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ALMOST PERIODICITY AND STABILITY FOR SOLUTIONS TO FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY*

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Abstract. We develop results on (a) almost periodicity properties and (b) (exponential) asymptotic stability of solutions to the functional differential equations with infinite delay (FDE) $\dot{x}(t) + \alpha x(t) + Bx(t) \ni F(x_t)$, $t \geq 0$, $x|_{\mathbb{R}^-} = \varphi \in E$ in a context which allows for a (generally) nonlinear and multivalued accretive state-responsive operator $B \subset X \times X$ in a Banach state space X and a locally defined and locally Lipschitz continuous history-responsive function $F : D(F) \subset E \to X$, in an appropriate initial history space E of continuous functions from $\mathbb{R}^-$ into X. Applications to models from population dynamics (delay logistic equation) and biology (Goodwin oscillator) are presented.

Introduction. The object of this paper is to study almost periodicity and stability properties of the solution $x_\varphi : \mathbb{R} \to X$ to the functional differential equation with infinite delay

$$\text{(FDE)} \quad \begin{cases} \dot{x}(t) + \alpha x(t) + Bx(t) \ni F(x_t), & t \geq 0 \\ x|_{\mathbb{R}^-} = \varphi \in E. \end{cases}$$

Here, α is a real constant, X a Banach (state) space, $B \subset X \times X$ a (generally) nonlinear and multivalued accretive operator in X, E a "suitably" chosen Banach space of continuous initial history functions $\varphi : \mathbb{R}^- \to X$, and $F : \hat{E} \subset E \to X$ a Lipschitz continuous mapping from a subset $\hat{E}$ of E into X. As usual, for a function $x : \mathbb{R} \to X$ and $t \geq 0$, the function $x_t : \mathbb{R}^- \to X$ is defined by $x_t(s) := x(t+s)$, $s \in \mathbb{R}^-$.

A number of authors have heretofore considered conditions under which solutions to (FDE) can be represented in terms of a strongly continuous operator semigroup (defined in the initial history space E); we refer to [4, 5, 6, 7], [13, 14], [17], [28, 29], [41], and [42, 43, 44, 45] for a representative sample of previous work in this direction. Most commonly, these efforts have required (a) the state-responsive operator B to be densely defined, m-accretive, and single valued (sometimes along with the additional requirements that $0 \in D(B)$ and $B(0) = 0$), and (b) the history-responsive operator F to be defined on E and globally Lipschitz (or at least Lipschitz continuous on norm-balls of E (cf. [7])). With regard to applications, these are severe restrictions. In

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problems from population dynamics, say, the history-responsive function F is usually defined only on a (positive) cone and Lipschitz continuous only on truncated cones. A more detailed discussion of such limitations can be found in [40, Sections 1 and 4].

For additional information concerning (FDE) in case B is single valued, see [8, 21, 24, 25] and the further references listed therein.

Recently in [40], we have established the framework for a local approach to semigroup representations of (global) solutions to (FDE) which gives wide latitude to both those subsets of the state space X and initial history space E where (a) the (generally multivalued) operator B is defined and accretive, and (b) $F : D(F) \subset E \to X$ is defined and Lipschitz continuous. Under suitable conditions on the initial history space $E, D(F) = \hat{E} \subset E$, the operator $B \subset X \times X$, and, possibly, the state space X (see Section 1 below), there exists a strongly continuous semigroup $(S(t))_{t\geq 0}$ in $\hat{E}$, $S(t) : C \subset \hat{E} \to C$, $t \geq 0$, such that, for $\varphi \in C$, the function

$$x_\varphi(t) = \begin{cases} \varphi(t) & t \leq 0 \\ (S(t)\varphi)(0) & t \geq 0 \end{cases}$$

is the unique solution to (FDE) [40, Theorem 2.5].

In the present paper, we use this semigroup representation under local conditions to derive (1) (asymptotic) almost periodicity properties and (2) stability results for solutions to (FDE) from corresponding asymptotic properties of the solution semigroup $(S(t))_{t\geq 0}$. This analysis is carried forward for a class of initial history spaces of "fading memory type"; namely, the weighted sup-norm spaces E_v of continuous functions from $\mathbb{R}^-$ into X corresponding to continuous weights $v : \mathbb{R}^- \to (0, 1]$.

We begin in Section 1 by briefly recalling our local approach to semigroup representation of solutions to (FDE) from [40], and set the context for the development that follows. Section 2 is devoted to a study of the relationship between asymptotic properties of solutions to (FDE) and those of the solution semigroup associated with (FDE). This will lead to various results on existence of (asymptotically) almost periodic and weakly almost periodic solutions to (FDE); cf. Corollary 2.8, Theorems 2.9 and 2.11, and Corollary 2.13. In Section 3, we apply our methods to derive (exponential) asymptotic stability results for solutions to (FDE). Applications of our results to concrete models – the delay logistic equation from population dynamics (Example 4.2) and the Goodwin oscillator from biology (Examples 4.3 and 4.4) – are presented in Section 4.

1. The operator semigroup associated with (FDE). We start with a brief outline of the local approach to (FDE) as developed in [40].

1.A. Initial history spaces of fading memory type.

Definition 1.1. Assume that $v : \mathbb{R}^- \to (0, 1]$ is a function with the following properties:

(v1) v is continuous, nondecreasing, and $v(0) = 1$;

(v2) $\lim_{u\to 0^-} \frac{v(s+u)}{v(s)} = 1$ uniformly over $s \in \mathbb{R}^-$.

We then put

(a) $Cv_b(\mathbb{R}^-, X) = \{\varphi \in C(\mathbb{R}^-, X) : v\varphi \text{ is bounded }\}$;

(b) $Cv_\circ(\mathbb{R}^-, X) = \{\varphi \in C(\mathbb{R}^-, X) : \lim_{s\to-\infty} v(s)\varphi(s) = 0\}$;
(c) $E_v = \{\varphi \in C(\mathbb{R}^-, X) : v\varphi$ is bounded and uniformly continuous $\}$;
(d) $E_{v_l} = \{\varphi \in E_v : \lim_{s\to-\infty} v(s)\varphi(s)$ exists $\}$;
(e) $E_{v_\circ} = \{\varphi \in E_v : \lim_{s\to-\infty} v(s)\varphi(s) = 0\}$, when v additionally satisfies $\lim_{s\to-\infty} v(s) = 0$.

Each of these spaces is equipped with the weighted sup-norm

$$\|\varphi\|_v = \sup_{s\le 0} v(s)\|\varphi(s)\|.$$

The Banach space E_v is sometimes called a UC_g- space ($v = 1/g$), and has been considered by various authors; e.g., see [1], [22, 23, 24], and the further references listed therein. However, we have chosen to view E_v and its relatives within the more familiar framework associated with the general theory of weighted sup-norm spaces.

In addition to the basic assumptions (v1) and (v2), which will be assumed throughout the paper, the following special properties of the weight functions v will play a role in the sequel:

$$(v3) \quad \lim_{s\to-\infty} v(s) = 0; \qquad\qquad (v3^*) \quad \lim_{t\to\infty} \sup_{s\le -t} \frac{v(s)}{v(s+t)} = 0.$$

Clearly, (v3*) implies (v3). If, for $\mu \ge 0$, $v_1(s) = e^{\mu s}$, and $v_2(s) = (1+|s|)^{-\mu}$, $s \le 0$, then

(a) v_1 and v_2 fulfill (v1) and (v2);
(b) if $\mu > 0$, v_1 and v_2 both fulfill (v3), v_1 even fulfills (v3*), but v_2 fails to satisfy (v3*).

1.B. Framework for (FDE). Throughout the paper, we consider (FDE) in the context of the initial history spaces E_v, E_{v_l}, and $E_{v_\circ}$ as in Definition 1.1. Given that E is any one of these spaces, we also assume that the following conditions are satisfied:

(A1) (i) $\hat{X}$ is a closed subset of X, and $\hat{E}$ is a closed and convex subset of E;
(ii) $B \subset X \times X$ is an accretive operator;
(iii) $F : \hat{E} \to X$ is Lipschitz continuous with Lipschitz constant $M \ge 0$;
(iv) $\alpha \in \mathbb{R}$, and $\gamma = \max\{0, M - \alpha\}$.

(A2) If $x \in \hat{X}$, $\psi \in \hat{E}$, $\lambda > 0$ with $\lambda\gamma < 1$, and φ_x is the solution to

$$\varphi - \lambda\varphi' = \psi, \quad \varphi(0) = x,$$

then $\varphi_x \in \hat{E}$.

(A3) If $\psi \in \hat{E}$ and $\lambda > 0$ with $\lambda\gamma < 1$, then

$$\frac{1}{1+\lambda\alpha}(\psi(0) + \lambda F(\varphi_x)) \in \Big(I + \frac{\lambda}{1+\lambda\alpha} B\Big)(D(B) \cap \hat{X})$$

for each $x \in \hat{X}$.

1.C. Solutions to (FDE). A continuous function $x : \mathbb{R} \longrightarrow X$ is said to be a *solution* to (FDE) if

(a) $x(s) = \varphi(s)$ for $s \in \mathbb{R}^-$;

(b) $x\,|_{\mathbb{R}^+}$ is locally absolutely continuous and differentiable almost everywhere;

(c) $x_t \in \hat{E}, x(t) \in D(B)$, and $\dot{x}(t) + \alpha x(t) - F(x_t) \in -B(x(t))$ for almost every $t \in \mathbb{R}^+$.

Theorem 1.2 (The solution semigroup for (FDE))**.** *In the context set forth in Section* 1.B, *the following assertions hold*:

(a) *If A is the operator in E defined by*

$$\begin{cases} D(A) = \{\varphi \in \hat{E} : \varphi' \in E, \varphi(0) \in D(B),\ \varphi'(0) \in F(\varphi) - \alpha\varphi(0) - B(\varphi(0))\} \\ A\varphi = -\varphi', \end{cases}$$

then $-A$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ of type γ on $clD(A)$: $\|S(t)\varphi - S(t)\psi\|_v \leq e^{\gamma t}\|\varphi - \psi\|_v$ for all $t \geq 0$ and all $\varphi, \psi \in clD(A)$.

(b) *If $\varphi \in clD(A)$, then $S(t)\varphi = (x_\varphi)_t$ for all $t \geq 0$, where*

$$x_\varphi(t) = \begin{cases} \varphi(t), & t \leq 0 \\ (S(t)\varphi)(0), & t \geq 0. \end{cases} \tag{1.1}$$

(c) *Given $\varphi \in clD(A)$, the function x_φ defined by* (1.1) *is the unique solution to* (FDE) *in each of the following situations* (*where we write $\hat{D}(A)$ to denote the generalized domain of A in the sense of* [9, Definition 1]) :

(c1) *X is reflexive and the norm of X is Fréchet differentiable at any $x \in X \setminus \{0\}$, $B \subset X \times X$ is maximal accretive, and $\varphi \in \hat{D}(A)$;*

(c2) *X has the Radon-Nikodym property, $B \subset X \times X$ is m-accretive, and $\varphi \in \hat{D}(A)$;*

(c3) *X has the Radon-Nikodym property, $D(B)$ is closed, B is single-valued with $B : D(B) \to X$ norm-weakly continuous, and $\varphi \in \hat{D}(A)$;*

(c4) *X is any Banach space, $D(B)$ is closed, B is single valued with $B : D(B) \to X$ continuous, and either* (i) *$\varphi \in \hat{D}(A)$ or* (ii) *$\varphi \in clD(A)$ and B maps bounded sets into bounded sets;*

(c5) *X is reflexive, $B : D(B) \to X$ is single valued and demiclosed* (*i.e., the graph of B is norm-weakly closed in $X \times X$*), *and $\varphi \in \hat{D}(A)$.*

These results follow from Theorem 2.1, Proposition 2.4, Theorem 2.5, and Remark 3.2 of [40], and from [30, Theorem 2.5].

Remark 1.3. The representation $x_\varphi(t) = (S(t)\varphi)(0)$, $t \geq 0$, given for the solution x_φ to (FDE) via the solution semigroup by (1.1) above allows asymptotic properties of the motion $S(\cdot)\varphi : \mathbb{R}^+ \to \hat{E}$ to directly carry over to corresponding asymptotic properties of $x_\varphi\,|_{\mathbb{R}^+}$. In this way, recent (weak) almost periodicity and stability results for motions of semigroups of (linear or nonlinear) operators (cf. [32, 33, 35, 37, 38, 39]) can be brought to bear on the analysis of corresponding properties for solutions to (FDE). Conversely, it is of interest to know the extent to which this relationship is strict; i.e., to know under what conditions these asymptotic properties hold for the solution $x_\varphi\mid_{\mathbb{R}^+}$ *only* if they hold as well for the semigroup motion $S(\cdot)\varphi$. These two points provide the basis for our investigation in the subsequent sections.

2. Almost periodicity properties of solutions to (FDE). In this section, we combine the semigroup representation (1.1) of Theorem 1.2 (for the case $E = E_v$) with our results [32, 33, 35, 37, 39] on asymptotic behavior of semigroup motions in order to derive almost periodicity properties of $x_\varphi\mid_{\mathbb{R}^+}$ for $\varphi \in clD(A)$. We start by recalling the requisite periodicity concepts. For $J \in \{\mathbb{R}, \mathbb{R}^+\}$, we let $C_b(J, X)$ denote the usual Banach space of bounded continuous functions from J into X under the supremum norm $\|\cdot\|_\infty$, while $BUC(\mathbb{R}^+, X)$, respectively $C_\circ(\mathbb{R}^+, X)$, denote the subspaces consisting of those $f \in C_b(\mathbb{R}^+, X)$ which are uniformly continuous, respectively, vanish at infinity. Further, given a function $f : J \to X$ and $\omega \in J$, the $\omega-$ translate f_ω of f is defined by $f_\omega(t) = f(t + \omega)$, $t \in J$, and $H(f) = \{f_\omega : \omega \in J\}$ will denote the set of all translates of f.

Definition 2.1. (a) A function $f \in C_b(\mathbb{R}, X)$ (respectively, $f \in C_b(\mathbb{R}^+, X)$) is said to be *almost periodic* (a.p.) (respectively, *asymptotically almost periodic* (a.a.p.)) if $H(f)$ is relatively compact in $C_b(\mathbb{R}, X)$ (respectively, $C_b(\mathbb{R}^+, X)$).

(b) A function $f \in C_b(\mathbb{R}^+, X)$ is said to be *Eberlein-weakly almost periodic* (E.-w.a.p.) if $H(f)$ is *weakly* relatively compact in $C_b(\mathbb{R}^+, X)$.

Of these notions, (a) dates back to Bohl, Bohr, Bochner (cf. [3]), and Favard [16] in the group case ($J = \mathbb{R}$), and Fréchet [18, 19] in the semigroup case ($J = \mathbb{R}^+$) with $dimX < \infty$ (cf. [32, 33, 34] and [46] for the case $J = \mathbb{R}^+$ and X a general Banach space); and (b) to Eberlein [15] for $dimX = 1$ (cf. [20], [27], and [36] for arbitrary X).

The spaces of X-valued functions defined in (a) and (b) of Definition 2.1 will be denoted respectively by (a) $AP(\mathbb{R}, X)$ ($AAP(\mathbb{R}^+, X)$, and (b) $W(\mathbb{R}^+, X)$. We also let $W_\circ(\mathbb{R}^+, X)$ denote the vector subspace of $W(\mathbb{R}^+, X)$ consisting of those $\varphi \in W(\mathbb{R}^+, X)$ for which the zero function belongs to the weak closure $w - clH(\varphi)$.

Of course, a.p. functions can as well be characterized in terms of relatively dense sets in $\mathbb{R}$ of ϵ-almost periods [3], and a corresponding formulation holds for a.a.p. functions [34]. For later use, we also need the following basic facts about the above listed concepts of almost periodicity (see Fréchet [18, 19], DeLeeuw-Glicksberg [11, 12] and [32, 34, 36]):

$f \in C_b(\mathbb{R}^+, X)$ is asymptotically almost periodic (respectively, Eberlein-weakly almost periodic) if and only if there exist unique functions $g \in AP(\mathbb{R}, X)$ and $\varphi \in C_\circ(\mathbb{R}^+, X)$ (respectively, $\varphi \in W_\circ(\mathbb{R}^+, X)$) such that $f = g\mid_{\mathbb{R}^+} + \varphi$.

Finally, we shall say that a function $f : \mathbb{R}^+ \to X$ is almost periodic if it is the restriction to $\mathbb{R}^+$ of an almost periodic function $g : \mathbb{R} \to X$.

In order to place the above periodicity properties into perspective (and motivate the study of these notions in the context of solutions to (FDE)), we make note of their relationship to other commonly studied modes of asymptotic behavior ((a), (d) and (e) below).

Let $f \in C_b(\mathbb{R}^+, X)$, and consider the following assertions:

(a) $\|\cdot\| - \lim_{t\to\infty} f(t)$ exists;

(b) $f \in AAP(\mathbb{R}^+, X)$;

(c) $f \in W(\mathbb{R}^+, X)$;
(d) $\|\cdot\| - \lim_{T\to\infty} \frac{1}{T}\int_0^T f(t)dt$ exists;
(e) f has weakly relatively compact range in X.

Then (a) implies (b), (b) implies (c), and (c) implies both (d) and (e). The fact that (c) implies (d) is the extension given in [38] of the classical ergodic theorem for scalar valued weakly almost periodic functions due to Eberlein [15] to the case of an arbitrary Banach range space.

For our study of the relationship between periodicity properties of the function $x_\varphi = S(\cdot)\varphi(0) : \mathbb{R}^+ \to X$ and the corresponding semigroup motion $S(\cdot)\varphi : \mathbb{R}^+ \to E_v$, the following technical result will be required.

Lemma 2.2. *Given $\varphi \in E_v$ and $t \geq 0$, define $\tilde{\varphi} : \mathbb{R}^+ \to C(\mathbb{R}^-, X)$ by*

$$\tilde{\varphi}(t)(r) = \begin{cases} \varphi(t+r) - \varphi(0), & r \leq -t \\ 0, & -t \leq r \leq 0. \end{cases}$$

Further, if $\varphi \in clD(A)$, let $g_\varphi(t) = S(t)\varphi - \tilde{\varphi}(t)$, $t \geq 0$. Then we have

(a) $\tilde{\varphi} \in BUC(\mathbb{R}^+, E_v)$;
(b) $\tilde{\varphi} \in C_\circ(\mathbb{R}^+, E_v)$ *if either v additionally satisfies* (v3*) *or v additionally satisfies* (v3) *and $\varphi \in Cv_\circ(\mathbb{R}^-, X)$;*
(c) *if $\varphi \in clD(A)$,*

$$g_\varphi(t)(r) = \begin{cases} \varphi(0), & r \leq -t \\ x_\varphi(t+r), & -t \leq r \leq 0, \end{cases}$$

for all $r \leq 0 \leq t$.

The proof is a matter of routine checking, and so we omit it.

Our first result on the relationship between properties of x_φ and the corresponding semigroup motion $S(\cdot)\varphi$ has a straightforward proof based on Lemma 2.2, which we thus omit as well.

Proposition 2.3. *Given $\varphi \in clD(A)$,*

(a) *$x_\varphi |_{\mathbb{R}^+} : \mathbb{R}^+ \to X$ is bounded if and only if $S(\cdot)\varphi : \mathbb{R}^+ \to E_v$ is bounded;*
(b) *$x_\varphi |_{\mathbb{R}^+} : \mathbb{R}^+ \to X$ is uniformly continuous if and only if $S(\cdot)\varphi : \mathbb{R}^+ \to E_v$ is uniformly continuous.*

Remark. For the case of a single valued accretive operator B with closed domain which is continuous and maps bounded sets into bounded sets, Theorem 1.2(c3) asserts that x_φ is then a solution to (FDE) for each $\varphi \in clD(A)$, and hence $x_\varphi |_{\mathbb{R}^+}$ will automatically be uniformly continuous whenever $x_\varphi |_{\mathbb{R}^+}$ is bounded. In this same context, moreover, we note for future reference that if, additionally, $D(B) = X$, then $clD(A) = E_v$ in view of [40, Proposition 2.6] since B is necessarily m-accretive under these conditions.

The next two results clarify the relationship between asymptotic almost periodicity, respectively, Eberlein-weak almost periodicity, of $x_\varphi |_{\mathbb{R}^+} : \mathbb{R}^+ \to X$ and the corresponding semigroup motion $S(\cdot)\varphi : \mathbb{R}^+ \to E_v$. Roughly speaking, they show that, under mild restrictions on either the weight v or the initial history function φ (see Remark 2.7.2 below), these two asymptotic properties for the function $x_\varphi |_{\mathbb{R}^+} : \mathbb{R}^+ \to X$ and the motion $S(\cdot)\varphi : \mathbb{R}^+ \to E_v$ are mutually equivalent.

Theorem 2.4. *Given $\varphi \in clD(A)$, the following assertions hold:*

(a) (1) *If $\{S(t)\varphi : t \geq 0\}$ is relatively compact in E_v, so is $x_\varphi(\mathbb{R}^+)$ in X.*
(2) *Conversely, if v additionally satisfies* (v3), *$\tilde{\varphi} \in C_\circ(\mathbb{R}^+, E_v)$, and $x_\varphi\,|_{\mathbb{R}^+}$ is uniformly continuous, then relative compactness of $x_\varphi(\mathbb{R}^+)$ in X implies relative compactness of $\{S(t)\varphi : t \geq 0\}$ in E_v.*

(b) (1) *If $S(\cdot)\varphi \in AAP(\mathbb{R}^+, E_v)$, then $x_\varphi\,|_{\mathbb{R}^+} \in AAP(\mathbb{R}^+, X)$.*
(2) *Conversely, if v additionally satisfies* (v3) *and $\tilde{\varphi} \in C_\circ(\mathbb{R}^+, E_v)$, then $x_\varphi\,|_{\mathbb{R}^+} \in AAP(\mathbb{R}^+, X)$ implies $S(\cdot)\varphi \in AAP(\mathbb{R}^+, E_v)$.*

Remarks 2.5. 1. In particular, Theorem 2.4 applies in situations such as that considered in [45, Section 5], where $X = \mathbb{R}^n$, $\hat{E} = E_v$ with $v(s) = \exp \mu s$ for some $\mu > 0$, $s \leq 0$, and $B : \mathbb{R}^n \to \mathbb{R}^n$ is continuous. According to [45, Proposition 5.1], if $\varphi \in E_v$ is such that $S(\cdot)\varphi : \mathbb{R}^+ \to E_v$ is bounded, then $\{S(t)\varphi : t \geq 0\}$ is relatively compact in E_v. In view of Lemma 2.2 and the Remark following Proposition 2.3 above, this is a special case of Theorem 2.4(a).

2. For related results on the relationship between relative compactness of $x_\varphi(\mathbb{R}^+)$ and that of $\{S(t)\varphi : t \geq 0\}$ when $X = \mathbb{R}^n$, we refer to [1, Theorem 2.2] and [22, Theorems 3.1 and 3.2].

Proof of Theorem 2.4. Parts (a) (1) and (b) (1) are obvious since point evaluation at $0 \in \mathbb{R}^-$ is continuous from E_v into X.

Part (a) (2): Given $\epsilon > 0$, there exists $t_\epsilon \geq 0$ such that $\|\tilde{\varphi}(t)\|_v < \epsilon$ for all $t \geq t_\epsilon$. Since $S(\cdot)\varphi = g_\varphi + \tilde{\varphi}$ and $S(\mathbb{R}^+)\varphi = S([0, t_\epsilon])\varphi \cup S([t_\epsilon, \infty))\varphi$, it remains to show that the set $\{g_\varphi(t) : t \geq t_\epsilon\}$ is relatively compact in E_v. To this end, let $H = \{g_\varphi(t) : t \geq 0\} \subset E_v$, and note the following:

(i)

$$v(r)\|g_\varphi(t)(r)\| = \left\{ \begin{array}{ll} v(r)\|\varphi(0)\|, & r \leq -t \\ v(r)\|x_\varphi(t + r)\|, & -t \leq r \leq 0 \end{array} \right\} \leq v(r)\|x_\varphi\,|_{\mathbb{R}^+}\|_\infty,$$

hence vH vanishes uniformly at $-\infty$.

(ii) $H(r) = \{g_\varphi(t)(r) : t \geq 0\} \subset x_\varphi(\mathbb{R}^+)$ is relatively compact in X for all $r \in \mathbb{R}^-$.

(iii)

$$\|g_\varphi(t)(r) - g_\varphi(t)(s)\| = \begin{cases} 0, & s < r \leq -t \\ \|\varphi(0) - x_\varphi(t + r)\|, & \\ & s \leq -t \leq r,\ 0 \leq (t + r) < r - s \\ \|x_\varphi(t + s) - x_\varphi(t + r)\|, & -t \leq s < r, \end{cases}$$

which shows that H is uniformly equicontinuous on $\mathbb{R}^-$. According to [34, Theorem 2.1], (i)–(iii) imply that H is relatively compact in $Cv_\circ(\mathbb{R}^-, X)$, thus completing this part of the proof.

Part (b) (2): Given $\epsilon > 0$, there exist P_ϵ relatively dense in $\mathbb{R}^+$ and $T_\epsilon > 0$ such that

$$\|x_\varphi(t + \tau) - x_\varphi(t)\| < \epsilon \quad \text{for all } \tau \in P_\epsilon \text{ and all } t \geq T_\epsilon, \text{ and}$$

$$v(r) < \frac{\epsilon}{2\|x_\varphi\mid_{\mathbb{R}^+}\|_\infty + 1} \quad \text{for all } r \leq -T_\epsilon.$$

As $S(\cdot)\varphi = g_\varphi + \tilde{\varphi}$, it is enough to show that $g_\varphi \in AAP(\mathbb{R}^+, E_v)$. Thus, let $\tau \in P_\epsilon$ and $t \geq 2T_\epsilon$. Then

$$v(r)\|g_\varphi(t+\tau)(r) - g_\varphi(t)(r)\|$$

$$= \begin{cases} 0, & r \leq -(t+\tau) \\ v(r)\|x_\varphi(t+\tau+r) - x_\varphi(t+r)\| < \epsilon, & r \geq T_\epsilon - t \geq -t \geq -(t+\tau) \\ v(r)\|x_\varphi(t+\tau+r) - x_\varphi(t+r)\| \leq 2v(-T_\epsilon)\|x_\varphi\mid_{\mathbb{R}^+}\|_\infty < \epsilon, & T_\epsilon - t \geq r \geq -t \geq -(t+\tau) \\ v(r)\|x_\varphi(t+\tau+r) - \varphi(0)\| \leq 2v(-T_\epsilon)\|x_\varphi\mid_{\mathbb{R}^+}\|_\infty < \epsilon, & -t \geq r \geq -(t+\tau). \end{cases}$$

This shows that, in fact, $g_\varphi \in AAP(\mathbb{R}^+, E_v)$, and thus completes the proof of Theorem 2.4.

Theorem 2.6. *Given* $\varphi \in clD(A)$, *we have*:

(a) (1) *If* $\{S(t)\varphi : t \geq 0\}$ *is weakly relatively compact in* E_v, *so is* $x_\varphi(\mathbb{R}^+)$ *in* X.
(2) *Conversely, if* v *additionally satisfies* (v3), $\tilde{\varphi} \in C_\circ(\mathbb{R}^+, E_v)$, *and* $x_\varphi\mid_{\mathbb{R}^+}$ *is uniformly continuous, then weak relative compactness of* $x_\varphi(\mathbb{R}^+)$ *in* X *implies weak relative compactness of* $\{S(t)\varphi : t \geq 0\}$ *in* E_v.

(b) (1) *If* $S(\cdot)\varphi \in W(\mathbb{R}^+, E_v)$, *then* $x_\varphi\mid_{\mathbb{R}^+} \in W(\mathbb{R}^+, X)$.
(2) *Conversely, if* v *additionally satisfies* (v3) *and* $\tilde{\varphi} \in C_\circ(\mathbb{R}^+, E_v)$, *then* $x_\varphi\mid_{\mathbb{R}^+} \in W(\mathbb{R}^+, X)$ *implies* $S(\cdot)\varphi \in W(\mathbb{R}^+, E_v)$.

Remark 2.7. 1. The additional requirements on v and $\tilde{\varphi}$ in (a)(2) and (b)(2) of both Theorems 2.4 and 2.6 are crucial. In Section 4, we give an example (Example 4.1.A) in the setting of $E = BUC(\mathbb{R}^-)$ (i.e., $v \equiv 1$) where, for some $\varphi \in clD(A), x_\varphi\mid_{\mathbb{R}^+} \in AAP(\mathbb{R}^+)$ (indeed, $\lim_{t\to\infty} x_\varphi(t)$ exists), but the set $\{S(t)\varphi : t \geq 0\}$ is not even weakly relatively compact in $BUC(\mathbb{R}^-)$.

2. We recall from Lemma 2.2 that the additional assumptions on v and $\tilde{\varphi}$ alluded to in the preceding remark are fulfilled (i) for any $\varphi \in E_v$ if v satisfies (v3*), or (ii) for any $\varphi \in Cv_\circ(\mathbb{R}^-, X)$ if v only satisfies (v3).

A particular case of (ii) occurs when v satisfies (v1)–(v3) and $\varphi \in BUC(\mathbb{R}^-, X)$.

Proof of Theorem 2.6. As in the proof of Theorem 2.4, parts (a) (1) and (b) (1) are obvious.

Part (a) (2): Again, as in the proof of the corresponding part of Theorem 2.4, we need only show that the set $H = \{g_\varphi(t) : t \geq 0\}$ is weakly relatively compact in E_v. In order to prove this fact, we use the weak compactness criterion of [36, Theorem 2.1], and hence assume we are given sequences $(t_n)_n \subset \mathbb{R}^+$ and $(r_m, x_m^*)_m \subset \mathbb{R}^- \times B_{X^*}$ such that $\alpha = \lim_m \lim_n \langle v(r_m) g_\varphi(t_n)(r_m), x_m^*\rangle$, and $\beta = \lim_n \lim_m \langle v(r_m) g_\varphi(t_n)(r_m), x_m^*\rangle$ both exist. (Here, B_{X^*} denotes the dual unit ball of the Banach space X.) According to [36, Theorem 2.1], we have to show that $\alpha = \beta$.

Case 1: $(t_n)_n$ is bounded. Then we can assume that $(t_n)_n \longrightarrow t \in \mathbb{R}^+$ and, since $(v(r_m)\delta_{r_m} \otimes x_m^*)_m \subset B_{{E_v}^*}$, that this latter sequence clusters weak-star at some $G \in {E_v}^*$. From the existence of the double limits, we thus conclude that

$$\alpha = \lim_m \langle g_\varphi(t), v(r_m)\delta_{r_m} \otimes x_m^*\rangle = \lim_n \langle g_\varphi(t_n), G\rangle = \beta.$$

Case 2: $(t_n)_n$ is unbounded. We can then assume that $(t_n)_n \nearrow \infty$. If $(r_m)_m$ is bounded, we can assume that $(r_m)_m \longrightarrow r \in \mathbb{R}^-$, as well as that $t_1 \leq t_n$ and $-t_1 \leq r_m$ for all $n, m \in \mathbb{N}$, whereby $g_\varphi(t_n)(r_m) = x_\varphi(t_n + r_m)$. Since $x_\varphi|_{\mathbb{R}^+}$ is uniformly continuous and $x_\varphi(\mathbb{R}^+)$ is weakly relatively compact in X, we can further assume that a subnet of $\big((x_\varphi|_{\mathbb{R}^+})_{t_n - t_1}\big)_n$ converges pointwise-weakly to some $h \in C(\mathbb{R}^+, X)$. Finally, some subnet of $(x_m^*)_m$ clusters weak-star at some $x^* \in B_{X^*}$. Hence,

$$\begin{aligned}\alpha &= \lim_m \lim_n \langle v(r_m)(x_{\varphi_{t_n - t_1}}(r_m + t_1), x_m^*\rangle \\ &= \lim_m \langle v(r_m) h(r_m + t_1), x_m^*\rangle = \langle v(r) h(r + t_1), x^*\rangle,\end{aligned}$$

and

$$\beta = \lim_n \langle v(r)(x_\varphi)_{t_n - t_1}(r + t_1), x^*\rangle = \langle v(r) h(r + t_1), x^*\rangle.$$

Otherwise, we can assume that $0 \geq r_m \longrightarrow -\infty$. Then, since v satisfies (v3), we obviously have $\alpha = 0 = \beta$, which completes the proof of part (a) (2).

Part (b) (2): In this case, $x_\varphi|_{\mathbb{R}^+} \in BUC(\mathbb{R}^+, X)$ [38, Proposition 2.1]. Thus, according to Proposition 2.3, we have $S(\cdot)\varphi \in BUC(\mathbb{R}^+, E_v)$, and hence the map

$$\mathbb{R}^+ \longrightarrow BUC(\mathbb{R}^+, E_v), \quad \omega \longmapsto S_\omega(\cdot)\varphi$$

is well defined and continuous. Given $\epsilon > 0$, choose $T_\epsilon \geq 0$ such that $\|\tilde{\varphi}(t)\|_v < \epsilon$ for all $t \geq T_\epsilon$. Then,

$$\|S_\omega(\cdot)\varphi - (g_\varphi)_\omega\|_\infty = \|\tilde{\varphi}_\omega\|_\infty < \epsilon \quad \text{for all} \quad \omega \geq T_\epsilon,$$

and $\{S_\omega(\cdot)\varphi : 0 \leq \omega \leq T_\epsilon\}$ is compact in $BUC(\mathbb{R}^+, E_v)$. Hence, it remains to show that $\{(g_\varphi)_\omega : \omega \geq 0\}$ is weakly relatively compact in $C_b(\mathbb{R}^+, E_v)$. From Lemma 2.2(c), it is obvious that

$$v(r)\|(g_\varphi)_\omega(t)(r)\| \leq v(r)\|x_\varphi|_{\mathbb{R}^+}\|_\infty \quad \text{for all} \;\; r \leq 0 \leq t,$$

and so we have that $\{(g_\varphi)_\omega : \omega \geq 0\}$ is bounded in $C_b(\mathbb{R}^+, Cv_\circ(\mathbb{R}^-, X))$. Thus, in order to show that it is also weakly relatively compact, we can use [36, Theorems 2.1 and 2.3]; i.e., given sequences $(\omega_n)_n \subset \mathbb{R}^+$, $(t_m, s_m, x_m^*)_m \subset \mathbb{R}^+ \times \mathbb{R}^- \times B_{X^*}$ such that

$$\begin{aligned}\alpha &= \lim_n \lim_m \langle v(s_m) g_\varphi(\omega_n + t_m)(s_m), x_m^*\rangle, \quad \text{and} \\ \beta &= \lim_m \lim_n \langle v(s_m) g_\varphi(\omega_n + t_m)(s_m), x_m^*\rangle\end{aligned}$$

both exist, we have to show that $\alpha = \beta$.

First, we observe that this is trivially true if either $(t_m)_m$ or $(\omega_n)_n$ is bounded, for g_φ is uniformly continuous and, according to the proof of proposition (a)(2) above, has weakly relatively compact range.

Next, if $0 \geq s_m \longrightarrow -\infty$, then $\alpha = 0 = \beta$ since v satisfies (v3).

Finally, if $0 \geq s_m \longrightarrow s \in \mathbb{R}^-$, and $t_m \longrightarrow \infty$, then we can assume that $t_m + s_m \geq 0$ for all $m \in \mathbb{N}$, and hence

$$\begin{aligned} v(s_m)\langle g_\varphi(\omega_n + t_m)(s_m), x_m^* \rangle &= v(s_m)\langle x_\varphi(\omega_n + t_m + s_m), x_m^* \rangle \\ &= \langle x_\varphi(\omega_n + (t_m + s_m)), v(s_m)x_m^* \rangle. \end{aligned}$$

Thus, since $(v(s_m)x_m^*)_m \subset B_{X^*}$ and $x_\varphi\,|_{\mathbb{R}^+} \in W(\mathbb{R}^+, X)$, we again have that $\alpha = \beta$. This completes the proof of Theorem 2.6. □

In order to formulate our next result, the following notation will be helpful.

Notation: Given a strongly continuous semigroup $(S(t))_{t\geq 0}$ of operators $S(t)$ on a subset C in a Banach space Y and $y \in C$,

$$\omega(y) = \{z \in Y : \exists 0 \leq t_n \longrightarrow \infty \ \text{ such that norm-}\lim_n S(t_n)y = z\}$$

denotes the $\omega-$ *limit set* of y, and

$$\omega_w(y) = \{z \in Y : \exists 0 \leq t_n \longrightarrow \infty \ \text{ such that weak-}\lim_n S(t_n)y = z\}$$

denotes the *weak-* $\omega-$ *limit set* of y.

The following decomposition result supplements the assertions of Theorem 2.4(b) and Theorem 2.6(b).

Corollary 2.8. *Assume that v satisfies* (v1)–(v3), *and take $\varphi \in clD(A)$ such that $\tilde{\varphi} \in C_\circ(\mathbb{R}^+, E_v)$. Then we have the following.*

(a) *If $x_\varphi\,|_{\mathbb{R}^+} \in AAP(\mathbb{R}^+, X)$, then there exist unique elements $\psi \in \omega(\varphi)$ and $\rho \in C_\circ(\mathbb{R}^+, X)$ such that*
 (i) *$x_\varphi\,|_{\mathbb{R}^+} = x_\psi\,|_{\mathbb{R}^+} + \rho$, and*
 (ii) *$x_\psi\,|_{\mathbb{R}^+}$ is almost periodic.*

(b) *If $x_\varphi\,|_{\mathbb{R}^+}$ is Eberlein- weakly almost periodic and, in addition, $clD(A)$ is convex, and φ is such that*
 (*) *weak convergence of $(S(t_n)\varphi)_n$ to $\psi \in clD(A)$ for some sequence $0 \leq t_n \longrightarrow \infty$ implies weak convergence of $(S(h + t_n)\varphi)_n$ to $S(h)\psi$ for all $h > 0$,*

 then there exist unique elements $\psi \in \omega_w(\varphi)$ and $\rho \in W_\circ(\mathbb{R}^+, X)$ such that
 (i) *$x_\varphi\,|_{\mathbb{R}^+} = x_\psi\,|_{\mathbb{R}^+} + \rho$, and*
 (ii) *$x_\psi\,|_{\mathbb{R}^+}$ is almost periodic.*

Furthermore, in both (a) *and* (b), *if $\varphi \in \hat{D}(A)$ and $\alpha \geq M$, then $\psi \in \hat{D}(A)$ as well.*

Proof. Part (a): If $x_\varphi\,|_{\mathbb{R}^+} \in AAP(\mathbb{R}^+, X)$, then Theorem 2.4(b)(2) implies that $S(\cdot)\varphi \in AAP(\mathbb{R}^+, E_v)$. Hence, according to [32, Theorem 2.2], there exists a unique $\psi \in \omega(\varphi) \subset clD(A)$ such that

$$S(\cdot)\psi = f\,|_{\mathbb{R}^+} \quad \text{for some} \quad f \in AP(\mathbb{R}, E_v), \quad \text{and} \quad \lim_{t\to\infty} \|S(t)\varphi - S(t)\psi\|_v = 0.$$

Clearly, the function $g(t) := f(t)(0)$, $t \in \mathbb{R}$, is an element of $AP(\mathbb{R}, X)$. Moreover, we have

$$x_\psi(t) = (S(t)\psi)(0) = f(t)(0) = g(t) \quad \text{for} \quad t \geq 0, \quad \text{and,}$$
$$\|x_\varphi(t) - x_\psi(t)\| = \|(S(t)\varphi)(0) - (S(t)\psi)(0)\| \leq \|S(t)\varphi - S(t)\psi\|_v \to 0 \text{ as } t \to \infty.$$

Part (b): If $x_\varphi |_{\mathbb{R}^+} \in W(\mathbb{R}^+, X)$, then Theorem 2.6(b)(2) implies $S(\cdot)\varphi \in W(\mathbb{R}^+, E_v)$. Thus, according to assumption (*) and [37, Theorem 2.4], there exist unique elements $\psi \in \omega_w(\varphi) \subset clD(A)$, $g \in W_\circ(\mathbb{R}^+, E_v)$, and $f \in AP(\mathbb{R}, E_v)$ such that $S(\cdot)\varphi = S(\cdot)\psi + g$ and $S(\cdot)\psi = f |_{\mathbb{R}^+}$. Clearly, $x_\psi |_{\mathbb{R}^+} = S(\cdot)\psi(0) = f(\cdot) |_{\mathbb{R}^+} (0)$ is almost periodic, $\rho := g(\cdot)(0) \in W_\circ(\mathbb{R}^+, X)$, and $x_\varphi |_{\mathbb{R}^+} = S(\cdot)\varphi(0) = S(\cdot)\psi(0) + g(\cdot)(0) = x_\psi |_{\mathbb{R}^+} + \rho$.

In order to prove the final statement of Corollary 2.8, assume that $\varphi \in \hat{D}(A)$ and $\alpha \geq M$. We have to show that $\psi \in \hat{D}(A)$. To this end, choose a sequence $0 \leq t_n \longrightarrow \infty$ such that weak-lim $S(t_n)\varphi = \psi$, and let $h > 0$. Then, weak-lim $S(h + t_n)\varphi = S(h)\psi$ (for case (b) use assumption (*)), and, since $\gamma = 0$,

$$\frac{1}{h}\|S(h)\psi - \psi\|_v \leq \liminf_n \frac{1}{h}\|S(h + t_n)\varphi - S(t_n)\varphi\|_v \leq \frac{1}{h}\|S(h)\varphi - \varphi\|_v.$$

According to [9, Theorem 1], we have

$$|A\psi| = L(\psi) := \liminf_{h \searrow 0} \frac{1}{h}\|S(h)\psi - \psi\|_v \leq L(\varphi) = |A\varphi| < \infty,$$

and thus $\psi \in \hat{D}(A)$ as well. This completes the proof of Corollary 2.8.

We now proceed to combine Theorem 2.4, Theorem 2.6, and Corollary 2.8 with our results from [32, 37] on the asymptotic behavior of motions of operator semigroups in order to derive sufficient conditions for the existence of asymptotically almost periodic, Eberlein-weakly almost periodic, and almost periodic solutions to (FDE).

Theorem 2.9. *Assume that v satisfies* (v1)–(v3), $\alpha \geq M$, *and either*

(i) $\varphi \in clD(A) \cap Cv_\circ(\mathbb{R}^-, X)$ *or*
(ii) $\varphi \in clD(A)$ *and v satisfies* (v3*).

Then we have the following.

(a) *If $x_\varphi(\mathbb{R}^+)$ is relatively compact in X, then $x_\varphi |_{\mathbb{R}^+}$ is asymptotically almost periodic.*
(b) *If $\dim X < \infty$, and there exists $\rho \in clD(A)$ such that x_ρ is bounded on $\mathbb{R}^+$, then $x_\varphi |_{\mathbb{R}^+}$ is asymptotically almost periodic.*
(c) *Under the conditions of either* (a) *or* (b), *there exists a unique element $\psi \in clD(A)$ such that*
  (i) *$x_\psi |_{\mathbb{R}^+}$ is almost periodic, and*
  (ii) $\lim_{t\to\infty}\|x_\varphi(t) - x_\psi(t)\| = 0$.

  Moreover, if $\varphi \in \hat{D}(A)$, then so is ψ.

Remark 2.10. As in Remark 2.5 above, we consider the particular example of [45, Section 5]. According to [45, Remark 5.1], if $M = \alpha$ and the solution semigroup $(S(t))_{t\geq 0}$ has a fixed point, then all trajectories $S(\cdot)\varphi : \mathbb{R}^+ \to E_v, \varphi \in E_v$, are bounded. (Recall that here $v(s) = e^{\mu s}$, for some $\mu > 0$, $s \leq 0$, $X = \mathbb{R}^n$, and $B : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous.) In view of the Remark following Proposition 2.3, Theorems 2.4(b) and 2.9(b) apply in this situation (and with $M \leq \alpha$) to show that all trajectories $S(\cdot)\varphi : \mathbb{R}^+ \to E_v, \varphi \in E_v$, are even asymptotically almost periodic. Also, the conclusions of 2.9(c) hold for each $\varphi \in E_v$.

Proof of Theorem 2.9. Part (a): Since $\gamma = 0$, Theorem 2.4(a) (in conjunction with Lemma 2.2(b)) applies to show that if $x_\varphi(\mathbb{R}^+)$ is relatively compact in X, then $S(\cdot)\varphi$ has relatively compact range in E_v. Hence, according to [32, Theorem 2.2], $S(\cdot)\varphi : \mathbb{R}^+ \to E_v$ is asymptotically almost periodic. Theorem 2.4(b)(1) then shows that $x_\varphi \mid_{\mathbb{R}^+}$ is asymptotically almost periodic as well.

Part (b) : Again since $\gamma = 0$, if $x_\rho(\mathbb{R}^+)$ is bounded in X, then so also is $x_\varphi(\mathbb{R}^+)$ for any $\varphi \in clD(A)$. Thus, since $dim\, X < \infty$, it suffices to apply 2.9(a).

Since part (c) is a consequence of Corollary 2.8, the proof of Theorem 2.9 is complete.

Theorem 2.11. *Assume that v satisfies* (v1)–(v3), $\alpha \geq M$, *and that* $(S(t))_{t\geq 0}$ *consists of bounded linear operators* (*i.e.*, $\hat{E}$ *is a closed linear subspace of* $E_v, F : \hat{E} \to X$ *is bounded and linear, and* $B : D(B) \subset X \to X$ *is linear*). *Moreover, assume that either*

(i) $\varphi \in clD(A) \cap Cv_\circ(\mathbb{R}^-, X)$ *or*

(ii) $\varphi \in clD(A)$ *and v satisfies* (v3*).

If $x_\varphi(\mathbb{R}^+)$ is weakly relatively compact in X, then $x_\varphi \mid_{\mathbb{R}^+} \in W(\mathbb{R}^+, X)$, and there exists a unique element $\psi \in clD(A)$ such that

(iii) $x_\psi \mid_{\mathbb{R}^+}$ *is almost periodic, and*

(iv) $(x_\varphi - x_\psi) \mid_{\mathbb{R}^+} \in W_\circ(\mathbb{R}^+, X)$.

Moreover, if $\varphi \in \hat{D}(A)$, then so is ψ.

Proof. In this situation, we conclude from Theorem 2.6(a)(2) that the motion $S(\cdot)\varphi : \mathbb{R}^+ \to E_v$ has weakly relatively compact range. Hence, since $\gamma = 0$, [37, Theorem 2.1] applies to show that $S(\cdot)\varphi \in W(\mathbb{R}^+, E_v)$. Theorem 2.6(b)(1) then implies that $x_\varphi \mid_{\mathbb{R}^+}$ is Eberlein-weakly almost periodic. The remaining assertions are special cases of Corollary 2.8, and so the proof is complete.

Remark 2.12. We recall that under each of the conditions on X and B specified in Theorem 1.2(c) (cf. [40, Theorem 2.5]), the function x_φ given by (1.1) in Section 1.C is the unique solution to (FDE) for all $\varphi \in \hat{D}(A)$, and sometimes even for all $\varphi \in clD(A)$ (cf. 1.2(c4)). Thus, in these settings, the above results can be used to show existence of solutions to (FDE) which exhibit various periodicity properties. We close this section with an example to this effect.

Corollary 2.13. *Assume that v satisfies* (v1)–(v3), $\alpha \geq M$, *and that any of the assumptions of* [40, Theorem 2.5] (Theorem 1.2(c) above) *on X and B are fulfilled. If either*

(i) $\varphi \in \hat{D}(A) \cap Cv_\circ(\mathbb{R}^-, X)$, *or*

(ii) $\varphi \in \hat{D}(A)$ *and v satisfies* (v3*), *then the following assertions hold*:

(a) x_φ *is the unique solution to* (FDE);

(b) *if* $x_\varphi(\mathbb{R}^+)$ *is relatively compact in* X, *then* $x_\varphi|_{\mathbb{R}^+}$ *is asymptotically almost periodic, and there exists* $\psi \in \hat{D}(A)$ *such that*

  (i) x_ψ *is the unique solution to* (FDE) *with initial history* ψ,

  (ii) $x_\psi|_{\mathbb{R}^+}$ *is almost periodic, and*

  (iii) $\lim_{t\to\infty}\|x_\varphi(t) - x_\psi(t)\| = 0$.

3. Asymptotic stability of solutions to (FDE). The prototype results on stability for solutions to (FDE) are Corollaries 3.3 and 3.7 of [2]. These are linear results set in the following context:

(i) $dim\, X < \infty$, and $\hat{E} = E = BC_l(\mathbb{R}^-, X)$, the space of bounded continuous functions from $\mathbb{R}^-$ into X having a limit at $-\infty$ (i.e., $E = E_{v_l}$ for $v \equiv 1$);

(ii) the accretive operator $B \in B(X)$, the space of bounded linear operators from X into X;

(iii) $F : \hat{E} \to X$ has the form

$$F(\varphi) = \int_{-\infty}^{0} k(s)\varphi(s)\, ds,$$

where $k \in L^1(\mathbb{R}^-, B(X))$.

If, in addition, for some $\mu > 0$, the function $\hat{k} : \mathbb{R}^- \to B(X)$, defined by $\hat{k}(s) = \exp(-\mu s)k(s)$, is an element of $L^1(\mathbb{R}^-, B(X))$ with

(iv) $\left\|\hat{k}\right\|_1 = M < \alpha$, then, given $\varphi \in \hat{E}$,

(v) $\|x_\varphi(t)\| \le e^{-\gamma t}\|\varphi\|_\infty$ for all $t \ge 0$,

where $\gamma = \min\{\mu, \alpha - M\}$, and x_φ is the unique solution to (FDE) with initial history $\varphi \in E$.

Although this linear result could easily be derived from [40, Theorem 2.5] (Theorem 1.2 above), it is more instructive to note that the crucial point underlying this result is the fact that condition (iv) allows F to be extended from $\hat{E} = BC_l(\mathbb{R}^-, X)$ to a Lipschitz continuous function with Lipschitz-constant $M_v < \alpha$ on E_v for $v(s) = e^{\mu s}, s \le 0$. This extension property for the history-responsive function F sets the tone for the first of our stability results in the general nonlinear and local context (Theorem 3.1).

In turn, the above finite dimensional and linear stability result will become a special case of the subsequent development (see Corollary 3.11 below).

Theorem 3.1. *Consider* (FDE) *in the context of* $E = E_u$, *where* u *satisfies* (v1) *and* (v2), *and assume that* $M < \alpha$. *Furthermore, assume that there exists* $v : \mathbb{R}^- \to (0, 1]$ *satisfying* (v1) *and* (v2) *such that*

(i) $v \le u$,

(ii) $\frac{v(r)}{u(r)} \longrightarrow 0$ *as* $r \longrightarrow -\infty$, *and*

(iii) $\|F\varphi - F\psi\| \le M_v\|\varphi - \psi\|_v$ *for all* $\varphi, \psi \in \hat{E}_u$, *with* $0 \le M_v < \alpha$.

Finally, let $\varphi, \psi \in clD(A) \subset \hat{E}_u$, *and assume that* X, B, φ *and* ψ *are such that* [40, Theorem 2.5] (Theorem 1.2(c) above) *applies.*

If the corresponding solutions x_φ *and* x_ψ *to* (FDE) *have* $x_\varphi(\mathbb{R}^+)$ *and* $x_\psi(\mathbb{R}^+)$ *relatively compact in* X, *then*

(a) $x_\varphi |_{\mathbb{R}^+}$ *and* $x_\psi |_{\mathbb{R}^+}$ *are asymptotically almost periodic, and*
(b) $\lim_{t\to\infty} \|S(t)\varphi - S(t)\psi\|_v = 0$. *In particular,*
(c) $\lim_{t\to\infty} \|x_\varphi(t) - x_\psi(t)\| = 0$.

Remark. We shall actually show that, in the notation of Theorem 3.1,

$$\lim_{t\to\infty} \|S(t)\varphi - S(t)\psi\|_w = 0, \qquad \text{where } w = \sqrt{uv}.$$

Proof. Step 1: Let $w = \sqrt{uv}$. Then
(1) $v \le w \le u$;
(2) w satisfies (v1)–(v3);
(3) $\frac{v(r)}{w(r)} \longrightarrow 0$ and $\frac{w(r)}{u(r)} \longrightarrow 0$ as $r \longrightarrow -\infty$.
From (1), it is easy to see that $E_u \subset E_w \subset E_v$ with continuous embeddings. We next put

$$\hat{E}_w := cl_{E_w}(\hat{E}_u), \qquad \text{and} \qquad \hat{E}_v = cl_{E_v}(\hat{E}_u).$$

Then $\hat{E}_u \subset \hat{E}_w \subset \hat{E}_v$, and it follows from (iii) that $F : \hat{E}_u \to X$ has a unique Lipschitz continuous extension to $\hat{E}_v$ which (along with its restriction to $\hat{E}_w$) we also denote by F; this extension $F : \hat{E}_v \to X$ has Lipschitz-constant (at most) M_v. We claim that assumptions (A1)–(A3) (of Section 1.B) continue to hold if we replace $\hat{E}_u$ by $\hat{E}_w$ (and F by its extension to $\hat{E}_w$).

Since (A1) is trivially satisfied, we turn to (A2). Given $x \in \hat{X}, \psi \in \hat{E}_w$, and $\lambda > 0$ with $\lambda\gamma < 1$, let φ_x be the solution to

$$\varphi - \lambda\varphi' = \psi, \quad \varphi(0) = x.$$

Then $\varphi_x \in E_w$ ([40, Remark 3.2]); we would show $\varphi_x \in \hat{E}_w$. Take $(\psi_n)_n \subset \hat{E}_u$ such that $\|\psi_n - \psi\|_w \longrightarrow 0$, and let φ_n^x be the solution to

$$\varphi - \lambda\varphi' = \psi_n, \quad \varphi(0) = x,$$

$n \in \mathbb{N}$. By (A2) in the $\hat{E}_u$-context, $(\varphi_n^x)_n \subset \hat{E}_u$. Moreover, for $r \le 0$, we have

$$\begin{aligned} w(r)\|\varphi_x(r) - \varphi_n^x(r)\| &\le w(r)\frac{\exp(\frac{r}{\lambda})}{\lambda}\int_r^0 \exp(\frac{-\xi}{\lambda})\,\|\psi(\xi) - \psi_n(\xi)\|d\xi \\ &= \frac{\exp(\frac{r}{\lambda})}{\lambda}\int_r^0 \exp(\frac{-\xi}{\lambda})\frac{w(r)}{w(\xi)}w(\xi)\|\psi(\xi) - \psi_n(\xi)\|d\xi \\ &\le \|\psi - \psi_n\|_w(1 - \exp(\frac{r}{\lambda})) \le \|\psi - \psi_n\|_w. \end{aligned}$$

Hence, $\varphi_x \in cl_{E_w}(\hat{E}_u) = \hat{E}_w$. Turning to (A3), we use the setting and notation of the foregoing argument. Then

$$y_n := \frac{1}{1+\lambda\alpha}(\psi_n(0) + \lambda F(\varphi_n^x)) \in (I + \frac{\lambda}{1+\lambda\alpha}B)(D(B) \cap \hat{X}), \quad n \in \mathbb{N},$$

and

$$\lim_n \left\| y_n - \frac{1}{1+\lambda\alpha}(\psi(0) + \lambda F(\varphi_x)) \right\| = 0.$$

If we choose any sequence $(x_n)_n \subset D(B) \cap \hat{X}$ such that $[x_n, y_n] \in (I + \frac{\lambda}{1+\lambda\alpha}B)$, then $\|x_n - x_m\| \le \|y_n - y_m\|$, whereby $(x_n)_n$ converges to some $x_\circ \in \hat{X}$. As B is closed under any of the assumptions of Theorem 1.2(c), we conclude that $x_\circ \in D(B)$ and

$$[x_\circ, \frac{1}{1+\lambda\alpha}(\psi(0) + \lambda F(\varphi_x))] \in (I + \frac{\lambda}{1+\lambda\alpha}B).$$

This shows that (A3) is fulfilled.

At this point, if we denote the solution semigroups and their generators obtained in $\hat{E}_u$ and $\hat{E}_w$ via Theorem 1.2 by $(S_u(t))_{t\ge 0}, (S_w(t))_{t\ge 0}$, and A_u, A_w, respectively, then it is easy to see that

(4) $D(A_u) \subset D(A_w), cl_{E_u} D(A_u) \subset cl_{E_w} D(A_w), \hat{D}(A_u) \subset \hat{D}(A_w)$, and

(5) $S_w(t)|_{cl_{E_u} D(A_u)} = S_u(t)$.

Thus, there is no need to distinguish between the operators in these two settings, and we shall delete the subscripts.

Step 2: Assume $\varphi, \psi \in clD(A) \subset \hat{E}_u$ have been chosen as prescribed by Theorem 3.1 with $x_\varphi(\mathbb{R}^+)$ and $x_\psi(\mathbb{R}^+)$ relatively compact in X. First, note that $E_u \subset Cw_\circ(\mathbb{R}^-, X)$ since, given any $\varphi \in E_u, w(r)\|\varphi(r)\| \le \frac{w(r)}{u(r)}\|\varphi\|_u \longrightarrow 0$ as $r \longrightarrow -\infty$ by (3). As w also satisfies (v3) (as noted in (2)), we conclude from Theorems 2.4 and 2.9 together with Corollary 2.8 (and its proof) that

(6) $S(\cdot)\varphi, S(\cdot)\psi : \mathbb{R}^+ \to E_w$ are asymptotically almost periodic, and

(7) there exist $\hat{\varphi}, \hat{\psi} \in clD(A_w)$ such that $S(\cdot)\hat{\varphi}, S(\cdot)\hat{\psi}$ are almost periodic, and $(S(\cdot)\varphi - S(\cdot)\hat{\varphi}), (S(\cdot)\psi - S(\cdot)\hat{\psi}) \in C_\circ(\mathbb{R}^+, E_w)$.

Moreover, $\hat{\varphi}, \hat{\psi} \in \hat{D}(A_w)$ whenever $\varphi, \psi \in \hat{D}(A_w)$. In particular, it follows from (4) that Theorem 1.2(c) applies to show that $x_{\hat{\varphi}}$ and $x_{\hat{\psi}}$ are the unique solutions to (FDE) for $\hat{\varphi}$ and $\hat{\psi}$, respectively.

We claim $x_{\hat{\varphi}}|_{\mathbb{R}^+} = x_{\hat{\psi}}|_{\mathbb{R}^+}$. Supposing that this is not the case, consider $x = (x_{\hat{\varphi}} - x_{\hat{\psi}})|_{\mathbb{R}^+} \in AP(\mathbb{R}, X)|_{\mathbb{R}^+}$ and $\rho = (\hat{\varphi} - \hat{\psi}) \in E_w$. Next, choose $R > 0$ such that (by (3))

(8) $\frac{v(r)}{w(r)} < \frac{\|x\|_\infty}{\|\rho\|_w}$ for all $r \le -R$.

For $t \geq R$ and $r \leq 0$,

$$v(r)\left\|S(t)\hat{\varphi}(r) - S(t)\hat{\psi}(r)\right\| = \begin{cases} v(r)\left\|\hat{\varphi}(t+r) - \hat{\psi}(t+r)\right\|, & r \leq -t \\ v(r)\left\|x_{\hat{\varphi}}(t+r) - x_{\hat{\psi}}(t+r)\right\|, & -t \leq r \leq 0 \end{cases}$$
$$\leq \begin{cases} \frac{v(r)}{w(r)}\frac{w(r)}{w(t+r)}w(t+r)\|\rho(t+r)\|, & r \leq -t \\ \|x\|_\infty, & -t \leq r \leq 0 \end{cases}$$
$$\leq \|x\|_\infty.$$

Hence, we have

(9) $\left\|S(t)\hat{\varphi} - S(t)\hat{\psi}\right\|_v \leq \|x\|_\infty$ for all $t \geq R$.

As $x_{\hat{\varphi}}, x_{\hat{\psi}}$ solve (FDE), we conclude from the arguments in the uniqueness proof of [40, Proposition 2.4] that

$$\|x(t)\|\frac{d}{ds}\|x(s)\|\,|_{s=t} \leq M_v\|x(t)\|\|x_t\|_v - \alpha\|x(t)\|^2, \qquad \text{and thus}$$

(10) $\frac{d}{dt}\left(e^{2\alpha t}\|x(t)\|^2\right) \leq 2M_v e^{2\alpha t}\|x(t)\|\|x_t\|$ for almost every $t \geq 0$.

Since $x \in AP(\mathbb{R}, X)\,|_{\mathbb{R}^+}$, there exists a sequence $(t_n)_n$ with $R \leq t_n \nearrow \infty$ such that

(11) $\|x(t_n)\| \to \|x\|_\infty$ as $n \to \infty$.

Integrating (10) from R to t_n, multiplying through by $e^{-2\alpha t_n}$ and invoking (9), we arrive at

$$\|x(t_n)\|^2 \leq e^{-2\alpha(t_n - R)}\|x(R)\|^2 + 2M_v\int_R^{t_n} e^{-2\alpha(t_n-\tau)}\|x(\tau)\|\,\|x_\tau\|_v d\tau$$
$$\leq e^{-2\alpha(t_n - R)}\|x(R)\|^2 + \frac{M_v}{\alpha}(1 - e^{-2\alpha(t_n-R)})\|x\|_\infty^2.$$

Taking the limit as $n \to \infty$, and invoking (11) and the fact that $M_v < \alpha$, leads to a contradiction. We have thus shown that $x_{\hat{\varphi}}\,|_{\mathbb{R}^+} = x_{\hat{\psi}}\,|_{\mathbb{R}^+}$.

Teamed with (9), this shows that $S(t)\hat{\varphi} = S(t)\hat{\psi}$ for all $t \geq R$. However, by (7), $S(\cdot)\hat{\varphi}$ and $S(\cdot)\hat{\psi}$ are (restrictions to $\mathbb{R}^+$ of) almost periodic functions into E_w, and thus $S(t)\hat{\varphi} = S(t)\hat{\psi}$ for all $t \in \mathbb{R}^+$. Invoking (7) once again, this reveals that

$$\lim_{t\to\infty}\|S(t)\varphi - S(t)\psi\|_w = 0.$$

This proves the assertion of the Remark following Theorem 3.1, and clearly implies the assertions (b) and (c) of Theorem 3.1, thus completing the proof.

Remark 3.2. In the context of Theorem 1.2 (for the case $E = E_v$), if $\alpha \geq M$, then (FDE) is always stable in the sense that

$$\|S(t)\varphi - S(t)\psi\|_v \leq \|\varphi - \psi\|_v$$

for all $t \geq 0$ and all $\varphi, \psi \in clD(A)$; Theorem 3.1 gives conditions under which (FDE) is asymptotically stable. As Example 4.1.A (in the following section) shows, however, even when $\alpha > M$, (FDE) can fail to be asymptotically stable if the history-responsive function $F : \hat{E} \to X$ does not have a Lipschitz continuous extension as specified in Theorem 3.1.

In special situations, Theorem 3.1 can be extended considerably. We next present one particular instance.

Corollary 3.3. *If, under the assumptions of Theorem* 3.1*, we additionally have*

(i) $\hat{E}_u = E_u$, *and*
(ii) $D(B) = X$, *and* $B : X \to X$ *is single-valued, continuous, accretive, and transforms bounded sets into bounded sets, then*
(a) $\lim_{t\to\infty} \|S(t)\varphi - S(t)\psi\|_v = 0$, *and, in particular,*
(b) $\lim_{t\to\infty} \|x_\varphi(t) - x_\psi(t)\| = 0$

for all $\varphi, \psi \in Cu_b(\mathbb{R}^-, X)$ *for which* $x_\varphi(\mathbb{R}^+)$ *and* $x_\psi(\mathbb{R}^+)$ *are relatively compact in* X. *Moreover, if* $B(0) = 0 = F(0)$, *then* $\lim_{t\to\infty} \|x_\varphi(t)\| = 0$ *for all* $\varphi \in Cu_b(\mathbb{R}^-, X)$ *for which* $x_\varphi(\mathbb{R}^+)$ *is relatively compact in* X.

Proof. As can be seen by a glance at the preceding proof, the hypotheses of Theorem 3.1 also hold if u is replaced by $w = \sqrt{uv}$. Under the conditions of 3.3, we have $\hat{E}_w = Cw_\circ(\mathbb{R}^-, X)$. According to the Remark following Proposition 2.3, moreover, $cl\,D(A_w) = \hat{E}_w$, and x_φ is the unique solution to (FDE) for all $\varphi \in \hat{E}_w$. Thus, it only remains to observe that $Cu_b(\mathbb{R}^-, X) \subset Cw_\circ(\mathbb{R}^-, X)$ in order to conclude the argument. □

We briefly pause to consider an example [23, Example 3.4] which illustrates the use of Theorem 3.1 and Corollary 3.3.

Example 3.4. Consider the scalar linear (FDE)

$$\begin{aligned} &\dot{x}(t) + \alpha x(t) = bx(t-r) + \int_{-\infty}^{t} C(t-s)x(s)\,ds, \;\; t \geq 0, \\ &x|_{\mathbb{R}^-} = \varphi, \end{aligned} \tag{3.1}$$

where $\alpha > 0$, $b \in \mathbb{R}$, $r \geq 0$, and $C \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ such that $(|b| + \|C\|_{L^1(\mathbb{R}^+)}) = m < \alpha$. In addition, given m^* such that $m < m^* < \alpha$, let u be a weight satisfying (v1) - (v3) with $u \equiv 1$ on $[-r, 0]$ such that

$$|b| + \int_{-\infty}^{0} |C(-s)| \frac{1}{u(s)} ds < m^* \tag{3.2}$$

(see [1]). Then, given any $\varphi \in Cu_b(\mathbb{R}^-)$, the unique solution x_φ to (3.1) satisfies $x_\varphi(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

Proof. As noted in [23, Example 3.4], there exists a weight u^* satisfying (v1)–(v3) with $u^* \equiv 1$ on $[-r, 0]$ such that

$$|b| + \int_{-\infty}^{0} |C(-s)| \frac{1}{u(s)u^*(s)}\, ds < m^*.$$

Taking $v := u\,u^*$, we conclude from (3.2) that the assumptions of Corollary 3.3 are fulfilled. As noted in the proof of 3.3, x_φ is the unique solution of (3.1) corresponding to any $\varphi \in Cu_b(\mathbb{R}^-)$. Moreover, since the zero function is a solution to (3.1), Theorem 2.9(b) shows that $x_\varphi(\mathbb{R}^+)$ is relatively compact for each $\varphi \in Cu_b(\mathbb{R}^-)$, and we thus arrive at the desired conclusion.

Remarks 3.5. 1. The use of Corollary 3.3 in the above proof avoids the more elaborate method of constructing appropriate Lyapunov-Razumikhin pairs for equation (3.1) used in [23].

2. Furthermore, our approach to (3.1) readily extends to yield the same conclusion for the more general problem

$$\begin{aligned} &\dot{x}(t) + \alpha x(t) + Bx(t) = Dx(t-r) + \int_{-\infty}^{t} C(t-s)x(s)\,ds, \quad t \geq 0, \\ &x|_{\mathbb{R}^-} = \varphi, \end{aligned} \tag{3.3}$$

where X is any finite dimensional Banach space, $\alpha > 0, r \geq 0$, $C \in C(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $D, B \in B(X)$, $\|D\| + \|C\|_{L^1(\mathbb{R}^+)} = m < \alpha$, B is accretive, and $\varphi \in Cu_b(\mathbb{R}^-, X)$, where the weight u is chosen as in Example 3.4.

We now turn to the question of exponential asymptotic stability. Here, we consider a restricted class of weights for which $v(s) = e^{\mu s}$ with $\mu > 0$, $s \leq 0$, is the basic example.

Theorem 3.6. *Assume that $v : \mathbb{R}^- \to (0,1]$ is a weight satisfying* (v1) *and* (v2) *such that $s \longmapsto v(s)e^{-\mu s}$ is nondecreasing on $\mathbb{R}^-$ for some $\mu > 0$, and put $\beta = \min\{\mu, \alpha - M\}$. Then the solution semigroup $(S(t))_{t\geq 0}$ for* (FDE) *satisfies*

$$\|S(t)\varphi - S(t)\psi\|_v \leq e^{-\beta t}\,\|\varphi - \psi\|_v$$

for all $t \geq 0$ and all $\varphi, \psi \in clD(A)$. In particular,

$$\|x_\varphi(t) - x_\psi(t)\| \leq e^{-\beta t}\,\|\varphi - \psi\|_v,$$

for all $t \geq 0$ and all $\varphi, \psi \in clD(A)$.

Corollary 3.7. *Under the assumptions of Theorem* 3.6, *if $\alpha > M$ (i.e., $\beta > 0$) and, in addition, there exists $\varphi \in clD(A)$ such that $S(\cdot)\varphi : \mathbb{R}^+ \to E_v$ is bounded, then there exists exactly one fixed point $\varphi_\circ \in clD(A)$ of $(S(t))_{t\geq 0}$. In particular, we have*

$$\|S(t)\varphi - \varphi_\circ\|_v \leq e^{-\beta t}\,\|\varphi - \varphi_\circ\|_v$$

and, for $x_\circ = \varphi_\circ(0)$,

$$\|x_\varphi(t) - x_\circ\| \leq e^{-\beta t}\,\|\varphi - \varphi_\circ\|_v$$

for all $t \geq 0$ and all $\varphi \in clD(A)$.

Example 3.8. 1. Example 4.1.A in the following section shows that, for general weights v, having $\alpha > M$ is not sufficient to even guarantee asymptotic stability of

(FDE). In Example 4.1.B, we further note that (FDE) can be asymptotically stable, but yet still fail to be exponentially asymptotically stable in the case $\alpha > M$.

2. For exponential asymptotic stability results related to Theorem 3.6 and Corollary 3.7, see [29, Section 4].

Proof of Theorem 3.6. According to Theorem 1.2 (also see [40, Theorem 2.1]), and referring to [10, Theorem I], it will suffice to show that the operator $(A-\beta I)$ is accretive in E_v. We start with the following observation. Given $x \in X, \lambda > 0$ and $\psi \in E_v$,

$$\varphi(s) = \exp(\frac{s}{\lambda})x + \frac{1}{\lambda}\exp(\frac{s}{\lambda})\int_s^0 \exp(-\frac{\xi}{\lambda})\psi(\xi)\,d\xi, \quad s \le 0,$$

is the solution to

$$\varphi - \lambda\varphi' = \psi, \quad \varphi(0) = x, \tag{1}$$

and, for $s \le 0$,

$$\begin{aligned} v(s)\|\varphi(s)\| &\le v(s)e^{-\mu s}\exp((\mu+\frac{1}{\lambda})s)\|x\| \\ &\quad + v(s)\exp(-\mu s)\frac{1}{\lambda}\exp((\mu+\frac{1}{\lambda})s)\int_s^0 \exp(-\frac{\xi}{\lambda})\|\psi(\xi)\|\,d\xi \\ &\le \exp((\mu+\frac{1}{\lambda})s)\|x\| + \frac{1}{\lambda}\exp((\mu+\frac{1}{\lambda})s)\int_s^0 \exp(-(\mu+\frac{1}{\lambda})\xi)v(\xi)\|\psi(\xi)\|\,d\xi \\ &\le \exp((\mu+\frac{1}{\lambda})s)\|x\| + (1-\exp((\mu+\frac{1}{\lambda})s))\frac{1}{1+\lambda\mu}\|\psi\|_v \end{aligned} \tag{2}$$

(since the function $s \mapsto v(s)e^{-\mu s}$ is nondecreasing on $\mathbb{R}^-$). Now, take $\lambda > 0$ such that $\lambda\beta < 1$, let $\varphi_i \in D(A)$, and put $\psi_i = (I+\lambda(A-\beta I))\varphi_i, i \in \{1,2\}$. Then

$$(\varphi_1-\varphi_2) - \frac{\lambda}{1-\lambda\beta}(\varphi_1-\varphi_2)' = \frac{1}{1-\lambda\beta}(\psi_1-\psi_2), \qquad \text{and} \tag{3}$$

$$\varphi_i(0) = (I+\frac{\lambda}{1+\lambda(\alpha-\beta)}B)^{-1}[\frac{1}{1+\lambda(\alpha-\beta)}(\psi_i(0)+\lambda F(\varphi_i))], \quad i \in \{1,2\}, \tag{4}$$

where (4) follows from the fact that $\varphi_i \in D(A)$, $i \in \{1,2\}$.

Setting $\varphi = (\varphi_1-\varphi_2), x = (\varphi_1(0)-\varphi_2(0))$, and $\psi = (\psi_1-\psi_2)$, we conclude from (1) - (3) that, for $s \le 0$,

$$\begin{aligned} v(s)\|\varphi(s)\| &\le \frac{1}{1+\lambda(\mu-\beta)}\|\psi\|_v \\ &\quad + \exp((\mu+\frac{1-\lambda\beta}{\lambda})s)(\|x\| - \frac{1}{1+\lambda(\mu-\beta)}\|\psi\|_v). \end{aligned} \tag{5}$$

If $\|x\| = \|\varphi(0)\| \le \frac{1}{1+\lambda(\mu-\beta)}\|\psi\|_v$, then we read from (5) that

$$\|\varphi\|_v \le \frac{1}{1+\lambda(\mu-\beta)}\|\psi\|_v \le \|\psi\|_v.$$

On the other hand, if $\|\varphi(0)\| = \|x\| > \frac{1}{1+\lambda(\mu-\beta)}\|\psi\|_v$, then we obtain from (5) and (4) that

$$\|\varphi\|_v \leq \|\varphi_1(0) - \varphi_2(0)\| \leq \frac{1}{1+\lambda(\alpha-\beta)}[\|\psi\|_v + \lambda M\|\varphi\|_v],$$

which implies that

$$[1 + \lambda((\alpha - M) - \beta)]\|\varphi\|_v \leq \|\psi\|_v.$$

Thus,

$$\|\varphi_1 - \varphi_2\|_v \leq \|\psi_1 - \psi_2\|_v$$

in any event; i.e., $(A - \beta I)$ is indeed accretive. This completes the proof of Theorem 3.6.

Proof of Corollary 3.7. In view of Theorem 3.6, we need only show the existence of a unique fixed point $\varphi_\circ$ for $(S(t))_{t\geq 0}$ in case $S(\cdot)\varphi : \mathbb{R}^+ \to E_v$ is bounded for some $\varphi \in clD(A)$. Starting from this assumption, let $(\omega_n)_n \subset \mathbb{R}^+$ be any sequence such that $0 < \omega_n \nearrow \infty$, and let $K = \sup\{\|S(t)\varphi\|_v : t \geq 0\}$. Given $\epsilon > 0$, choose $n_\circ \in \mathbb{N}$ such that $2Ke^{-\beta\omega_n} < \epsilon$ for all $n \geq n_\circ$. Then, for $n \geq m \geq n_\circ$,

$$\|S(\omega_n)\varphi - S(\omega_m)\varphi\|_v \leq e^{-\beta\omega_m}\|S(\omega_n - \omega_m)\varphi - \varphi\|_v \leq 2Ke^{-\beta\omega_m} < \epsilon,$$

and hence $\|S(\omega_n) - \varphi_\circ\|_v \longrightarrow 0$ for some $\varphi_\circ \in clD(A)$. It is now easy to see that $\varphi_\circ$ is a fixed point of $(S(t))_{t\geq 0}$, and the only such, which completes the proof.

The following is an obvious consequence of Corollary 3.7 and [40, Proposition 2.9.2].

Corollary 3.9. *Assume that the weight v fulfills the assumptions of Theorem 3.6 (in particular, let $v(s) = e^{\mu s}$, $s \leq 0$, for some $\mu > 0$), and assume that $\alpha > M$. If $0 \in D(B)$ with $0 \in B(0)$, and $0 \in \hat{E}$ with $F(0) = 0$, then*

$$\|x_\varphi(t)\| \leq e^{-\beta t}\|\varphi\|_v \quad \textit{for all} \quad t \geq 0 \quad \textit{and all} \quad \varphi \in clD(A),$$

where $\beta = min\{\mu, \alpha - M\} > 0$. In particular, this conclusion holds in the linear case; i.e., if $\hat{E}$ is a closed linear subspace of E, and both $B : D(B) \to X$ and $F : \hat{E} \to X$ are linear.

Taken together with techniques from the proof of Theorem 3.1, Theorem 3.6 and Corollary 3.7 lead to the following exponential asymptotic stability result in the context $E = BUC(\mathbb{R}^-, X)$.

Theorem 3.10. *Consider* (FDE) *in the context $E = BUC(\mathbb{R}^-, X)$, and assume that $B \subset X \times X$ is closed. Moreover, assume that there exist constants $\mu > 0$ and $M < \alpha$ such that, for $v(s) = e^{\mu s}$, $s \leq 0$,*

$$\|F\varphi - F\psi\| \leq M\|\varphi - \psi\|_v \quad \textit{for all} \ \ \varphi, \psi \in \hat{E}.$$

Then, for $\beta := \min\{\mu, \alpha - M\} > 0$, we have the following:

(a) $\|S(t)\varphi - S(t)\psi\|_v \leq e^{-\beta t}\|\varphi - \psi\|_\infty$ *and*
$\|x_\varphi(t) - x_\psi(t)\| \leq e^{-\beta t}\|\varphi - \psi\|_\infty$
for all $t \geq 0$ and all $\varphi, \psi \in clD(A) \subset \hat{E}$.

(b) *If there exists* $\varphi \in clD(A)$ *such that* $S(\cdot)\varphi : \mathbb{R}^+ \to E_v$ *is bounded (in particular, if* A *is linear), then there exist unique elements* $x_\circ \in X$ *and* $\varphi_\circ \in cl_{E_v}(\hat{E})$ *(with* $x_\circ = 0 \in X$ *and* $\varphi_\circ = 0 \in \hat{E}$ *if* A *is linear) such that*

$$\|x_\varphi(t) - x_\circ\| \le e^{-\beta t} \|\varphi - \varphi_\circ\|_v$$

for all $\varphi \in clD(A)$ *and all* $t \ge 0$.

For a proof, it is enough to note that F can be extended to $\hat{E}_v = cl_{E_v}(\hat{E})$ so that (A1)–(A3) also hold in the context of E_v (compare the proof of Theorem 3.1), and then to invoke Theorem 3.6 and Corollary 3.7.

We close this section with an obvious consequence of Theorem 3.10 which extends Corollaries 3.3 and 3.7 of [2] mentioned at the beginning of the section.

Corollary 3.11. *Let* X *be a Banach space,* $B : D(B) \subset X \to X$ *a linear m-accretive operator, and* $\hat{E} = E = BUC(\mathbb{R}^-, X)$. *Moreover, assume that* $K : \mathbb{R}^- \to B(X)$ *(the bounded linear operators on* X*) is such that*

(i) $K(\cdot)x : \mathbb{R}^- \to X$ *is continuous for each* $x \in X$, *and*
(ii) *there exist* $\mu > 0$ *and* $w \in L^1(\mathbb{R}^-)$ *with* $\|w\|_{L^1(\mathbb{R}^-)} = M$ *such that* $\|e^{-\mu s}K(s)\| \le w(s)$ *for all* $s \le 0$.

Finally, take $\alpha > M$, *and put* $\beta = \min\{\mu, \alpha - M\}$. *If, for* $\varphi \in E$, *the triple* $\{X, B, \varphi\}$ *fulfills any of the assumptions of* [40, Theorem 2.5] (*Theorem* 1.2(c) *above*), *then the unique solution* $x_\varphi : \mathbb{R} \to X$ *of*

$$\begin{aligned} &\dot{x}(t) + \alpha x(t) + Bx(t) = \int_{-\infty}^{t} K(s-t)x(s)\,ds, \quad t \ge 0, \\ &x|_{\mathbb{R}^-} = \varphi \end{aligned} \tag{3.4}$$

has the property

$$\|x_\varphi(t)\| \le e^{-\beta t}\|\varphi\|_\infty \qquad \text{for all} \;\; t \ge 0.$$

In particular, if B *is a bounded linear operator, this assertion holds for all* $\varphi \in E$.

4. Examples and applications. In this section, we apply our stability results to concrete models – the delay logistic equation from population dynamics (Example 4.2) and the Goodwin oscillator from biology (Examples 4.3 and 4.4). The solution semigroup approach to these models has been discussed in [40, Section 4], and we largely take that development as the point of departure for our present work.

We start, however, with a two-part technical example that illustrates the need for various assumptions in the results established above.

Example 4.1. (Compare [40, Example 4.1]) **4.1 A**: Consider $u(r) \equiv 1$, $r \le 0$, and define $F : E_u = BUC(\mathbb{R}^-) \to \mathbb{R}$ by $F(\varphi) = \|\varphi\|_\infty$. Then, given $\varphi, \psi \in E_u$,

$$|F\varphi - F\psi| = |\|\varphi\|_\infty - \|\psi\|_\infty| \le \|\varphi - \psi\|_\infty,$$

whereby $M = 1$. For any $v : \mathbb{R}^- \to (0, 1]$ satisfying (v1)–(v3), however, if we put

$$\varphi_n(r) = \begin{cases} \frac{1}{v(r)}, & -n \leq r \leq 0 \\ \frac{1}{v(-n)}, & r \leq -n, \end{cases}$$

then $\varphi_n \in E_u \subset E_v$ and $\|\varphi_n\|_v = 1$, $n \in \mathbb{N}$, but $\lim_n \|\varphi_n\|_\infty = \infty$. Thus, given any $\alpha > 1$, there does not exist $M_v < \alpha$ such that

$$|F(0) - F(\varphi_n)| = \|\varphi_n\|_\infty \leq M_v \|\varphi_n\|_v.$$

Furthermore, taking $\hat{E} = E_u$ and $\hat{X} = \mathbb{R}$,

$$\dot{x}(t) + \frac{3}{2}x(t) = F(x_t),\ t \geq 0, \quad x\,|_{\mathbb{R}^-} = \varphi \tag{4.1}$$

is an instance of (FDE) which is not asymptotically stable. Indeed, given any $\beta > 0$, if we take

$$E_\beta = \{\varphi \in E_u : \|\varphi\|_\infty \leq \beta\} \quad \text{and} \quad X_\beta = \{x \in \mathbb{R} : |x| \leq \beta\}$$

in place of $\hat{E}$ and $\hat{X}$, Theorem 1.2 again applies to show that $\|(x_\varphi)_t\|_\infty \leq \beta$ for any $\varphi \in E_\beta$ and all $t \geq 0$. Hence, given any $\varphi \in E_u = clD(A)$,

$$x_\varphi(t) = (\varphi(0) - \frac{2}{3}\|\varphi\|_\infty) \exp(-\frac{3}{2}t) + \frac{2}{3}\|\varphi\|_\infty, \quad t \geq 0.$$

In addition to providing the example promised in Remark 3.2 (also compare Theorem 3.10), equation (4.1) also serves to illustrate Remark 2.7.1. For instance, if we take $\varphi(r) \equiv 1$, $r \leq 0$, then $\varphi \in E_u = clD(A)$ and the corresponding solution x_φ to (4.1) satisfies $\lim_{t\to\infty} x_\varphi(t) = \frac{2}{3}$ (hence $x_\varphi\,|_{\mathbb{R}^+} \in AAP(\mathbb{R}^+)$), but $\{S(t)\varphi : t \geq 0\}$ is not even weakly relatively compact in $E_u = BUC(\mathbb{R}^-)$. To see this, let $(t_n)_n$ be a sequence in $\mathbb{R}^+$ with $t_n \longrightarrow \infty$, and take $(r_m)_m$ to be a sequence in $\mathbb{R}^-$ with $r_m \longrightarrow -\infty$. Then

$$\lim_m \lim_n S(t_n)\varphi(r_m) = \lim_m \lim_{n \geq n_m} x_\varphi(t_n + r_m) = \frac{2}{3},$$

where $n \geq n_m \Rightarrow t_n + r_m \geq 0$, while

$$\lim_n \lim_m S(t_n)\varphi(r_m) = \lim_n \lim_{m \geq m_n} \varphi(t_n + r_m) = 1,$$

where $m \geq m_n \Rightarrow t_n + r_m \leq 0$. Thus, the double limits criterion [36, Theorem 2.1] fails to hold, which establishes our claim.

4.1.B: Again consider $u(r) \equiv 1$, $r \leq 0$, $\hat{E} = E_u = BUC(\mathbb{R}^-)$, and $\hat{X} = \mathbb{R}$, but now define $F : E_u \to \mathbb{R}$ by

$$F(\varphi) = \int_{-\infty}^0 \frac{\varphi(r)}{1 + r^2}\, dr,$$

and fix $\alpha > \frac{\pi}{2}$. Then

$$\dot{x}(t) + \alpha x(t) = F(x_t), \quad t \geq 0, \quad x|_{\mathbb{R}^-} = \varphi \tag{4.2}$$

is also an instance of (FDE). Moreover, taking $\varphi(r) \equiv 1$, $r \leq 0$, it follows that

$$\liminf_{t\to\infty} t|x_\varphi(t)| \geq \frac{1}{\alpha}$$

(cf. [29, Example 2, pp. 106-107]), whereby (4.2) is not exponentially asymptotically stable. On the other hand, it is known (cf. [29]) that (4.2) is asymptotically stable, and hence (4.2) together with (4.1) provide the examples advertised in Remark 3.8.2.

Contrary to the situation in [29], however, the asymptotic stability of (4.2) does follow from our methods. First note that $x_\varphi|_{\mathbb{R}^+}$ is bounded for each $\varphi \in E_u$. Since [1, Theorem 3.1] implies that there exists $v : \mathbb{R}^- \to (0,1]$ satisfying (v1)–(v3) such that

$$\int_{-\infty}^{0} \frac{1}{(1+r^2)v(r)}\, dr < \alpha,$$

Corollary 3.3 then applies to show that $\lim_{t\to\infty} \|x_\varphi(t)\| = 0$ for each $\varphi \in E_u = BUC(\mathbb{R}^-)$.

Example 4.2. Consider the delay logistic equation

$$\dot{x}(t) = x(t)(a - bx(t) - H(x_t)), \quad t \geq 0, \quad x|_{\mathbb{R}^-} = \varphi \in E, \tag{4.3}$$

where $a, b > 0$, $E = BUC(\mathbb{R}^-)$, $\varphi \in E$, and $H : E \to \mathbb{R}$ is defined by $H(\varphi) = \int_{-\infty}^{0} k(-r)\varphi(r)\, dr$ for fixed $k \in L^1(\mathbb{R}^+)$. Assume that $5\|k\|_1 < b$, and choose $\beta \in [\frac{a}{b-\|k\|_1}, \frac{a}{4\|k\|_1})$. Then $\beta \geq \frac{a}{b}$ with (i) $l(\beta) = \beta\|k\|_1 < a$ and (ii) $a + l(\beta) \leq b\beta$. Thus, according to [40, Example 4.2], equation (4.3) can be viewed as an instance of (FDE) in our local context, where $\alpha = 2(a - l(\beta))$, $\hat{E} = \{\varphi \in E : 0 \leq \varphi \leq \beta$ and $\frac{\alpha}{2b} \leq \varphi(0) \leq \beta\}$, $\hat{X} = [\frac{\alpha}{2b}, \beta]$, $B = bx^2 - \alpha x$ for $x \in [\frac{\alpha}{2b}, \infty)$, $F : \hat{E} \to \mathbb{R}$ is defined by $F(\varphi) = \varphi(0)(a - H(\varphi))$, and F has Lipschitz constant (at most) $M(\beta) = a + 2\beta\|k\|_1 < \alpha$. Moreover, Theorem 1.2(c4) applies for every $\varphi \in clD(A) = \hat{E}$, and the solution semigroup $(S(t))_{t\geq 0}$ is a contraction semigroup. Theorem 3.1 will show that much more is true.

According to [1, Theorem 3.1], given $\epsilon > 0$, there exists $v : \mathbb{R}^- \to (0,1]$ satisfying (v1)–(v3) such that

$$\int_{-\infty}^{0} \frac{|k(-r)|}{v(r)}\, dr < \|k\|_1 + \epsilon.$$

Thus, Theorem 3.1 will apply provided there exists $M_v < \alpha$ such that

$$|F\varphi - F\psi| \leq M_v \|\varphi - \psi\|_v \quad \text{for all } \varphi, \psi \in \hat{E}.$$

For $\varphi, \psi \in \hat{E}$, however, a straightforward calculation shows that

$$|F\varphi - F\psi| < (M(\beta) + \epsilon\beta)\|\varphi - \psi\|_v,$$

and $M_v = M(\beta) + \epsilon\beta < \alpha$ provided $\epsilon < \frac{\alpha - M(\beta)}{\beta}$. It remains to note that if

$$\varphi_e(s) = \frac{a}{b + \int_{-\infty}^{0} k(-r)dr}, \quad s \in \mathbb{R}^-,$$

then $\varphi_e \in \hat{E}$ with $S(t)\varphi_e = \varphi_e$, $t \geq 0$. Consequently, since the solution x_φ to (4.3) is clearly bounded for each $\varphi \in \hat{E}$, Theorem 3.1 yields that

$$\lim_{t \to \infty} x_\varphi(t) = \frac{a}{b + \int_{-\infty}^{0} k(-r)\, dr}$$

for every $\varphi \in \hat{E}$.

More generally, given any $H : \hat{E} \to \mathbb{R}$ and functions $l, m : \mathbb{R}^+ \to \mathbb{R}^+$ such that (H1) and (H2) of [40, Example 4.2] are satisfied, assume that

(1) there exists $\beta \geq \frac{a}{b}$ such that
  (i) $l(\beta) \leq a$,
  (ii) $a + l(\beta) \leq b\beta$, and
  (iii) $M(\beta) = a + l(\beta) + \beta m(\beta) < 2(a - l(\beta))$;

(2) for $\alpha = 2(a - l(\beta))$ and some $\epsilon \in (0, \frac{\alpha - M(\beta)}{\beta})$, there exists $v : \mathbb{R}^- \to (0, 1]$ satisfying (v1)–(v3) such that

$$|H(\varphi) - H(\psi)| \leq (m(\beta) + \epsilon)\|\varphi - \psi\|_v$$

for all $\varphi, \psi \in \hat{E}$.

Then we still have an instance of (FDE) (as described above), x_φ is a solution to (4.3) for every $\varphi \in \hat{E}$, there exists a (constant) function $\varphi_e \in \hat{E}$ such that $S(t)\varphi_e = \varphi_e$, $t \geq 0$, and

$$\lim_{t \to \infty} x_\varphi(t) = \varphi_e(0) \in [\frac{\alpha}{2b}, \beta] \qquad \text{for every } \ \varphi \in \hat{E}.$$

For further results related to Example 4.2, compare [21, Section 14.3].

Example 4.3. Here, we consider the Goodwin oscillator with infinite delay (cf. [26]) as a model for biochemical reaction sequences with end product inhibition described for $t \geq 0$ by the system

$$\begin{aligned} \dot{x}_1(t) + a_1 x_1(t) &= b_1 \Big[1 + \Big(\int_{-\infty}^{0} k(-s) x_n(t+s)\, ds \Big)^m \Big]^{-1}, \\ \dot{x}_i(t) + a_i x_i(t) &= b_i \int_{-\infty}^{0} k(-s) x_{i-1}(t+s)\, ds, \quad i \in \{2, \dots, n\}, \end{aligned} \tag{4.4}$$

where $a_i, b_i > 0$ for $i \in \{1, \dots, n\}$, $k \in L^1(\mathbb{R}^+) \cap C(\mathbb{R}^+)$ with $k \geq 0$, $m \in \mathbb{N}$, and $x_i|_{\mathbb{R}^-} \in BUC(\mathbb{R}^-, \mathbb{R}^+)$, $i \in \{1, \dots, n\}$. As in [40, Example 4.3], we shall restrict attention to chains of two reactions ($n = 2$) since the pattern remains the same for $n > 2$.

Taking $X = \mathbb{R}^2$ and $E = BUC(\mathbb{R}^-, X)$, we recall from [40, Example 4.3] that (4.4) is again a special case of (FDE) in the context of Section 1, where $\alpha = \min\{a_1, a_2\}$,

$$B = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} - \alpha I,$$

$\hat{E} = E^+$ is the cone of all $\varphi = \binom{\varphi_1}{\varphi_2} \in E$ such that $\varphi_1, \varphi_2 \geq 0$, $\hat{X} = X^+ = \{\binom{x_1}{x_2} \in \mathbb{R}^2 : x_1, x_2 \geq 0\}$, $F : \hat{E} \to \mathbb{R}^2$ is defined by

$$F\begin{pmatrix}\varphi_1 \\ \varphi_2\end{pmatrix} = \begin{pmatrix} b_1 \left[1 + (\int_{-\infty}^0 k(-s)\varphi_2(s)\,ds)^m\right]^{-1} \\ b_2 \int_{-\infty}^0 k(-s)\varphi_1(s)\,ds \end{pmatrix},$$

and F has Lipschitz constant (at most) $M = \|k\|_1 \left[(b_1 c(m))^2 + b_2^2\right]^{\frac{1}{2}}$ with $c(1) = 1$ and $c(m) = \frac{(m+1)^2}{4m}\left(\frac{m-1}{m+1}\right)^{\frac{m-1}{m}}$, $m > 1$. According to assertions 1 and 2 of [40, Example 4.3], moreover, Theorem 1.2(c4) applies for every $\varphi \in clD(A) = E^+$, and $(S(t))_{t\geq 0}$ has a unique fixed point $\varphi_e \in E^+$.

We now assume that $M < \alpha$, and choose

$$\epsilon \in \left(0, \frac{\alpha - M}{\left[(b_1 c(m))^2 + b_2^2\right]^{\frac{1}{2}}}\right).$$

Again applying [1, Theorem 3.1], there exists $v : \mathbb{R}^- \to (0, 1]$ satisfying (v1)–(v3) such that

$$\int_{-\infty}^0 \frac{k(-r)}{v(r)}\,dr < \|k\|_1 + \epsilon.$$

Furthermore, given $\varphi, \psi \in E^+$,

$$\|F(\varphi) - F(\psi)\|^2 \leq (b_1 c(m))^2 \left[\int_{-\infty}^0 k(-s)|\varphi_2(s) - \psi_2(s)|\,ds\right]^2 + b_2^2\Big[\int_{-\infty}^0 k(-s)|\varphi_1(s) - \psi_1(s)|\,ds\Big]^2,$$

whereby

$$\|F(\varphi) - F(\psi)\| \leq \left(M + \left[(b_1 c(m))^2 + b_2^2\right]^{\frac{1}{2}} \epsilon\right) \|\varphi - \psi\|_v.$$

In view of assertion 3 of [40, Example 4.3], if $\varphi \in E^+$, then the corresponding solution x_φ to (4.4) is bounded in case $M < \alpha$. Theorem 3.1 now shows that, in fact,

$$\lim_{t\to\infty} x_\varphi(t) = \varphi_e(0)$$

for every $\varphi \in E^+$.

Example 4.4. We consider the Goodwin oscillator with feedback (again for the case $n = 2$) as treated in [40, Example 4.4]. Here, the setting is that specified in the preceding example except that $F : E^+ \to \mathbb{R}^2$ is now defined by

$$F\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix} b_1\left[1 + (\int_{-\infty}^0 k(-s)\varphi_2(s)\,ds)^m\right]^{-1} \\ b_2\varphi_2(0)\int_{-\infty}^0 k(-s)\varphi_1(s)\,ds\end{pmatrix}.$$

To place this problem in the context of (FDE) with $M < \alpha$, put $l = 2b_1b_2\|k\|_1$, $p = b_1\|k\|_1 c(m)$, and assume that

$$p^2 + 2(p^4 + 3l^2)^{\frac{1}{2}} < 3\alpha^2.$$

Then $l < \alpha^2$, and there exists $\beta > 0$ such that

$$\alpha^2 - (\alpha^4 - l^2)^{\frac{1}{2}} \leq 2\left(b_2\|k\|_1\right)^2\beta^2 < \frac{1}{2}(\alpha^2 - p^2).$$

Now, again choosing $\hat{E} = \{\varphi \in E^+ : \|\varphi\|_\infty \leq \beta\}$ and $\hat{X} = \{x \in X^+ : \|x\| \leq \beta\}$, we again have an instance of (FDE), where $F|_{\hat{E}}$ has Lipschitz constant (at most) $M = \|k\|_1\left[(b_1c(m))^2 + (2b_2\beta)^2\right]^{\frac{1}{2}} < \alpha$, Theorem 1.2(c4) applies for every $\varphi \in clD(A) = \hat{E}$, and $(S(t))_{t\geq 0}$ has a unique fixed point $\varphi_e \in \hat{E}$ given by

$$\varphi_e(r) = \begin{pmatrix} b_1/a_1 \\ 0\end{pmatrix}, \quad r \leq 0.$$

To see that Theorem 3.1 applies, take

$$\epsilon \in \left(0, \frac{\alpha - M}{\left[(b_1c(m))^2 + (2b_2\beta)^2\right]^{\frac{1}{2}}}\right).$$

According to [1, Theorem 3.1], there exists $v : \mathbb{R}^- \to (0, 1]$ satisfying (v1)–(v3) such that

$$\int_{-\infty}^0 \frac{k(-r)}{v(r)}\,dr < \|k\|_1 + \epsilon.$$

Thus, for $\varphi, \psi \in \hat{E}$, since

$$\|F(\varphi) - F(\psi)\| \leq \left[(b_1c(m))^2 + (2b_2\beta)^2\right]^{\frac{1}{2}}(\|k\|_1 + \epsilon)\,\|\varphi - \psi\|_v,$$

and since x_φ is clearly bounded for each $\varphi \in \hat{E}$, we conclude from Theorem 3.1 that, given any $\varphi \in \hat{E}$,

$$\lim_{t\to\infty} x_\varphi(t) = \begin{pmatrix} b_1/a_1 \\ 0\end{pmatrix}.$$

Note added in proof. With regard to Theorem 1.2 above, it will be shown in [31] that in the general context of assumptions (A1)–(A3) of section 1 the function x_φ of (1.1) in Theorem 1.2 always is the unique mild solution to (FDE).

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