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# SUFFICIENT CONDITIONS FOR MINIMA OF SOME TRANSLATION INVARIANT FUNCTIONALS

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**I. Introduction.** We consider the question of minimizing certain functionals subject to a constraint. For the problems we are going to study, a very powerful method, concentration-compactness, introduced by P.L. Lions ([1]), has been developed. This method consists in showing that strict subadditivity is a necessary and sufficient condition for precompactness of minimizing sequences; of course the strict subadditivity has to be verified and this has been done for many important problems in both Pure and Applied Mathematics.

In this paper we present a slightly different point of view. What we do is to replace strict subadditivity by the verification of the assumption $H_4$ in part II or $H_5$ in part III. Our results are just sufficient for the convergence of minimizing sequences and a motivation for them is that they cover some examples for which the verification of the strict subadditivity, apparently, would not be easy. Of course we can not say that it would be impossible because it is a necessary condition. In [2] we have already used this method in a simpler problem and here we consider functionals with nonlocal terms and functionals with two dependent variables (giving rise to elliptic systems). All problems we have treated have just one constraint; in the case of more than one (as in [6]) it seems difficult to use our method.

There is a very large literature about the topic we treat but our list of references contains only the papers which are more closely related to ours.

**II. Functionals with nonlocal terms.** In this section we consider the question of minimizing the functional

$$V(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\operatorname{grad} u(x)|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N\times\mathbb{R}^N} k(x-y)H(u(x))H(u(y))\,dx\,dy + \int_{\mathbb{R}^N} F(u(x))\,dx \tag{II.1}$$

in the set of nonnegative elements $u$ in $H^1(\mathbb{R}^N)$ such that

$$I(u) = \int_{\mathbb{R}^N} u^2(x)\,dx = \lambda > 0. \tag{II.2}$$

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Related problems have been studied in references [1], [3], [4], [5], [6] and [7], among others.

For $N \geq 2$ we set $2^* = \frac{2N}{N-2}$ and we denote by $h(u)$ and $f(u)$ the derivatives of $H(u)$ and $F(u)$ respectively, and by $M(\lambda)$ the admissible set $\{u \in H^1(\mathbb{R}^n) : I(u) = \lambda\}$. Our basic assumptions are the following:

$H_1$) the functions $H(u)$ and $F(u)$ are even functions of the real variable $u$, $H(u)$ can be written as $H(u) = cu^2 + H_1(u)$, where $H_1(u)$ and $F(u)$ are $C^2$ functions vanishing at $u = 0$ as well their first derivatives and with second derivatives satisfying the growth conditions $|H_1''(u)| \leq M(|u|^{q-2} + |u|^{p-2})$, for $2 < q \leq p < 2^*$ and $|F''(u)| \leq M(|u|^{\bar{q}-2} + |u|^{\bar{p}-2})$, for $2 < \bar{q} \leq \bar{p} < 2^*$;

$H_2$) $k : \mathbb{R}^N \to \mathbb{R}$ is an even function and for some $r$ in the range $0 < \frac{1}{r} < 2 - \frac{2p}{2^*}$, the convolution operator having $k(z)$ as kernel maps $L_s(\mathbb{R}^N)$ into $L_t(\mathbb{R}^N)$ continuously for $1 < s \leq t < \infty$ and $\frac{1}{t} = \frac{1}{s} + \frac{1}{r} - 1$;

$H_3$) $V$ is bounded below on the admissible set and any minimizing sequence is bounded in $H^1(\mathbb{R}^N)$.

Assumptions $H_1 - H_3$ are standard. Next we state the crucial assumption that makes the method work.

$H_4$) For any nontrivial nonnegative solution $u \in H^1(\mathbb{R}^N)$ of the equation

$$-\Delta u(x) + \Big[\int_{\mathbb{R}^N} k(x-y)H(u(y))\,dy\Big]h(u(x)) + f(u(x)) + \alpha u(x) = 0 \tag{II.3}$$

with $\alpha \geq 0$, there is an element $\varphi \in H^1(\mathbb{R}^N)$ such that

$$\begin{aligned} W(u)(\varphi,\varphi) = &\int_{\mathbb{R}^N} |\mathrm{grad}\,\varphi(x)|^2\,dx \\ &+ \int_{\mathbb{R}^N\times\mathbb{R}^N} k(x-y)H(u(y))h'(u(y))\varphi^2(x)\,dx\,dy \\ \int_{\mathbb{R}^N\times\mathbb{R}^N} &k(x-y)h(u(x))h(u(y))\varphi(x)\varphi(y)\,dx\,dy \\ &+ \int_{\mathbb{R}^N} (f'(u(x)) + \alpha)\varphi^2(x)\,dx < 0. \end{aligned}$$

**Remarks.** 1) The condition $f'(0) = 0$ which has been imposed is just a normalization.

2) With some modifications, the results we prove can be extended to the case where the constraint is given by the integral of a function $G(u(x))$ more general than $u^2(x)$. We can also remove the evenness assumption on $F$, $G$ and $H$.

3) Assumption $H_2$ allows $k(z)$ to be singular; for instance, if $N = 3$ and $k(z) = \frac{1}{|z|}$, then $r = 3$ and $p < 5$.

4) If the nonlocal term is absent, then assumption $H_4$ is automatically satisfied [2]; later we give sufficient conditions for it. That assumption seems to be very far

from being necessary; in fact, there are some problems for which strict subadditivity can be verified very easily and it is not clear whether $H_4$ holds. We will come back to this point in Section III.

We denote by $V(\lambda)$ the infimum of $V(u)$ for $u$ on the admissible set $M(\lambda)$.

**Theorem II.4.** *Under assumptions $H_1$ to $H_4$, if $V(\lambda) < 0$ and $(u_n)$ is a minimizing sequence then, except for translation in the $x$-variable and passing to a subsequence if necessary, there is an element $u \in H^1(\mathbb{R}^N)$ such that $u_n \to u$ in $L_s(\mathbb{R}^N)$, $2 < s < 2^*$ and grad $u_n \to$ grad $u$ in $L_2(\mathbb{R}^N)$.*

Let us emphasize the following point: the condition $V(\lambda) < 0$ has to be satisfied only for the given level $\lambda$; in particular, we do not have to assume that $V(\lambda) < 0$ for all $\lambda > 0$.

Theorem II.4 says that $V(u) \leq V(\lambda)$ but $u$ is not necessarily a minimizer because the convergence in $L_2(\mathbb{R}^N)$ is not guaranteed. We discuss this point later.

It is easy to construct an admissible sequence $(u_n)$ such that $|\text{grad} u_n|_{L_2} \to 0$ and $|u_n|_{L_\infty} \to 0$; for this sequence we have $\lim V(u_n) = 0$ and this shows that the condition $V(\lambda) < 0$ is necessary for the convergence of minimizing sequences.

Parts of the proof of Theorem II.4 are similar to ones presented in [2]. First we quote a few statements which will be labeled for future references. Their proof can be supplied without difficulty.

**S.1.** The functionals $V,\ I : H^1(\mathbb{R}^N) \to \mathbb{R}$ are of class $C^2$ and have first and second derivatives uniformly bounded sets and uniformly continuous on bounded sets.

**S.2.** If $u : (-\delta,\ \delta) \to H^1(\mathbb{R}^N)$ is a $C^2$ admissible curve with $u(0) = u$ then the vectors $\dot{h} = \dot{u}(0)$ and $\ddot{h} = \ddot{u}(0)$ (the dot means derivative with respect to $t$) satisfy the compatibility conditions

$$\int_{\mathbb{R}^N} u(x)\dot{h}(x)dx = 0 \tag{II.5}$$

and

$$\int_{\mathbb{R}^N} [u(x)\ddot{h}(x) + \dot{h}^2(x)]dx = 0. \tag{II.6}$$

**S.3.** If $(u_n)$ is a sequence of elements of $H^1(\mathbb{R}^N)$ converging weakly in $H^1(\mathbb{R}^N)$ to $u \not\equiv 0$ and $(\dot{h}_n)$ and $(\ddot{h}_n)$ are bounded sequences of $H^1(\mathbb{R}^N)$ satisfying the compatibility conditions (II.5) and (II.6) above, then there are $\delta > 0$ and a sequence $u_n : (-\delta, \delta) \to H^1(\mathbb{R}^N)$ of $C^2$ admissible curves with first and second derivatives uniformly continuous in $t \in (-\delta, \delta)$, uniformly in $n$, such that $\dot{u}_n(0) = \dot{h}_n$ and $\ddot{u}_n(0) = \ddot{h}_n$.

**S.4.** If $(u_n)$ is a minimizing sequence converging weakly in $H^1(\mathbb{R}^N)$ to $u$, then $|V'(u_n)| \to 0$ (the norm being calculated on the admissible $\dot{h}$) and

$$\liminf \frac{d^2}{dt^2} V(u_n(t))\big|_{t=0} \geq 0$$

for any $C^2$ sequence of admissible curves $u_n : (-\delta, \delta) \to H^1(\mathbb{R}^N)$ with second derivatives uniformly in $t$, uniformly in $n$, and such that $\dot{u}_n(0)$ and $\ddot{u}_n(0)$ are bounded; moreover, $W(u)(\dot{h}, \dot{h}) \geq 0$ for any admissible $\dot{h}$.

**S.5.** If $u$ is a generic admissible element of $H^1(\mathbb{R}^N)$ then there are real numbers $\alpha$ and $\gamma$ and an element $\bar{\varphi}$ of $H^1(\mathbb{R}^N)$ such that $|\bar{\varphi}|_{H^1(\mathbb{R}^N)} = 1$ and

$$\begin{aligned}\int_{\mathbb{R}^N} \langle \, \mathrm{grad} u(x), \ \mathrm{grad}\, \varphi(x) \rangle \, dx + \int_{\mathbb{R}^N \times \mathbb{R}^N} k(x-y) H((u(y)) h(u(x)) \varphi(x) \, dx \, dy \\ + \int_{\mathbb{R}^N} f(u(x)) \varphi(x) dx + \alpha \int_{\mathbb{R}^N} u(x) \varphi(x) \, dx \\ + \gamma \int_{\mathbb{R}^N} (\langle \mathrm{grad} \bar{\varphi}(x), \ \mathrm{grad} \varphi(x) \rangle + \bar{\varphi}(x) \varphi(x)) \, dx = 0\end{aligned} \tag{II.7}$$

for any $\varphi$ in $H^1(\mathbb{R}^N)$ and

$$|V'(u)| = -\gamma. \tag{II.8}$$

In particular

$$\begin{aligned}-\Delta(u(x) + \gamma \bar{\varphi}(x)) + \big[ \int_{\mathbb{R}^N} k(x-y) H(u(y)) \, dy \big] h(u(x)) \\ + f(u(x)) + \alpha u(x) + \gamma \bar{\varphi}(x) = 0.\end{aligned} \tag{II.9}$$

Furthermore, if $u : (-\delta, \delta) \to H^1(\mathbb{R}^N)$ is a $C^2$ admissible curve with $u(0) = u$, $\dot{u}(0) = \dot{h}$ and $\ddot{u}(0) = \ddot{h}$, then

$$\begin{aligned}\frac{d^2 V(u(t))}{dt^2}\bigg|_{t=0} &= \int_{\mathbb{R}^N} |\mathrm{grad} \ \dot{h}(x)|^2 \, dx \\ &+ \int_{\mathbb{R}^N \times \mathbb{R}^N} k(x-y) H(u(y)) h'(u(x)) \dot{h}^2(x) \, dx \, dy \\ &+ \int_{\mathbb{R}^N \times \mathbb{R}^N} k(x-y) h(u(x)) h(u(y)) \dot{h}(x) \dot{h}(y) \, dx \, dy \\ &+ \int_{\mathbb{R}^N} (f'(u(x)) + \alpha) \dot{h}^2(x) \, dx \, dy \\ &- \gamma \int_{\mathbb{R}^N} (\langle \mathrm{grad} \bar{\varphi}(x), \, \mathrm{grad}\, \ddot{h}(x) \rangle + \bar{\phi}(x) \ddot{h}(x)) \, dx.\end{aligned} \tag{II.10}$$

The verification of S.4 is just an "elementary" differential calculus proof. The precise meaning of $|V'(u_n)|$ is made clear in statement S.5 because what we do in statement S.5 is to compute the minimum of $V'(u)(\varphi)$ subject to $\int_{\mathbb{R}^n} u(x) \varphi(x) = 0$ and $|\varphi|^2_{H^1(\mathbb{R}^N)} = 1$.

We pass to the proof itself. For simplicity we break it into several steps.

**Step 1.** *If $(u_n)$ is a minimizing sequence, then except for a translation in the $x$-variable, $(u_n)$ has a subsequence converging weakly to some $u \not\equiv 0$.*

**Proof.** Otherwise, by Lieb's lemma ([8]) $(u_n)$ would be convergent to zero in $L_s(\mathbb{R}^N)$ for $s$ in the open interval $(2, 2^*)$ and this would imply $\lim V(u_n) \geq 0$, a contradiction.

So, if $(u_n)$ is a minimizing sequence, making a translation in the $x$-variable and passing to a subsequence, we can assume $(u_n)$ converges weakly in $H^1(\mathbb{R}^N)$ to some $u \not\equiv 0$.

According to statements S.4 and S.5 there are sequences $\alpha_n$, $\gamma_n$ and $\bar{\varphi}_n$ with $\gamma_n \to 0$ and $|\varphi_n|_{H^1(\mathbb{R}^N)} = 1$ such that

$$\begin{aligned}&\int_{\mathbb{R}^N} \langle \text{grad }\ u_n(x),\ \text{grad}\varphi(x)\rangle\, dx + \int_{\mathbb{R}^N\times\mathbb{R}^N} k(x-y)H(u_n(y))h(u_n(x))\varphi(x)\, dx\, dy \\ &+ \int_{\mathbb{R}^N} f(u_n(x))\varphi(x)\, dx + \alpha_n \int_{\mathbb{R}^N} u_n(x)\varphi(x)\, dx \\ &- \gamma_n \int_{\mathbb{R}^N} (\langle \text{grad}\bar{\varphi}_n(x),\ \text{grad }\varphi(x)\rangle + \bar{\varphi}_n(x)\varphi(x))dx = 0 \end{aligned} \tag{II.11}$$

for any $\varphi$ in $H^1(\mathbb{R}^N)$ and

$$\begin{aligned} -\Delta(u_n(x) + \gamma_n\bar{\varphi}_n(x)) + \Big(\int_{\mathbb{R}^N} k(x-y)H(u_n(y))\, dy\Big)h(u_n(x)) \\ + f(u_n(x)) + \alpha_n u_n(x) + \gamma_n\bar{\varphi}(x) = 0. \end{aligned} \tag{II.12}$$

**Step 2.** *$\alpha_n$ is bounded.*

**Proof.** If not, passing to a subsequence if necessary, we can assume $\alpha_n \to \infty$. Dividing (II.10) by $\alpha_n$ and passing to the limit, we get $\int_{\mathbb{R}^N} u(x)\varphi(x)dx = 0$ for any $\varphi$ in $H^1(\mathbb{R}^N)$, a contradiction. So, passing to a subsequence if necessary, we can assume that $\alpha_n \to \alpha$ and then, from (II.10), $u$ satisfies

$$-\Delta u(x) + \left[\int_{\mathbb{R}^N} k(x-y)H(u(y))\, dy\right]h(u(x)) + f(u(x)) + \alpha u(x) = 0. \tag{II.13}$$

The assumptions $H_1$ and $H_2$ together with the usual bootstrap argument give that $u$ belongs to $W^{2,s}(\mathbb{R}^N)$ for any $s \in [2, +\infty)$; in particular, it is continuous and tends to zero at infinity. From statement S.4 we know that $W(u)(\dot{h}, \dot{h}) \geq 0$ for any admissible $\dot{h}$ and all this together implies $\alpha \geq 0$.

**Step 3.** *$u_n \to u$ in $L_s(\mathbb{R}^N)$ for any $s$ in the interval $(2, 2^*)$.*

**Proof.** Otherwise by Lieb's lemma and passing to a subsequence if necessary, we know that there is a sequence $(d_n)$, $d_n \in \mathbb{R}^N$ with $|d_n| \to +\infty$ such that the sequence $v_n(x) = u_n(x + d_n)$ converges weakly in $H^1(\mathbb{R}^N)$ to some $v \not\equiv 0$ that satisfies

$$-\Delta v(x) - \left[\int_{\mathbb{R}^N} k(x-y)H(v(y))\,dy\right]h(v(x)) + f(v(x)) + \alpha v(x) = 0.$$

From (II.12) and (II.13) and assumption $H_4$, we know that there are $\varphi$ and $\psi$ such that $W(u)(\varphi,\varphi) < 0$ and $W(v)(\psi,\psi) < 0$. For simplicity we take $\varphi$ and $\psi$ smooth with compact support and $L_2$-normalized.

Next define $\dot{h}_n(x) = a_{1,n}\varphi(x) + a_{2,n}\psi(x - d_n)$ imposing $\int_{\mathbb{R}^N} u_n(x)\dot{h}_n(x)dx = 0$ and $a_{1,n}^2 + a_{2,n}^2 = 1$; for $\ddot{h}_n$ we make the choice $\ddot{h}_n(x) = c_n\eta(x)$, where $\eta(x)$ is a smooth function with compact support such that $\int_{\mathbb{R}^N} u(x)\eta(x)dx \neq 0$ and $c_n$ is chosen to make $\ddot{h}_n$ admissible. Let $u_n : (-\delta,\delta) \to H^1(\mathbb{R}^N)$ be a sequence of admissible curves according to statement S.3. A short computation shows that $\liminf \frac{d^2}{dt^2}V(u_n(t))\big|_{t=0} < 0$ and this contradicts statement S.4 and proves Step 3.

**Remark.** Although it is not clearly stated, the version of Lieb's lemma that we have used has been proved in [1].

**End of Proof.** If we subtract (II.12) from (II.13), multiply by $(u_n - u)$ and integrate we get

$$\begin{aligned}&\int_{\mathbb{R}^N} |\mathrm{grad} u_n(x) - \mathrm{grad} u(x)|^2\,dx + \alpha\int_{\mathbb{R}^N} |u_n(x) - u(x)|^2\,dx \\ &= -\gamma_n \int_{\mathbb{R}^N} \langle \mathrm{grad}\bar{\varphi}_n, \mathrm{grad}(u_n - u)\rangle\,dx + \int_{\mathbb{R}^N} (u_n - u)Q\,dx,\end{aligned} \tag{II.15}$$

where $Q$ is the obvious (big) expression. Using the fact that $\gamma_n \to 0$, $\alpha_n \to \alpha$ and $u_n \to u$ in $L_s(\mathbb{R}^N)$, $2 < s < 2^*$ and Holder's inequality, we can see that the right-hand side of (II.15) goes to zero and this implies grad $u_n \to \mathrm{grad} u$ in $L_2(\mathbb{R}^N)$ (because $\alpha \geq 0$) and the theorem is proved. □

We now give sufficient conditions for $H_4$.

(i) (The attractive case) Condition $H_4$ is satisfied if $k(z) \leq 0$ and $h(u) \geq 0$.

In fact, suppose $W(u)(\varphi,\varphi) \geq 0$ for any $\varphi \in H^1(\mathbb{R}^N)$. Differentiating (II.3) with respect to, say, $x_1$, we get $W(u)(\varphi_1,\varphi_1) = 0$, where $\varphi_1 = \frac{\partial u}{\partial x_1}$. We rewrite $W(u)(\varphi,\varphi)$ as

$$\begin{aligned}W(u)(\varphi,\varphi) = \int_{\mathbb{R}^N} |\mathrm{grad}\ \varphi(x)|^2 - \int_{\mathbb{R}^N\times\mathbb{R}^N} k(x,y)\varphi(x)\varphi(y)\,dx\,dy \\ + \int_{\mathbb{R}^N} p(x)\varphi^2(x)\,dx,\end{aligned}$$

where $k(x,y)$ and $p(x)$ have an obvious meaning; in particular $k(x,y) \geq 0$. We also have $W(|\varphi_1|, |\varphi_1|) \geq 0$ and this implies it is zero because

$$\int_{\mathbb{R}^N \times \mathbb{R}^N} k(x,y)\varphi_1(x)\varphi_1(y)\,dx\,dy \leq \int_{\mathbb{R}^N \times \mathbb{R}^N} k(x,y)|\varphi_1(x)|\;|\varphi_1(y)|\,dx\,dy;$$

then,

$$-\Delta(|\varphi_1|) + p(x)|\varphi_1| = \int_{\mathbb{R}^N} k(x,y)|\varphi_1(x)|\;|\varphi_1(y)|\,dx\,dy \geq 0$$

and the maximum principle implies either $|\varphi_1(x)| > 0$ everywhere or $|\varphi_1(x)| \equiv 0$, a contradiction.

(ii) If $H(u) = u^2$ and there is a $\beta \leq 3$ such that

$$f'(u)u^2 \leq \beta u f(u) \tag{II.16}$$

for $u \geq 0$ (if $\beta = 3$ the inequality has to be strict for $u > 0$), then, independently of the sign of $k(z)$, assumption $H_4$ is satisfied.

In fact, if we multiply II.3 by $u$, integrate and eliminate the term containing $k(x-y)$ (whose sign is, in principle, unknown), we get $W(u)(u,u) < 0$.

**Remark.** In [5] the condition

$$F(tu) \leq t^4 F(u), \quad t \geq 1, \tag{II.17}$$

was shown to be sufficient for the existence of minimum. Clearly (II.16) is much more severe than (II.17) because it requires two derivatives but, strictly speaking, (II.16) does not imply (II.17), although (II.16) is not sufficient to guarantee the existence of minimum because, as we have pointed out, we still have to verify the convergence of the minimizing sequence in the $L_2$ norm. Notice that both conditions exclude the possibility of having $F(u) = c|u|^p$ with $c > 0$ and $p > 4$.

The following remarks have been made by P.L. Lions.

1) The proof of Theorem II.4 says that the condition $V(\lambda) < V(\alpha)$, $0 \leq \alpha < \lambda$, is a necessary and sufficient condition for the compactness, up to translation, of minimizing sequence. In addition, if it does not hold then, up to translation, any minimizing sequence converges in $\mathcal{D}^{1,2} \cap L_p$, $2 < p \leq 2^*$ ($\mathcal{D}^{1,2}$ is the completion of $\mathcal{D}(\mathbb{R}^N)$ for the norm $(\int_{\mathbb{R}^N} |grad\, u(x)|^2 dx)^{\frac{1}{2}}$) to some $u$ satisfying

$$\int_{\mathbb{R}^N} u^2(x)dx = \alpha < \lambda, \quad V(u) = V(\alpha) = V(\mu), \alpha \leq \mu \leq \lambda$$

and $V(\lambda - \alpha) = 0$ (because $V(\lambda) \leq V(\mu) + V(\lambda - \mu)$ and $V(\mu) \leq 0$ for $0 \leq \mu \leq \lambda$).

2) For the type of constraint we are using, namely $\int_{\mathbb{R}^N} u^2(x)\,dx = \lambda$, if $\dot{h}(x)$ satisfies (II.5), there is the natural admissible curve $u(t) = \sqrt{\lambda}\frac{u+t\dot{h}}{||u+t\dot{h}||_{L_2}}$ and so we

do not have to worry about the compatibility of the second derivative (but, if we had $\int_{\mathbb{R}^N} G(u(x))dx = \lambda$ as constraint, probably we should worry about it).

3) Instead of using the condition $|V'(u_n)| \to 0$ we can use a perturbation principle due to Ekeland that has been used in [6].

4) According to the previous remarks we can give the following alternative proof for Theorem II.4 using the method developed in [6]: if a nontrivial dichotomy occurs then there are $u_1, u_2$, and $y_n$ such that

$$|y_n| \to \infty, \quad \int_{\mathbb{R}^N} u_1^2(x)\,dx = \alpha_1, \quad \int_{\mathbb{R}^N} u_2^2(x)\,dx = \alpha_2, \quad \alpha_1 \neq 0 \neq \alpha_2,$$

$u_1$ and $u_2$ solve $V'(u) + \theta u = 0$ with the same $\theta \geq 0$ (this is precisely equations (II.12) and (II.14)), $V''(u_1(\cdot) + u_2(\cdot + y_n)) + \theta + \varepsilon_n \geq 0$, where $\varepsilon_n \to 0$ as $n \to +\infty$ and

$$V(\alpha_1 + \alpha_2) = V(\alpha_1) + V(\alpha_2) = \lim_{n \to +\infty} (V(u_1(\cdot)) + V(u_2(\cdot + y_n))).$$

Let $\varphi(x), \psi(x), \dot{h}_n(x), a_{1n}$ and $a_{2n}$ be as in our proof and define

$$U_n(t) = \sqrt{\lambda} \frac{u_n + t\dot{h}_n}{||u_n + t\dot{h}_n||_{L_2}}.$$

Then we have $\lim_{t \to 0}(\lim_{n \to +\infty} V(U_n(t))) = V(\alpha_1 + \alpha_2)$, uniformly on $t$, and on the other hand as $n \to +\infty$ we also have uniformly in $t$ small $V(U_n(t)) = V(\alpha_1 + \alpha_2) + \frac{t^2}{2} a_{1n}^2 V''(u_1)(\psi, \psi) + \frac{t^2}{2} a_{2n}^2 V''(u_2)(\psi, \psi) + \text{h.o.t} < V(\alpha_1 + \alpha_2)$, a contradiction.

Now we discuss the convergence in the $L_2$ norm.

As we have pointed out we know (see [1]) that $V(\lambda) \leq V(\lambda - \mu) + V(\mu)$ for $0 < \mu < \lambda$; if we had $\lim \int_{\mathbb{R}^N} u_n^2(x)dx = \mu < \lambda$ this would imply $V(\lambda) = V(\mu)$ and we can exclude this possibility assuming that $V(\lambda) < 0$ for $\lambda > 0$. This condition is satisfied if, for instance, $k(z) \leq 0$, $H(u) \geq 0$ and $\lim_{u \to 0} F(u)|u|^{-\ell} = +\infty$ where $\ell = \frac{2N+4}{N}$ (see [1]).

According to II.15 another possibility of having convergence in $L_2(\mathbb{R}^N)$ is to guarantee in advance that $\alpha > 0$ and this can be done if we assume that the following condition holds:

(C) $V(u) \geq 0$ for any solution of

$$-\Delta u + \Big(\int_{\mathbb{R}^N} k(x-y)H(u(y))\,dy\Big)h(u(x)) + f(u(x)) = 0. \tag{II.18}$$

In order to give sufficient conditions for condition $C$ we consider first the case where $k(z)$ is homogeneous of degree $-\gamma$, $0 < \gamma < N$; for those kernels we see that any solution of (II.18) satisfies Pohozaev's identity

$$(2-N)\int_{\mathbb{R}^N} |\text{grad} u(x)|^2\,dx - (\gamma - 2N)\int_{\mathbb{R}^N \times \mathbb{R}^N} k(x-y)H(u(y))H(u(x))\,dx\,dy$$
$$- 2N\int_{\mathbb{R}^N} F(u(x))\,dx = 0.$$

Formally this identity is obtained making $\frac{d}{d\sigma}V(\sigma x)|_{\sigma=1}=0$.

Using that identity we get

$$V(u)=\frac{(\gamma-N-2)}{2(\gamma-2N)}\int_{\mathbb{R}^N}|\text{grad }u(x)|^2\,dx+\frac{(\gamma-N)}{(\gamma-2N)}\int_{\mathbb{R}^N}F(u)(x))\,dx$$
$$=\frac{1}{N}\int_{\mathbb{R}^N}|\text{grad }u(x)|^2-\frac{(\gamma-N)}{(\gamma-2N)}\int_{\mathbb{R}^N\times\mathbb{R}^N}k(x-y)H(u(x))H(u(y))\,dx\,dy.$$

So, for homogeneous Kernel of degree $-\gamma$, $0<\gamma<N$, condition $C$ is satisfied if either $F(u)\geq 0$ or $H(u)\geq 0$ for any $u\geq 0$ and $k(z)\geq 0$ for any $z\in\mathbb{R}^N$.

Next we deal with the case where $F(u)$ and $G(u)$ behave like homogeneous functions. First we multiply (II.18) by $u$ and integrate.

Suppose that $F(u)$ and $H(u)$ satisfy the condition

$$\text{(i)}\qquad F(u)\geq\frac{1}{\beta}uf(u),\quad H(u)\geq 0,\quad H(u)\leq\frac{2}{\beta}uh(u)\quad\text{and}\quad k(z)\geq 0$$

for any $z\in\mathbb{R}$, any $u\geq 0$ and some constant $\beta\geq 2$.

Under (i) for any solution $u$ of (II.8) we have

$$\int_{\mathbb{R}^N}F(u(x))\,dx\geq-\frac{1}{\beta}\Big[\int_{\mathbb{R}^N}|\text{grad }u(x)|^2$$
$$+\int_{\mathbb{R}^N\times\mathbb{R}^N}k(x-y)H(u(y))u(x)h(u(x))\,dx\,dy\Big]$$

and then

$$V(u)\geq(\frac{1}{2}-\frac{1}{\beta})\int_{\mathbb{R}^N}|\text{grad }u(x)|^2$$
$$+\int\int_{\mathbb{R}^N\times\mathbb{R}^N}k(x-y)[H(u(y))(\frac{1}{\beta}u(x)h(u(x))-\frac{1}{2}H(u(x))]dxdy\geq 0$$

and then condition C is satisfied.

Suppose now that $F(u)$ and $H(u)$ satisfy

$$\text{(ii)}\qquad uh(u)=\beta H(u)\quad\text{and}\quad F(u)\geq\frac{1}{2\beta}uf(u)$$

for any $u\geq 0$ and some constant $\beta\geq 1$. Under these conditions for any solution $u$ of (II.8) we have

$$\int_{\mathbb{R}^N\times\mathbb{R}^N}k(x-y)H(u(y))u(x))\,dx\,dy$$
$$=\frac{1}{\beta}\left[\int_{\mathbb{R}^N}|\text{grad }u(x)|^2\,dx+\int_{\mathbb{R}^N}u(x)f(u(x))\,dx\right]$$

and then

$$V(u) \geq (\frac{1}{2} - \frac{1}{\beta}) \int_{\mathbb{R}^N} |\text{grad } u(x)|^2 + \int_{\mathbb{R}^N} \big(F(u) - \frac{1}{2\beta} u(x) f(u(x))\big)\, dx \geq 0$$

and condition C is satisfied.

**III. The case of two dependent variables.** The problem we discuss in this section is to minimize the functional

$$V(u,v) = \frac{1}{2} \int_{\mathbb{R}^N} (|\text{grad} u(x)|^2 + |\text{grad} v(x)|^2)\, dx + \int_{\mathbb{R}^N} F(u(x), v(x)\, dx$$

subject to

$$I(u,v) = \int_{\mathbb{R}^N} G(u(x), v(x))\, dx = \lambda \neq 0$$

and we make the following assumptions:

$H_1$) $F(u,v)$ and $G(u,v)$ can be written as $F(u,v) = Q_1(u,v) + F_1(u,v)$ and $G(u,v) = Q_0(u,v) + G_1(u,v)$ where $Q_1(u,v)$ and $Q_0(u,v)$ are the quadratic parts of $F(u,v)$ and $G(u,v)$ at $(0,0)$ and $F_1(u,v)$ and $G_1(u,v)$ are $C^2$ functions vanishing at $(0,0)$ as well their first derivatives and whose second derivatives are bounded by $k(|u|^{q-2} + |v|^{q-2} + |u|^{p-2} + |v|^{p-2})$, for some constant $k$ and $2 < q \leq p < 2^*$.

$H_2$) The admissible set $\{(u,v) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N) : I(u,v) = \lambda\}$ is not empty.

$H_3$) $V$ is bounded below on the admissible set and any minimizing sequence is bounded in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

$H_4$) If $(u(\cdot), v(\cdot)) \not\equiv (0,0)$ then $(G_u(u(\cdot), v(\cdot)), G_v(u(\cdot), v(\cdot)) \not\equiv (0,0)$.

$H_5$) For any nontrivial solution $(u(\cdot), v(\cdot)) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ of

$$\begin{aligned} &-\Delta u + F_u(u,\ v) + \alpha G_u(u,\ v) = 0 \\ &-\Delta v + F_v(u,\ v) + \alpha G_v(u,\ v) = 0, \end{aligned}$$

where $\alpha$ is such that the hessian matrix of $F(u,v) + \alpha G(u,v)$ at $(0,0)$ is positive, there is a pair $(h,k) \in H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ such that

$$\begin{aligned} W(u,v)(h,h,k,k) = &\int_{\mathbb{R}^N} (|\text{grad} h|^2 + |\text{grad} k|^2) dx + \int_{\mathbb{R}^N} \langle (h,k), A(u,v)(h,k)\rangle\, dx \\ &< 0, \end{aligned}$$

where $A(u,\ v)$ is the hessian matrix of $F(u,v) + \alpha G(u,v)$.

**Theorem III.4.** *Under assumptions $H_1$ to $H_5$, if $(u_n, v_n)$ is a minimizing sequence and $(u_n, v_n)$ converges weakly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ to $(u,v) \not\equiv (0,0)$, then $(u_n, v_n)$ converges to $(u,v)$ strongly in $L_r(\mathbb{R}^N) \times L_r(\mathbb{R}^N)$, $2 < r < 2^*$, and* (grad $u_n$, grad $v_n$) *converges to* (grad $u$, grad $v$) *in $L_2(\mathbb{R}^N) \times L_2(\mathbb{R}^N)$ and $(u,v)$ satisfies* (II.3.)

The proof of Theorem III.4 is similar to the one of Theorem II.4 and we will not repeat it.

**Remark.** If $F \equiv 0$ then the strict subadditivity holds ([1]). In this case it is not clear how to verify $H_5$; in fact, it is not clear whether it holds or not.

In order to give sufficient conditions for $H_5$ we notice that

$$W(u,\ v)\ (\frac{\partial u}{\partial x_1},\ \frac{\partial u}{\partial x_1},\ \frac{\partial v}{\partial x_1},\ \frac{\partial v}{\partial x_1}) = 0$$

and so, defining $H(u,v) = F(u,v) + \alpha G(u,v)$, we see that assumption $H_5$ is satisfied if the cooperative condition $H_{uv}(u,v) \geq 0$ holds. Since rotations in the $(u,v)$ variables preserve the form of the functional, the cooperative condition can be improved by saying that there are constants $a$ and $b$ with $a^2 + b^2 \neq 0$ such that $a(H_{uu} - H_{vv}) + bH_{uv} \geq 0$. As we indicated, the assumption $H_5$ has to be verified for the $\alpha$'s for which the hessian matrix of $F(u,v) + \alpha(u,v)$ at $(0,0)$ is positive and, depending on the sign of the eigenvalues of the hessian matrix of $G(u,v)$ at $(0,0)$, this restricts the range of the possible $\alpha$'s. As far as the verification of the cooperative condition is concerned, the case $G_{uv} = 0$ is particularly simpler.

A different strategy to verify $H_5$ is to impose $W(u,v)(u,u,v,v) < 0$ and for this it is sufficient to have

$$\langle (u,v), A(u,v)(u,v)\rangle - u(F_u(u,v) + \alpha G_u(u,v)) - v(F_v(u,v) + \alpha G_v(u,v)) < 0$$

for $(u,v) \neq (0,0)$.

We still have to discuss whether the minimizing sequence is not vanishing as well as its convergence in the $L_2$ norm. We consider three cases.

**First case.** *The quadratic from $Q_0(u,v)$ is positive definite and $\lambda > 0$.*

First of all we normalize $Q_1(u,v)$ by replacing $F(u,v)$ by $F(u,v) + \eta G(u,v)$ in such a way that the new $Q_1(u,v)$ has the form $(au+bv)^2$. After this normalization we can see that there is an admissible vanishing sequence such that $V(u_n, v_n)$ tends to zero. So, $V(\lambda) < 0$ is a necessary condition for convergence of minimizing sequences, modulo translation in $x$.

Next we show that it is sufficient. In fact, arguing exactly as in the proof of theorem II.4, we get an equality similar to II.5 and then we conclude that all we have to do is to show that the quadratic form $Q_1 + \alpha Q_0$ is positive definite, where $\alpha$ is the multiplier corresponding to $(u(\cdot), v(\cdot))$ and, since $Q_1(h,k)$ has the form $(au+bu)^2$, we have to show that $\alpha > 0$. But, we know that $\alpha \geq 0$ (because $Q_0(h,k)$ is positive definite) and if it was equal to zero then Pohozaev's identity would imply $V(\lambda) \geq 0$, a contradiction.

The conclusion is: under assumptions $H_1$ to $H_5$, if the quadratic form $Q_0(u,v)$ is positive definite and $\lambda > 0$, then, under the normalization condition above, any minimizing sequence is precompact (modulo translation) in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ if and only if $V(\lambda) < 0$.

**Second case.** *The quadratic form $Q_0$ is positive definite, $\lambda < 0$ and $Q_1$ vanishes identically.*

In this case, if $(u_n, v_n)$ is a minimizing sequence then, except for translation and passing to a subsequence, it converges weakly in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ to some $(u, v) \not\equiv (0, 0)$ because, otherwise, the constraint would be violated. So, according to Theorem III.4, grad $u_n$ converges to grad $u$ in $L_r(\mathbb{R}^n)$ and $u_n$ converges to $u$ in $L_2(\mathbb{R}^N), 2 < r < \frac{2N}{N-2}$. In order to show the convergence in the $L_2$ norm all we have to do is to show that we cannot have $\alpha = 0$ and $|u|_{L_2} < \lim |u_n|_{L_2}$ or $|v|_{L_2} < \lim |v_n|_{L_2}$. If that was so (we can assume that $Q_0$ is in its diagonal form) we would have $\int_{\mathbb{R}^N} G(u(x), v(x))\, dx < \lambda < 0$ and if $N \geq 2$ this leads us to a contradiction. The argument is the same we have used in ([2]) but we repeat it for sake of completeness. Suppose first $N \geq 3$. Defining $\sigma^N = \lambda / \int_{\mathbb{R}^N} G(u(x), v(x))\, dx$ and $\bar{u}(x) = u\big(\frac{x}{\sigma}\big), \bar{v}(x) = v\big(\frac{x}{\sigma}\big)$, we see that $(\bar{u}, \bar{v})$ is admissible and then $V(u, v) \leq V(\bar{u}, \bar{v})$, that is

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^N} (|\text{grad} u(x)|^2 + |\text{grad} v(x)|^2)\, dx + \int_{\mathbb{R}^N} F(u(x),\ v(x))\, dx \\ &\leq \frac{\sigma^{N-2}}{2}\int_{\mathbb{R}^N} (|\text{grad} u(x)|^2 + |\text{grad} v(x)|^2)\, dx + \sigma^N \int_{\mathbb{R}^N} F(u(x), v(x))\, dx.\end{aligned}$$

Moreover, since

$$\begin{aligned}&-\Delta u + F_u(u, v) = 0, \\ &-\Delta v + F_v(u, v) = 0\end{aligned}$$

we have

$$\frac{N-2}{2}\int_{\mathbb{R}^N} (|\text{grad} u(x)|^2 + |\text{grad} v(x)|^2) dx = -N \int_{\mathbb{R}^N} F(u(x),\ v(x))\, dx$$

and then

$$\begin{aligned}&\big(\frac{1}{2} + \frac{2-N}{2N}\big)\int_{\mathbb{R}^N} (|\text{grad} u(x)|^2 + |\text{grad} v(x)|^2)\, dx \\ &\leq \big(\frac{\sigma^{N-2}}{2} + \frac{(2-N)}{2N}\sigma^N\big)\int_{\mathbb{R}^N} (|\text{grad} u(x)|^2 + |\text{grad} v(x)|^2)\, dx\end{aligned}$$

and this implies $\text{grad} u(x) \equiv \text{grad} v(x) \equiv 0$, a contradiction.

If $N = 2$, we have $\int_{\mathbb{R}^N} F(u(x), v(x))\, dx = 0$ and this shows $V(\bar{u}, \bar{v}) = V(u, v)$; hence $V$ has a minimum at $(\bar{u}, \bar{u})$ and then

$$\begin{aligned}&-\Delta \bar{u} + F_u(\bar{u}, \bar{v}) = \beta G_u(\bar{u}, \bar{v}), \\ &-\Delta \bar{v} + F_v(\bar{u}, \bar{v}) = \beta G_v(\bar{u}, \bar{v}).\end{aligned}$$

Using Pohozaev's identity again we get $\beta = 0$ and then

$$-\frac{1}{\sigma^2}\Delta u + F_u(u,v) = 0,$$
$$-\frac{1}{\sigma^2}\Delta v + F_v(u,v) = 0.$$

This implies $\Delta u \equiv \Delta v \equiv 0$, a contradiction.

The conclusion is that under assumptions $H_1$ to $H_5$ if $Q_0$ is positive definite, $Q_1$ vanishes identically, $\lambda < 0$ and $N \geq 2$, then, modulo translation, any minimizing sequence is precompact in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

**Third case.** *$Q_1$ is positive definite and $Q_2$ vanishes identically.*

Using an equation similar to (II.15) we can see that in this case and under assumptions $H_1$ to $H_5$ and except for translation in the $x$-variable, any minimizing sequence is precompact in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$.

**Forth case.** $Q_0(u,v) = u^2$, $Q_1(u,v) = v^2$, $G_1(u,v) \equiv 0$, $F_1(u,v) = F_2(u) + cu^2v + F_3(v)$*, where $c > 0$ is constant and $F_2(u)$ is an even function of $u$.*

In order to satisfy assumption $H_3$ the condition $F_3(v) + v^2 \geq 0$, for any $v$, is necessary because $v$ does not appear in the constraint. It is not difficult to give conditions on $F_2(u)$ to ensure $H_3$.

Since we have assumed that $F_2(u)$ is even, we may restrict ourselves to nonnegative functions $u(x)$ and for those the cooperative condition $F_{uv}(u,v) \geq 0$ is satisfied and so, arguing as in the first case the conclusion is: any minimizing sequence $(u_n, v_n), u_n \geq 0$, is precompact in $H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ modulo translation if and only if $\inf V(u,v) < 0$.

The existence of minima in the forth case is related to the existence of solitary waves for some classical model in field theory (see [9] and [10]).

**Added in proof.** We have shown that, basically, assumption $H_5$ in section III is always satisfied.

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