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ON HOMOCLINIC AND HETEROCLINIC ORBITS FOR HAMILTONIAN SYSTEMS

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Abstract. We extend some earlier results on existence of homoclinic solutions for a class of Hamiltonian systems. We also study heteroclinic solutions. We use a variational approach.

1. Introduction. Recently variational techniques have been used in a number of papers to obtain existence of homoclinic and heteroclinic orbits of the Hamiltonian systems

$$u'' - L(t)u + V_u(t,u) = 0; \tag{1.1}$$

see, e.g., A. Ambrosetti and M.L. Bertotti ([1]), P.H. Rabinowitz ([7]), W. Omana and M. Willem ([5]), and P. Korman and A.C. Lazer ([3]). Here $L(t)$ is a given positive definite $n \times n$ matrix, the potential $V(t,u)$ is assumed to be superquadratic in u, and the solution is sought in the class $H^1(R,R^n)$, which implies that it is homoclinic at zero; i.e., $\lim_{t\to\pm\infty} u(t) = 0$. The approach used in [1], [5] and [3], was to restrict the problem (1.1) to a bounded interval $(-T,T)$ with Dirichlet boundary conditions $u(-T) = u(T) = 0$, show existence of solutions using the mountain-pass lemma, and then let $T \to \infty$. The crucial observation made in [1], and independently in [3], is that in addition to existence of solutions, the mountain-pass lemma allows one to obtain a uniform-in-T estimate of H^1 norm of the solution. It is then straightforward, via the usual diagonal process, to show existence of a homoclinic solution of (1.1). The problem is to show that this solution is nontrivial. P.H. Rabinowitz and K. Tanaka proved existence of a solution under the condition that the smallest eigenvalue of $L(t)$ tends to ∞ as $|t| \to \infty$; see [8], and also [5], where

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an alternative proof is given. The above condition does not seem to be natural, and in fact in [3], P. Korman and A.C. Lazer showed that it can be dropped if $L(t)$ and $V(t,u)$ are even functions in t. In the present paper we prove a similar result for a broad class of problems without assuming evenness. In the case of one equation, we prove sharper results, and moreover obtain positive homoclinics.

In Section 3 we use a similar approach and elementary techniques to show existence and uniqueness of an odd heteroclinic solution for a class of equations.

2. Positive homoclinics for a class of equations. In this section we shall prove existence of positive homoclinics for a model equation with a polynomial nonlinearity. Namely, we are looking for a positive solution of

$$u'' - a(x)u + b(x)u^p = 0, \quad -\infty < x < \infty, \quad 1 < p < \infty, \tag{2.1}$$

$$u(-\infty) = u'(-\infty) = u(\infty) = u'(\infty) = 0. \tag{2.2}$$

We assume that the functions $a(x), b(x) \in C^1(-\infty,\infty)$ are strictly positive on $(-\infty,\infty)$; i.e., $a(x) \geq a_0 > 0$ and $b(x) \geq b_0 > 0$.

As in [3], we shall obtain a solution of (2.1)–(2.2) as the limit when $T \to \infty$ of the solutions of

$$u'' - a(x)u + b(x)u^p = 0 \quad \text{for} \quad x \in (-T,T), \quad u(-T) = u(T) = 0. \tag{2.3}$$

Let u_T denote a solution of (2.3). To show that a subsequence of $\{u_T\}$ converges to a positive solution of (2.1)–(2.2) as $T \to \infty$, we need to exclude the possibility of this subsequence converging to zero. Let x_0 be the point of global maximum of $u(x)$. From (2.3), since $u''(x_0) \leq 0$, it follows that

$$u(x_0) \geq \left(\frac{a(x_0)}{b(x_0)}\right)^{\frac{1}{p-1}}, \tag{2.4}$$

and hence if we can show that x_0 stays in a bounded interval as $T \to \infty$, it will exclude the possibility of $\{u_{T_k}\} \to 0$. We shall give two sets of conditions, which constrain x_0 to a bounded interval. But first we recall the existence result from [3]. Since we intend to send T to infinity, we shall restrict ourselves to $T \geq 1$ in (2.3).

Lemma 2.1 ([3])**.** *The problem* (2.3) *has under our conditions a positive solution for any $T \geq 1$, which is obtained by a variational technique. Moreover, for this* (*variational*) *solution we have an estimate*

$$\int_{-T}^{T} ({u'}^2(x) + a(x)u^2)\, dx \leq c \quad \textit{uniformly in} \quad T \geq 1. \tag{2.5}$$

We recall that in the process of proving this lemma it was shown that

$$c_T = \int_{-T}^{T} \left[\frac{{u'_T}^2}{2} + a(x)\frac{u_T^2}{2} - b(x)\frac{u_T^{p+1}}{p+1}\right] dx$$

is nonincreasing in T, which implies that $c_T \le c_1$ for all $T > 1$. Multiplying the equation (2.3) by u and integrating, we easily express

$$\int_{-T}^{T} \big(\frac{u_T'^2}{2} + a(x)\frac{u_T^2}{2}\big)\,dx = \frac{(p+1)}{p-1}c_T. \tag{2.6}$$

Lemma 2.2. *Assume that*

$$xa'(x) \ge 0 \quad \textit{and} \quad xb'(x) \le 0 \quad \textit{for all } x. \tag{2.7}$$

Let $u(x)$ be a positive solution of (2.1)–(2.2), x_0 its point of maximum. Assume that the following two conditions hold:

$$\lim_{x\to\pm\infty} \frac{\big(\sqrt{a(0)} + \sqrt{a(x)}\big)}{2}\big[\frac{(p+1)a(x)}{b(x)}\big]^{\frac{2}{p-1}} > \lim_{T\to\infty} \frac{(p+1)}{p-1}c_T. \tag{2.8}$$

Then x_0 belongs to a bounded interval uniformly in $T > 1$.

Proof. We recall that it was proved in Korman-Ouyang ([4]) that $u(x)$ has only one point of local maximum, which is the point of global maximum, which we denote by x_0, and we assume without loss of generality that $x_0 \ge 0$. Multiplying the equation (2.1) by u' and integrating over (x_0, T) gives (using that $a(x)$ and $-b(x)$ take their minimum at x_0)

$$u(x_0) \ge \big[\frac{(p+1)a(x_0)}{2b(x_0)}\big]^{\frac{1}{p-1}} \tag{2.9}$$

(which is stronger than the estimate (2.4) obtained by maximum principle). For any $T > 1$ we have by (2.6)

$$\int_{-T}^{T} \sqrt{a(x)}|uu'|\,dx \le \int_{-T}^{T} \big(\frac{1}{2}au^2 + \frac{1}{2}u'^2\big)\,dx = \frac{(p+1)}{p-1}c_T. \tag{2.10}$$

On the other hand, using (2.9),

$$\begin{aligned}
\frac{(p+1)}{p-1}c_T &> \int_{-T}^{T} \sqrt{a(x)}|uu'|\,dx \\
&= \int_{-T}^{x_0} \sqrt{a(x)}\big(\frac{u^2}{2}\big)'\,dx - \int_{x_0}^{T} \sqrt{a(x)}\big(\frac{u^2}{2}\big)'\,dx \\
&\ge \sqrt{a(0)}\frac{u^2(x_0)}{2} + \sqrt{a(x_0)}\frac{u^2(x_0)}{2} \\
&\ge \frac{(\sqrt{a(0)} + \sqrt{a(x_0)})}{2}\big[\frac{(p+1)a(x_0)}{2b(x_0)}\big]^{\frac{2}{p-1}}.
\end{aligned}$$

By (2.8) it then follows that x_0 belongs to a bounded interval.

Remark 1. Condition (2.8) is satisfied if, for example, $\lim_{|x|\to\infty} a(x) = \infty$ and $b(x)$ is bounded.

Remark 2. Instead of (2.7) we could allow a more general condition: $(x-c)a'(x) \geq 0$ and $(x-c)b'(x) \leq 0$ for some $c \in R$ and all x.

A similar result can be given without any symmetry assumptions on $a(x)$ and $b(x)$. Recall that the total variation of the function $f(x)$ on $[a,b]$ is $\int_a^b |f'(x)|\,dx$.

Lemma 2.3. *Assume that*

$$\liminf_{|x|\to\infty} \sqrt{a_0}\Big(\frac{a(x)}{b(x)}\Big)^{\frac{1}{p-1}} > \lim_{T\to\infty} \frac{(p+1)}{p-1} c_T. \tag{2.11}$$

Then x_0 belongs to a bounded interval.

Proof. Proceeding as in the proof of the previous lemma, we have, using (2.4),

$$\begin{aligned}
\frac{(p+1)}{p-1} c_T &> \int_{-T}^{T} \sqrt{a(x)}\,\Big|\Big(\frac{u^2}{2}\Big)'\Big|\,dx \\
&\geq \min_{[-T,x_0]} \sqrt{a(x)} \int_{-T}^{x_0} \Big|\Big(\frac{u^2}{2}\Big)'\Big|\,dx + \min_{[x_0,T]} \sqrt{a(x)} \int_{x_0}^{T} \Big|\Big(\frac{u^2}{2}\Big)'\Big|\,dx \\
&\geq \sqrt{a_0}\, u^2(x_0) \geq \sqrt{a_0}\Big(\frac{a(x_0)}{b(x_0)}\Big)^{\frac{1}{p-1}}.
\end{aligned}$$

In view of (2.11) the lemma follows.

Theorem 2.1. *Assume that $a(x)$ and $b(x)$ satisfy either the conditions of Lemma* 2.2 *or of Lemma* 2.3. *Then the problem* (2.1)–(2.2) *has a positive solution.*

Proof. Take a sequence $\{T_n\} \to \infty$, and denote by u_n the corresponding positive variational solution of the problem (2.3), which exists by Lemma 2.1. Using the estimate (2.5), which implies a uniform bound in H^1, one shows exactly in the same way as in [3] that a subsequence of $\{u_n(x)\}$ converges uniformly on bounded intervals to a function $u(x) \in C^2(-\infty,\infty)$, which is a solution of the equation (2.1) for all $x \in (-\infty,\infty)$. Clearly, $u(x) \geq 0$ for all x.

We claim that

$$u(x) > 0 \quad \text{for all} \quad x \in (-\infty,\infty). \tag{2.12}$$

Indeed, denoting x_{0n} the point of maximum of $u_n(x)$ we have by Lemmas 2.2 and 2.3 that $\{x_{0n}\}$ belong to a bounded interval, call it I. Along a subsequence $x_{on_k} \to y \in I$ and by (2.4) $u(y) \geq \min_I \big(\frac{a(x)}{b(x)}\big)^{\frac{1}{p-1}} > 0$. Since $u(x)$ is nonnegative and nontrivial, it is positive by the maximum principle.

The rest of the proof is exactly the same as in [3].

Example. Consider (a is a constant)

$$u'' - a^2 u + 2u^3 = 0, \quad -\infty < x < \infty, \quad u(\pm\infty) = u'(\pm\infty) = 0. \tag{2.13}$$

Multiplying (2.13) by u' and integrating, we obtain a homoclinic solution $u(x) = \frac{a}{\cosh ax}$. In fact, there is an infinite family of homoclinics $u(x) = \frac{a}{\cosh a(x-\gamma)}$ for any constant γ.

3. Odd heteroclinic solutions. We begin with a simple problem,

$$u'' + u - u^3 = 0 \quad \text{for} \quad x \in (-\infty, \infty), \quad u(\pm\infty) = \pm 1, u'(\pm\infty) = 0. \tag{3.1}$$

Multiplying (3.1) by u' and integrating, we easily compute an odd heteroclinic solution $u = \tanh \frac{x}{\sqrt{2}}$.

Our goal is to obtain a similar result for the problem

$$\begin{aligned} &u'' + a(x)(u - |u|^{p-1}u) = 0 \quad \text{for} \quad x \in (-\infty, \infty), \\ &u(\pm\infty) = \pm 1, \; u'(\pm\infty) = 0. \end{aligned} \tag{3.2}$$

We assume that $p > 1$ is a real number and the function $a(x)$ is even of class $C^1(-\infty, \infty)$, with

$$a'(x) < 0 \quad \text{for almost all} \quad x > 0, \tag{3.3}$$

$$a(\infty) > 0. \tag{3.4}$$

We shall obtain the solution of (3.1) as a limit when $T \to \infty$ of solutions of

$$u'' + a(x)(u - |u|^{p-1}u) = 0 \quad \text{for} \quad x \in (-T, T), \quad u(\pm T) = \pm 1. \tag{3.5}$$

The solution of (3.5) will in turn depend on the problem

$$u'' + a(x)(u - |u|^{p-1}u) = 0 \quad \text{on} \quad (0, T), \quad u(0) = 0, u(T) = 1. \tag{3.6}$$

Lemma 3.1. *The problem* (3.6) *has for each* $T > 0$ *a unique positive solution, which is an increasing function.*

Proof. The function $u \equiv 1$ is a supersolution of (3.6), while $u = \alpha x$ is a subsolution, when the constant α is sufficiently small. It follows that (3.6) has a positive solution ($0 < u < 1$ on $(0, T)$). By the maximum principle any solution of (3.6) satisfies $0 < u < 1$ on $(0, T)$. Turning to the uniqueness, recall that the method of super-subsolutions implies existence of a maximal solution $u(x)$; i.e., $u(x) \geq v(x)$ for all $x \in (0, T)$, if $v(x)$ is any other solution of (3.6). Multiplying (3.6) by v, and the same equation for v by u, subtracting and integrating,

$$\int_0^T a(x)uv(v^{p-1} - u^{p-1})\, dx + u'(T) - v'(T) = 0,$$

which implies that $v \equiv u$.

Finally, assume that $u(x)$ is not monotone. Then it has a point $\overline{x}$ of local minimum on $(0, T)$, at which $u''(\overline{x}) \geq 0$ and $u(\overline{x}) - u^p(\overline{x}) > 0$, which implies a contradiction in (3.6).

Lemma 3.2. *The problem* (3.5) *has under our conditions a unique solution, which is an odd and increasing function.*

Proof. Let $u(x)$ be the solution of (3.6) for $x \in [0,T]$, obtained in the previous lemma. We extend it to $[-T,0]$ as $-u(-x)$. The resulting function is an odd and increasing solution of (3.5). Uniqueness follows as above (-1 and $+1$ are respectively sub- and supersolution).

Theorem 3.1. *The problem* (3.2) *has, under the conditions* (3.3) *and* (3.4), *a unique solution, which is an odd and strictly increasing function.*

Proof. Take a sequence $T_n \to \infty$, and consider the problem (3.5) on the interval $(-T_n, T_n)$; i.e., consider

$$\begin{aligned} &u'' + a(x)(u - |u|^{p-1}u) = 0 \quad \text{on} \quad (-T_n, T_n), \\ &u(-T_n) = -1, u(T_n) = 1. \end{aligned} \tag{3.7}$$

By Lemma 3.2 the problem (3.11) has a unique solution $u_n(x)$. Since $|u_n(x)| < 1$, we conclude that

$$|u_n''(x)| \le c \quad \text{for all} \quad x \in (-T_n, T_n) \quad \text{uniformly in} \quad n. \tag{3.8}$$

Since $u_n(x)$ is monotone the estimate (3.12) implies

$$|u_n'(x)| \le c \quad \text{for all} \quad x \in (-T_n, T_n) \quad \text{uniformly in} \quad n. \tag{3.9}$$

(If $u_n'(x)$ were to become large at some x, then by (3.8) $u_n'(x)$ would stay large over a long interval, which would contradict the total variation of $u_n(x)$ being equal to 2.)

Arguing as in [3], we see via the usual diagonal process that there is a function $u(x) \in C^2(-\infty, \infty)$ such that along a subsequence we have for all $x \in (-\infty, \infty)$

$$u_{n_k}(x) \to u(x) \quad \text{and} \quad u_{n_k}'(x) \to u'(x) \quad \text{uniformly on bounded intervals,} \tag{3.10}$$

and that $u(x)$ is a solution of (3.2).

We claim that there is a constant $c_0 > 0$ such that

$$u_n'(0) \ge c_0 \quad \text{uniformly in} \quad n. \tag{3.11}$$

Indeed, introducing the "energy" function for $x \ge 0$ (where $u_n(x) \ge 0$)

$$E(x) = \frac{1}{2} u_n'^2 + a(x)\left(\frac{u_n^2}{2} - \frac{u_n^{p+1}}{p+1}\right),$$

we compute using (3.5)

$$E'(x) = a'(x)\left(\frac{u_n^2}{2} - \frac{u_n^{p+1}}{p+1}\right) < 0.$$

Therefore

$$E(0) = \frac{1}{2}u_n'^2(0) > E(T_n) > a(\infty)\frac{p-1}{2(p+1)},$$

and (3.11) follows. It follows that $u(x) \not\equiv 0$.

By (3.10) $u'(x) \geq 0$. Since also $-1 \leq u(x) \leq 1$, it follows that $\lim_{x\to\pm\infty} u(x)$ exist, and the only possibility in view of (3.4) is that $\lim_{x\to\pm\infty} u(x) = \pm 1$ (since $u''(x)$ must be small for x large). Since $u(x)$ is nondecreasing it follows that $\lim_{x\to\pm\infty} u'(x) = 0$. Notice that $u(x)$ is, in fact, strictly increasing, since otherwise we would have $u'(x_0) = 0$ at some $x_0 > 0$, and then integrating the equation (3.2) over (x_0, ∞), we would get a contradiction.

Turning to the uniqueness, let $v(x)$ be another solution of (3.2). We consider four possible cases.

(i) $u(x)$ and $v(x)$ intersect at least twice on $[0, \infty)$. That is, we can find $0 \leq x_1 < x_2 < \infty$, such that $u(x_1) = v(x_1) \equiv u_1$, $u(x_2) = v(x_2) \equiv u_2$ and say $u(x) > v(x)$ on (x_1, x_2). As in Lemma 3.1 we obtain

$$u_2(u'(x_2) - v'(x_2)) - u_1(u'(x_1) - v'(x_1)) + \int_{x_1}^{x_2} auv(v^{p-1} - u^{p-1})\,dx = 0,$$

which is impossible, since $u'(x_1) > v'(x_1)$ and $u'(x_2) < v'(x_2)$.

(ii) $u(x)$ and $v(x)$ intersect exactly once on $[0, \infty)$, say at $x_1 \geq 0$. Integrating over (x_1, R) and letting $R \to \infty$, we obtain the same contradiction.

(iii) $u(x)$ and $v(x)$ have only negative points of intersection. By considering $-u(-x)$ and $-v(-x)$ (which are also solutions of (3.2)) we reduce this case to one of the previous cases.

(iv) $u(x)$ and $v(x)$ never intersect. Integrating over $(-R, R)$ and letting $R \to \infty$, we again obtain a contradiction.

Clearly, we have also proved the following theorem.

Theorem 3.2. *Consider the problem*

$$\begin{aligned} &u'' + a(x)(u - u^p) = 0 \quad \text{for} \quad x \in (0, \infty) \\ &u(0) = 0, \ \ u(\infty) = 1, \ \ u'(\infty) = 0, \end{aligned} \tag{3.12}$$

with $a(x) \in C^1[0, \infty)$ satisfying the conditions (3.3) and (3.4), and p is a real number with $p \geq 1$. Then the problem (3.12) has a unique positive solution, which is a strictly increasing function.

4. Homoclinic solutions for a class of Hamiltonian systems. We are looking for nontrivial solutions $u(t) \in H^1(R, R^n)$ of the system

$$u'' - L(t)u + V_u(t,u) = 0 \quad -\infty < t < \infty, \tag{4.1}$$

$$u(\pm\infty) = u'(\pm\infty) = 0. \tag{4.2}$$

Here V_u is the gradient of V with respect to u variables. We assume that

$$\begin{aligned} L(t) = [\ell_{ij}(t)] \quad &\text{is a positive definite matrix of class} \\ &C^1(R), \quad \text{and there is } \alpha(t) \in C(R,R) \text{ such that} \\ &\alpha(t) \geq \alpha_0 > 0 \quad \text{for all } t \in R \text{ and } (L(t)u,u) \geq \alpha(t)|u|^2; \end{aligned} \tag{4.3}$$

$$\begin{aligned} &V(t,u) \in C^1(R \times R^n, R), \ \text{ and for some constant } \gamma > 2 \\ &0 < \gamma V(t,\xi) \leq (V_\xi(t,\xi),\xi) \ \text{ for all } \xi \in R^n \backslash \{0\} \text{ and } t \in R. \end{aligned} \tag{4.4}$$

As in Section 2 we approximate (4.1)–(4.2) by the problem (with say $T > 1$)

$$u'' - L(t)u + V_u(t,u) = 0 \quad \text{for} \quad t \in (-T,T), \ \ u(-T) = u(T) = 0. \tag{4.5}$$

We recall that under our conditions the problem (4.5) has a nontrivial solution $u = u_T$, which is a critical point of the functional

$$J(u) = \int_{-T}^{T} \left[\frac{1}{2}|u'|^2 + \frac{1}{2}(L(t)u,u) - V(t,u)\right] dt,$$

and that $c_T \equiv J(u_T)$ is nonincreasing in T; see [3]. Let t_0 denote (any) point of global maximum of $|u_T|$. Similarly to the scalar case, we wish to constrain t_0 to a bounded region. To this end we assume existence of a function $\beta : R \to R$ and a constant $t_1 > 0$, such that for $|t| > t_1$,

$$(L(t)u,u) > (V_u(t,u),u) \quad \text{provided that} \quad |u|^2 \leq \beta(t). \tag{4.6}$$

Remark 3. It was shown in [3] that under the condition (4.4) the function $V(t,u)$ is superquadratic in u near the origin. While the condition (4.6) does not seem to follow from (4.4), it is clear that it is not a very restrictive condition.

Theorem 4.1. *For the problem* (4.1)–(4.2) *assume that conditions* (4.3), (4.4) *and* (4.6) *hold, and in addition assume that*

$$\liminf_{t\to\infty} \alpha_0 \beta(t) > \lim_{T\to\infty} \frac{2\gamma}{\gamma - 2} c_T. \tag{4.7}$$

Then the problem (4.1)–(4.2) *has a nontrivial solution.* (*Keep in mind that* c_T *is decreasing in* T. *So that* (4.7) *will follow, if for example,* $\liminf_{t\to\infty} \alpha_0\beta(t) > \frac{2\gamma}{\gamma-2}c_1$.)

Proof. As in the previous section (and as in [3]) we approximate our problem by (4.5) and let $T_k \to \infty$. In [3] it was shown that H^1 norm of solutions u_{T_k} is bounded uniformly in k. As before this allows us to conclude that a subsequence of $\{u_{T_k}\}$ converges uniformly on bounded intervals to a function $u(x) \in C^2(R, R^n)$, which is a solution of (4.1). It remains to show that $u(t)$ is nontrivial (that $u(t)$ satisfies (4.2) follows exactly as in [3]).

Define $q(t) = |u(t)|^2$. Compute

$$q''(t) = 2|u'|^2 + 2u \cdot u''. \tag{4.8}$$

It t_0 is the point of maximum of $q(t)$, then $q''(t_0) \le 0$, and it follows from (4.8) that

$$u(t_0) \cdot u''(t_0) \le 0. \tag{4.9}$$

We may assume that $|t_0| > t_1$, since otherwise t_0 already belongs to a bounded interval.

Multiplying the i-th equation in (4.1) by u_i and summing, we obtain in view of (4.9)

$$-(L(t_0)u(t_0), u(t_0)) + (V_u(t_0, u(t_0)), u(t_0)) \ge 0.$$

Comparing this with (4.6) we conclude

$$|u(t_0)|^2 > \beta(t_0). \tag{4.10}$$

We recall that it was shown in [3] that

$$\int_{-T}^{T} \big[\frac{1}{2}|u_T'|^2 + \frac{1}{2}(L(t)u_T, u_T)\big]\, dt \le \frac{2\gamma}{\gamma-2}c_T \le \frac{2\gamma}{\gamma-2}c_1, \tag{4.11}$$

where as before $c_T = J(u_T)$.

On the other hand, proceeding as in Lemma 2.3, and using (4.10) and (4.11),

$$\begin{aligned} \frac{2\gamma}{\gamma-2}c_T &\ge \int_{-T}^{T} \sum_{i=1}^{n} \sqrt{\alpha(t)}|u_i u_i'|\, dt \\ &\ge \alpha_0 \int_{-T}^{T} \big|\frac{d}{dt}\frac{1}{2}|u|^2\big|\, dt \ge \alpha_0|u(t_0)|^2 > \alpha_0\beta(t_0). \end{aligned} \tag{4.12}$$

Condition (4.7) then implies that t_0 stays in a bounded interval as $T_k \to \infty$. As in Theorem 2.1 we show existence of $\bar{t}$ such that

$$|u(\bar{t})|^2 > \liminf_{t\to\infty} \beta(t) > 0.$$

(For the second inequality use (4.7) and that $c_T > 0$, since c_T is the value of $J(u)$ at the mountain pass.)

Hence $u(t)$ is a nontrivial solution of (4.1). As in [3] one sees that it also satisfies (4.2), completing the proof.

Remark 4. Condition (4.7) can be generalized to read

$$\liminf_{t\to\infty} \beta(t) \min_{(-t,t)} \alpha(s) > \lim_{T\to\infty} \frac{2\gamma}{\gamma - 2} c_T.$$

Remark 5. If $|V_u(t,u)| < c_0 u^{1+\delta}$ for some constants $c_0, \delta > 0$ uniformly in $t \in R$, then

$$(L(t)u, u) \geq \alpha(t)|u|^2 \geq c_0|u|^{2+\delta} > (V_u(t,u), u),$$

provided $\alpha(t) \geq c_0|u|^\delta$. Hence we can take $\beta(t) = \left(\frac{\alpha(t)}{c_0}\right)^{2/\delta}$, and if we are given that $\lim_{|t|\to\infty} \alpha(t) = \infty$, then condition (4.7) holds and our theorem applies. This corollary appears to be roughly equivalent to the theorem of P.H. Rabinowitz and K. Tanaka (see [5, page 1116]). Our result is considerably more general than this corollary.

Remark 6. Our numerical calculations for the problem

$$u'' - 2u + u^3 = 0 \quad \text{on} \quad (-T, T), \;\; u(-T) = u(T) = 0$$

suggest that $\lim_{T\to 0} c_T = \infty$, while $\lim_{T\to\infty} c_T > 0$.

5. A curious maximum principle for elliptic systems. Our argument in Section 4 suggests a maximum principle for elliptic systems, which is quite unlike the classical one in [6] or its recent generalizations; see, e.g., [2]. In particular we do not require the system to be of cooperative type.

Let Ω be a bounded domain in R^d. We consider the system of m weakly coupled equations with m unknown functions $u^k(x)$, $k = 1, \dots, m$,

$$\sum_{i,j=1}^{d} a_{ij}(x) u^k_{ij} + \sum_{\ell=1}^{m} b_{k\ell}(x) u^\ell = f_k(x,u), \quad x \in \Omega, \quad k = 1, \dots, m. \tag{5.1}$$

Here $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$, and we assume that for some constant $\theta > 0$

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \theta|\xi|^2 \quad \text{for all} \quad x \in \Omega \quad \text{and} \quad \xi \in R^d. \tag{5.2}$$

We denote $u = (u^1, \dots, u^m)$.

We do not impose any smoothness assumptions on $a_{ij}(x), b_{k\ell}(x)$ and $f_k(x,u)$; however, we do assume that we have a classical solution of (5.1); i.e., $u^k \in C^2(\Omega)$. Let B be the $m \times m$ matrix, $B = [b_{k\ell}(x)]$.

Theorem 5.1. *Assume that* $\frac{1}{2}\big(B+B^T\big)$ *is negative semidefinite; i.e.,*

$$\sum_{k,\ell=1}^{m} b_{k\ell}(x)u^k u^\ell \leq 0 \quad \textit{for all} \quad u \in R^m \quad \textit{and} \quad x \in \Omega. \tag{5.3}$$

We assume also

$$\sum_{k=1}^{m} f_k(x,u)u^k \geq 0 \quad \textit{for all} \quad u \in R^m \quad \textit{and} \quad x \in \Omega. \tag{5.4}$$

Assume finally that at each $x \in \Omega$ *at least one of the above two inequalities is strict. Then* $|u(x)|^2 = \sum_{k=1}^m u^{k^2}(x)$ *has no points of maximum inside* Ω*.*

Proof. Denote $q(x) = |u(x)|^2$ and let $x_0 \in \Omega$ be its point of maximum. Compute

$$q_{ij}(x) = 2\sum_{k=1}^{m} u_i^k u_j^k + 2\sum_{k=1}^{m} u^k u_{ij}^k. \tag{5.5}$$

Since $\displaystyle\sum_{i,j=1}^{d} a_{ij}(x_0)q_{ij}(x_0) \leq 0$, and

$$\sum_{i,j=1}^{d} a_{ij}(x_0)u_i^k u_j^k \geq \theta|\nabla u^k|^2 \geq 0,$$

we conclude using (5.5)

$$\sum_{k=1}^{m}\sum_{i,j=1}^{d} a_{ij}(x_0)u^k(x_0)u_{ij}^k(x_0) \leq 0. \tag{5.6}$$

We now multiply the k-th equation in (5.1) by u^k and sum. In view of (5.3), (5.4) and (5.6) we have a contradiction at $x = x_0$.

Corollary 1. *Assume that homogeneous Dirichlet conditions are imposed:*

$$u^k(x) = 0 \quad \textit{for} \quad x \in \partial\Omega, \quad k = 1,\dots,m. \tag{5.7}$$

Then the trivial solution (*if it exists*) *is the only possible solution of* (5.1), (5.7).

Remark 7. If nonnegative solutions of (5.1) are considered, i.e. $u^k(x) \geq 0$ for all $x \in \Omega$ and $k = 1,\dots,m$, then (5.4) will follow from the condition

$$f_k(x,u) \geq 0 \quad \text{for all} \quad u \in R_+^m, \quad k = 1,\dots,m, \quad \text{and} \quad x \in \Omega.$$

Remark 8. For the corresponding parabolic system one can prove the same way that $|u(x,t)|^2$ can have points of maximum only on the parabolic boundary. In [6, page 194] there are references to some related results.

REFERENCES

1 A. Ambrosetti and M.L. Bertotti, *Homoclinics for second order conservative systems,* in "Partial Differential Equations and Related Subjects," M. Miranda (Ed.), Longman Scientific & Technical, 1992.

2 D.G. de Figueiredo and E. Mitidieri, *A maximum principle for an elliptic system and applications to semilinear problems,* SIAM J. Math. Anal., 17, (1986), 836–849.

3 P. Korman and A.C. Lazer, *Homoclinic orbits for a class of symmetric Hamiltonian systems,* EJDE, Volumes/1994/01-Korman.

4 P. Korman and T. Ouyang, *Solution curves for two classes of boundary-value problems,* Nonlinear Analysis TMA, to appear.

5 W. Omana and M. Willem, *Homoclinic orbits for a class of Hamiltonian systems,* Differential and Integral Equations, 5 (1992), 1115–1120.

6 M.H. Protter and H. Weinberger, "Maximum Principles in Differential Equations," Prentice Hall, Englewood Cliffs, 1967.

7 P.H. Rabinowitz, *Some recent results on heteroclinic and other connecting orbits of Hamiltonian systems,* in "Progress in Variational Methods in Hamiltonian Systems and Elliptic Equations," M. Girardi, M. Matzeu and F. Pacella (Eds.), Longman Scientific & Technical, 1992.

8 P.H. Rabinowitz and K. Tanaka, *Some results on connecting orbits for a class of Hamiltonian systems,* Math Z., 206 (1991), 473–499.