0 \in H^1(\mathbb{R}^n)$ satisfies $\|u_0 - \phi_c\|_1 < \delta$ then there exists $c' \in I_c$ such that $q'(c) = Q(u_0)$ and by (4.10), (4.11), we conclude that $\|\phi_{c'} - \phi\|_1 < \frac{\varepsilon}{2}$ and $\|u_0 - \phi_{c'}\|_1 < \|\phi_{c'}\|_1$. Since $q(c') = Q(u_0)$ and $q'(c') > 0$ then by the initial discussion we have

$$\inf_{y\in\mathbb{R}^n} \|u(\cdot, t) - \phi_{c'}(\cdot - y)\|_1 < \frac{\varepsilon}{2}, \qquad \forall\, 0 \le t < +\infty,$$

where as before $u(\cdot, t)$ denotes the solution of (2.13) corresponding to initial data $u_0$. Using the triangle inequality we obtain

$$\inf_{y\in\mathbb{R}^n} \|u(\cdot, t) - \phi_c(\cdot - y)\|_1 \le \inf_{y\in\mathbb{R}^n} \|u(\cdot, t) - \phi_{c'}(\cdot - y)\|_1 + \|\phi_{c'} - \phi_c\|_1 < \varepsilon,$$

for all $0 \le t < +\infty$. This completes the proof of Theorem 4.2.

**5. Proof of Lemma 4.5.** Before proving Lemma 4.5 we will state and prove some preparatory lemmas. We will denote by $[w_1, \cdots, w_k]$ the subspace spanned by the vectors $w_1, \cdots, w_k$.

**Lemma 5.1.** *Let $\beta > 0$ be as in Lemma 3.1 and let $W = [\psi_-, \frac{\partial\phi_c}{\partial x_1}, \cdots, \frac{\partial\phi_c}{\partial x_n}]^\perp$, where $\perp$ denotes orthogonal complement in $L^2(\mathbb{R}^n)$ and $\psi_-$ denotes an eigenfunction associated to the eigenvalue $\alpha$. Then, $F_c(w, w) \ge \beta\|w\|_1^2$, for all $w \in W \cap H^1(\mathbb{R}^n)$.*

**Proof.** Denote by $(E_\lambda)_{\lambda\in\mathbb{R}}$ the spectral family associated to $\mathcal{L}_c$. Since $[(-\infty, \alpha) \cup (\alpha, 0) \cup (0, \beta)] \subset \rho(\mathcal{L}_c)$ applying the spectral theorem we see that

$$(\mathcal{L}_c w, w) \ge (\alpha - \beta)\left((E_\alpha - E_{\alpha-0})\, w, w\right) - \beta\big((E_0 - E_{0-0})\, w, w\big) + \beta\|w\|^2,$$

for all $w \in H^2(\mathbb{R}^n)$. By density it follows that

$$F_c(w, w) \ge (\alpha - \beta)\left((E_\alpha - E_{\alpha-0})\, w, w\right) - \beta\big((E_0 - E_{0-0})\, w, w\big) + \beta\|w\|^2,$$

for all $w \in H^1(\mathbb{R}^n)$.

In particular, $F_c(w, w) \ge \beta\|w\|^2$ for all $w \in W \cap H^1(\mathbb{R}^n)$. □

In the next lemma we will assume that $\psi_-$ is chosen so that $\|\psi_-\| = 1$.

**Lemma 5.2.** *Suppose that $c > c_0$ satisfies $q'(c) > 0$. Then $F_c(h,h) > 0$ for all $h \neq 0$ in $H^1(\mathbb{R}^n)$ satisfying*

$$(h, \phi_c)_1 = 0 \qquad \text{and} \qquad \big(h, \frac{\partial \phi_c}{\partial x_j}\big)_1 = 0, \qquad j = 1, \cdots, n. \tag{5.1}$$

**Proof.** We will borrow some ideas of the proof of Theorem 3.3 in [8]. For each $s > c_0$ by (2.4) we have

$$s\int_{\mathbb{R}^n} \nabla\phi_s \cdot \nabla w \, dx + s\int_{\mathbb{R}^n} \phi_s w \, dx - c_0 \int_{\mathbb{R}^n} \phi_s w\, dx - \int \frac{\phi_s^{p+1}}{p+1} w\, dx = 0, \quad \forall w \in H^1(\mathbb{R}^n).$$

Differentiation of this expression with respect to $s$ yields

$$(\phi_s, w) = -sF_s(\phi_s', w), \qquad \forall w \in H^1(\mathbb{R}^n), \quad \text{where} \quad \phi_s' = \frac{d\phi_s}{ds}.$$

In particular,

$$F_c(\phi_c', h) = 0, \qquad \text{for all } h \text{ satisfying (5.1)}. \tag{5.2}$$

We also have

$$F_c(\phi_c', \phi_c') = -\frac{1}{c}(\phi_c, \phi_c')_1 = -\frac{1}{c}\, q'(c) < 0. \tag{5.3}$$

Now, choose $v_1, \cdots, v_n$ in $H^2(\mathbb{R}^n)$ mutually orthogonal on $H^1(\mathbb{R}^n)$ so that $[v_1, \cdots, v_n]$ $= [\frac{\partial\phi_c}{\partial x_1}, \cdots, \frac{\partial\phi_c}{\partial x_n}]$ and write

$$\phi_c' = b_0\psi_- + b_1 v_1 + \cdots + b_n v_n + w_0, \quad h = \widetilde{b}_0\psi_- + \widetilde{b}_1 v_1 + \cdots + \widetilde{b}_n v_n + w_1,$$

where $b_j$, $\widetilde{b}_j$ are real constants; $w_0$, $w_1$ are in $W \cap H^1(\mathbb{R}^n)$ and $h \in H^1(\mathbb{R}^n)$ satisfies (5.1). Then

$$F_c(\phi_c', \phi_c') = b_0^2\alpha + F_c(w_0, w_0), \quad F_c(\phi_c', h) = b_0\widetilde{b}_0\alpha + F_c(w_0, h),$$

from which we deduce using (5.2) and (5.3) that

$$F_c(h,h) = \widetilde{b}_0^2\alpha + F_c(w_1, w_1) \geq \widetilde{b}_0^2\alpha + \frac{F_c(w_0, w_1)}{F_c(w_0, w_0)} > \widetilde{b}_0^2\alpha - \frac{(-b_0\widetilde{b}_0\alpha)^2}{b_0^2\alpha} = 0.$$

**Lemma 5.3.** *Let $c > c_0$ satisfy $q'(c) > 0$. Then, there exists a positive constant $A_0 = A_0(c)$ depending only on $c$ such that $F_c(h,h) \geq A_0\|h\|^2$ for all $h \in H^1(\mathbb{R}^n)$ satisfying* (5.1)*.*

**Proof.** Since $F_c$ is bilinear it is sufficient to prove the lemma for $h \in H^1(\mathbb{R}^n)$ satisfying (5.1) and $\|h\| = 1$. Under these hypotheses suppose that there is no constant $A_0 > 0$

such that $F_c(h,h) \geq A_0\|h\|^2$. Then, there exists a sequence $(h_m)$ in $H^1(\mathbb{R}^n)$, $\|h_m\| = 1$, for all $m$, such that

$$(h_m, \phi_c)_1 = 0, \qquad \big(h_m, \frac{\partial \phi_c}{\partial x_j}\big)_1 = 0, \qquad j = 1, \cdots, n, \tag{5.4}$$

$$F_c(h_m, h_m) < \frac{1}{m}, \qquad \forall m = 1, 2, \cdots. \tag{5.5}$$

From (5.5) it follows that $(h_m)$ is bounded in $H^1(\mathbb{R}^n)$; therefore we can select a subsequence which we will also denote by $(h_m)$ such that

$$\begin{aligned} h_m &\rightharpoonup h \qquad \text{weak in} \quad H^1(\mathbb{R}^n), \\ \frac{\partial h_m}{\partial x_j} &\rightharpoonup \frac{\partial h}{\partial x_j} \qquad \text{weak in} \quad L^2(\mathbb{R}^n). \end{aligned} \tag{5.6}$$

The boundedness of $(h_m)$ in $H^1(\mathbb{R}^n)$, the decaying property (2.8) of $\phi_c$ and the Sobolev's compact imbedding $H^1(B_R(0)) \hookrightarrow L^2(\mathbb{R}^n)$, where $B_R(0) = \big\{x \in \mathbb{R}^n : |x| < R\big\}$, imply that

$$\lim_{m \to +\infty} \int_{\mathbb{R}^n} \phi_c^p h_m^2 dx = \int_{\mathbb{R}^n} \phi_c^p h^2 dx. \tag{5.7}$$

Thus, from (5.5), (5.6) and (5.7) we have

$$F_c(h,h) \leq \underline{\lim}\, F_c(h_m, h_m) \leq 0 \quad \text{and} \quad h \neq 0.$$

Passing to the limit in (5.4) and taking into account the convergence (5.6) we obtain $(h, \phi_c)_1 = 0$ and $(h, \frac{\partial \phi_c}{\partial x_j})_1 = 0$, $j = 1, \cdots, n$, which contradicts Lemma 5.2.

**Lemma 5.4.** *Suppose that $c > c_0$ satisfies $q'(c) > 0$. Then, there exists a positive constant $A = A(c)$ such that $F_c(h,h) \geq A\|h\|_1^2$ for all $h \in H^1(\mathbb{R}^n)$ satisfying* (5.1)*. Moreover, if $\mu > 0$ is such that $c - \mu > c_0$ and $I_c = [c-\mu, c+\mu]$ then $A_\mu = \inf_{s \in I_c} A(s) > 0$.*

**Proof.** Let $\widetilde{A}(c) = \inf_{h \in Y} \frac{F_c(h,h)}{\|h\|^2}$, where $Y = \{h \in H^1(\mathbb{R}^n) : h \neq 0 \text{ and satisfies (5.1)}\}$. $\widetilde{A}(c) > 0$ by Lemma 5.3. We also have

$$F_c(h,h) \geq \big(1 - \frac{c_0}{c}\big) - \frac{1}{c}\, |\phi_c|_\infty^p \widetilde{A}(c)^{-1} F_c(h,h), \qquad \forall h \in Y.$$

Then, for all $h \in H^1(\mathbb{R}^n)$ satisfying (5.1) we have $F_c(h,h) \geq A\|h\|_1^2$, where

$$A = A(c) = \frac{c - c_0}{c + (c - c_0)|\psi|_\infty^p \widetilde{A}(c)^{-1}} > 0.$$

Now, let $\widetilde{A}_\mu = \inf_{s \in I_c} \widetilde{A}(s)$, where $\mu > 0$ is such that $c - \mu > c_0$ and $I_c = [c-\mu, c+\mu]$. Clearly $\widetilde{A}_\mu \geq 0$. By repeating the arguments of Lemma 5.2 we see that $\widetilde{A}_\mu > 0$. Finally,

$$A_\mu = \inf_{s \in I_c} A(s) \geq \inf_{s \in I_c} \frac{s - c_0}{s + (s - c_0)|\psi|_\infty^p \widetilde{A}_\mu^{-1}} = \frac{c - \mu - c_0}{(c - \mu) + (c - \mu - c_0)|\psi|_\infty^p \widetilde{A}_\mu^{-1}} > 0.$$

**Proof of Lemma 4.5.** Recall that $h = h(\cdot,t) = u(\cdot + z(t), t) - \phi_c$, where $c > c_0$ is such that $q'(c) > 0$, $q(c) = Q(u_0)$ and $z(t)$ satisfies (4.1). From $q(c) = Q(u_0)$ it follows that

$$(h, \phi_c)_1 = -\frac{1}{2}\,\|h\|_1^2 \tag{5.8}$$

and from (4.1) we have $(h, \frac{\partial \phi_c}{\partial x_j})_1 = 0$, for all $j = 1, \cdots, n$. Thus, writing $h = h_1 + h_2$ with

$$h_1 = h_1(\cdot,t) = \frac{(h(\cdot,t),\phi_c)_1}{\|\phi_c\|_1^2}\,\phi_c \quad \text{and} \quad h_2 = h_2(\cdot,t) = h(\cdot,t) - \frac{(h(\cdot,t),\phi_c)_1}{\|\phi_c\|_1^2}$$

we obtain $(h_2, \frac{\partial \phi_c}{\partial x_j})_1 = 0$, for all $j = 1, \cdots, n$, and $(h_2, \phi_c)_1 = 0$.

Hence, $F_c(h_2, h_2) \geq A\|h_2\|_1^2$ by Lemma 5.4. Moreover, by (5.8) and the Sobolev imbeddings $H^1(\mathbb{R}^n) \hookrightarrow L^{p+2}(\mathbb{R}^n)$, $H^2(\mathbb{R}^n) \hookrightarrow L^{2(p+2)}(\mathbb{R}^n)$ for $p$ as in (1.4) it follows that

$$F_c(h_1,h_1) = \frac{1}{4}\,F_c(\phi_c,\phi_c)\,\frac{\|h\|_1^4}{\|\phi_c\|_1^4} \geq \frac{K_1^{p+2}}{4c_0}\,\|\phi_c\|_1^{p-2}\|h\|_1^4,$$

$$F_c(h_1,h_2) = -\frac{1}{2}\,F_c(\phi_c,h_2)\,\frac{\|h\|_1^2}{\|\phi_c\|_1^2} \geq -\big(1 + \frac{2K_2^p}{c_0}\,\|\phi_c\|_1^p\big)\,\frac{\|h\|_1^3}{\|\phi_c\|_1},$$

where $K_1$, $K_2$ are positive constants depending only on $p$ and $n$. We also have

$$F_c(h_2,h_2) \geq A\|h_2\|_1^2 = A\big(\|h\|_1^2 - \frac{1}{4}\,\frac{\|h\|_1^4}{\|\phi_c\|_1^2}\big).$$

Hence,

$$\frac{c}{2}F_c(h,h) = \frac{c}{2}F_c(h_1,h_1) + cF_c(h_1,h_2) + \frac{c}{2}F_c(h_2,h_2) \geq A_1\|h\|_1^2 - A_2\|h\|_1^3 - A_3\|h\|_1^4,$$

where

$$A_1 = A_1(c) = \frac{c}{2}\,A(c) > 0,$$

$$A_2 = A_2(c) = \frac{c}{\|\phi_c\|_1}\,\big(1 + \frac{2K_2p}{c_0}\,\|\phi_c\|_1^p\big) > 0,$$

$$A_3 = A_3(c) = \frac{c}{4}\,\big(\frac{K_1^{p+2}}{2c_0}\,\|\phi_c\|_1^{p-2} + A(c)\|\phi_c\|_1^{-2}\big) > 0.$$

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