GLOBAL EXISTENCE, UNIQUENESS AND REGULARITY
OF SOLUTIONS TO A VON KÁRMÁN SYSTEM WITH
NONLINEAR BOUNDARY DISSIPATION

ANGELO FAVINI$^1$, MARY ANN HORN$^2$†, IRENA LASIECKA$^3$‡, AND DANIEL TATARU$^4$

1 Dipartimento di Matematica, University di Bologna, Piazza di Porta S. Donato 5, 40127 Bologna, Italy
2 School of Mathematics, University of Minnesota, 127 Vincent Hall, Minneapolis, Minnesota 55455
3 Department of Applied Mathematics, Thornton Hall, University of Virginia, Charlottesville, VA 22903
4 Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, Illinois 60208

(Submitted by: Y. Giga)

Abstract. Systems of nonlinear elasticity described by Von Karman equations with nonlinear boundary dissipation are considered. Global existence, uniqueness of weak solutions as well as the regularity of solutions with “smooth” data is established. Thus the paper solves, in particular, an outstanding problem of uniqueness of weak solutions to Von Karman system, which has been open in the literature even in the case of the homogeneous boundary data. This is accomplished by proving “sharp” regularity results of the Airy stress function.

1. Introduction.
1.1. Statement of the problem. Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^2$ with boundary $\Gamma$. In $\Omega \times (0, T)$, where $T > 0$ is given, we consider the following von Kármán system with nonlinear boundary conditions:

$$
\begin{align*}
\frac{\partial^2 w}{\partial t^2} + \Delta^2 w &= [\mathcal{F}(w), w] \quad \text{in } Q_T = (0, T) \times \Omega \\
w(0, \cdot) &= w_0, \quad w_t(0, \cdot) = w_1 \quad \text{in } \Omega \\
\Delta w + (1 - \mu) B_1 w &= -h(\frac{\partial}{\partial \nu} w_t) \quad \text{on } \Sigma_T = (0, T) \times \Gamma \\
\frac{\partial}{\partial \nu} \Delta w + (1 - \mu) B_2 w - w &= g(w_t) \quad \text{on } \Sigma_T = (0, T) \times \Gamma,
\end{align*}
$$

and

$$
\Delta^2 \mathcal{F}(w) = -[w, w] \quad \text{in } (0, T) \times \Omega, \quad \mathcal{F}|_\Gamma = \frac{\partial}{\partial \nu} \mathcal{F}|_\Gamma = 0 \quad \text{on } (0, T) \times \Gamma,
$$

Received for publication September 1994.
†This material is based upon work partially supported under a National Science Foundation Mathematical Sciences Postdoctoral Research Fellowship.
‡Research supported by National Science Foundation Grant DMS 9204338.
AMS Subject Classifications: 35.
where \([u, v] \equiv u_{xx}v_{yy} + u_{yy}v_{xx} - 2u_{xy}v_{xy}\). In (1.1), the boundary operators \(B_1\) and \(B_2\) are given by

\[
B_1w = 2n_1n_2w_{xy} - \eta_1^2w_{yy} - \eta_2^2w_{xx},
\]

\[
B_2w = \frac{\partial}{\partial \tau} [(\eta_1^2 - \eta_2^2)w_{xy} + n_1n_2(w_{yy} - w_{xx})],
\]

where \(\nu = (n_1, n_2)\) denotes an outward normal to the boundary and \(\tau\) denotes tangential direction. In the boundary conditions, \(0 < \mu < \frac{1}{2}\) is Poisson's ratio, and the functions, \(h\) and \(g\) are continuously differentiable, monotone increasing, real-valued functions and are subject to the following constraints:

\[
h(s)s \geq 0, \quad g(s)s \geq 0, \quad m \leq h'(s) \leq M, \quad m \leq g'(s) \leq M(|s|^{r} + 1)
\]

(H-1)

where \(0 < m \leq M, r\) is any positive constant, and

\[
g'(s)\text{ is locally Lipschitz.}
\]

(H-2)

This system describes the transversal displacement \(w\) and the Airy-stress function \(F(w)\) of a vibrating plate, whose boundary is subject to nonlinear damping in the form of moments and forces/shears applied to the edge of the plate.

Vibrating structures with boundary damping have attracted considerable attention in recent years in the context of the study of boundary stabilizability (see [12], [13], [9], etc.). Solution to the stabilization problem depends inherently on an adequate existence, uniqueness and regularity theory (see [12]) for the underlying dynamics. This is the motivation for our analysis.

Questions related to the existence of solutions for the von Kármán system (1.1.a), (1.1.b), (1.2), with the homogenous boundary conditions:

\[
w|_{\Gamma} = \frac{\partial}{\partial \nu}w|_{\Gamma} = 0 \quad \text{on} \quad (0, T) \times \Gamma,
\]

have received considerable attention in the literature. Indeed, the existence of global weak (i.e., \((w(t), w_{\tau}(t)) \in H^2(\Omega) \times L_2(\Omega)\)) solutions for (1.1.a), (1.1.b), (1.2), (1.4) has been proven by Faedo-Galerkin methods in Lions ([16]) and Vorovič ([27]). In [25], [26], von Wahl gives a proof of existence and uniqueness of a local solution with higher regularity. Existence of local classical solutions has been established by Stahel in [21]. Arguments of [21] also prove the uniqueness property which is valid on a small time interval. More recently, Chueskov ([3]) and Koch and Stahel ([11]) were able to establish appropriate a priori bounds for their classical solutions, hence proving the global existence of classical solutions. Global existence and uniqueness results for strong (i.e., \(H^4(\Omega) \times H^2(\Omega)\)) solutions to (1.1.a), (1.1.b), (1.2), (1.4) was proven by Chueskov in [3]. Existence and uniqueness of solutions in fractional Sobolev spaces \(H^{2+s}(\Omega); s > 0\) were established in [28].

As already stated, the results quoted above refer to a situation when the boundary conditions are homogenous, or, more precisely, of the form as in (1.4). Moreover, none of these references had addressed the problem of uniqueness of weak \((H^2(\Omega))\) solutions, which has been an open problem in the literature (see [17], [13], [28]). The techniques
used in these references can be adapted to treat the case of boundary conditions as in (1.1.c) and (1.1.d), but with linear boundary dissipation; i.e., $g$ and $h$ are linear (this fact was already noted by Lagnese in [12]). In fact, existence and regularity of solutions to a full von Kármán system with linear dissipation was proven in [19]. However, the presence of nonlinear functions in boundary conditions (1.1.c) and (1.1.d) raises a number of technical difficulties (including passage to the limit on boundary nonlinearities) since the methods developed previously are not applicable any longer.

To our knowledge, the only results available in the literature and dealing with the nonlinear boundary damping are due to Lagnese and Leugering ([13]), where the one-dimensional model has been treated, and in [7], [14], where the rotational inertia of the plate, which induces a regularizing effect on the velocity, was accounted for in the model. In this last case, the mathematical nature of the problem is, obviously, very different.

Thus, the main distinctive feature of our paper is that we treat a two-dimensional von Kármán system (1.1), (1.2) with fully nonlinear boundary damping. The main result of our paper is a global existence and uniqueness of (i) regular solutions and (ii) weak solutions. (Weak solutions are defined, as usual, by a variational form.) We note that the uniqueness result for weak solutions is new even in the case of linear homogeneous boundary conditions. Precise statements of these results are given below.

**Theorem 1.1** (Existence of smooth solutions). Assume that the functions $h$ and $g$ satisfy hypotheses (H-1) and (H-2). Then for all initial data, $w_0 \in H^4(\Omega), w_1 \in H^2(\Omega)$, such that

$$
\Delta w_0 + (1 - \mu)B_1 w_0 = -h\left(\frac{\partial}{\partial \nu} w_1\right) \quad \text{on} \quad \Gamma
$$

$$
\frac{\partial}{\partial \nu}\Delta w_0 + (1 - \mu)B_2 w_0 - w_0 = g(w_1) \quad \text{on} \quad \Gamma,
$$

there exists a solution, $(w, w_t) \in C(0, T; H^3(\Omega) \times H^2(\Omega))$, where $T > 0$ is arbitrary.

**Theorem 1.2** (Uniqueness of weak solutions). Assume that the functions $f$ and $g$ are monotone increasing. Let $(w, w_t)$ be any weak solution corresponding to (1.1) which satisfies $(w, w_t) \in C(0, T; H^2(\Omega) \times L_2(\Omega))$. Then, such a solution is unique.

**Theorem 1.3** (Existence of weak solutions). In addition to hypotheses (H-1) and (H-2), we assume the following growth condition:

$$
g(s) \geq m|s|^{r+1} \quad \text{for} \quad |s| > R,
$$

where $R$ is a large number. Then, for all initial data $(w_0, w_1) \in H^2(\Omega) \times L_2(\Omega)$, there exists a unique global weak solution such that $(w, w_t) \in C(0, T; H^2(\Omega)) \times C(0, T; L_2(\Omega))$ and $w_t|_{\Gamma}, \frac{\partial}{\partial \nu} w_t|_{\Gamma} \in L^2(\Sigma T)$.

**Theorem 1.4** (Intermediate solutions). Assume hypotheses (H-1)–(H-3) hold with the value $r \leq 1$. Then for $0 \leq \theta \leq 1$ and all initial data $w_0 \in H^{2 + \theta}(\Omega), w_1 \in H^{2\theta}(\Omega)$ subject to compatibility conditions (1.5) satisfied for $\theta > \frac{1}{4}$ and (1.6) satisfied for $\theta > \frac{3}{4}$, we have that $(w, w_t) \in C(0, T; H^{2 + \theta}(\Omega)) \times C(0, T; H^{2\theta}(\Omega))$.

**Remark 1.1.** The results presented in Theorems 1.1–1.4 also remain valid in the case when there is no damping on the boundary (i.e., $h = g = 0$). This in particular
refers to the uniqueness statement in Theorem 1.2 which is new even in this special case. (See [12], [17] and most recently [19] where a uniqueness result is proven for regular solutions only.) In this particular case, the statement of Theorem 1.3 will not guarantee the additional boundary regularity.

**Remark 1.2.** In the special case when $h = 0$ in (1.1.c), the regularity result of Theorem 1.1 reads: $(w, w_t) \in C(0, T; H^4(\Omega) \times H^2(\Omega))$. A similar comment applies to Theorem 1.4.

**Remark 1.3.** The result of Theorem 1.3 holds true under weaker hypotheses imposed on $h, g$. For instance, it would suffice to assume the growth condition (H-1) valid for the functions (rather than the derivatives). Also, it is enough to assume (H-1) for large values of “$s$” only. In order to focus the attention on the key difficulties of the problem, we shall dispense with this level of generality.

The outline of the paper is as follows. Sections 2–4 deal with the existence of regular (smooth) solutions. Here the result of Theorem 1.1 is proven via a suitable application of Schaeffer’s Theorem. Section 5 provides an uniqueness result for weak solutions. The proof of this uniqueness (Theorem 1.2) is based on a new “sharp” regularity result for the Airy’s stress function (see Theorem 5.1). Proof of Theorem 5.1 employs pseudodifferential calculus combined with compensated compactness methods and is given in Section 8. Section 6 deals with the existence of weak solutions (Theorem 1.3), where the “sharp” regularity of the Airy’s stress function is used again. The existence of the solutions in the intermediate spaces (Theorem 1.4) is obtained in Section 7 through an application of a nonlinear interpolation theorem due to Tartar ([23]).

2. **Technical lemmas.** In this section, we shall prove a number of a priori estimates for the following nonlinear equation:

$$
w_{tt} + \Delta^2 w = [f, w] \quad \text{in } Q_T = (0, T) \times \Omega$$

$$w(0, \cdot) = w_0, \quad w_t(0, \cdot) = w_1 \quad \text{in } \Omega$$

$$\Delta w + (1 - \mu)B_1 w = -h(\frac{\partial}{\partial \nu} w_t) \quad \text{on } \Sigma_T = (0, T) \times \Gamma \tag{2.1}$$

$$\frac{\partial}{\partial \nu} \Delta w + (1 - \mu)B_2 w - w = g(w_t) \quad \text{on } \Sigma_T = (0, T) \times \Gamma,$$

where the function $f$ is a given element of $L^1(0, T; W^2_{\infty}(\Omega))$.

For the convenience of the reader, we list several known properties of the bracket, $[v, w]$ (see [3], [11], [16]) which will be used throughout our proofs. The constant $C$ throughout the paper denotes a generic constant.

**Properties of $[v, w]$:** Let $0 < \theta < 1$. Then

$$\|[v, w]\|_{H^{-1-\theta}(\Omega)} \leq C \|[v]\|_{H^2(\Omega)} \|w\|_{H^2(\Omega)}, \tag{2.2}$$

$$\|[v, w]\|_{L^2(\Omega)} \leq C \|[v]\|_{W^2_{\infty}(\Omega)} \|w\|_{W^2_{\infty}(\Omega)}, \tag{2.3}$$

where \(\frac{1}{q} + \frac{1}{q} = 1\),

$$\|[v, w]\|_{H^{-\theta}(\Omega)} \leq C \|[v]\|_{H^{3-\theta}(\Omega)} \|w\|_{H^2(\Omega)}, \tag{2.4}$$

$$\|[v, w]\|_{H^{2-\theta}(\Omega)} \leq C \|[v]\|_{H^2(\Omega)} \|w\|_{H^{3-\theta}(\Omega)}, \tag{2.5}$$

$$\|[v, w]\|_{H^{-2}(\Omega)} \leq C \|[v]\|_{H^2(\Omega)} \|w\|_{H^{2+\theta}(\Omega)}, \tag{2.6}$$

$$\|[v, w]\|_{H^{-1-\theta}(\Omega)} \leq C \|[v]\|_{H^2(\Omega)} \|w\|_{H^2(\Omega)}. \tag{2.7}$$
We shall start the proofs of our a priori estimates with a simple preliminary result.

**Proposition 2.1.** For any \( f \in L^1(0,T;W^2_\infty(\Omega)) \), \( g \) and \( h \) subject to hypothesis (H-1), and \( w_0 \in H^2(\Omega) \), \( w_1 \in L_2(\Omega) \), there exists a unique solution, \((w,w_1)\) to (2.1) such that

\[
\|w(t)\|^2_{H^2(\Omega)} + \|w_1(t)\|^2_{L_2(\Omega)} + \int_0^t \int_\Gamma h(\frac{\partial}{\partial \nu} w_t) \frac{\partial}{\partial \nu} w_t \, d\Gamma \, dt + \int_0^t \int_\Gamma g(w_t)w_t \, d\Gamma \, dt \\
\leq C_T \exp(\|f\|_{L^1(0,T;W^2_\infty(\Omega))}) \left[\|w_0\|^2_{H^2(\Omega)} + \|w_1\|^2_{L_2(\Omega)}\right] \equiv C_0(f,w_0,w_1) \quad \forall t \leq T,
\]

and

\[
\|w_t\|^2_{L_2(\Sigma_T)} + \left\| \frac{\partial}{\partial \nu} w_t \right\|^2_{L_2(\Sigma_T)} \leq C_0(f,w_0,w_1). \tag{2.8}
\]

**Proof.** It follows from [15] that problem (2.1) with \( f \equiv 0 \) generates a nonlinear semigroup of contractions on \( H^2(\Omega) \times L_2(\Omega) \). To see this, we need to put problem (2.1) into an abstract framework of [15]. We define the following operators: \( \mathcal{A} : L_2(\Omega) \rightarrow L_2(\Omega) \) is defined by:

\[
\mathcal{A}u \equiv \Delta^2 u;
\tag{2.9}
\]

\[
\mathcal{D}(\mathcal{A}) = \{ u \in H^2(\Omega) : \Delta u + (1-\mu)B_1u|_\Gamma = 0, \frac{\partial}{\partial \nu}\Delta u + (1-\mu)B_2u|_\Gamma = 0 \},
\]

\( G_i : L_2(\Gamma) \rightarrow L_2(\Omega) \), \( i = 1,2 \), are defined by

\[
G_1g \equiv v \iff \Delta^2 v = 0 \quad \text{in} \quad \Omega
\]

\[
\Delta v + (1-\mu)B_1v = g \quad \text{on} \quad \Gamma, \quad \frac{\partial}{\partial \nu}\Delta v + (1-\mu)B_2v = 0 \quad \text{on} \quad \Gamma,
\]

\[
G_2g \equiv v \iff \Delta^2 v = 0 \quad \text{in} \quad \Omega
\]

\[
\Delta v + (1-\mu)B_1v = 0 \quad \text{on} \quad \Gamma, \quad \frac{\partial}{\partial \nu}\Delta v + (1-\mu)B_2v - v = g \quad \text{on} \quad \Gamma.
\]

With the above notation, equation (2.1) with \( f = 0 \) can be rewritten (on \( \mathcal{D}(\mathcal{A})' \)) as

\[
w_{tt} + \mathcal{A}w + AG_1h(G_1^* Aw_t) + AG_2g(G_2^* Aw_t) = 0, \tag{2.10}
\]

or equivalently,

\[
w_{tt} + \mathcal{A}w + \mathcal{B}\partial \Phi(B^* w_t) \ni 0, \tag{2.11}
\]

where \( \mathcal{B} : L_2(\Gamma) \times L_2(\Gamma) \rightarrow \mathcal{D}(\mathcal{A}^{1/2})' \) is given by

\[
\mathcal{B}(g_1,g_2) = AG_1g_1 + AG_2g_2, \tag{2.12}
\]

and the adjoint of \( \mathcal{B} \), defined by

\[
(g,B^* v)_{L_2(\Gamma) \times L_2(\Gamma)} = (B g, v)_{\mathcal{D}(\mathcal{A}) \times \mathcal{D}(\mathcal{A})'}, \tag{2.13}
\]

can be written as

\[
B^* = [G_1^* Av, G_2^* Av] = [\frac{\partial}{\partial \nu} v, v], \tag{2.14}
\]
by using Green’s formula (see [14]).

With $U_0 \equiv L_2(\Gamma) \times L_2(\Gamma)$, $\partial \Phi \equiv [h, g] \in U_0 \times U_0$ and, by monotonicity of $h$ and $g$, is a subgradient of a proper, convex function $\Phi : U_0 \rightarrow R$. By using trace theory ([18]), one easily shows that $B^*$ is bounded and surjective between the spaces $H^2(\Omega) \rightarrow H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$. Thus, hypothesis (H-1) of [15] is satisfied with $U \equiv H^{3/2}(\Gamma) \times H^{1/2}(\Gamma)$, $U \subset U_0 \subset U'$ and $D(A^{1/2}) = H^2(\Omega)$ (see [8]). Hence, we are in a position to apply Theorem 2.2 in [15] which claims the generation of a nonlinear semigroup of contractions for the operator

$$C \equiv \begin{bmatrix} 0 & -I \\ A & B \partial \Phi B^* \end{bmatrix}$$

(2.15)

on $\overline{D(C)} \subset H^2(\Omega) \times L_2(\Omega)$. One can easily check that, in our case, $\overline{D(C)} = H^2(\Omega) \times L_2(\Omega)$, hence equation (2.1) (written as a system of equations) generates a nonlinear semigroup of contractions on $H^2(\Omega) \times L_2(\Omega)$.

On the other hand, the term $F(w) \equiv [f, w]$ is Lipschitz from $H^2(\Omega) \rightarrow L_2(\Omega)$. Indeed,

$$\|F(w_1) - F(w_2)\|_{L_2(\Omega)} = \|[f, w_1 - w_2]\|_{L_2(\Omega)} \leq C\|f\|_{W^2_2(\Omega)} \|w_1 - w_2\|_{H^2(\Omega)},$$

where we have used the property (2.3) of the bracket, $[ , ]$.

Applying a standard perturbation theorem for nonlinear semigroups (see Barbu, [2]), we obtain the existence and uniqueness of the solutions to (2.1). The usual semigroup estimates applied to (2.1) yield

$$\|w_t(t)\|^2_{L_2(\Omega)} + a(w(t), w(t)) + 2 \int_0^t \int_{\Gamma} h(\frac{\partial}{\partial \nu}, w_t) \frac{\partial}{\partial \nu} w_t d\Gamma dt + 2 \int_0^t \int_{\Omega} g(w_t) w_t d\Omega dt = \|w_1\|^2_{L_2(\Omega)} + a(w_0, w_0) + 2 \int_0^t \int_{\Omega} [f, w] w_t d\Omega dt,$$  

(2.16)

where

$$a(w, w) \equiv \int_{\Gamma} w^2 d\Gamma + \int_{\Omega} [(\Delta w)^2 + (1 - \mu)(2w_x^2 - 2w_{xx}w_{yy})] d\Omega,$$  

(2.17)

and

$$m_1 \|w\|^2_{H^2(\Omega)} \leq a(w, w) \leq M_1 \|w\|^2_{H^2(\Omega)}.$$  

(2.18)

On the other hand, by (2.3),

$$\left| \int_{\Omega} [f, w] w_t d\Omega \right| \leq \|w_t\|_{L_2(\Omega)} \|w\|_{H^2(\Omega)} \|f\|_{W^2_2(\Omega)}$$

$$\leq C\|f\|_{W^2_2(\Omega)} \|w_t\|^2_{L_2(\Omega)} + \|w\|^2_{H^2(\Omega)}.$$  

(2.19)

Combining (2.16) with (2.18) and (2.19), applying Gronwall’s inequality with $L^1$ kernel, and recalling (H-1), yields the result of our proposition, obtained first for smooth data and then extended (by virtue of stability of the norms involved) to all data in $H^2(\Omega) \times L_2(\Omega)$.

The following higher regularity result is critical.
Lemma 2.1. Let $w$ be a solution to (2.1). Let $f \in L^1(0,T;W^{2}_{\infty}(\Omega))$, $f_t \in L^{\infty}(0,T;H^{2+\varepsilon}(\Omega))$, where $\varepsilon > 0$ is arbitrarily small, $(w_0,w_1) \in H^4(\Omega) \times H^2(\Omega)$, and $g$ and $h$ satisfy hypothesis (H-1). Then the following estimate holds for all $t \leq T$:

$$
\|w(t)\|_{H^3(\Omega)}^2 + \|w_t(t)\|_{H^2(\Omega)}^2 + \|w_{tt}(t)\|_{L^2(\Omega)}^2 + \int_0^t \int_\Gamma \left( \frac{\partial}{\partial \nu}w_{tt} + w_{ttt} \right)^2 d\Gamma dt
\leq C_T(\|f\|_{L^1(0,T;W^{2}_{\infty}(\Omega))}, \|f_t\|_{L^{\infty}(0,T;H^{2+\varepsilon}(\Omega))}) \left[ \|w_0\|_{H^4(\Omega)}^2 + \|w_1\|_{H^2(\Omega)}^2 \right],
$$

where $C_T(u,v)$ is a continuous function and $w_0$, $w_1$ satisfy the compatibility conditions (1.5) (1.6).

Proof. We consider the following equation for $z$.

$$
z_{tt} + \Delta^2 z = [f_t,w] + [f,z] \quad \text{in} \quad Q_T = (0,T) \times \Omega
$$

$$
z(0,\cdot) = w_t(0) = w_1 \in H^2(\Omega) \quad \text{in} \quad \Omega
$$

$$
z_t(0,\cdot) = -\Delta w_0 + [f(0),w_0] \in L_2(\Omega) \quad \text{in} \quad \Omega
$$

$$
\Delta z + (1 - \mu)B_1 z = -h'(\frac{\partial}{\partial \nu}w_t) \frac{\partial}{\partial \nu}z_t \quad \text{on} \quad \Sigma_T = (0,T) \times \Gamma
$$

$$
\frac{\partial}{\partial \nu} \Delta z + (1 - \mu)B_2 z - z = g'(w_t)z_t \quad \text{on} \quad \Sigma_T = (0,T) \times \Gamma.
$$

Notice that from inequality (2.8), we have that

$$
\frac{\partial}{\partial \nu}w_t \in L_2(\Sigma_T), \quad w_t \in L_2(\Sigma_T),
$$

thus $h'(\frac{\partial}{\partial \nu}w_t)$ and $g'(w_t)$ are well-defined-almost-everywhere functions on $\Sigma_T$. Since by (H-1),

$$
h'(\frac{\partial}{\partial \nu}w_t) \geq m, \quad g'(w_t) \geq m \quad \text{a.e. in} \quad \Sigma,
$$

and problem (2.21) is linear in $z$ with dissipative (linear) boundary conditions, the existence and uniqueness of the solution, $z$, to (2.21) in the space $C(0,T;H^2(\Omega)) \times C(0,T;L_2(\Omega))$ and such that $z_t, \frac{\partial}{\partial \nu}z_t \in L^2(\Sigma_T)$ follows from standard linear theory. Energy methods applied to equation (2.21) yield

$$
\|z_t(t)\|_{L_2(\Omega)}^2 + a(z(t),z(t)) + 2 \int_0^t \int_\Gamma h'(\frac{\partial}{\partial \nu}w_t)(\frac{\partial}{\partial \nu}z_t)^2 d\Gamma dt + 2 \int_0^t \int_\Gamma g'(w_t)z_t^2 d\Gamma dt
\leq \|z_t(0)\|_{L_2(\Omega)}^2 + a(z(0),z(0)) + 2 \int_0^t \int_\Omega ([f_t,w]z_t + [f,z]z_t) d\Omega dt.
$$

On the other hand, from (2.3),

$$
\left| \int_\Omega [f,z]z_t d\Omega \right| \leq (\|z_t\|_{L_2(\Omega)}^2 + \|z\|_{H^2(\Omega)}^2) \|f\|_{W^{2}_{\infty}(\Omega)}.
$$
For $\frac{1}{q} + \frac{1}{q} = 1$,
\[
\int_\Omega ||[f_t, w]|^2 d\Omega|^{1/2} \leq C ||w||_{W^q_2(\Omega)} ||f_t||_{W^q_2(\Omega)},
\]
and by Sobolev's imbeddings, $\forall \epsilon > 0$, $\exists q > 1$ such that
\[
\int_\Omega ||[f_t, w]|^2 d\Omega|^{1/2} \leq C ||w||_{H^3(\Omega)} ||f_t||_{H^{2+\epsilon}(\Omega)}.
\]
Hence,
\[
\int_\Omega ||[f_t, w]|z_t| d\Omega \leq C ||z_t||_{L^2(\Omega)} ||w||_{H^3(\Omega)} ||f_t||_{H^{2+\epsilon}(\Omega)},
\]
\[
\int_0^t \int_\Omega ||[f_t, w]|z_t| d\Omega dt \leq C \sup_{t \geq 0} ||z_t(t)||_{L^2(\Omega)} ||f_t(t)||_{H^{2+\epsilon}(\Omega)} \int_0^t ||w(\tau)||_{H^3(\Omega)} d\tau
\]
\[
\leq \epsilon_0 ||z_t(t)||^2_{L^2(0, T; L^2(\Omega))} + \frac{C}{\epsilon_0} \sup_{t \geq 0} ||f_t(t)||^2_{H^{2+\epsilon}(\Omega)} \left( \int_0^t ||w(\tau)||_{H^3(\Omega)} d\tau \right)^2.
\]
Selecting $\epsilon_0$ suitably small, combining (2.24), (2.25), and (2.26), and recalling (2.18) and hypothesis (II-1) yields
\[
||z_t(t)||^2_{L^2(\Omega)} + ||z(t)||^2_{H^2(\Omega)} + \int_0^t \frac{\partial}{\partial \nu} z_t ||z_t||^2_{L^2(\Omega)} dt + \int_0^t ||z_t||^2_{L^2(\Gamma)} dt \leq C \{ ||z_t(0)||^2_{L^2(\Omega)} + \sup_{t \geq 0} ||f_t(t)||^2_{H^{2+\epsilon}(\Omega)} \left( \int_0^t ||w(\tau)||_{H^3(\Omega)} d\tau \right)^2
\]
\[
+ \int_0^t ||f(\tau)||_{W^q_2(\Omega)} (||z(\tau)||^2_{L^2(\Omega)} + ||z(\tau)||^2_{H^2(\Omega)}) d\tau \}.
\]
Since both $\frac{\partial}{\partial \nu} z_t$, $z_t \in L^2(\Sigma_T)$, one can easily show by using compatibility conditions (1.5) that $z \equiv w_t$.

Our next step is to estimate the $H^3(\Omega)$ norm of $w(t)$. This is done by using elliptic theory applied to (2.1). Indeed, from (2.1) and [18], we obtain
\[
||w(t)||_{H^3(\Omega)} \leq C \{ ||w(t)||_{L^2(\Omega)} + ||f(t), w(t)||_{L^2(\Omega)}
\]
\[
+ ||h(\frac{\partial}{\partial \nu} w(t))||_{H^{1.5}(\Gamma)} + \|g(w_t(t))||_{H^{-1.5}(\Gamma)},
\]
\[
||f(t), w(t)||_{L^2(\Omega)} \leq C \|w(t)||_{H^3(\Omega)} ||f(t)||_{W^q_2(\Omega)},
\]
\[
||g(w_t)||_{H^{-1.5}(\Gamma)} \leq ||g(w_t)||_{L^1(\Gamma)} \leq \left( \int_\Gamma |g(w_t)| |w_t|^{r+1} \text{d} \Gamma \right)^{\frac{1}{r+1}}
\]
\[
\leq \left( \int_\Gamma g(w_t) w_t \text{d} \Gamma \right)^{\frac{1}{r+1}},
\]
where the first inequality follows because

\[ H^{1/2}(\Gamma) \subset L^p(\Gamma) \quad \text{for any} \quad p \Rightarrow L^{p'}(\Gamma) \subset H^{-1/2}(\Gamma) \quad \text{for any} \quad p' > 1, \]

and the last equality is obtained by choosing \( \epsilon = \frac{1}{r+1} \).

Let \((Hu)(x) \equiv h(u(x))\), where \( h \) satisfies (H-1). From the definition of the fractional Sobolev space \( H^{1/2}(\Gamma) \) and from hypothesis (H-1), it follows immediately that

\[ \|Hu\|_{H^{1/2}(\Gamma)} \leq C\|u\|_{H^{1/2}(\Gamma)}. \]  

(2.31)

From (2.31),

\[ \|h(\frac{\partial}{\partial \nu} w_t(t))\|_{H^{1/2}(\Gamma)} \leq C\|\frac{\partial}{\partial \nu} w_t(t)\|_{H^{1/2}(\Gamma)} \leq C\|w_t(t)\|_{H^2(\Omega)}, \]  

(2.32)

where the last inequality follows by trace theory. Collecting (2.28)–(2.30), and (2.32),

\[ \|w(t)\|_{H^1(\Omega)} \leq C\{\|w(t)\|_{H^2(\Omega)}\|f(t)\|_{W^1_2(\Omega)} + \|w_t(t)\|_{L^2(\Omega)} \]  

\[ + \int_{\Gamma} g(w_t)w_t \, d\Gamma + \|w_t(t)\|_{H^2(\Omega)} \}, \]  

(2.33)

and from Proposition 2.1, recalling \( w_t = z \),

\[ \int_0^t \|w(t)\|_{H^1(\Omega)} \, dt \leq C \int_0^t \|z_t(t)\|_{L^2(\Omega)} + \|w(t)\|_{H^2(\Omega)}\|f(t)\|_{W^1_2(\Omega)} \]  

(2.34)

\[ + \|z(t)\|_{H^2(\Omega)} \) \, dt + C \int_0^t \int_{\Gamma} g(w_t)w_t \, d\Gamma \, dt \]  

\[ \leq C_T \int_0^t \|z_t(t)\|_{L^2(\Omega)} + \|z(t)\|_{H^2(\Omega)} \) \, dt + (\|f\|_{L^1(0,T;W^1_2(\Omega))} + 1) C_0(f, w_0, w_1). \]

Going back to (2.27),

\[ \|z_t(t)\|_{L^2(\Omega)}^2 + \|z(t)\|_{H^2(\Omega)}^2 \leq C(\|z_t(0)\|_{L^2(\Omega)}^2 + \|z(0)\|_{H^2(\Omega)}^2) \]  

\[ + C_T \int_0^t \|f(t)\|_{L^\infty(\Omega)}\|z_t(t)\|_{L^2(\Omega)}^2 + \|z(t)\|_{H^2(\Omega)}^2 \) \, dt \]  

(2.35)

\[ + \|f\|_{L^\infty(0,T;H^{2+s}(\Omega))} \int_0^t \|z_t(t)\|_{L^2(\Omega)}^2 + \|z(t)\|_{H^2(\Omega)}^2 \) \, dt \]  

\[ + \|f\|_{L^1(0,T;W^1_2(\Omega))} C_0^2(f, w_0, w_1). \]

Gronwall’s inequality applied to (2.35) yields the estimate in (2.20) for \( w_t \) and \( w_{tt} \). Combining this result with the estimate for \( w \) in (2.33) and using the inequality

\[ \int_{\Gamma} w_t g(w_t) \, d\Gamma \leq C \int_{\Gamma} |w_t|^{r+2} \, d\Gamma \leq C \|w_t\|_{C(0,T;H^2(\Omega))}^{r+2}, \]  

(2.36)

we obtain the final result of Lemma 2.1.
3. A priori bounds.

Lemma 3.1. Let $w$ be a solution to (1.1) such that $w_0 \in H^4(\Omega), w_1 \in H^2(\Omega)$ and

$$w \in C(0,T;H^{3-\epsilon}(\Omega)) \quad \text{and} \quad w_t \in C(0,T;H^2(\Omega)) \quad \text{and} \quad w_{tt} \in C(0,T;L_2(\Omega)),$$

where $0 < \epsilon < \frac{1}{2}$. Then the following a priori bounds hold:

$$\|w(t)\|_{C(0,T;H^4(\Omega))} + \|w_t(t)\|_{C(0,T;H^2(\Omega))} + \|w_{tt}(t)\|_{C(0,T;L_2(\Omega))} \leq C_T(\|w_0\|_{H^4(\Omega)}, \|w_1\|_{H^2(\Omega)}),$$

where $C_T(u,v)$ is bounded for any arbitrary $T > 0, u, v \in \mathbb{R}^1$.

Proof. We first show that energy estimates applied to (1.1) give a priori bounds in $H^2(\Omega) \times L_2(\Omega)$. Notice that with $w \in C(0,T;H^{3-\epsilon}(\Omega))$, we have

$$\mathcal{F}(w) = -G[w,w] \in H^4(\Omega) \quad \text{and} \quad [\mathcal{F}(w),w] \in L_2(\Omega),$$

where $G$ is defined by

$$Gf = g \iff \Delta^2 g = f \quad \text{in} \quad \Omega, \quad g = 0, \quad \partial_\nu g = 0 \quad \text{on} \quad \Gamma.$$

Applying estimate (2.16) with $f(t) = [\mathcal{F}(w(t)),w(t)]$ yields

$$a(w(t),w(t)) + \|w_t(t)\|^2_{L_2(\Omega)} + 2 \int_0^t \int_{\Gamma} h' \left( \frac{\partial}{\partial \nu} w_t \right) \frac{\partial}{\partial \nu} w_t \, d\Gamma \, dt + 2 \int_0^t \int_{\Omega} g(w_t) \, w_t \, d\Omega \, dt$$

$$\leq a(w_0,w_0) + \|w_1\|^2_{L_2(\Omega)} + 2 \int_0^t \int_{\Omega} [\mathcal{F}(w),w] \, w_t \, d\Omega \, dt$$

$$= a(w_0,w_0) + \|w_1\|^2_{L_2(\Omega)} - 2 \|\Delta \mathcal{F}(w(t))\|^2_{L_2(\Omega)} + 2 \|\Delta \mathcal{F}(w(0))\|^2_{L_2(\Omega)}.$$ (3.1)

Hence, since $\|\Delta \mathcal{F}(w(0))\|_{L_2(\Omega)} \leq C\|w_0\|^2_{H^2(\Omega)},$

$$\|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{L_2(\Omega)} + \int_0^t \int_{\Gamma} (\frac{\partial}{\partial \nu} w_t)^2 + w_t^2) \, d\Gamma \, dt \leq C(\|w_0\|_{H^2(\Omega)}, \|w_1\|_{L_2(\Omega)}),$$ (3.2)

where hypothesis (H-1) has again been applied.

Our next step is to obtain bounds for higher norms. We return to equation (2.21) which, for a fixed $w$, $w_t$ and $\frac{\partial}{\partial \nu} w_t|\Gamma$ (well defined globally by (3.2)), is a linear equation in $z = w_t$. Thus with $f_t(t) \in L_2(Q_T)$, by standard energy estimates, which are justified for the linear problem, we have

$$a(z(t),z(t)) + \|z_t(t)\|^2_{L_2(\Omega)} + 2 \int_0^t \int_{\Gamma} h' \left( \frac{\partial}{\partial \nu} w_t \right) \frac{\partial}{\partial \nu} z_t \, d\Gamma \, dt + 2 \int_0^t \int_{\Omega} g'(w_t) z_t^2 \, d\Omega \, dt$$

$$= a(z(0),z(0)) + \|z_t(0)\|^2_{L_2(\Omega)} + 2 \int_0^t \int_{\Omega} f_t(t) z_t \, d\Omega \, dt.$$ (3.3)
Let $f_t(t) \equiv \frac{d}{dt} [\mathcal{F}(w(t)), w(t)]$. Then $f_t(t)$ can be rewritten as

$$f_t(t) = \frac{d}{dt} \mathcal{F}(w(t), w(t)) + [\mathcal{F}(w(t)), w_t(t)]$$

$$= -2[G[w, w_t], w(t)] - [G[w, w], w_t]. \quad (3.4)$$

Clearly, if $w_t \in C(0, T; H^2(\Omega))$, $w \in C(0, T; H^{3-\epsilon}(\Omega))$, then, by the properties of the bracket and elliptic regularity, $f_t(t) \in C(0, T; L_2(\Omega))$, therefore (3.3) holds with $f_t(t)$ defined as above.

To simplify the integral in (3.3) corresponding to $f_t(t)$, the following estimates are useful:

$$\int_\Omega [G[w, w_t], w(t)] z_t(t) d\Omega = \int_\Omega G[w, w_t][w(t), z_t(t)] d\Omega$$

$$= \int_\Omega G[w, z] \frac{d}{dt} [z, w] d\Omega - \int_\Omega [z, z] G[w, z] d\Omega$$

$$= \int_\Omega G[w, z] \frac{d}{dt} \Delta^2 G[w, z] d\Omega - \int_\Omega [z, z] G[w, z] d\Omega$$

$$= \frac{1}{2} \frac{d}{dt} \int_\Omega |\Delta G[w, z]|^2 d\Omega - \int_\Omega [z, z] G[w, z] d\Omega, \quad (3.5)$$

where the last line follows by using the boundary conditions associated with the operator $G$. In addition, we have

$$\int_\Omega [\mathcal{F}(w), w_t] z_t d\Omega$$

$$= \frac{d}{dt} \int_\Omega [\mathcal{F}(w), z] z d\Omega - \int_\Omega [\mathcal{F}(w), z_t] z d\Omega - \int_\Omega \frac{d}{dt} [\mathcal{F}(w), z] z d\Omega$$

$$= \frac{d}{dt} \int_\Omega [\mathcal{F}(w), z] z d\Omega - \int_\Omega [\mathcal{F}(w), z] z_t d\Omega + 2 \int_\Omega [G[w, z], z] z d\Omega.$$

Hence, rearranging terms in the above identity,

$$\int_\Omega [\mathcal{F}(w), w_t] z_t d\Omega = \frac{1}{2} \frac{d}{dt} \int_\Omega [\mathcal{F}(w), z] z d\Omega + \int_\Omega [z, z] G[w, z] d\Omega. \quad (3.6)$$

Combining (3.4), (3.5), and (3.6) yields

$$\int_\Omega f_t(t) z_t d\Omega = - \frac{d}{dt} \int_\Omega |\Delta G[w, z]|^2 d\Omega + 3 \int_\Omega [z, z] G[w, z] d\Omega$$

$$- \frac{1}{2} \frac{d}{dt} \int_\Omega [\mathcal{F}(w), z] z d\Omega. \quad (3.7)$$

Substituting (3.7) into (3.3) gives

$$a(z(t), z(t)) + \|z_t(t)\|_{L_2(\Omega)}^2 + 2\|\Delta G[w, z](t)\|_{L_2(\Omega)}^2$$

$$\leq a(z(0), z(0)) + \|z_t(0)\|_{L_2(\Omega)}^2 + 2\|\Delta G[w, z](0)\|_{L_2(\Omega)}^2 + 6 \int_0^t \int_\Omega [z, z] G[w, z] d\Omega$$

$$+ \int_\Omega [\mathcal{F}(w(t)), z(t)] z(t) d\Omega + \int_\Omega [\mathcal{F}(w(0)), z(0)] z(0) d\Omega.$$
Proposition 3.1.

\[
| \int_\Omega [F(w(t)), z(t)] z(t) \, d\Omega | \leq C(\|w_0\|_{H^2(\Omega)}, \|w_1\|_{L^2(\Omega)}) \|z(t)\|_{H^2(\Omega)}^{3/2},
\]

(3.9)

\[
| \int_\Omega [z(t), z(t)] G[w(t), z(t)] \, d\Omega | \leq C(\|w_0\|_{H^2(\Omega)}, \|w_1\|_{L^2(\Omega)}) \|z(t)\|_{H^2(\Omega)}^2.
\]

(3.10)

Proof of Proposition 3.1. Step 1: Proof of (3.9). We already know (see [12]) \([\|F(w)\|_{H^{3-1}(\Omega)} \leq C\|w\|_{H^2(\Omega)}^2].\) Therefore,

\[
| \int_\Omega [F(w), z] z \, d\Omega | = | \int_\Omega [z, z] F(w) \, d\Omega | \leq C\|F(w)\|_{H^1_\Omega} \|z\|_{H^2(\Omega)} \|z\|_{H^2(\Omega)}^{1/2},
\]

where 0 < \(\theta < 1\) and the last inequality follows by property (2.6) of the bracket. By interpolation inequalities, \([\|z\|_{H^\theta(\Omega)} \leq C\|z\|_{L^2(\Omega)}^{1/2} \|z\|_{H^2(\Omega)}^{1/2},\) 0 < \(\theta < 2\). Thus,

\[
\|z\|_{H^2(\Omega)} \leq C\|z\|_{L^2(\Omega)} \|z\|_{H^2(\Omega)}^{\frac{1}{2}} \|z\|_{H^2(\Omega)}^{\frac{1}{2}} \|z\|_{H^2(\Omega)}^{\frac{1}{2}}.
\]

(3.11)

\[
\|z\|_{H^{1+\theta}(\Omega)} \leq C\|z\|_{L^2(\Omega)} \|z\|_{H^2(\Omega)}^{\frac{3}{2}}.
\]

(3.12)

Hence,

Collecting equations (3.11) and (3.12), we find that

\[
| \int_\Omega [F(w), z] z \, d\Omega | \leq C\|w\|_{H^2(\Omega)}^2 \|z\|_{L^2(\Omega)} \|z\|_{H^2(\Omega)}^{3/2}.
\]

(3.13)

Now (3.9) follows from (3.13) combined with (3.2).

Step 2: Proof of (3.10). Recall from (2.6),

\[
\|[z, w]\|_{H^{-1-\theta}(\Omega)} \leq C\|z\|_{H^2(\Omega)} \|w\|_{H^2(\Omega)},
\]

0 < \(\theta < 1\). Hence,

\[
\|G[z, w]\|_{H^{3-\theta}(\Omega)} \leq C\|z\|_{H^2(\Omega)} \|w\|_{H^2(\Omega)}.
\]

(3.14)

By property (2.4) of the bracket and (3.14),

\[
\|[G[z, w], z]\|_{H^{-\theta}(\Omega)} \leq C\|z\|_{H^2(\Omega)} \|G[z, w]\|_{H^{3-\theta}(\Omega)} \leq C\|z\|_{H^2(\Omega)} \|z\|_{H^2(\Omega)} \|w\|_{H^2(\Omega)}.
\]

Therefore,

\[
| \int_\Omega [z, z] G[w, z] \, d\Omega | = | \int_\Omega [G[w, z], z] z \, d\Omega | \leq C\|[G[w, z], z]\|_{H^{-\theta}(\Omega)} \|z\|_{H^\theta(\Omega)} \leq C\|z\|_{H^2(\Omega)} \|w\|_{H^2(\Omega)} \|z\|_{L^2(\Omega)},
\]

(3.15)
where the last inequality follows by application of the interpolation inequality,
\[ \|z\|_{H^{2-\epsilon}(\Omega)} \leq C \|z\|_{L^2(\Omega)} \|z\|_{H^2(\Omega)}. \]  
(3.15)

Combining (3.2) with (3.15) yields (3.10).

**Proof of Lemma 3.1** (continued). From (3.8), (3.9), and (3.10) applied at \( t = 0 \) and
Gronwall’s inequality, we find
\[ \|z(t)\|_{H^2(\Omega)}^2 + \|z_\epsilon(t)\|_{L^2(\Omega)}^2 \leq C_0 \{ C_1 + \int_\Omega [\mathcal{F}(w(t)), z(t)] \, d\Omega \} e^{C_0 t}, \]
(3.16)
where the constants depend on the following norms:
\[ C_1 \equiv C_1(\|w_0\|_{H^4(\Omega)} + \|w_1\|_{H^2(\Omega)}), \]
\[ C_0 \equiv C_0(\|w(t)\|_{H^2(\Omega)}, \|w_\epsilon(t)\|_{L^2(\Omega)}) \leq C_0(\|w_0\|_{H^2(\Omega)}, \|w_1\|_{L^2(\Omega)}). \]

Inserting (3.9) into (3.16), we find
\[ \|z(t)\|_{H^2(\Omega)}^2 + \|z_\epsilon(t)\|_{L^2(\Omega)}^2 \leq C_0 C_1 (1 + \|z(t)\|_{H^2(\Omega)}^{3/2}) \leq C_0 (C_1 + \|z(t)\|_{H^2(\Omega)}^{3/2}) e^{C_0 t}, \]
by the a priori bound in (3.2). Since the dependence on \( \|z(t)\|_{H^2(\Omega)} \) is subquadratic, it follows that
\[ \|z(t)\|_{H^2(\Omega)}^2 + \|z_\epsilon(t)\|_{L^2(\Omega)}^2 \leq C_0 C_1, \quad t \leq T. \]  
(3.17)

To achieve the desired result of Lemma 3.1, elliptic regularity results applied to
\[ \Delta^2 w = -w_{tt} + [\mathcal{F}(w), w] \quad \text{in} \quad \Omega, \quad \Delta w + (1 - \mu) B_1 w = -h(\frac{\partial}{\partial \nu} w_\epsilon) \quad \text{on} \quad \Gamma, \]
\[ \frac{\partial}{\partial \nu} \Delta w + (1 - \mu) B_2 w - w = g(w_\epsilon) \quad \text{on} \quad \Gamma, \]
yields, with \( \epsilon < \frac{1}{2} \),
\[ \|w(t)\|_{H^3(\Omega)} \leq \{ \|w_{tt}(t)\|_{H^{-\epsilon}(\Omega)} + \|[\mathcal{F}(w(t)), w(t)]\|_{H^{-\epsilon}(\Omega)} \]
\[ + \|h(\frac{\partial}{\partial \nu} w_\epsilon)\|_{H^{1/2}(\Gamma)} + \|g(w_\epsilon)\|_{H^{-1/2}(\Gamma)} \} \]
\[ \leq C \{ \|w_{tt}(t)\|_{H^{-\epsilon}(\Omega)} + \|w(t)\|_{H^2(\Omega)} + \|w_\epsilon(t)\|_{H^2(\Omega)} + \|w(t)\|_{L^{r+1}(\Omega)} + \|w_\epsilon(t)\|_{L^{r+1}(\Omega)} \}, \]
(3.18)
where we have used \( L^{1+\epsilon}(\Gamma) \subset H^{-1/2}(\Gamma) \). By Sobolev’s imbeddings, trace theory, interpolation inequalities and (3.17) with (3.2),
\[ \|w\|_{L^{r+1}(\Gamma)} \leq C \|w\|_{H^2(\Omega)}^{r+1} \leq C \|w\|_{H^2(\Omega)} \|w\|_{L^2(\Omega)}^{r+1} \leq (C_0 C_1)^{r+1} C_0^{r+1}. \]  
(3.19)
Combining (3.17), (3.18)–(3.19) and (3.2) proves the desired result of Lemma 3.1.

4. *Existence of regular solutions: Proof of Theorem 1.1.* We shall use Schaeffer’s Theorem (see Kesavan, [10, page 221]). To accomplish this, we construct a map
\[ v \longrightarrow T v, \]  
(4.1)
defined on a Banach space, \( X \), where
\[ X = C(0, T; H^{3-\epsilon}(\Omega)) \cap C^1(0, T; H^{2-\epsilon}(\Omega)), \]  
(4.2)
and is a fixed number, and \( T v \) is defined as the solution to (2.1) with \( f(t) \) given by
\[ f(t) = \mathcal{F}(v(t)). \]  
(4.3)
Proposition 4.1. With reference to (4.3), \(0 < \epsilon < \frac{1}{2}\), we have
\[
\|f(t)\|_{H^{4-\epsilon}(\Omega)} \leq C\|v(t)\|_{H^{3-\epsilon}(\Omega)}\|v(t)\|_{H^{2}(\Omega)},
\]
(4.4)
\[
\|f_t(t)\|_{H^{2-\epsilon}(\Omega)} \leq C\|v(t)\|_{H^{3-\epsilon}(\Omega)}\|v(t)\|_{H^{2-\epsilon}(\Omega)}.
\]
(4.5)

Proof. From property (2.4) of the bracket,
\[
\|[v, v]\|_{H^{-\epsilon}(\Omega)} \leq C\|v(t)\|_{H^{3-\epsilon}(\Omega)}\|v(t)\|_{H^{2}(\Omega)}.
\]
(4.6)

By elliptic regularity and (4.6),
\[
\|\mathcal{F}(v)\|_{H^{4-\epsilon}(\Omega)} \leq \|G[v, v]\|_{H^{4-\epsilon}(\Omega)} \leq C\|[v, v]\|_{H^{-\epsilon}(\Omega)} \leq C\|v\|_{H^{3-\epsilon}(\Omega)}\|v\|_{H^{2}(\Omega)},
\]
(4.7)
proving (4.4). As for (4.5), we use property (2.5) of the bracket to find
\[
\|[v, v_t]\|_{H^{2-\epsilon}(\Omega)} \leq C\|v_t\|_{H^{2-\epsilon}(\Omega)}\|v\|_{H^{3-\epsilon}(\Omega)},
\]
(4.8)

which, in turn, by elliptic regularity gives
\[
\|G[v, v_t]\|_{H^{4-\epsilon}(\Omega)} \leq C\|v_t\|_{H^{2-\epsilon}(\Omega)}\|v\|_{H^{3-\epsilon}(\Omega)},
\]
(4.9)

which, in particular, proves (4.5). \(\square\)

Since, by Sobolev imbeddings (see [1])
\[
H^{4-\epsilon}(\Omega) \subset W^2_\infty(\Omega) \quad \forall \ 0 < \epsilon < \frac{1}{2},
\]
(4.10)

the result of Lemma 2.1 together with Proposition 4.1 assert that the map \(T\) is bounded from \(X\) into
\[
X_1 \equiv L^\infty(0, T; H^3(\Omega)) \cap W^1_\infty(0, T; H^2(\Omega)) \cap W^2_\infty(0, T; L_2(\Omega)).
\]

Thus, by the compactness of imbeddings, \(H^3(\Omega) \subset H^{3-\epsilon}(\Omega)\) and \(H^2(\Omega) \subset H^{2-\epsilon}(\Omega)\), and by the result due to Simon ([20], Corollary 4), we obtain that
\[
T : X \rightarrow X \quad \text{is compact.}
\]
(4.11)

Our next step is to show that the map \(T : X \rightarrow X\) is continuous. Denote \(F \equiv L^1([0, T]; W^2_\infty(\Omega)) \cap L^\infty([0, T], H^{2+\epsilon}(\Omega))\). By the same arguments as those in Proposition 4.1, one easily shows that the map \(F : X \rightarrow F\) is continuous. Thus, it suffices to prove the continuity of the map \(f \rightarrow (w, w_t)\) from \(F \rightarrow X\), where \((w, w_t)\) is a solution to (2.1) corresponding to \(f\). Let \(f_n \in F\) be such that \(f_n \rightarrow f\) in \(F\). We need to show that \((w_n, w_{n,t}) \rightarrow (w, w_t)\) in \(X\), where \(w_n\) is a solution to (2.1) corresponding
to \( f_n \). Consider the system solved by \( \tilde{w} \equiv w_n - w \). Multiplying the result by \( \tilde{w}_t \) and integrating by parts, we find

\[
\| \tilde{w}(t) \|_{H^2(\Omega)}^2 + \| \tilde{w}_t(t) \|_{L_2(\Omega)}^2 + \int_0^t \int_{\Gamma} [h(\frac{\partial}{\partial \nu} w_{n,t}) - h(\frac{\partial}{\partial \nu} w_t)] \frac{\partial}{\partial t} \tilde{w} \, d\Gamma \, dt
\]

\[
+ \int_0^t \int_{\Gamma} [g(w_{n,t}) - g(w_t)] \tilde{w}_t \, d\Gamma \, dt \leq C \int_0^t \int_{\Omega} \left\{ \| [f_n - f, w_n] \|_{L_2(\Omega)} + \| [f, \tilde{w}] \|_{L_2(\Omega)} \right\} \| \tilde{w}_t \|_{L_2(\Omega)} \, dt
\]

\[
\leq C \int_0^t \int_{\Omega} \left\{ \| f_n - f \|_{W^2_2(\Omega)} \| w_n \|_{H^2(\Omega)} + \| w_{n,t} \|^2_{L_2(\Omega)} + \| w_t \|^2_{L_2(\Omega)} \right\} \, dt
\]

\[
+ \int_0^t \| f \|_{W^2_2(\Omega)} \| \tilde{w} \|^2_{H^2(\Omega)} + \| \tilde{w}_t \|^2_{L_2(\Omega)} \right\} \, dt
\]

Applying Proposition 2.1 and Gronwall’s inequality and noting that the two integral terms on the left-hand side of (4.12) are positive by the monotonicity of \( h \) and \( g \), we obtain

\[
\| \tilde{w}(t) \|^2_{H^2(\Omega)} + \| \tilde{w}_t(t) \|^2_{L_2(\Omega)} \leq C(\| f \|_{L_2(\Omega)} \| w_0 \|^2_{H^2(\Omega)} + \| w_1 \|_{L_2(\Omega)}) \int_0^t \| f_n - f \|_{W^2_2(\Omega)} \, dt.
\]

Hence, as \( \| f_n - f \|_{L_2(\Omega)} \rightarrow 0 \), \( \| \tilde{w}(t) \|^2_{H^2(\Omega)} + \| \tilde{w}_t(t) \|^2_{L_2(\Omega)} \rightarrow 0 \).

Next, we consider the continuity of the higher norms. Returning to equation (2.25), we proceed as before. Let \( z_n \) denote the solution of equation (2.25) with \( f \) replaced by \( f_n \). Consider the system solved by \( \tilde{z} \equiv z_n - z \). Multiplying the result by \( \tilde{z}_t \), integrating by parts, and recalling hypothesis (H-1), we find

\[
\| \tilde{z}(t) \|^2_{H^2(\Omega)} + \| \tilde{z}_t(t) \|^2_{L_2(\Omega)} + \int_0^t \int_{\Gamma} \left( \frac{\partial}{\partial \nu} \tilde{z}_t \right)^2 \, d\Gamma \, dt + \int_0^t \int_{\Omega} \left( \frac{\partial}{\partial t} \tilde{z}_t \right)^2 \, dt \, dt
\]

\[
\leq C \int_0^t \int_{\Gamma} [h'(\frac{\partial}{\partial \nu} w_t) - h'(\frac{\partial}{\partial \nu} w_{n,t})] \frac{\partial}{\partial t} \tilde{z}_t \, d\Gamma \, dt
\]

\[
+ C \int_0^t \int_{\Omega} \left\{ \| f_n - f, w_n \| + \| f_n - f, z_n \| \right\} \tilde{z}_t \, dt \leq C_{T, \rho} \int_0^t \int_{\Gamma} \left[ h'(\frac{\partial}{\partial \nu} w_t) - h'(\frac{\partial}{\partial \nu} w_{n,t}) \right]^2 \, d\Gamma \, dt
\]

\[
+ \int_0^t \int_{\Gamma} \left[ g'(w_t) - g'(w_{n,t}) \right]^2 \, d\Gamma \, dt + \rho \int_0^t \int_{\Gamma} \left[ \| \frac{\partial}{\partial \nu} \tilde{z}_t \|^2 + \tilde{z}_t^2 \right] \, d\Gamma \, dt
\]

\[
+ C \int_0^t \int_{\Omega} \left\{ \| f_n - f, w_n \| + \| f_n - f, z_n \| \right\} \tilde{z}_t \, dt \}
\]

where \( \rho > 0 \) can be taken arbitrarily small.
Using the properties of the bracket, (2.2)–(2.7), we obtain the following bounds:

\[
\int_0^t \int_\Omega [f_{n,t} - f_t, w_n] \tilde{z}_t \, d\Omega \, dt \\
\leq C \|\tilde{z}_t\|_{L^\infty (0,T;L^2(\Omega))} \|w_n\|_{L^1(0,T;H^3(\Omega))} \|f_{1,t} - f_{2,t}\|_{L^\infty (0,T;H^{2+\epsilon}(\Omega))},
\]

\[
\int_0^t \int_\Omega [f_t, \tilde{w}] \tilde{z}_t \, d\Omega \, dt \\
\leq C \|\tilde{z}_t\|_{L^\infty (0,T;L^2(\Omega))} \|\tilde{w}\|_{L^1(0,T;H^3(\Omega))} \|f_{2,t}\|_{L^\infty (0,T;H^{2+\epsilon}(\Omega))},
\]

(4.15)

\[
\int_0^t \int_\Omega [f_n - f, z_n] \tilde{z}_t \, d\Omega \, dt \\
\leq C (\|\tilde{z}_t\|_{L^\infty (0,T;L^2(\Omega))} + \|z_n\|_{L^\infty (0,T;H^2(\Omega))}) \|f_n - f\|_{L^1(0,T;W^2_2(\Omega))},
\]

\[
\int_0^t \int_\Omega [f, z_n - \tilde{z}] \tilde{z}_t \, d\Omega \, dt \\
\leq C (\|\tilde{z}_t\|_{L^\infty (0,T;L^2(\Omega))} + \|\tilde{z}\|_{L^\infty (0,T;H^2(\Omega))}) \|f\|_{L^1(0,T;W^2_2(\Omega))}.
\]

On the other hand, by using elliptic theory to estimate \(\|\tilde{w}(t)\|_{H^3(\Omega)}\), we obtain

\[
\|\tilde{w}(t)\|_{H^3(\Omega)} \leq C (\|\tilde{w}t(t)\|_{L^2(\Omega)} + \|[f(t) - f(t), w_n(t)]\|_{L^2(\Omega)} + \|[f, \tilde{w}(t)]\|_{L^2(\Omega)}
\]

\[
+ \|h(\frac{\partial}{\partial v} w_n(t)) - h(\frac{\partial}{\partial v} w(t))\|_{H^{1/2}(\Gamma)} + \|g(w_n(t)) - g(w(t))\|_{H^{-1/2}(\Gamma)}).
\]

(4.16)

By an application of Theorem 4 in [23] and arguments similar to those in (2.31)–(2.32), combined with trace theory, we obtain

\[
\|h(\frac{\partial}{\partial v} w_n(t)) - h(\frac{\partial}{\partial v} w(t))\|_{H^{1/2}(\Gamma)} \leq C \|\frac{\partial}{\partial v} \tilde{w}_t(t)\|_{H^{1/2}(\Gamma)} \leq C \|\tilde{w}_t(t)\|_{H^2(\Omega)}.
\]

The local Lipschitz property of the function \(g\), hypothesis (H-1), and Sobolev’s embeddings give us

\[
\|g(w_n(t)) - g(w(t))\|_{H^{1/2}(\Gamma)} \leq C (\|w_n(t)\|_{H^2(\Omega)}, \|w(t)\|_{H^2(\Omega)}) \|\tilde{w}_t(t)\|_{H^2(\Omega)}
\]

\[
\leq C_T (\|f\|_{L^2}, \|w_0\|_{H^4(\Omega)}, \|w_1\|_{H^2(\Omega)}) \|\tilde{w}_t(t)\|_{H^2(\Omega)},
\]

(4.17)

where we have used the result of Lemma 2.1. Combining (4.16)–(4.17), yields, for all \(t \leq T\),

\[
\|\tilde{w}(t)\|_{H^3(\Omega)} \leq C_T (\|f\|_{L^2}, \|w_0\|_{H^4(\Omega)}, \|w_1\|_{H^2(\Omega)}) (\|\tilde{w}_t(t)\|_{L^2(\Omega)} + \|\tilde{w}_t(t)\|_{H^2(\Omega)}
\]

\[
+ \|\tilde{w}(t)\|_{H^2(\Omega)} + \|f_n - f(t)\|_{W^2_2(\Omega)}).
\]

(4.18)

Going back to (4.14), using (4.15), (4.18), Gronwall’s inequality and selecting \(\rho\) in (4.14) suitably small yields

\[
\|\tilde{w}(t)\|_{H^3(\Omega)} + \|\tilde{w}_t(t)\|_{H^2(\Omega)} + \|\tilde{w}_{tt}(t)\|_{L^2(\Omega)}
\]

\[
\leq C_T (\|f\|_{L^2}, \|w_0\|_{H^4(\Omega)}, \|w_1\|_{H^2(\Omega)}) (\|f_n - f\|_{L^2}^2
\]

\[
+ C_{\rho} \int_0^t \int_{\Gamma} |h'((\frac{\partial}{\partial v} w_n(t)) - h'((\frac{\partial}{\partial v} w(t)))|^2 |\frac{\partial}{\partial v} z(t)|^2 \, d\Gamma \, dt
\]

\[
+ \int_0^t \int_{\Gamma} |g'(w_n(t)) - g'(w(t))|^2 z(t)^2 \, d\Gamma \, dt).
\]

(4.19)
Next, we must analyze the effects of the boundary integrals in (4.19).

**Analysis of** $A_n = \int_0^t \int_\Gamma \left[ h'(\frac{\partial}{\partial \nu} w_{n,t}) - h'(\frac{\partial}{\partial \nu} w_t) \right] \left| \frac{\partial}{\partial \nu} z_t \right|^2 d\Gamma dt$. Notice that from Lemma 2.1 and hypothesis (H-1), the integrand satisfies the following estimate uniformly in $n$:

$$|h'(\frac{\partial}{\partial \nu} w_{n,t}) - h'(\frac{\partial}{\partial \nu} w_t)| \left| \frac{\partial}{\partial \nu} z_t \right|^2 \leq C \left| \frac{\partial}{\partial \nu} z_t \right|^2 \in L_1(\Sigma_T);$$  

(4.20)

therefore Lebesgue’s dominated convergence theorem applies, provided the integrand goes to zero as $f_n \to f$ in $F$. Recall the estimate for $\tilde{w}$, (4.12). We know

$$\int_0^t \int_\Gamma \left[ h(\frac{\partial}{\partial \nu} w_{n,t}) - h(\frac{\partial}{\partial \nu} w_t) \right] \frac{\partial}{\partial \nu} \tilde{w}_t d\Gamma dt \leq C_0(f, w_0, w_1) \int_0^t \|f_n - f\|_{W^{2,1}_2(\Omega)} dt,$$

(4.21)

therefore, by hypothesis (H-1),

$$\int_0^t \int_\Gamma \left| \frac{\partial}{\partial \nu} (w_{n,t} - w_t) \right| d\Gamma dt \to 0; \text{ i.e., } \frac{\partial}{\partial \nu} \tilde{w}_t \to 0 \text{ a.e.,}$$  

(4.22)

hence,

$$h'(\frac{\partial}{\partial \nu} w_{n,t}) - h'(\frac{\partial}{\partial \nu} w_t) \to 0 \text{ a.e.,}$$  

(4.23)

and Lebesgue’s dominated convergence theorem applies, allowing us to conclude $A_n \to 0$ when $f_n \to f$ in $F$.

**Analysis of** $B_n = \int_0^t \int_\Gamma |g'(w_t) - g'(w_{n,t})|^2 z_t^2 d\Gamma dt$. We bound $B_n$ as follows:

$$\int_0^t \int_\Gamma |g'(w_t) - g'(w_{n,t})|^2 z_t^2 d\Gamma dt \leq C \left\| g'(w_t) - g'(w_{n,t}) \right\|_{C(0,T;T;\Gamma)} \left\| z_t \right\|_{L^2(0,T;\Gamma)}^2$$

$$\leq C_T \left( \|f\|_F, \|w_0\|_{H^4(\Omega)}, \|w_1\|_{H^2(\Omega)} \right) \|g'(w_t) - g'(w_{n,t})\|_{C(0,T;\Gamma)}^2$$

$$\leq C_T \left( \|f\|_F, \|w_0\|_{H^4(\Omega)}, \|w_1\|_{H^2(\Omega)} \right)$$

$$\times L \left( \|w_{n,t}\|_{L^\infty(0,T;H^2(\Omega))}, \|w_t\|_{L^\infty(0,T;H^2(\Omega))} \right) \|w_{n,t} - w_t\|_{C(0,T;\Gamma)}^2$$

$$\leq C_T \left( \|f\|_F, \|w_0\|_{H^4(\Omega)}, \|w_1\|_{H^2(\Omega)} \right) \|w_{n,t} - w_t\|_{C(0,T;\Gamma)}^2$$

(4.24)

$$\leq C_T \left( \|f\|_F, \|w_0\|_{H^4(\Omega)}, \|w_1\|_{H^2(\Omega)} \right)$$

where the second inequality follows from Lemma 2.1, the third by the local Lipschitz property of $g'$, Sobolev’s imbeddings and, again, Lemma 2.1, and the fifth by interpolation inequality with $0 < \theta < 1$. Let $\theta = \frac{\gamma + 1}{2}$. Using $ab \leq \epsilon_0 a^p + C_\epsilon b^p$, where $\frac{1}{p} + \frac{1}{q} = 1$, we obtain that $\forall \epsilon_0 > 0$,

$$B_n \leq C_T \left( \|f\|_F, \|w_0\|_{H^4(\Omega)}, \|w_1\|_{H^2(\Omega)} \right) \left\| \tilde{w}_t \right\|_{C(0,T;L^2(\Omega))}^2 + \epsilon_0 \left\| \tilde{w}_t \right\|_{L^2(0,T;H^2(\Omega))}^2.$$

(4.25)

By (4.19), (4.23) and (4.25), after taking $\epsilon_0$ small enough, we obtain

$$\left\| \tilde{w}(t) \right\|_{H^2(\Omega)} + \left\| \tilde{w}_t(t) \right\|_{L^2(\Omega)}$$

$$\leq C_T \left( \|f\|_F, \|w_0\|_{H^4(\Omega)}, \|w_1\|_{H^2(\Omega)} \right) \left\| f_n - f \right\|_F^2 + A_n + \left\| \tilde{w}_t \right\|_{C(0,T;L^2(\Omega))} \to 0 \text{ as } n \to \infty,$$

(4.26)
where the last conclusion follows from (4.23) and (4.13). This concludes the proof of the continuity of the map, \( T \); i.e., as \( \| f_n - f \|_F \to 0 \), \( \| \ddot{z}(t) \|_{H^2(\Omega)} + \| \ddot{z}_t(t) \|_{L^2(\Omega)} \to 0 \). Thus, the map \( T : X \to X \) is continuous.

The a priori bounds of Lemma 3.1 imply that if \( w \) is a solution to \( w = \delta T(w) \), where \( \delta < 1 \), then
\[
\| w \|_X \leq C_0(\| w_0 \|_{H^3(\Omega)}, \| w_1 \|_{H^2(\Omega)}).
\] (4.27)

This, together with (4.11) and the continuity of \( T \) allows for an application of Schaeffer’s Theorem ([10, page 221]) to conclude that there is a solution, \( w \), to (1.1) which belongs to the space \( X \). Once again using the result of Lemma 3.1 allows us to “boost” (by \( \epsilon \)) the regularity of the solution \( w \) from \( X \) to \( C(0, T; H^3(\Omega)) \cap C^1(0, T; H^2(\Omega)) \).

5. Uniqueness: Proof of Theorem 1.2. The following result is critical to the proof.

**Theorem 5.1.** (i) The map \((u, v) \to G[u, v]\) is bounded from \(H^2(\Omega) \times H^2(\Omega) \to H^3(\Omega) \cap W^1_2(\Omega), \ \Omega \subset \Omega\).

(ii) The map \((f, u) \to [f, u]\) is bounded from \(H^3(\Omega) \times H^2(\Omega) \to L^2(\Omega)\).

As the proof of Theorem 5.1, based on the compensated compactness method, is technical, it is relegated to Section 8.

**Remark 5.1.** Notice that the result of Theorem 5.1 improves by “\( \epsilon \)” a known regularity result stating that this map is bounded from \(H^2(\Omega) \times H^2(\Omega) \to H^3-\epsilon(\Omega)\). As we shall see later, this improvement by “\( \epsilon \)” is critical.

Let \( w_1 \) and \( w_2 \) be any two solutions of (1.1) such that \( w_i \in C(0, T; H^2(\Omega)) \times C^1(0, T; L^2(\Omega)) \). Set \( \tilde{w} \equiv w_1 - w_2 \). Then \( \tilde{w} \) satisfies
\[
\begin{cases}
\tilde{w}_{tt} + \Delta^2 \tilde{w} = [\mathcal{F}(w_1) - \mathcal{F}(w_2), w_1] + [\mathcal{F}(w_2), \tilde{w}] & \text{in } Q_T \\
\Delta \tilde{w} + (1 - \mu)B_1 \tilde{w} = -[h\left( \frac{\partial}{\partial \nu} w_1, t \right) - h\left( \frac{\partial}{\partial \nu} w_2, t \right)] & \text{on } \Sigma_T \\
\frac{\partial}{\partial \nu} \Delta \tilde{w} + (1 - \mu)B_2 \tilde{w} = g(w_1, t) - g(w_2, t) & \text{on } \Sigma_T.
\end{cases}
\] (5.1)

Energy estimates applied to (5.1) yield
\[
\| \tilde{w}(t) \|_{H^2(\Omega)}^2 + \| \tilde{w}_t(t) \|_{L^2(\Omega)}^2 + \int_0^t \int_{\Gamma} [h\left( \frac{\partial}{\partial \nu} w_1, t \right) - h\left( \frac{\partial}{\partial \nu} w_2, t \right)] \frac{\partial}{\partial \nu} \tilde{w} \\
+ [g(w_1, t) - g(w_2, t)] \tilde{w}_t \} d\Gamma dt \\
= \int_0^t \int_{\Omega} \{[\mathcal{F}(w_1) - \mathcal{F}(w_2), w_1] \tilde{w}_t + [\mathcal{F}(w_2), \tilde{w}] \tilde{w}_t \} d\Omega dt.
\] (5.2)

Indeed, the formal computation leading to (5.2) can be made rigorous by treating the nonlinear bracket, \([w, \chi(w)]\), as a perturbation of a nonlinear semigroup of contractions generated by (2.17) and applying the usual (see [2]) energy estimates in the context of nonlinear semigroup theory.

From monotonicity of \( h \) and \( g \) and using the Cauchy-Schwarz inequality, we obtain
\[
\| \tilde{w}(t) \|_{H^2(\Omega)}^2 + \| \tilde{w}_t(t) \|_{L^2(\Omega)}^2 \\
\leq C \int_0^t \{ [\mathcal{F}(w_1) - \mathcal{F}(w_2), w_1] \|_{L^2(\Omega)}^2 + [\mathcal{F}(w_2), \tilde{w}] \|_{L^2(\Omega)}^2 \} dt.
\] (5.3)
From part (i) of Theorem 5.1,
\[ \|F(w_2)\|_{H^3(\Omega)} \leq C\|w_2\|_{H^2(\Omega)}\|w_2\|_{H^2(\Omega)}. \]
(5.4)

Moreover,
\[ \|F(w_2), \tilde{w}\|_{L_2(\Omega)} \leq C\|\tilde{w}\|_{H^2(\Omega)}\|F(w_2)\|_{H^3(\Omega)} \leq C\|\tilde{w}\|_{H^2(\Omega)}\|w_2\|_{H^2(\Omega)}\|w_2\|_{H^2(\Omega)}. \]
(5.5)

We can rewrite \( F(w_1) - F(w_2) \) in the following way:
\[ F(w_1) - F(w_2) = G([w_1, w_1] - [w_2, w_2]) = G[\tilde{w}, w_1 + w_2]. \]
(5.6)
Hence, again from Theorem 5.1,
\[ \|G[\tilde{w}, w_1 + w_2]\|_{H^3(\Omega)} \leq C\|\tilde{w}\|_{H^2(\Omega)}\|w_1\|_{H^2(\Omega)} + \|w_2\|_{H^2(\Omega)}, \]
(5.7)
\[ \implies \|F(w_1) - F(w_2), w_1\|_{L_2(\Omega)} \leq C\|w_1\|_{H^2(\Omega)}\|F(w_1) - F(w_2)\|_{H^3(\Omega)} \leq C\|w_1\|_{H^2(\Omega)}\|\tilde{w}\|_{H^2(\Omega)}\|w_1\|_{H^2(\Omega)} + \|w_2\|_{H^2(\Omega)}]. \]
(5.8)

Combining (5.2), (5.5), and (5.8) proves that the solution must be unique.

**Remark 5.2.** Notice that the proof of Theorem 1.1 could be simplified if we were to use the regularity result of Theorem 5.1. The reason why we chose not to evoke this result at that point is that the present proof of Theorem 1.1 is easily adaptable to other von Kármán systems (e.g., the full von Kármán system considered in [19]) where “sharp” regularity on the von Kármán nonlinearity is unavailable. In fact, the proof of Theorem 1.1, based on the application of Schaeffer’s Theorem, provides the existence of “smooth” solutions for a full von Kármán system with nonlinear boundary dissipation. (The arguments of [19], based on a classical contraction fixed-point method do not appear adaptable to treat the case of nonlinear boundary damping.)


**Lemma 6.1** (Local existence). Assume hypothesis (H-1). Then there exists \( T_0 > 0 \) such that for any initial data \( w_0 \in H^2(\Omega) \), \( w_1 \in L_2(\Omega) \), there exists a unique solution to (1.1), \( w \in C(0, T_0; H^2(\Omega)), w_t \in C(0, T_0; L_2(\Omega)) \) and \( w_t, \frac{\partial}{\partial t} w_t \in L^2(\Sigma T_0) \).

**Proof.** It suffices to construct a unique fixed point for the map \( T \) introduced in (4.1) and defined on \( C(0, T_0; B_R) \), where \( B_R \equiv \{ u \in H^2(\Omega) \times L_2(\Omega) : \|u\|_{H^2(\Omega) \times L_2(\Omega)} \leq R \} \). The boundary regularity will follow from energy estimates.

We shall prove that for sufficiently small values of \( T_0 \) and sufficiently large values of \( R \), the map \( T \) is a contraction on \( C(0, T_0; B_R) \). To accomplish this, we first note that by virtue of Theorem 5.1, the Airy’s stress function \( F(w) \) is locally Lipschitz: \( H^2(\Omega) \to H^3(\Omega) \). Indeed, from Theorem 5.1, it follows that
\[ \|F(w_1) - F(w_2)\|_{H^3(\Omega)} = \|G[w_1 - w_2, w_1 + w_2]\|_{H^3(\Omega)} \leq C\|w_1 - w_2\|_{H^2(\Omega)}\|w_1\|_{H^2(\Omega)} + \|w_2\|_{H^2(\Omega)}. \]
(6.1)
Applying (2.16) with \( w \equiv T v \), where \( v \in C(0, T; B_R) \), and \( f \equiv F(v) \) and recalling (H-1), we obtain for \( t \leq T_0 \),

\[
\|Tv(t)\|^2_{H^2(\Omega)} + \frac{d}{dt} \|Tv(t)\|^2_{L^2(\Omega)} + \frac{d}{dt} \|Tv\|^2_{L^2(\Sigma_t)} + \frac{\partial}{\partial t} \frac{d}{dt} \|Tv\|^2_{L^2(\Sigma_t)} \\
\leq C\{ \|w_0\|^2_{H^2(\Omega)} + \|w_1\|^2_{L^2(\Omega)} + \int_0^t \|\mathcal{F}(v(\tau)), (w(\tau))\|^2_{L^2(\Omega)} d\tau \} \\
\leq C\{ \|w_0\|^2_{H^2(\Omega)} + \|w_1\|^2_{L^2(\Omega)} + \int_0^t \|w(\tau)\|^4_{H^2(\Omega)} \|Tv(\tau)\|^2_{H^2(\Omega)} d\tau \} \\
\leq C\{ \|w_0\|^2_{H^2(\Omega)} + \|w_1\|^2_{L^2(\Omega)} + R^4T_0 \|Tv\|^2_{C(0,T_0;B_R)} \},
\]

(6.2)

where the first inequality follows from Theorem 5.1 and (2.3). Selecting \( R \) so that

\[
\|w_0\|^2_{H^2(\Omega)} + \|w_1\|^2_{L^2(\Omega)} \leq \frac{R^2}{4C},
\]

and taking \( T_0 \) sufficiently small yields \( T(C(0,T_0;B_R)) \subset C(0,T_0;B_R) \). On the other hand, estimate (4.12) applied with \( f_n \equiv \mathcal{F}(v_1), f \equiv \mathcal{F}(v_2) \) gives

\[
\|Tv_1 - Tv_2\|^2_{H^2(\Omega)} + \frac{d}{dt} (Tv_1 - Tv_2)(t) \leq C \int_0^t \{ \|\mathcal{F}(v_1) - \mathcal{F}(v_2)\|^2_{H^2(\Omega)} \|Tv_1\|^2_{H^2(\Omega)} + \|\mathcal{F}(v_2)\|^2_{H^2(\Omega)} \|Tv_1 - Tv_2\|^2_{H^2(\Omega)} \} dt \\
\leq CR^4T_0 \{ \|v_1 - v_2\|^2_{C(0,T_0;H^2(\Omega))} + \|Tv_1 - Tv_2\|^2_{C(0,T_0;H^2(\Omega))} \},
\]

where we have used (5.1), (5.2) and Theorem 5.1. Hence,

\[
\|Tv_1 - Tv_2\|^2_{C(0,T_0;B_R)} \leq \frac{CR^4T_0}{1 - CR^4T_0} \|v_1 - v_2\|^2_{C(0,T_0;H^2(\Omega))}.
\]

(6.4)

Taking \( T_0 \) small enough yields the contraction property for the map \( T \). The result of Lemma 6.1 now follows from the Contraction Mapping Principle and inequality (6.2).

To complete the proof of Theorem 1.3, it suffices to establish the following a priori bound.

**Lemma 6.2** (A priori bounds). Assume hypotheses (H-1)–(H-3) hold. Let \( (w, w_t) \) be any local solution to (1.1) such that \( w \in C(0,T_0;H^2(\Omega)), w_t \in C(0,T_0;L^2(\Omega)) \) and \( w_t|_{\Gamma}, \frac{\partial}{\partial \nu} w_t|_{\Gamma} \in L^2(\Sigma_{T_0}) \). Then the following a priori bound holds.

\[
\|w(t)\|^2_{H^2(\Omega)} + \|w_t(t)\|^2_{L^2(\Omega)} + \|w_t\|^2_{L^2(\Sigma_t)} + \|\frac{\partial}{\partial \nu} w_t\|^2_{L^2(\Sigma_t)} \\
\leq C(\|w_0\|^2_{H^2(\Omega)} + \|w_1\|^2_{L^2(\Omega)}) \quad t \geq 0.
\]

(6.5)

**Proof.** The a priori bound in (6.5) was already proven for “smooth” solutions (see (3.2)). We need to extend this bound to hold for all weak solutions. While this is standard in the case of homogeneous or, more generally, linear boundary conditions (see
[11], [19]), in the case of nonlinear boundary damping, the treatment of the boundary terms requires special care. To this end, we select a suitable approximation of the initial data \( w_0 \in H^2(\Omega), w_1 \in L_2(\Omega) \) such that

\[
H^4(\Omega) \ni w_{0m} \to w_0 \text{ in } H^2(\Omega), \quad H^2(\Omega) \ni w_{1m} \to w_1 \text{ in } L_2(\Omega),
\]

and \( w_{0m}, w_{1m} \) satisfy compatibility conditions (1.5). Let \( w_m(t) \) denote a solution to (1.1) corresponding to initial data \( (w_{0m}, w_{1m}) \).

From Theorem 1.1, we infer that \( w_m(t) \) satisfies the regularity properties listed in (3.1). Hence, inequalities (3.1) and (3.2) apply and

\[
\|w_m(t)\|_{H^2(\Omega)}^2 + \|w_{m,t}(t)\|_{L_2(\Omega)}^2 + \int_0^t \int_\Gamma h(\frac{\partial}{\partial \nu}w_{m,t}) \frac{\partial}{\partial \nu}w_{m,t} \, d\Gamma \, dt \leq C(\|w_0\|_{H^2(\Omega)}, \|w_1\|_{L_2(\Omega)}).
\]

(6.7)

We shall show that

\[
w_m \to w \text{ in } C(0,T_0; H^2(\Omega)), \quad w_{m,t} \to w_t \text{ in } C(0,T_0; L_2(\Omega)),
\]

(6.8)

where, we recall, \( w \) is a weak solution to (1.1).

Estimates (4.12) and (4.13) applied with \( f_n \equiv \mathcal{F}(w_n), f \equiv \mathcal{F}(w_m) \) and Theorem 5.1 yield

\[
\|(w_n - w_m)\|_{H^2(\Omega)}^2 + \|(w_{n,t} - w_{m,t})\|_{L_2(\Omega)}^2 + \int_0^t \int_\Gamma \left[ h(\frac{\partial}{\partial \nu}w_{n,t}) - h(\frac{\partial}{\partial \nu}w_{m,t}) \right] \frac{\partial}{\partial \nu}(w_{n,t} - w_{m,t}) \, d\Gamma \, dt
\]

\[
+ \int_0^t \int_\Gamma [g(w_{n,t}) - g(w_{m,t})](w_{n,t} - w_{m,t}) \, d\Gamma \, dt \leq C\{\|w_{0n} - w_{0m}\|_{H^2(\Omega)}^2 + \|w_{1n} - w_{1m}\|_{L_2(\Omega)}^2\}
\]

\[
+ C\int_0^t \|\mathcal{F}(w_n) - \mathcal{F}(w_m)\|_{H^2(\Omega)}^2 \|w_n\|_{H^2(\Omega)} d\tau + C\int_0^t \|\mathcal{F}(w_m)\|_{H^2(\Omega)}^2 \|w_n - w_m\|_{H^2(\Omega)}^2 d\tau.
\]

(6.9)

From (6.1) and (6.7),

\[
\int_0^t \|\mathcal{F}(w_n) - \mathcal{F}(w_m)\|_{H^2(\Omega)}^2 \|w_n\|_{H^2(\Omega)} d\tau \leq C(\|w_0\|_{H^2(\Omega)}, \|w_1\|_{L_2(\Omega)}) \int_0^t \|w_n(\tau) - w_m(\tau)\|_{H^2(\Omega)}^2 d\tau.
\]

(6.10)

From Theorem 5.1 and (6.11),

\[
\int_0^t \|\mathcal{F}(w_m(\tau))\|_{H^2(\Omega)}^2 \|w_n(\tau) - w_m(\tau)\|_{H^2(\Omega)}^2 d\tau \leq C(\|w_0\|_{H^2(\Omega)}, \|w_1\|_{L_2(\Omega)}) \int_0^t \|w_n(\tau) - w_m(\tau)\|_{H^2(\Omega)}^2 d\tau.
\]

(6.11)
Inserting (6.10) and (6.11) into (6.9) and applying Gronwall’s inequality with (6.6) gives

\[ w_m \to w^* \quad \text{in} \quad C(0,T;H^2(\Omega)), \quad w_{m,t} \to w_t^* \quad \text{in} \quad C(0,T;L_2(\Omega)), \]  

(6.12)

\[
\lim_{n,m \to \infty} \left\{ \int_0^t \int_{\Gamma} [h(\frac{\partial}{\partial \nu} w_{n,t}) - h(\frac{\partial}{\partial \nu} w_{m,t})] \frac{\partial}{\partial \nu} (w_{n,t} - w_{m,t}) \, d\Gamma \, dt \\
+ \int_0^t \int_{\Gamma} [g(w_{n,t}) - g(w_{m,t})](w_{n,t} - w_{m,t}) \, d\Gamma \, dt \right\} \to 0.
\]

We need to show that \( w^* \) coincides with \( w \), the weak solution to (1.1).

From (6.1), (2.3), and (6.12), it follows that \([\mathcal{F}(w_m(t)), w_m(t)] \to [\mathcal{F}(w^*(t)), w^*(t)]\) in \( L_2(\Omega) \). This result allows us to deduce (by standard arguments) that \( w^* \) satisfies (1.1.a), (1.1.b) in the sense of distributions. In order to show that \( w^* \) also satisfies boundary conditions (1.1.c) and (1.1.d), it suffices to prove that

\[
h(\frac{\partial}{\partial \nu} w_{m,t}) \to h(\frac{\partial}{\partial \nu} w_t^*) \quad \text{in} \quad L^2(0,T_0;\Gamma), \\
g(w_{m,t}) \to g(w_t^*) \quad \text{in} \quad L^2(0,T_0;\Gamma).
\]

(6.13)

Indeed, once (6.13) is established, the uniqueness of solutions to (1.1) in \( C(0,T_0;H^2(\Omega)) \times C(0,T_0;L_2(\Omega)) \) asserts that \( w \equiv w^* \), as desired for (6.8). To show (6.13), we return to (6.7) which in view of (H-3), in particular, implies

\[
\| \frac{\partial}{\partial \nu} w_{m,t} \|_{L^2(0,T_0;\Gamma)} + \| w_{m,t} \|_{L^{r+2}(0,T_0;\Gamma)} \leq C(\| w_0 \|_{H^2(\Omega)}, \| w_1 \|_{L_2(\Omega)}).
\]

Hence,

\[
\frac{\partial}{\partial \nu} w_{m,t} \to \frac{\partial}{\partial \nu} w_t^* \quad \text{weakly in} \quad L^2(0,T_0;\Gamma), \\
w_{m,t} \to w_t^* \quad \text{weakly in} \quad L^{r+2}(0,T_0;\Gamma).
\]

On the other hand, by recalling hypothesis (H-1), we obtain

\[
h(\frac{\partial}{\partial \nu} w_{m,t}) \to h_0 \quad \text{weakly in} \quad L^2(0,T_0;\Gamma), \\
g(w_{m,t}) \to g_0 \quad \text{weakly in} \quad L^{r+2}_{\Gamma}(0,T_0;\Gamma).
\]

Applying Lemma 1.3 in [2] after recalling the monotonicity of \( h \) and \( g \) yields

\[
h_0 \equiv h(\frac{\partial}{\partial \nu} w_t^*), \\
g_0 \equiv g(w_t^*).
\]

Thus, \( w^* \equiv w \) is a unique solution to (1.1).

Convergence in (6.8) together with passage to the limit on inequality (6.7) yields the desired a priori bound in Lemma 6.2.
7. Proof of Theorem 1.4. We shall use interpolation Theorem 2 due to Tartar ([23]). To this end, we define a nonlinear map $\mathcal{K}(w_0, w_1) = (w(t), w_t(t))$, where $w(t)$ solves (1.1). Denote

$$A_1 \equiv H^2(\Omega) \times L_2(\Omega), \quad B_1 \equiv C(0, T; H^2(\Omega) \times L_2(\Omega)),$$

$$A_2 \equiv \{w = (w_0, w_1) \in H^4(\Omega) \times H^2(\Omega) :$$

$$(w_0, w_1) \text{ satisfy compatibility relations (1.5)},$$

$$B_2 \equiv C(0, T; H^3(\Omega) \times L_2(\Omega)).$$

By virtue of Theorems 1.1 and 1.3, we know that $\mathcal{K}$ is well defined from $A_1 \to B_1$ and from $A_2 \to B_2$. Moreover, estimate (3.2) yields the following bound:

$$\|\mathcal{K}(w_0, w_1)\|_{B_1} \leq C(\|w_0\|_{H^2(\Omega)} \|w_0\|_{H^2(\Omega)} + \|w_1\|_{L_2(\Omega)})$$

$$\leq C(\|(w_0, w_1)\|_{A_1}) \|(w_0, w_1)\|_{A_1}. \quad (7.1)$$

By tracing the constants in estimates (3.17)–(3.19) and using the fact that $r \leq 1$, we also obtain

$$\|\mathcal{K}(w_0, w_1)\|_{B_2} \leq C(\|w_0\|_{H^2(\Omega)} \|w_1\|_{L_2(\Omega)}(\|w_0\|_{H^2(\Omega)} + \|w_1\|_{H^2(\Omega)})$$

$$\leq C(\|(w_0, w_1)\|_{A_1}) \|(w_0, w_1)\|_{A_2}. \quad (7.2)$$

In order to apply nonlinear interpolation Theorem 2 from [23], we need to show that $\mathcal{K}$ is locally Lipschitz from $A_1 \to B_1$. To accomplish this, we use inequality (4.12) applied with (i) $\tilde{w} = w - v$, $w_n = w$, where $w$ (respectively, $v$) is a solution to (1.1) corresponding to initial data $(w_0, w_1)$ (respectively, $(v_0, v_1)$), (ii) $f_n = F(w), f = F(v)$. This yields

$$\|w(t) - v(t)\|^2_{H^2(\Omega)} + \|w_t(t) - v_t(t)\|^2_{L_2(\Omega)} \quad (7.3)$$

$$\leq C\left\{ \int_0^t \int_\Omega \left\{ \|F(w(\tau)) - F(v(\tau)), w(\tau))|^2 + \|F(v(\tau)), w(\tau) - v(\tau)|^2 \right\} d\Omega d\tau$$

$$+ \|w_0 - v_0\|^2_{H^2(\Omega)} + \|w_1 - v_1\|^2_{L_2(\Omega)} \right\}$$

$$\leq C\left\{ \int_0^t \|w(\tau) - v(\tau)\|^2_{H^2(\Omega)} \|w(\tau)\|^4_{H^2(\Omega)} + \|v(\tau)\|^4_{H^2(\Omega)} d\tau$$

$$+ \|w_0 - v_0\|^2_{H^2(\Omega)} + \|w_1 - v_1\|^2_{L_2(\Omega)} \right\}$$

$$\leq C\left\{ \|w_0\|_{H^2(\Omega)} \|v_0\|_{H^2(\Omega)}, \|w_1\|_{L_2(\Omega)}, \|v_1\|_{L_2(\Omega)} \right\} \int_0^t \|w(\tau) - v(\tau)\|^2_{H^2(\Omega)} d\tau,$$

where we have used (6.1), Theorem 5.1, and Lemma 6.2. Gronwall’s inequality applied to (7.3) yields

$$\|w(t) - v(t)\|^2_{H^2(\Omega)} + \|w_t(t) - v_t(t)\|^2_{L_2(\Omega)}$$

$$\leq Ce^{C(\|(w_0, w_1)\|_{A_1}, \|(v_0, v_1)\|_{A_1}) t} \left\{ \|w_0 - v_0\|^2_{H^2(\Omega)} + \|w_1 - v_1\|^2_{L_2(\Omega)} \right\}.$$
Hence,
\[
\|\mathcal{K}(w_0, w_1) - \mathcal{K}(v_0, v_1)\|_{B_1} \leq C(\|w_0, w_1\|_{A_1}, \|v_0, v_1\|_{A_1}, \|(w_0, w_1) - (v_0, v_1)\|_{A_1}).
\]

Now, by virtue of (7.1), (7.2) and (7.4), the conclusion of Theorem 2 in [23] applies to yield the final result of Theorem 1.4.

8. Proof of Theorem 5.1.
8.1. Proof of Part (i). Let \( f = G[u, v] \); i.e.,
\[
\Delta^2 f = [u, v] \quad \text{in } \Omega; \quad f = 0, \quad \frac{\partial}{\partial \nu} f = 0 \quad \text{on } \Gamma.
\]

By the Closed Graph Theorem, it suffices to prove that for \( u, v \in H^2(\Omega), f \in H^3(\Omega) \cap W^4_1(\Omega) \). By using partition of unity, it is enough to consider the case when both \( u \) and \( v \) are supported in the neighborhood of the point \( x_0 \in \Omega \). We shall consider two cases:
(a) \( x_0 \in \text{Int}(\Omega) \), (b) \( x_0 \in \Gamma \).

Case (a): We introduce a bilinear continuous operator \( B \) from \( C^0_0(\mathbb{R}^2) \times C^0_0(\mathbb{R}^2) \) into \( \mathcal{D}'(\mathbb{R}^2) \) defined as \( B(u, v) \equiv [\Delta^{-1}u, \Delta^{-1}v] \), where \( \Delta^{-1} \) denotes the inverse of the Laplacian on \( \mathbb{R}^2 \). Let \( b(\xi, \eta) \) be a symbol associated with \( B \) (see [4], page 29) defined by
\[
B(e^{i\xi x}, e^{i\eta x}) = b(\xi, \eta) e^{i(\xi + \eta)x}, \quad \xi \in \mathbb{R}^2, \quad \eta \in \mathbb{R}^2, \quad x \in \mathbb{R}^2.
\]

Then
\[
b(\xi, \eta) = \frac{\xi_1^2 \eta_2^2 + \xi_2^2 \eta_1^2 - 2\xi_1 \xi_2 \eta_1 \eta_2}{(\xi_1^2 + \xi_2^2)(\eta_1^2 + \eta_2^2)}. \tag{8.1}
\]

Since \( b(\xi, -\xi) = 0 \) for all \( \xi \neq 0 \), the result of Theorem V.1 in [4] applies and tells us that \( B \) is bounded from \( L^2(\mathbb{R}^2) \times L^2(\mathbb{R}^2) \to H^1(\mathbb{R}^2) \), where \( H^1 \) is a real Hardy space (see [5]). This means that for \( u \in H^2(\Omega), v \in H^2(\Omega) \) (where, we recall, \( u \) and \( v \) are supported away from \( \Gamma \)),
\[
[u, v] \in H^1. \tag{8.2}
\]

Let \( f_0 \equiv G_0[u, v] \), where \( G_0 \in OPS^{-4} \) with a symbol \( g_0 = \frac{1}{(\xi_1^2 + \xi_2^2)^2} \). By \( OPS^n \), we denote, as usual (see [24]), a class of pseudodifferential operators of order \( n \).

Since \([u, v]\) is supported away from \( \Gamma \), \( f_0 \) differs from \( f \) by a \( C^\infty \) function. From
\[
D^\alpha G_0 \in OPS^0 \quad \text{for } |\alpha| \leq 4, \tag{8.3}
\]

where \( D^\alpha \) stands for the differential operator of order \( |\alpha| \), we obtain, by Theorem 26 of [5] (see page 121), (8.2) and (8.3), that
\[
D^\alpha G_0[u, v] \subset L^1(\mathbb{R}^2), \quad \text{for } |\alpha| \leq 4. \tag{8.3}
\]

By Sobolev’s Imbedding (see [1], page 97, (7)), \( f_0 \in W^4_1(\Omega) \subset H^3(\Omega) \) and \( f \in H^3(\Omega) \cap W^4_1(\Omega) \).

Case (b): Let \( x_0 \in \Gamma \). We extend functions \( u, v \) to \( H^2(\mathbb{R}^2) \) such that their support is compact. We shall denote the extensions by \( \tilde{u}, \tilde{v} \). By the same argument as the one used for (8.2), we have
\[
[\tilde{u}, \tilde{v}] \in H^1(\mathbb{R}^2). \tag{8.4}
\]

We shall prove the following imbedding later.
Proposition 8.1.\[
H_1(\mathbb{R}^2) \subset H^{-1}(\mathbb{R}^2). \tag{8.5}
\]
Define the operator \( \tilde{G}(\tilde{u}, \tilde{v}) : H^2(\mathbb{R}^2) \times H^2(\mathbb{R}^2) \to H^2(\mathbb{R}^2) \) by the formula
\[
\Delta^2 \tilde{G}(\tilde{u}, \tilde{v}) = [\tilde{u}, \tilde{v}] \quad \text{in} \quad \mathbb{R}^2. \tag{8.5}
\]
From (8.4), (8.5) and standard elliptic regularity, we obtain
\[
\tilde{G}(\tilde{u}, \tilde{v}) \in H^3(\mathbb{R}^2). \tag{8.6}
\]
Since for \( x \in \Omega, f = \tilde{G}(\tilde{u}, \tilde{v}) - f^* \), where \( f^* \) satisfies
\[
\Delta^2 f^* = 0 \quad \text{in} \quad \Omega, \quad f^* = G(\tilde{u}, \tilde{v}) \quad \text{on} \quad \Gamma,
\]
\[
\frac{\partial}{\partial \nu} f^* = \frac{\partial}{\partial \nu} \tilde{G}(\tilde{u}, \tilde{v}) \quad \text{on} \quad \Gamma, \tag{8.7}
\]
we conclude, by (8.6), that \( f \in H^3(\Omega) \) if and only if \( f^* \in H^3(\Omega) \).

From (8.2) and trace theory, \( G(\tilde{u}, \tilde{v}) \in H^{5/2}(\Gamma), \frac{\partial}{\partial \nu} G(\tilde{u}, \tilde{v}) \in H^{3/2}(\Omega) \). Standard elliptic regularity gives \( f^* \in H^3(\Omega) \), hence \( f \in H^3(\Omega) \), as desired for part (i).

8.2. Proof of Part (ii). We extend \( f \in H^3(\Omega) \) outside \( \Omega \), denoting the extension by \( \tilde{f} \), such that \( \text{supp} \, \tilde{f} \) is compact in \( \mathbb{R}^2 \). It suffices to prove that for all \( \tilde{u} \in H^2(\mathbb{R}^2) \),
\[
[\tilde{f}, \tilde{u}] = \tilde{f}_{xx} \tilde{u}_{yy} + \tilde{f}_{yy} \tilde{u}_{xx} - 2 \tilde{f}_{xy} \tilde{u}_{xy} \in L^2(\mathbb{R}^2). \tag{8.8}
\]
Let \( L \in S_0^2(\mathbb{R}^2) \) (see [24]) with symbol \( l(\xi) = |\xi|^2 = \xi_1^2 + \xi_2^2 \). We write \([\tilde{f}, \tilde{u}] = K_1 + K_2 - 2K_3\), where
\[
K_1 \equiv \tilde{f}_{xx} \tilde{u}_{yy} - L \tilde{f} L^{-1} \tilde{u}_{xxyy}, \quad K_2 \equiv \tilde{f}_{yy} \tilde{u}_{xx} - L \tilde{f} L^{-1} \tilde{u}_{yyxx},
\]
\[
K_3 \equiv \tilde{f}_{xy} \tilde{u}_{xy} - L \tilde{f} L^{-1} \tilde{u}_{xyxy},
\]
To prove part (ii), it suffices to show that \( K_i \in L_2(\Omega), i = 1, 2, 3 \). Notice that
\[
K_1 = [\tilde{f}_{xx} - L \tilde{f} L^{-1} \frac{\partial^2}{\partial x^2}] \tilde{u}_{yy} = [\frac{\partial^2}{\partial x^2} L^{-1} L \tilde{f} - L \tilde{f} L^{-1} \frac{\partial^2}{\partial x^2}] \tilde{u}_{yy} = \{ \frac{\partial^2}{\partial x^2} L^{-1}, L \tilde{f} \} \tilde{u}_{yy},
\]
where \( \{A, B\} \) stands for the commutator. Similarly,
\[
K_2 \equiv \{ \frac{\partial^2}{\partial y^2} L^{-1}, L \tilde{f} \} \tilde{u}_{xx}, \quad K_3 \equiv \{ \frac{\partial^2}{\partial x \partial y} L^{-1}, L \tilde{f} \} \tilde{u}_{xy}. \tag{8.9}
\]
If we denote by \( R_1 \) (respectively, \( R_2 \)) the Riesz transform (see [5]) \( \frac{\partial}{\partial x} \sqrt{(-\Delta)^{-1}} \) (respectively, \( \frac{\partial}{\partial y} \sqrt{(-\Delta)^{-1}} \)), with symbol \( \frac{\xi}{\sqrt{\xi_1^2 + \xi_2^2}} \), then
\[
\frac{\partial^2}{\partial x^2} L^{-1} = \text{Op}(\frac{\xi_1^2}{\sqrt{\xi_1^2 + \xi_2^2}}) = R_1^2, \quad \frac{\partial^2}{\partial y^2} L^{-1} = \text{Op}(\frac{\xi_2^2}{\sqrt{\xi_1^2 + \xi_2^2}}) = R_2^2,
\]
\[
\frac{\partial^2}{\partial x \partial y} L^{-1} = \text{Op}(\frac{\xi_1 \xi_2}{\sqrt{\xi_1^2 + \xi_2^2}}) = R_1 R_2.
\]
Applying Theorem 25 in [5] together with Proposition 8.1, we obtain $H^1(\mathbb{R}^2) \subset \text{BMO}(\mathbb{R}^2)$. On the other hand, since $\tilde{f} \in H^3(\Omega)$, $\tilde{L}f \in H^1(\Omega)$,

$$L\tilde{f} \in \text{BMO}(\mathbb{R}^2).$$

Thus each $K_i$ corresponds to a commutator of powers of Riesz transforms with BMO functions; i.e.,

$$K_1 = \{R_1^2, L\tilde{f}\} u_{xx}, \quad K_2 = \{R_2^2, L\tilde{f}\} u_{yy}, \quad K_3 = \{R_1 R_2, L\tilde{f}\} u_{xy},$$

where $L\tilde{f} \in \text{BMO}$.

From inequality (25), page 257 of [4], we infer that

$$\|\{R_i, b\} f\|_{L^p(\mathbb{R}^2)} \leq C\|b\|_{\text{BMO}}\|f\|_{L^p(\mathbb{R}^2)}, \quad 1 < p < \infty,$$

and since (see [5]) $R_i \in \mathcal{L}(L^p(\mathbb{R}^2))$, $1 < p < \infty$, we conclude that $\forall i, j = 1, 2$ and $b \in \text{BMO}$, $\{R_i R_j, b\} = R_i \{R_j, b\} + \{R_i, b\} R_j$ is bounded from $L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2)$. In particular,

$$\|\{R_i R_j, b\} f\|_{L^2(\mathbb{R}^2)} \leq C\|b\|_{\text{BMO}}\|f\|_{L^2(\mathbb{R}^2)}.$$

Combining (8.10), (8.11), (8.12) yields $\{R_i R_j, L\tilde{f}\} \in \mathcal{L}(L^2(\mathbb{R}^2))$, and, consequently, $K_i \in \mathcal{L}(H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2))$, as desired for (8.8).

It remains to prove Proposition 8.1.

8.3. **Proof of Proposition 8.1.** Let $T \in \mathcal{S}_0^{-1}(\mathbb{R}^2 \times \mathbb{R}^2)$ be a pseudodifferential operator associated with a symbol $a(\xi_1, \xi_2) = (\xi_1^2 + \xi_2^2)^{-1/2}$. Then by [22, page 235], $T$ has a singular integral realization,

$$(T_a f)(x) = \int_{\mathbb{R}^2} K(z) f(x - z) \, dz,$$

where

$$a(\xi) = \int_{\mathbb{R}^2} K(z) e^{-2\pi i z \xi} \, dz \equiv \hat{K}(\xi),$$

where $\hat{K}$ denotes the Fourier transform.

On the other hand (see [6, page 156]),

$$\left(\frac{1}{|x|^{n-\alpha}}\right)^* = C(n, \alpha) \frac{1}{|\xi|^{\alpha}},$$

where $C(n, \alpha) = 2^{\alpha-n/2} \frac{\Gamma(n/2)}{\Gamma(\alpha/2)}$. Applying (8.14) with $\alpha = 1$, $n = 2$ gives $K(x) = \frac{1}{C(2,1)|x|}, C(2,1) = \frac{\Gamma(1/2)}{\Gamma(1/2)} = 1$, and by (8.13),

$$(T_a f)(x) = \int_{\mathbb{R}^2} \frac{1}{|z|} f(x - z) \, dz.$$
VON KARMAN SYSTEM

293

Now we are in position to apply the Hardy-Littlewood-Sobolev Inequality (see [22, page 136]), which states that

\[ \|T_a f\|_{H_0(\mathbb{R}^2)} \leq C \|f\|_{H_p(\mathbb{R}^2)}, \quad \frac{1}{q} = \frac{1}{p} - \frac{1}{2} \implies p = 1, \ q = 2, \]

and

\[ \|T_a f\|_{L^2(\mathbb{R}^2)} \leq C \|f\|_{H_1(\mathbb{R}^2)}. \]

Since \( T_a \) is a canonical isomorphism from \( H^{-1}(\mathbb{R}^2) \to L^2(\Omega) \), we obtain

\[ \|f\|_{H^{-1}(\mathbb{R}^2)} \leq C \|T_a f\|_{L^2(\mathbb{R}^2)} \leq C \|f\|_{H_1(\mathbb{R}^2)}, \]

as desired.

REFERENCES


