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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF CONSTRAINED SYSTEMS OF INTEGRAL EQUATIONS

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Abstract. The existence and uniqueness of a solution of a constrained system of integral equations is considered and the asymptotic behavior of this solution is studied.

1. Introduction. Differential algebraic equations (DAEs) arise as mathematical models of many dynamical processes, as, for instance, those occurring in electrical networks or constrained mechanical systems. They also appear as modifications of other mathematical models such as certain singularly perturbed ordinary differential equations (ODE's) or optimal control problems, see e.g. [1].

As pointed out in [13] DAE's should not always be considered stiff. But, certainly, stiffness does play an important role in the solution of DAE's as it does for ODE's. In [13] and [14] the relations between stiff DAEs and singularly perturbed DAEs are considered. The asymptotic behavior of solutions of the following singularly perturbed DAEs is studied in [13]:

$$\begin{aligned} x' &= f_1(x, y, z, \epsilon) \\ \epsilon y' &= f_2(x, y, z, \epsilon) \\ 0 &= f_3(x, y, z, \epsilon) \end{aligned} \tag{1.1a}$$

together with the initial conditions

$$x(0, \epsilon) = \xi(\epsilon), \quad y(0, \epsilon) = \eta(\epsilon), \quad z(0, \epsilon) = \zeta(\epsilon), \tag{1.1b}$$

where $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $z \in \mathbb{R}^k$ and $\epsilon \in \mathbb{R}^1$. More specifically, we introduce the following basic assumption about this system.

Assumption: There are non-empty open sets $\mathcal{D}_x \subset R^m$, $\mathcal{D}_y \subset R^n$, $\mathcal{D}_z \subset R^k$ and $\mathcal{J}_\epsilon \subset R^1$, $\mathcal{J}_\epsilon = \{\epsilon : |\epsilon| < \epsilon',\ \epsilon' > 0\}$, such that the mappings

$$\begin{aligned} &f_1 : \mathcal{D} \times \mathcal{J}_\epsilon \subset R^m \times R^n \times R^k \times R \to R^m, \quad \mathcal{D} = \mathcal{D}_x \times \mathcal{D}_y \times \mathcal{D}_z, \\ &f_2 : \mathcal{D} \times \mathcal{J}_\epsilon \subset R^m \times R^n \times R^k \times R \to R^n, \\ &f_3 : \mathcal{D} \times \mathcal{J}_\epsilon \subset R^m \times R^n \times R^k \times R \to R^k, \\ &\xi : \mathcal{J}_\epsilon \to \mathcal{D}_x, \quad \eta : \mathcal{J}_\epsilon \to \mathcal{D}_y, \quad \zeta : \mathcal{J}_\epsilon \to \mathcal{D}_z, \end{aligned}$$

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are continuous on the indicated domains. Moreover, for fixed $\epsilon \in \mathcal{J}_\epsilon$, f_1, f_2, f_3 are of class C^∞ on $\mathcal{D}$ and the initial point $(\xi(\epsilon),\ \eta(\epsilon),\ \zeta(\epsilon))$ satisfies the compatibility condition

$$f_3(\xi(\epsilon), \eta(\epsilon), \zeta(\epsilon), \epsilon) = 0.$$

Finally, assume that $f_2(x, y, z, 0) \not\equiv 0$ and $f_3(x, y, z, 0) \not\equiv 0$ in the domain $\mathcal{D}$, and that the Jacobian matrix

$$D_z f_3(\xi_0, \eta_0, \zeta_0, 0) \tag{1.2}$$

is nonsingular, where $\xi_0 = \xi(0),\ \eta_0 = \eta(0),\ \zeta_0 = \zeta(0)$.

In order to study the structure of this system, a so-called outer problem was introduced in [13], which is defined as

$$\begin{aligned} X' &= f_1(X, Y, Z, \epsilon), \\ \epsilon Y' &= f_2(X, Y, Z, \epsilon), \\ 0 &= f_3(X, Y, Z, \epsilon), \end{aligned} \tag{1.3a}$$

with some initial condition

$$X(0, \epsilon) = \xi^*(\epsilon) \tag{1.3b}$$

and the limiting assumption

$$X(t, 0) = X_0(t),\ Y(t, 0) = Y_0(t),\ Z(t, 0) = Z_0(t). \tag{1.3c}$$

Here $(X_0(t), Y_0(t), Z_0(t)),\ 0 \le t \le T$, is a solution of the following reduced problem:

$$\begin{aligned} X_0' &= f_1(X_0, Y_0, Z_0, 0), \\ 0 &= f_2(X_0, Y_0, Z_0, 0), \\ 0 &= f_3(X_0, Y_0, Z_0, 0), \end{aligned} \tag{1.4a}$$

with which we associate initial conditions of the form

$$X_0(0) = \xi_0, \quad Y_0(0) = Y_0^0, \quad Z_0(0) = Z_0^0. \tag{1.4b}$$

It can be proved that the solution of this reduced problem has a strong influence on the solution of (1.1) as ϵ becomes very small. But to prove it, we need to show that there exists an outer solution $(X(t,\epsilon), Y(t,\epsilon), Z(t,\epsilon))$ which can be expanded in terms of ϵ:

$$\begin{aligned} X(t, \epsilon) &= \sum_{i=0}^{N} X_i(t)\epsilon^i + O(\epsilon^{N+1}) \quad \text{as } \epsilon \to 0 \\ Y(t, \epsilon) &= \sum_{i=0}^{N} Y_i(t)\epsilon^i + O(\epsilon^{N+1}) \quad \text{as } \epsilon \to 0 \\ Z(t, \epsilon) &= \sum_{i=0}^{N} Z_i(t)\epsilon^i + O(\epsilon^{N+1}) \quad \text{as } \epsilon \to 0, \end{aligned} \tag{1.5}$$

which is assumed to hold uniformly for $0 \le t \le T$, $0 < \epsilon \le \epsilon_1$ $(\epsilon_1 \le \epsilon')$. Here the coefficient $(X_i(t), Y_i(t), Z_i(t))$, $i = 1, \dots, N$, satisfies a linear DAE of the form

$$\begin{aligned} \frac{dX_r}{dt} &= f_{1x}(t)X_r + f_{1y}(t)Y_r + f_{1z}(t)Z_r + p_r(t), \\ \frac{dY_{r-1}}{dt} &= f_{2x}(t)X_r + f_{2y}(t)Y_r + f_{2z}(t)Z_r + q_r(t), \\ 0 &= f_{3x}(t)X_r + f_{3y}(t)Y_r + f_{3z}(t)Z_r + r_r(t), \\ X_r(0) &= \xi_r^*, \end{aligned} \tag{1.6$_r$}$$

where

$$\begin{aligned} f_{ix}(t) &= D_x f_i(X_0(t), Y_0(t), Z_0(t), 0), \\ f_{iy}(t) &= D_y f_i(X_0(t), Y_0(t), Z_0(t), 0), \quad i = 1, 2, 3. \\ f_{iz}(t) &= D_z f_i(X_0(t), Y_0(t), Z_0(t), 0), \end{aligned}$$

and the terms $p_r(t), q_r(t)$ and $r_r(t)$ are polynomials in $X_1, Y_1, Z_1, \dots, X_{r-1}, Y_{r-1}, Z_{r-1}$ for which the coefficients are higher derivatives of the functions f_1, f_2, f_3 at the point $(X_0(t),\ Y_0(t),\ Z_0(t), 0)$. The right side ξ_r^* of the initial condition is the corresponding coefficient in the asymptotic expansion of $\xi^*(\epsilon)$ and p_r, q_r and r_r, $r = 0, 1, \dots, N$, are obtained recursively. Therefore $(p_r(t), q_r(t), r_r(t))$ is well defined on the interval $0 \le t \le T$ if the previous terms, $X_1, Y_1, Z_1, \dots, X_{r-1}, Y_{r-1}, Z_{r-1}$, are well defined on $[0, T]$. In order to prove the existence of such outer solution, we introduce a change of variables defined by the affine mapping

$$\mathcal{T}_t : R^m \times R^n \times R^k \to R^m \times R^n \times R^k; \quad \mathcal{T}_t(u, v, w) = (X, Y, Z), \tag{1.7a}$$

$$\begin{aligned} X &= u + \sum_{r=0}^{N} X_r(t)\epsilon^r, \\ Y &= v + \sum_{r=0}^{N} Y_r(t)\epsilon^r + A_1(t)u, \\ Z &= w + \sum_{r=0}^{N} Z_r(t)\epsilon^r + A_2(t)u + B_1(t)v. \end{aligned} \tag{1.7b}$$

Here A_1, A_2 and B_1 are chosen as

$$\begin{aligned} A_1(t) &= B(t)^{-1}(f_{2z}(t)(f_{3z}(t))^{-1} f_{3x}(t) - f_{2x}(t)), \\ A_2(t) &= -(f_{3z}(t))^{-1} f_{3y}(t) A_1(t) - (f_{3z}(t))^{-1} f_{3x}(t), \\ B_1(t) &= -(f_{3z}(t))^{-1} f_{3y}(t), \\ B(t) &= D_y f_2(\xi_0, Y_0(t), Z_0(t), 0) - D_z f_2(\xi_0, Y_0(t), Z_0(t), 0) \\ &\quad (D_z f_3(\xi_0, Y_0(t), Z_0(t), 0))^{-1} D_y f_3(\xi_0, Y_0(t), Z_0(t), 0) \end{aligned} \tag{1.8}$$

under the mapping $\mathcal{T}_t$ the system (1.3) becomes

$$\begin{aligned}\frac{du}{dt} &= C_1(t)u + L_1(t)v + E_1(t)w + \hat{F}_1(t,u,v,w,\epsilon)\\ \epsilon\frac{dv}{dt} &= B(t)v + E_2(t)w + \hat{F}_2(t,u,v,w,\epsilon)\\ 0 &= E_3(t)w + \hat{F}_3(t,u,v,w,\epsilon)\\ u(0,\epsilon) &= \xi^*(\epsilon) - \sum_{r=0}^{N}\xi_r^*\epsilon^r = \theta_N(\epsilon) = O(\epsilon^{N+1}).\end{aligned} \tag{1.9}$$

Here $E_2(t) = D_z f_2(\Omega_0(t))$, $E_3(t) = D_z f_3(\Omega_0(t))$, and $\hat{F}_1$, $\hat{F}_2$ and $\hat{F}_3$ satisfy Hypothesis (H), which will be defined later. Thus, we need to show that there exists a solution $(u(t,\epsilon), v(t,\epsilon), w(t,\epsilon))$ of the system (1.9) which satisfies

$$u(t,\epsilon) = O(\epsilon^{N+1}),\quad v(t,\epsilon) = O(\epsilon^{N+1}),\quad w(t,\epsilon) = O(\epsilon^{N+1}),\quad \text{as } \epsilon \to 0 \tag{1.10}$$

uniformly for $0 \le t \le T$. Instead of (1.9) we consider the constrained system of integral equations

$$\begin{aligned}u(t,\epsilon) &= \Phi(t)(\theta_N(\epsilon) + \int_0^t \Phi^{-1}(s)(L_1(s)v(s,\epsilon) + E_1(s)w(s,\epsilon)\\ &\qquad + \hat{F}_1(s,u(s,\epsilon),v(s,\epsilon),w(s,\epsilon),\epsilon))ds),\\ v(t,\epsilon) &= \int_0^t \frac{\Psi(t,s,\epsilon)}{\epsilon}(E_2(s)w(s,\epsilon) + \hat{F}_2(s,u(s,\epsilon),v(s,\epsilon),w(s,\epsilon),\epsilon))ds,\\ w(t,\epsilon) &= -(E_3(t))^{-1}\hat{F}_3(t,u(t,\epsilon),v(t,\epsilon),w(t,\epsilon),\epsilon),\end{aligned} \tag{1.11}$$

where $\Phi(t)$ satisfies

$$\frac{d\Phi(t)}{dt} = C_1(t)\Phi(t),\quad 0 \le t \le T,\quad \Phi(0) = I$$

while $\Psi(t,s,\epsilon)$ is the solution of the following system

$$\frac{d\Psi}{dt} = \frac{1}{\epsilon}B(t)\Psi,\ \ 0 \le s \le t \le T,\quad \Psi(t,s,\epsilon)|_{t=s} = I.$$

Note that the system (1.11) is not equivalent with the system (1.9). But, obviously, if $(u(t,\epsilon),\ v(t,\epsilon),\ w(t,\epsilon))$ solves (1.11), then it solves (1.9) as well. Under the assumption that for any $t \in [0,T]$ all eigenvalues of $B(t)$ remain strictly in the left half plane, we obtain

$$||\Psi(t,s,\epsilon)|| \le Ke^{-\mu(t-s)/\epsilon}$$

for $0 \le s \le t \le T$, where K is a constant independent of ϵ (for the proof see, e.g. [7]).

Thus we are led to consider the following more general constrained system of integral equations

$$\begin{aligned} u(t,\epsilon) &= \theta_1(t,\epsilon) + \int_0^t (G_1(t,s,\epsilon)v(s,\epsilon) + H_1(t,s,\epsilon)w(s,\epsilon) \\ &\qquad + F_1(s,u(s,\epsilon),v(s,\epsilon),w(s,\epsilon),\epsilon))\,ds, \\ v(t,\epsilon) &= \theta_2(t,\epsilon) + \frac{1}{\epsilon}\int_0^t K(t,s,\epsilon)(H_2(t,s,\epsilon)w(s,\epsilon) \\ &\qquad + F_2(s,u(s,\epsilon),v(s,\epsilon),w(s,\epsilon),\epsilon))\,ds, \\ w(t,\epsilon) &= F_3(t,u(t,\epsilon),v(t,\epsilon),w(t,\epsilon),\epsilon), \end{aligned} \tag{1.12}$$

where $u \in R^m, v \in R^n, w \in R^k$ and ϵ is a small real parameter.

In this paper, we address the existence and uniqueness of a solution of (1.12) and the asymptotic behavior of this solution as $\epsilon \to 0$. The principal results are contained in Theorem 1 and Theorem 2, which are based on the following assumption:

Hypothesis (H). (a) *There are non-empty open sets $\mathcal{D}_u \subset R^m$, $\mathcal{D}_v \subset R^n$, $\mathcal{D}_w \subset R^k$, $\mathcal{J}_\epsilon \subset R^1$, $\mathcal{J}_\epsilon = \{\ \epsilon \mid 0 < \epsilon \le \epsilon_0\ \}$, such that $(0,0,0) \in \hat{\mathcal{D}} = \mathcal{D}_u \times \mathcal{D}_v \times \mathcal{D}_w$, and the mappings*

$$\begin{aligned} F_1 &: [0,T] \times \hat{\mathcal{D}} \times \mathcal{J}_\epsilon \subset R^m \times R^n \times R^k \times R \to R^m, \\ F_2 &: [0,T] \times \hat{\mathcal{D}} \times \mathcal{J}_\epsilon \subset R^m \times R^n \times R^k \times R \to R^n, \\ F_3 &: [0,T] \times \hat{\mathcal{D}} \times \mathcal{J}_\epsilon \subset R^m \times R^n \times R^k \times R \to R^k, \end{aligned}$$

are of class C^1 on the indicated domains.

(b) *Functions F_1, F_2, F_3, satisfy the following asymptotic relations,*

(i)

$$F_i(t,0,0,0,\epsilon) = O(\epsilon^N), \quad \text{as } \epsilon \to 0,\ i = 1,2,3 \tag{1.13a}$$

uniformly for $0 \le t \le T$, where N is a positive integer;

(ii)

$$\left.\begin{aligned} D_u F_i(t,u,v,w,\epsilon) &= O(\epsilon + |u| + |v| + |w|) \\ D_v F_i(t,u,v,w,\epsilon) &= O(\epsilon + |u| + |v| + |w|) \\ D_w F_i(t,u,v,w,\epsilon) &= O(\epsilon + |u| + |v| + |w|) \end{aligned}\right\} \quad \text{as } \epsilon, |u|, |v|, |w| \to 0 \tag{1.13b}$$

uniformly for $0 \le t \le T$, $i = 1,2,3$.

(c) *The functions*

$$\begin{aligned} \theta_1 &: [0,T] \times \mathcal{J}_\epsilon \to R^m, & \theta_2 &: [0,T] \times \mathcal{J}_\epsilon \to R^n, \\ G_1 &: [0,T] \times [0,T] \times \mathcal{J}_\epsilon \to R^{m\times n}, & H_1 &: [0,T] \times [0,T] \times \mathcal{J}_\epsilon \to R^{m\times k}, \\ H_2 &: [0,T] \times [0,T] \times \mathcal{J}_\epsilon \to R^{n\times k}, & K &: [0,T] \times [0,T] \times \mathcal{J}_\epsilon \to R^{m\times m}, \end{aligned}$$

are continuous on their domains.

2. Lemma and theorems. The following generalization of Gronwall's lemma is needed in the proof of Theorem 1. It uses the standard componentwise partial ordering $\leq$ on R^n and on the space $R^{m\times n}$ of $m\times n$ real matrices. As usual, a matrix $A\in R^{m\times n}$ is called nonnegative if all its entries are nonnegative. This is denoted by $A\geq 0$, and more generally, for any two nonnegative matrices A, $B\in R^{m\times n}$, $A\leq B$ is equivalent to $B-A\geq 0$.

Lemma. *Suppose that*

(i) A, $B\in R^{n\times n}$ *are two commutative nonnegative matrices, and the spectral* $\rho(B)$ *of* B *is strictly less than* 1*;*

(ii) $d^k:[0,T]\to R^n$, $k=0,1,\ldots,$ *is a sequence of nonnegative, continuous functions such that*

$$d^{k+1}(t)\leq A\int_0^t d^k(s)\,ds+Bd^k(t),\quad k=0,1,\ldots. \tag{2.1}$$

Then $\lim_{k\to\infty}d^k(t)=0$ *exists uniformly in* $0\leq t\leq T$*, and the estimate*

$$\max_{0\leq t\leq T}\|d^k(t)\|=O((\rho(B)+\delta)^k)\|\max_{0\leq t\leq T}d^0(t)\|\quad \text{as } k\to\infty$$

holds for any δ, $0<\delta<1-\rho(B)$*, and with*

$$\max_{0\leq t\leq T}d^0(t)=\begin{pmatrix}\max_{0\leq t\leq T}d_1^0(t)\\ \vdots\\ \max_{0\leq t\leq T}d_n^0(t)\end{pmatrix},$$

where the norm is l^2 *norm.*

Proof. We set $e^0=\max_{0\leq t\leq T}d^0(t)$ and consider the integral equations

$$\begin{aligned}e^0(t)&=e^0,\quad t\in[0,T],\\ e^{l+1}(t)&=A\int_0^t e^l(s)\,ds+Be^l(t),\quad t\in[0,T],\quad l=0,1,\ldots.\end{aligned} \tag{2.2}$$

By induction, it follows that

$$d^l(t)\leq e^l(t),\quad \text{for all}\quad l\geq 0,\quad 0\leq t\leq T. \tag{2.3}$$

Once more by induction, we find that the solution of (2.2) is

$$e^k(t)=\sum_{s=0}^{k}\binom{k}{s}B^sA^{k-s}\frac{t^{k-s}}{(k-s)!}e^0,\quad k=0,1,2,\ldots,\quad 0\leq t\leq T. \tag{2.4}$$

For simplicity we write

$$\rho = \rho(B), \quad a = \|A\|.$$

Since $\rho < 1$, there exists a constant $C > 0$ such that $\|B^k\| \leq C(\rho + \delta/2)^k$ for all $k \geq 0$ and any $\delta > 0$ for which $\rho + \delta < 1$. From (2.4), it follows that

$$\begin{aligned} \|e^k(t)\| &\leq \sum_{s=0}^{k} \binom{k}{s} \|B^s\| \, \|A^{k-s}\| \frac{t^{k-s}}{(k-s)!} \|e^0\| \qquad (2.5) \\ &\leq C\|e^0\| \sum_{s=0}^{k} \binom{k}{s} (\rho + \delta/2)^s \frac{(at)^{k-s}}{(k-s)!} = C\|e^0\| \sum_{s=0}^{k} \binom{k}{s} \frac{(at)^s}{s!} (\rho + \delta/2)^{k-s}. \end{aligned}$$

Because of

$$\lim_{s \to \infty} \frac{(aT)^s}{s!} (\frac{2}{\delta})^s = 0,$$

there exists a positive integer k_0 such that

$$\frac{(aT)^s}{s!} (\frac{2}{\delta})^s \leq 1, \quad \forall s \geq k.$$

Thus

$$\sum_{s=0}^{k} \binom{k}{s} \frac{(aT)^s}{s!} (\rho + \delta/2)^{k-s} \leq e^{aT} (\rho + \delta/2)^{-k_0} / k_0! k^{k_0} (\rho + \delta/2)^k + (\rho + \delta)^k \quad (2.6)$$

and, since

$$\lim_{k \to \infty} k^{k_0} \frac{(\rho + \delta/2)^k}{(\rho + \delta)^k} = 0,$$

with (2.5) and (2.6), we find that

$$\|e^k(t)\| = O((\rho + \delta)^k) \|e^0\|, \quad \text{as } k \to \infty,$$

uniformly in $0 \leq t \leq T$. Hence from (2.3) it follows that

$$\|d^k(t)\| \leq \|e^k(t)\| = O((\rho + \delta)^k) \| \max_{0 \leq t \leq T} d^0(t) \|, \quad \text{as } k \to \infty$$

or

$$\max_{0 \leq t \leq T} \|d^k(t)\| = O((\rho + \delta)^k) \| \max_{0 \leq t \leq T} d^0(t) \|$$

which completes the proof of Lemma.

Theorem 1. *Under the Hypothesis* (H) *suppose that G_1, H_1, H_2 are bounded on their domains, and that the kernel K and the constant terms θ_1, θ_2 satisfy*

$$\theta_1(t,\epsilon) = O(\epsilon^N), \qquad \theta_2(t,\epsilon) = O(\epsilon^N), \qquad K(t,s,\epsilon) = O(e^{-\mu(t-s)/\epsilon}), \quad \text{as } \epsilon \to 0$$

uniformly in $0 \le s \le t \le T$. Then there exists a positive $\hat{\epsilon} \in \mathcal{J}_\epsilon$ such that the integral equation (1.12) *has a solution $(u(t,\epsilon), v(t,\epsilon), w(t,\epsilon)) \in \mathcal{D}_u \times \mathcal{D}_v \times \mathcal{D}_w$ for all $0 < \epsilon \le \hat{\epsilon}$, which satisfies*

$$u(t,\epsilon) = O(\epsilon^N),\ v(t,\epsilon) = O(\epsilon^N),\ w(t,\epsilon) = O(\epsilon^N) \quad \text{as } \epsilon \to 0 \tag{2.7}$$

uniformly for $0 \le t \le T$.

Proof. We use the method of successive approximations to prove that there exists a sufficiently small $\hat{\epsilon} \in \mathcal{J}_\epsilon$ such that the integral equation (1.12) has a solution $(u(t,\epsilon), v(t,\epsilon), w(t,\epsilon))$ of (1.12) on the interval $0 \le t \le T$ for all $0 < \epsilon \le \hat{\epsilon}$, which satisfies the asymptotic relations (2.7).

When started from $u^0(t,\epsilon) \equiv 0$, $u^0(t,\epsilon) \equiv 0$, $u^0(t,\epsilon) \equiv 0$, the successive iterates (u^k, v^k, w^k), $k = 1, 2, \ldots$ for (1.12) are recursively defined by

$$\begin{aligned}
u^{k+1}(t,\epsilon) &= \theta_1(t,\epsilon) + \int_0^t (G(t,s,\epsilon)v^k(s,\epsilon) + H_1(t,s,\epsilon)w^k(s,\epsilon) \\
&\qquad + F_1(s, u^k(s,\epsilon), v^k(s,\epsilon), w^k(s,\epsilon), \epsilon))\, ds, \\
v^{k+1}(t,\epsilon) &= \theta_2(t,\epsilon) + \frac{1}{\epsilon}\int_0^t K(t,s,\epsilon)(H_2(t,s,\epsilon)w^k(s,\epsilon) \\
&\qquad + F_2(s, u^k(s,\epsilon), v^k(s,\epsilon), w^k(s,\epsilon), \epsilon))\, ds, \\
w^{k+1}(t,\epsilon) &= F_3(t, u^k(t,\epsilon), v^k(t,\epsilon), w^k(t,\epsilon), \epsilon).
\end{aligned} \tag{2.8}$$

By assumption we have

$$\int_0^t |K(t,s,\epsilon)|\, ds = \int_0^t O(e^{-\mu(t-s)/\epsilon})\, ds \le C\epsilon, \quad t \in [0,T],$$

where C is a constant independent of ϵ. Moreover, Hypothesis (H) and our assumptions on θ_1, θ_2, ensure that the following asymptotic relations

$$u^1(t,\epsilon) = O(\epsilon^N),\ v^1(t,\epsilon) = O(\epsilon^N),\ w^1(t,\epsilon) = O(\epsilon^N), \quad \text{as } \epsilon \to 0 \tag{2.9a}$$

hold uniformly for $0 \le t \le T$, and that

$$(u^1(t,\epsilon), v^1(t,\epsilon), w^1(t,\epsilon)) \in \mathcal{D}_u \times \mathcal{D}_v \times \mathcal{D}_w \tag{2.9b}$$

for all $0 \le t \le T$ and $0 < \epsilon \le \epsilon_1$ ($\epsilon_1 \in \mathcal{J}_\epsilon$). For any given $\delta > 0$, there exists some $r(\delta) > 0$ such that

$$B(0, r(\delta)) \times B(0, r(\delta)) \times B(0, r(\delta)) \times B(0, r(\delta)) \subset \mathcal{D}_u \times \mathcal{D}_v \times \mathcal{D}_w \times \mathcal{J}_\epsilon$$

and that

$$\left.\begin{aligned}|D_uF_i(t,u,v,w,\epsilon)| < \delta,\\ |D_vF_i(t,u,v,w,\epsilon)| < \delta,\\ |D_wF_i(t,u,v,w,\epsilon)| < \delta,\end{aligned}\right\} \quad i = 1,2,3, \tag{2.10}$$

whenever $\epsilon, |u|, |v|, |w| < r(\delta)$. Hence from

$$\begin{aligned}F_i(t,u,v,w,\epsilon) = F_i(t,0,0,0,\epsilon) + \int_0^t (D_uF_i(t,\lambda u,\lambda v,\lambda w,\epsilon)u\\ + D_vF_i(t,\lambda u,\lambda v,\lambda w,\epsilon)v + D_wF_i(t,\lambda u,\lambda v,\lambda w,\epsilon)w)\,d\lambda\end{aligned}$$

together with $F_i(t,0,0,0,\epsilon) = O(\epsilon^N)$ it follows that

$$|F_i(t,u,v,w,\epsilon)| \le C_3\epsilon^N + \delta(|u| + |v| + |w|) \quad i = 1,2,3, \tag{2.11}$$

whenever $\epsilon, |u|, |v|, |w| < r(\delta)$. For sufficiently small $\epsilon_1 > 0$, we have

$$\begin{aligned}&|u^0(t,\epsilon)| = |v^0(t,\epsilon)| = |w^0(t,\epsilon)| \equiv 0 < r(\delta),\\ &|u^1(t,\epsilon)| = O(\epsilon^N) \le C_1\epsilon^N < r(\delta),\\ &|v^1(t,\epsilon)| = O(\epsilon^N) \le C_1\epsilon^N < r(\delta),\\ &|w^1(t,\epsilon)| = O(\epsilon^N) \le C_1\epsilon^N < r(\delta),\end{aligned}$$

uniformly for $0 \le t \le T$, $0 < \epsilon \le \epsilon_1$. We prove by induction that for sufficiently small δ and ϵ the following estimates

$$|u^i(t,\epsilon)| < r(\delta), \quad |v^i(t,\epsilon)| < r(\delta), \quad |w^i(t,\epsilon)| < r(\delta), \tag{2.12}$$

hold for all $i \ge 0$ and $0 \le t \le T$. Since (2.12) was seen to be valid for $i = 0,1$, suppose that it remains true for $i = 0,1,2,\dots,k$. From (2.8) and inequality (2.11) it then follows that

$$\begin{aligned}|u^{i+1}(t,\epsilon)| &\le C_2\epsilon^N + M\int_0^t (|v^i(s,\epsilon)| + |w^i(s,\epsilon)| + C_3\epsilon^N\\ &\quad + \delta(|u^i(s,\epsilon)| + |v^i(s,\epsilon)| + |w^i(s,\epsilon)|)\,ds\\ &\le (C_2 + TMC_3)\epsilon^N + M\int_0^t (|v^i(s,\epsilon)| + |w^i(s,\epsilon)|\\ &\quad + \delta(|u^i(s,\epsilon)| + |v^i(s,\epsilon)| + |w^i(s,\epsilon)|)\,ds\\ |v^{i+1}(t,\epsilon)| &\le CM\max_{0\le t\le T}|w^i(t,\epsilon)| + (CC_3 + C_2)\epsilon^N\\ &\quad + C\delta(\max_{0\le t\le T}|u^i(t,\epsilon)| + \max_{0\le t\le T}|v^i(t,\epsilon)| + \max_{0\le t\le T}|w^i(t,\epsilon)|)\\ |w^{i+1}(t,\epsilon))| &\le C_3\epsilon^N + \delta(|u^i(t,\epsilon)| + |v^i(t,\epsilon)| + |w^i(t,\epsilon)|)\end{aligned} \tag{2.13}$$

for $i = 0, 1, \dots, k$, where M, C_2 are constants such that

$$\|G_1(t,\epsilon))\|,\ \|H_1(t,\epsilon))\|,\ \|H_2(t,\epsilon))\| \le M, \quad \|\theta_1(t,\epsilon))\|,\ \|\theta_2(t,\epsilon))\| \le C_2\epsilon^N$$

for all $0 \le t \le T$, $0 \le \epsilon \le \epsilon_1$. The third inequality of (2.13) implies that

$$\begin{aligned} |w^i(t,\epsilon)| &\le C_3\epsilon^N + \delta(\max_{0\le t\le T}|u^{i-1}(t,\epsilon)| + \max_{0\le t\le T}|v^{i-1}(t,\epsilon)| \\ &\quad + \max_{0\le t\le T}|w^{i-1}(t,\epsilon)|) \le C_3\epsilon^N + 3\delta r(\delta), \quad i = 0, 1, \dots, k, \end{aligned} \tag{2.14}$$

whence we obtain for $v^{i+1}(t,\epsilon)$ and $w^{i+1}(t,\epsilon)$ from the last two inequalities of (2.13) that for $i = 0, 1, \dots, k$

$$\begin{aligned} |v^{i+1}(t,\epsilon)| &\le MC\max_{0\le t\le T}|w^i(t,\epsilon)| + (CC_3 + C_2)\epsilon^N + C\delta r(\delta) \\ &\le 3(M+1)C\delta r(\delta) + ((M+1)CC_3 + C_2)\epsilon^N. \end{aligned} \tag{2.15}$$

Therefore, let δ be such that

$$\delta < \min\left\{\frac{1}{6}, \frac{1}{6(M+1)C}\right\} \tag{2.16}$$

and select a sufficiently small $\hat{\epsilon}$, $0 < \hat{\epsilon} \le \epsilon_1$ so that

$$C_3\epsilon^N < \frac{1}{2}r(\delta), \quad ((M+1)CC_3 + C_2)\epsilon^N < \frac{1}{2}r(\delta), \quad \forall\epsilon,\ 0 < \epsilon \le \hat{\epsilon}. \tag{2.17}$$

Then it follows from (2.14) and (2.15) for $i = k$ that

$$|v^{k+1}(t,\epsilon)| < r(\delta), \quad |w^{k+1}(t,\epsilon)| < r(\delta).$$

Now from the first inequality of (2.13) together with (2.14) for $i = k$, and (2.15) for $i = k - 1$ we obtain that

$$|u^{k+1}(t,\epsilon)| \le (C_2(1+MT) + TM(C_3 + (M+2)CC_3))\epsilon^N + MT(3(M+1)C + 6)\delta r(\delta). \tag{2.18}$$

Hence, suppose that δ is chosen such that not only (2.15) holds but that also

$$\delta < \frac{1}{6MT((M+1)C+2)}. \tag{2.19}$$

Moreover, choose $\hat{\epsilon}$ sufficiently small such that, in addition to (2.17), the relation

$$(C_2(1+MT) + TM(C_3 + (M+2)CC_3))\epsilon^N < \frac{r(\delta)}{2}, \quad 0 < \epsilon \le \hat{\epsilon},$$

is satisfied. Then we obtain

$$|u^{k+1}(t,\epsilon)| < r(\delta), \quad 0 \le t \le T, \quad 0 < \epsilon \le \hat{\epsilon},$$

which shows that (2.12) holds for all $i \ge 0$.

In order to prove that the approximation $(u^i(t,\epsilon), v^i(t,\epsilon), w^i(t,\epsilon))$ generated by (2.8) converges for $i \to \infty$, let

$$\begin{aligned} d_i(t) = {} & \max_{0\le s\le t}\{|u^i(s,\epsilon) - u^{i-1}(s,\epsilon)|\} + \max_{0\le s\le t}\{|v^i(s,\epsilon) - v^{i-1}(s,\epsilon)|\} \\ & + \max_{0\le s\le t}\{|w^i(s,\epsilon) - w^{i-1}(s,\epsilon)|\}. \end{aligned}$$

Then from (2.8) and the inequalities (2.12), it follows by subtraction of (2.8) for $k = i+1$ from (2.8) for $k = i$ that for all $i \ge 0$,

$$\begin{aligned} &|u^{i+1}(t,\epsilon) - u^i(t,\epsilon)| \le (M+\delta)\int_0^t d_i(s)\,ds \\ &|v^{i+1}(t,\epsilon) - v^i(t,\epsilon)| \le MC \max_{0\le s\le t}\{|w^i(s,\epsilon) - w^{i-1}(s,\epsilon)|\} + \delta C d_i(t) \\ &|w^{i+1}(t,\epsilon) - w^i(t,\epsilon)| \le \delta d_i(t). \end{aligned} \tag{2.20}$$

From the third inequality of (2.20) we obtain that

$$\max_{0\le s\le t}|w^i(s,\epsilon) - w^{i-1}(s,\epsilon)| \le \delta d_{i-1}(t) \tag{2.21}$$

and by substitution into the second inequality of (2.20), that

$$|v^{i+1}(t,\epsilon) - v^i(t,\epsilon)| \le MC\delta d_{i-1}(t) + C\delta d_i(t). \tag{2.22}$$

Hence, (2.20) and (2.22) together yield

$$\begin{aligned} d_{i+1}(t) &\le (M+\delta)\int_0^t d_i(s)\,ds + M'\delta(d_i(t) + d_{i-1}(t) \\ &\le M_1\int_0^t d_i(s)\,ds + \alpha(d_i(t) + d_{i-1}(t)), \end{aligned} \tag{2.23}$$

where $M' = \max\{MC, C+1\}$, $M_1 = 2M$, $\alpha = M'\delta$. With

$$\hat{d}_i(t) = \begin{pmatrix} d_{i+1}(t) \\ d_i(t) \end{pmatrix}$$

we write (2.23) in the form

$$\hat{d}_i(t) \le \int_0^t \begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix} \hat{d}_{i-1}(s)\,ds + \begin{pmatrix} \alpha & \alpha \\ 1 & 0 \end{pmatrix} \hat{d}_{i-1}(t). \tag{2.24}$$

Note here that the coefficient matrices

$$\begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \alpha & \alpha \\ 1 & 0 \end{pmatrix}$$

are not commutative. Thus, in order to apply the Lemma, we modify (2.24) as follows

$$\begin{aligned} \hat{d}_i(t) &\leq \int_0^t \begin{pmatrix} M_1 & 0 \\ 0 & 0 \end{pmatrix} \hat{d}_{i-1}(s)\,ds + \begin{pmatrix} \alpha & \alpha \\ 1 & 0 \end{pmatrix} \hat{d}_{i-1}(t) \\ &\leq \int_0^t \begin{pmatrix} M_1 & 0 \\ 0 & M_1 \end{pmatrix} \hat{d}_{i-1}(s)\,ds + \begin{pmatrix} \alpha & \alpha \\ 1 & 0 \end{pmatrix} \hat{d}_{i-1}(t) = A\int_0^t \hat{d}_{i-1}(s)\,ds + B\hat{d}_{i-1}(t), \end{aligned}$$

where

$$A = M_1 I_{2\times 2}, \qquad B = \begin{pmatrix} \alpha & \alpha \\ 1 & 0 \end{pmatrix},$$

for which now $AB = BA$. The eigenvalues of B are

$$\lambda_1(B) = \frac{\alpha + \sqrt{\alpha^2 + 4\alpha}}{2}, \quad \lambda_2(B) = \frac{\alpha - \sqrt{\alpha^2 + 4\alpha}}{2},$$

whence we have $\rho(B) < 1$ for $\alpha > 0$ if and only if $\alpha < 1/2$. Since $\alpha = \delta M'$ we can choose δ such that $\alpha < 1/2$ and that the other requirements (2.16) and (2.19) hold. Then by the lemma, and with $\lambda_0 > 0$ such that $\lambda_0 + \rho(B) < 1$, we find that

$$\max_{0\leq t\leq T} |\hat{d}_i(t)| = O((\lambda_0 + \rho(B))^i) | \max_{0\leq t\leq T} \hat{d}_0(t)|$$

which implies that $d_i(T) = O((\lambda_0 + \rho(B))^i d_0(T)$. Therefore the series

$$\sum_0^\infty |u^{i+1}(t,\epsilon) - u^i(t,\epsilon)|, \quad \sum_0^\infty |v^{i+1}(t,\epsilon) - v^i(t,\epsilon)|, \quad \sum_0^\infty |w^{i+1}(t,\epsilon) - w^i(t,\epsilon)|,$$

converge uniformly for $0 \leq t \leq T$ and $0 < \epsilon \leq \hat{\epsilon}$, which guarantees the existence of functions $u(t,\epsilon)$, $v(t,\epsilon)$, $w(t,\epsilon)$ such that

$$\begin{aligned} \lim_{i\to\infty} u^i(t,\epsilon) &= \sum_{i=0}^\infty (u^{i+1}(t,\epsilon) - u^i(t,\epsilon)) = u(t,\epsilon), \\ \lim_{i\to\infty} v^i(t,\epsilon) &= \sum_{i=0}^\infty (v^{i+1}(t,\epsilon) - v^i(t,\epsilon)) = v(t,\epsilon), \\ \lim_{i\to\infty} w^i(t,\epsilon) &= \sum_{i=0}^\infty (w^{i+1}(t,\epsilon) - w^i(t,\epsilon)) = w(t,\epsilon), \end{aligned} \tag{2.25}$$

uniformly in $0 \leq t \leq T$ and $0 < \epsilon \leq \hat{\epsilon}$. Since the convergence is uniform for $0 \leq t \leq T$ the limiting functions $u(t,\epsilon)$, $v(t,\epsilon)$, $w(t,\epsilon)$ are continuous in $t \in [0,T]$. Moreover, since (2.12) holds for all $i \geq 0$, it follows that

$$|u(t,\epsilon)| \leq r(\delta), \quad |v(t,\epsilon)| \leq r(\delta), \quad |w(t,\epsilon)| \leq r(\delta), \qquad 0 \leq t \leq T, \quad 0 < \epsilon \leq \hat{\epsilon}. \tag{2.26}$$

This means that the functions $(u(t,\epsilon),\, v(t,\epsilon),\, w(t,\epsilon))$ belong to $\mathcal{D}_u \times \mathcal{D}_v \times D_w$ for all $0 \leq t \leq T$, $0 < \epsilon \leq \hat{\epsilon}$.

On the other hand, since

$$\begin{aligned} |u(t,\epsilon)| &\leq \sum_{i=0}^{\infty} |u^{i+1}(t,\epsilon) - u^i(t,\epsilon)| \leq \sum_{i=0}^{\infty} d_i(t) \leq C_4 d_0(T), \\ |v(t,\epsilon)| &\leq \sum_{i=0}^{\infty} d_i(t) \leq C_4 d_0(T), \\ |W(t,\epsilon)| &\leq \sum_{i=0}^{\infty} d_i(t) \leq C_4 d_0(T), \end{aligned} \tag{2.27}$$

and

$$d_0(T) = \max_{0 \leq s \leq T}\{|u^1(s,\epsilon)|\} + \max_{0 \leq s \leq T}\{|v^1(s,\epsilon)|\} + \max_{0 \leq s \leq T}\{|w^1(s,\epsilon)|\} = O(\epsilon^N),$$

we find that

$$u(t,\epsilon) = O(\epsilon^N), \quad v(t,\epsilon) = O(\epsilon^N), \quad w(t,\epsilon) = O(\epsilon^N), \tag{2.28}$$

uniformly in $0 \leq t \leq T$, and $0 < \epsilon < \hat{\epsilon}$. By letting $k \to \infty$ in (2.7) we see that the limiting function $(u(t,\epsilon), v(t,\epsilon), w(t,\epsilon))$ certainly solves (1.12) since the convergence of $(u^k(t,\epsilon), v^k(t,\epsilon), w^k(t,\epsilon))$ is uniform in t for $0 \leq t \leq T$.

Theorem 2. *Under the assumptions of Theorem* 1*, the solution of* (1.12) *is unique if the functions*

$$\theta_1(0,\epsilon) = u(0,\epsilon),\ \theta_2(0,\epsilon) = v(0,\epsilon),\ \theta_3(\epsilon) = w(0,\epsilon),$$

satisfy the compatibility condition $F_3(0,\theta_1(0,\epsilon),\, \theta_2(0,\epsilon),\, \theta_3(\epsilon),\epsilon) = \theta_3(\epsilon)$ *as well as* $\theta_3(\epsilon) = O(\epsilon)$ *as* $\epsilon \to 0$.

Proof. Suppose that there exists another solution $(\tilde{u}(t,\epsilon),\, \tilde{v}(t,\epsilon),\, \tilde{w}(t,\epsilon))$ of (1.12) which satisfies the initial conditions

$$\tilde{u}(0,\epsilon) = \theta_1(0,\epsilon),\ \tilde{v}(0,\epsilon) = \theta_2(0,\epsilon),\ \tilde{w}(0,\epsilon) = \theta_3(\epsilon),$$

where $\theta_3(\epsilon) = O(\epsilon)$. First we show that $w(0,\epsilon) = \tilde{w}(0,\epsilon)$ for all sufficiently small ϵ. Indeed, from

$$w(0,\epsilon) = F_3(0,\theta_1(0,\epsilon),\, \theta_2(0,\epsilon),\, w(0,\epsilon),\epsilon), \tag{2.29a}$$

$$\tilde{w}(0,\epsilon) = F_3(0,\theta_1(0,\epsilon),\, \theta_2(0,\epsilon),\, \tilde{w}(0,\epsilon),\epsilon), \tag{2.29b}$$

we find, by subtracting (2.29b) from (2.29a), that

$$w(0,\epsilon)-\tilde{w}(0,\epsilon)=\int_0^1 D_wF_3(0,\theta_1(0,\epsilon),\theta_2(0,\epsilon),\tilde{w}(0,\epsilon) \\ +\lambda(w(0,\epsilon)-\tilde{w}(0,\epsilon)),\epsilon)\,d\lambda(w(0,\epsilon)-\tilde{w}(0,\epsilon)). \tag{2.30}$$

Because of $\theta_1(0,\epsilon)$, $\theta_2(0,\epsilon)$, $w(0,\epsilon)$, $\tilde{w}(0,\epsilon)\to 0$ as $\epsilon\to 0$ and in view of the property (1.13b) of F_3 it follows that

$$\begin{aligned}&|D_wF_3(0,\theta_1(0,\epsilon),\theta_2(0,\epsilon),\tilde{w}(0,\epsilon)+\lambda(w(0,\epsilon)-\tilde{w}(0,\epsilon)),\epsilon)|\\ &\quad=O(\epsilon+|\theta_1(0,\epsilon)|+|\theta_2(0,\epsilon)|+|w(0,\epsilon)|+|\tilde{w}(0,\epsilon)|)\\ &\quad=O(\epsilon),\qquad \text{as } \epsilon\to 0.\end{aligned} \tag{2.31}$$

Hence, (2.30) and (2.31) give

$$|w(0,\epsilon)-\tilde{w}(0,\epsilon)|\le O(\epsilon)|w(0,\epsilon)-\tilde{w}(0,\epsilon)|$$

which implies that there exists ϵ_3, $\epsilon_3\le\hat{\epsilon}$ such that $w(0,\epsilon)-\tilde{w}(0,\epsilon)=0$ for all $0<\epsilon\le\epsilon_3$.

We set

$$\begin{aligned}e_u(t,\epsilon)&=\max_{0\le s\le t}|u(s,\epsilon)-\tilde{u}(s,\epsilon)|\\ e_v(t,\epsilon)&=\max_{0\le s\le t}|v(s,\epsilon)-\tilde{v}(s,\epsilon)|\\ e_w(t,\epsilon)&=\max_{0\le s\le t}|w(s,\epsilon)-\tilde{w}(s,\epsilon)|\\ e(t,\epsilon)&=e_u(t,\epsilon)+e_v(t,\epsilon)+e_w(t,\epsilon).\end{aligned}$$

Then for any fixed ϵ $(0<\epsilon\le\epsilon_3)$ we have

$$\begin{aligned}e_u(t,\epsilon)&\le MT(1+\delta)\int_0^1 e(s,\epsilon)\,ds\\ e_v(t,\epsilon)&\le CMe_w(t,\epsilon)+MC\delta e(t,\epsilon)\\ e_w(t,\epsilon)&\le\delta e(t,\delta)\end{aligned} \tag{2.32}$$

as long as

$$|\tilde{u}(t,\epsilon)|<r(\delta),\quad |\tilde{v}(t,\epsilon)|<r(\delta),\quad |\tilde{w}(t,\epsilon)|<r(\delta), \tag{2.33}$$

where $\epsilon_4(\le\epsilon_3)$ is chosen such that

$$\left.\begin{aligned}|u(t,\epsilon)|&=O(\epsilon^N)<r(\delta),\\ |v(t,\epsilon)|&=O(\epsilon^N)<r(\delta),\\ |w(t,\epsilon)|&=O(\epsilon^N)<r(\delta),\end{aligned}\right\}\quad \forall 0\le t\le T,\quad 0<\epsilon\le\epsilon_4. \tag{2.34}$$

The existence of this ϵ_4 is guaranteed by (2.28). The strict inequalities in (2.34), extending the earlier estimate (2.26), will play an important role in the following proof.

Since for $0 < \epsilon \leq \epsilon_4$

$$\tilde{u}(0,\epsilon) = u(0,\epsilon), \quad \tilde{v}(0,\epsilon) = v(0,\epsilon), \quad \tilde{w}(0,\epsilon) = w(0,\epsilon),$$

it follows that (2.33) holds for $0 \leq t \leq t_0$ with sufficiently small t_0. Now (2.32) implies that

$$e(t,\epsilon) \leq MT(1+\delta)\int_0^t e(s,\epsilon)\,ds + 2CM\delta e(t,\epsilon)$$

or

$$e(t,\epsilon) \leq \frac{MT(1+\delta)}{1-2CM\delta}\int_0^t e(s,\epsilon)\,ds \tag{2.35}$$

as long as

$$|\tilde{u}(t,\epsilon)|, |\tilde{v}(t,\epsilon)|, |\tilde{w}(t,\epsilon)| < r(\delta).$$

From Gronwall's inequality it follows that $e(t,\epsilon) = 0$ as long as

$$|\tilde{u}(t,\epsilon)|, |\tilde{v}(t,\epsilon)|, |\tilde{w}(t,\epsilon)| < r(\delta).$$

In order to prove that this implies that $e(t,\epsilon) = 0$ for all $0 \leq t \leq T$, $0 < \epsilon \leq \epsilon_4$, let

$$\mathcal{B} = \{b \bigm| b \in [0,T] \text{ and } e(t,\epsilon) = 0,\ \forall t \in [0,b]\}$$

which is obviously nonempty because of $t_0 \in \mathcal{B}$, and set

$$t^* = \sup_{b\in\mathcal{B}}\{b\}. \tag{2.36}$$

We wish to show that $t^* = T$. If $t^* < T$, then, by the continuity of $e(t,\epsilon)$ and the closeness of $\mathcal{B}$, we have $t^* \in \mathcal{B}$ which implies that $e(t,\epsilon) = 0$, $\forall t \in [0,t^*]$, and therefore

$$\tilde{u}(t,\epsilon) = u(t,\epsilon),\ \tilde{v}(t,\epsilon) = v(t,\epsilon),\ \tilde{w}(t,\epsilon) = w(t,\epsilon), \quad \forall t \in [0,t^*].$$

Thus, from

$$|u(t,\epsilon)|, |v(t,\epsilon)|, |w(t,\epsilon)| < r(\delta), \quad \forall t \in [0,t^*], \tag{2.37}$$

it follows that

$$|\tilde{u}(t,\epsilon)|, |\tilde{v}(t,\epsilon)|, |\tilde{w}(t,\epsilon)| < r(\delta) \tag{2.38}$$

for all $t \in [0,t^*]$. Hence, by the continuity of $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ on $[0,T]$, there exists a sufficiently small $\hat{\sigma} > 0$ such that (2.38) holds for all $t \in [0,t^* + \hat{\sigma}] \subset [0,T]$. From this, it follows that (2.35) holds for all $t \in [0,t^* + \hat{\sigma}]$. By Gronwall's inequality, this means that $e(t,\epsilon) = 0$ for all $t \in [0,t^* + \hat{\sigma}]$ which implies that $t^* + \hat{\sigma} \in \mathcal{B}$. This contradicts the definition (2.37) of t^*. Hence $e(t,\epsilon) = 0$ for all $t \in [0,T]$, and the solution of the system (1.12) is unique.

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