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# ASYMPTOTIC ESTIMATES AND CONVEXITY OF LARGE SOLUTIONS TO SEMILINEAR ELLIPTIC EQUATIONS

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**Abstract.** We investigate explosive solutions of the equation $\Delta u = f(u)$ in a bounded convex domain $D$. We find some asymptotic estimates and prove the log-convexity of the solution $u(x)$ under suitable conditions on $f$.

**1. Introduction.** In this paper we investigate asymptotic estimates and convexity of classical solutions of the boundary value problem

$$\Delta u = f(u) \quad \text{in} \quad D, \quad u(x) \to \infty \quad \text{as} \quad x \to \partial D. \tag{1.1}$$

Such solutions are called large solutions [3, 4] or explosive solutions [1, 8]. Here $D \subset R^N$, $N > 1$, is a bounded convex smooth domain, $f(t)$ is a differentiable positive nondecreasing function on $(t_0, \infty)$ satisfying $\lim_{t\to t_0} f(t) = 0$ and $\lim_{t\to t_0} f'(t) < \infty$. It is well known [12] that under the growth condition

$$\int_a^\infty \frac{dt}{\sqrt{F(t)}} < \infty, \quad F(t) = \int_{t_0}^t f(\tau)\, d\tau, \quad a > t_0, \tag{1.2}$$

problem (1.1) has a classical solution. The special case when $f(t) = e^t$ and $D$ is a plane domain has been investigated by Bieberbach [5], who proved that $u(x) + 2\log\delta(x)$ is bounded in $D$; $\delta(x)$ denotes the distance from $x$ to the boundary of $D$. The result of Bieberbach has been extended to domains in $R^N$, $N > 1$, by Lazer and McKenna [15]. In a recent paper [16] Lazer and McKenna have shown that if $f(t)$ is convex for large $t$ and if

$$\lim_{t\to\infty} \frac{f'(t)}{\sqrt{F(t)}} = \infty, \tag{1.3}$$

then the solution $u(x)$ of problem (1.1) satisfies

$$\lim_{\delta(x)\to 0} [u(x) - \phi(\delta(x))] = 0, \tag{1.4}$$

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where $\phi(s)$ is the function defined as

$$\int_{\phi(s)}^{\infty} \frac{dt}{\sqrt{2F(t)}} = s.$$

The estimate (1.4) may be used for investigating the convexity of $u(x)$ [3, 4]. Condition (1.3) is fulfilled by $f(t) = e^t$ and by $f(t) = t^p$, $p > 3$. In [16], Lazer and McKenna observe that condition (1.3) is optimal for the asymptotic estimation (1.4) to hold.

In the first part of the present paper we investigate the behaviour of $u(x)-\phi(\delta(x))$ near the boundary under assumptions different from (1.3). More precisely, let $R$ be the minimum radius of curvature of the boundary of $D$, and let $d$ be its maximum radius of curvature. We prove that if for some $L \geq 1$ and $t_1 \geq t_0$ one has

$$\frac{f(t+L)-f(t)}{\sqrt{2F(t)}} > \frac{N}{R}, \quad \forall t > t_1,$$

then any solution $u(x)$ of (1.1) satisfies, near to $\partial D$,

$$0 < u(x) - \phi(\delta(x)) \leq M, \tag{1.5}$$

where $M$ is a constant depending on $N/R$. This is the case for $f(t) = t^p$, $p \geq 3$.

Furthermore, we prove that if for some $l > 0$ and $t_1 \geq t_0$ one has

$$\frac{f(t+l)-f(t)}{\sqrt{2F(t)}} < \frac{N-1}{d}, \quad \forall t > t_1,$$

then there exists a positive constant $\sigma$, depending on $(N-1)/d$, such that

$$u(x) - \phi(\delta(x)) \geq \sigma.$$

Moreover, we show that if

$$\lim_{t\to\infty} \frac{f'(t)}{\sqrt{F(t)}} = 0, \tag{1.6}$$

then

$$\lim_{\delta(x)\to 0} [u(x) - \phi(\delta(x))] = \infty. \tag{1.7}$$

Condition (1.6) is fulfilled when $f(t) = t^p$, $1 < p < 3$.

We recall another type of asymptotic estimate. Under some restrictions on $f(t)$, the solution $u(x)$ to problem (1.1) satisfies

$$\lim_{\delta(x)\to 0} \frac{u(x)}{\phi(\delta(x))} = 1. \tag{1.8}$$

Of course, (1.8) is weaker than (1.5) or (1.4); nevertheless it is sufficient to prove uniqueness for problem (1.1) [8]. The asymptotic estimate (1.8) was obtained by C. Loewner and L. Nirenberg [17] in the special case $f(t) = t^{(N+2)/(N-2)}$, $N > 2$. For $f(t) = t^p$, $p > 1$, the same estimate was proved by C. Bandle and M. Marcus [3] and, independently, by L. Véron [19]. See also G. Diaz and R. Letelier [8]. In [3] this estimate has been established for a more general class of functions $f$ defined in $[0, \infty)$ and satisfying the condition

$$\exists\ \mu > 1,\ t_1 > 0 : f(\tau t) \leq \tau^{\mu} f(t), \quad \forall \tau \in (0,1),\ \forall t > t_1. \tag{1.9}$$

In section 3 we investigate the convexity of the solution $u(x)$ to problem (1.1). We recall that $D$ is assumed to be convex. In [4] it was proved that if (1.4) holds, if $f(t)$ is nondecreasing and if $1/f(t)$ is convex, then $u(x)$ is convex. This is the case when $f(t) = e^t$ or $f(t) = t^p,\ p > 3$. The case $f(t) = t^p,\ 1 < p < 3$ is not included in [4] because (1.4) does not hold in this situation. In this paper we prove that when (1.9) holds, when $f(t)/t$ is nondecreasing and when $e^s/f(e^s)$ is convex, then $\log u(x)$ is convex. This is the case for $f(t) = t^p$, $p > 1$. We note that log-convexity is stronger than convexity.

**2. Asymptotic behaviour.** Let $f \in C^1(t_0, \infty)$, where $t_0$ is either a real number or $-\infty$. If $t_0 = -\infty$ we suppose that

$$\int_{-\infty}^{t} f(\tau)\, d\tau < \infty.$$

For $t \geq \lambda \geq t_0$ define

$$F_\lambda(t) = \int_\lambda^t f(\tau)\, d\tau, \quad F(t) = \int_{t_0}^t f(\tau)\, d\tau.$$

Assume

$$f(t) > 0, \quad \lim_{t \to t_0} f(t) = 0, \quad f'(t) \geq 0, \quad \lim_{t \to t_0} f'(t) < \infty, \quad \int_a^\infty \frac{dt}{\sqrt{F(t)}} < \infty, \tag{2.1}$$

where $a > t_0$. We will use the function $\phi\big(\delta(x)\big)$, where $\delta(x)$ denotes the distance from $x$ to $\partial D$, and $\phi(s)$ is defined as

$$\int_{\phi(s)}^\infty \frac{dt}{\sqrt{2F(t)}} = s. \tag{2.2}$$

**Proposition A.** *Let $f(t)$ satisfy* (2.1)*, and let $\phi(s)$ be defined as in* (2.2)*. For $R>0$, the function $w(r)=\phi(R-r)$, $r<R$, satisfies*

$$w' = \sqrt{2F(w)}, \quad w'' = f(w), \quad \lim_{r\to R} w(r) = \infty. \tag{2.3}$$

**Proof.** This is an immediate consequence of the definition of $\phi$.

**Proposition B.** *Let $f(t)$ satisfy* (2.1)*. For any $R>0$ there exists a solution $v(r)$ to the problem*

$$v''(r) + \frac{N-1}{r}v'(r) = f(v), \quad 0<r<R, \quad v'(0)=0, \quad \lim_{r\to R} v(r) = \infty. \tag{2.4}$$

**Proof.** We refer to [12]. □

We prove some comparison results between the functions $w(r)$ and $v(r)$ introduced in Propositions A and B, respectively. Let $v(r)$ be a solution to problem (2.4) and let $\lambda = v(0)$. Since $v'(r)>0$, from (2.4) it follows that $v''<f(v)$. Multiplying by $v'$ and integrating on $(0,r)$ we get

$$\frac{(v')^2}{2} < \int_\lambda^v f(t)\,dt = F_\lambda(v).$$

Integrating once more we find

$$\int_{v(r)}^\infty \frac{ds}{\sqrt{2F_\lambda(s)}} < R-r = \int_{w(r)}^\infty \frac{ds}{\sqrt{2F(s)}}.$$

Since $F(s) > F_\lambda(s)$ we infer that $v(r)>w(r)$ on $(0,R)$.

**Lemma 2.1.** *Let $f(t)$ satisfy* (2.1) *and suppose there exist $L\ge 1$ and $t_1 \ge t_0$ such that*

$$f(t+L) - f(t) \ge \frac{N}{R}\sqrt{2F(t)}, \quad \forall t > t_1. \tag{2.5}$$

*Then we have*

$$0 < v(r) - w(r) \le M, \quad \forall r \in (0,R), \tag{2.6}$$

*for some* (*finite*) *$M$ depending on $N/R$.*

**Proof.** Condition (2.5) and Proposition A imply the existence of $r_0<R$ such that

$$f(w(r)+L) - f(w(r)) > \frac{N-1}{r}w'(r), \quad \forall r \in (r_0,R). \tag{2.7}$$

Following [16], take $\epsilon < R-r_0$ and define

$$z(r) = w(r+\epsilon) + L,$$

where $L$ is as in (2.7). From (2.3) and (2.7) it follows that

$$z'' + \frac{N-1}{r} z' = f(z) - f(w+L) + f(w) + \frac{N-1}{r} w' < f(z), \quad \forall r \in (r_0, R-\epsilon).$$

The function $w$ in the last inequality is evaluated at $r+\epsilon$. Now increase $L$ if necessary so that $v(r_0) \le w(r_0) + L$. Then $v(r_0) < z(r_0)$. At $R-\epsilon$, $v$ is finite, whereas $z(R-\epsilon) = \infty$. By the comparison theorem we must have $v(r) < z(r)$ on $(r_0, R-\epsilon)$. Since $\epsilon$ is arbitrary we have $v(r) - w(r) \le L$ in $(r_0, R)$. The lemma follows.

**Lemma 2.2.** *Let $f(t)$ satisfy* (2.1) *and suppose there exist $l > 0$ and $t_1 \ge t_0$ such that*

$$f(t+l) - f(t) \le \frac{N-1}{R} \sqrt{2F(t)}, \quad \forall t > t_1. \tag{2.8}$$

*Then we have*

$$v(r) - w(r) \ge \sigma, \quad \forall r \in (0, R), \tag{2.9}$$

*for some positive $\sigma$ depending on $(N-1)/R$.*

**Proof.** Proposition A and condition (2.8) imply the existence of $r_0 < R$ such that

$$f(w(r)+l) - f(w(r)) < \frac{N-1}{r} w'(r), \quad \forall r \in (r_0, R). \tag{2.10}$$

Take $\epsilon < R - r_0$ and define

$$z(r) = w(r-\epsilon) + l,$$

where $l$ is as in (2.10). From (2.3) and (2.10) it follows, for $\epsilon$ small enough,

$$z'' + \frac{N-1}{r} z' = f(z) - f(w+l) + f(w) + \frac{N-1}{r} w' > f(z), \quad \forall r \in (r_0, R).$$

The function $w$ in the last inequality is evaluated at $r-\epsilon$. Now decrease $l$ if necessary up to a positive number $\sigma$ so that $v(r_0) \ge w(r_0) + \sigma$ (this is possible because $v(r_0) > w(r_0)$). Then, $v(r_0+\epsilon) > z(r_0+\epsilon)$. We have $v(R) = \infty$, whereas $z(R) = w(R-\epsilon)+\sigma$ is finite. By the comparison theorem we must have $v(r) > z(r) = w(r-\epsilon) + \sigma$ in $(r_0+\epsilon, R)$. Since $\epsilon$ is arbitrary we have $v(r) - w(r) \ge \sigma$ in $(r_0, R)$. The lemma follows.

**Lemma 2.3.** *Let $f(t)$ satisfy* (2.1) *and let*

$$\lim_{t\to\infty} \frac{f'(t)}{\sqrt{F(t)}} = 0. \tag{2.11}$$

*Then we have*

$$\lim_{r\to R}[v(r)-w(r)]=\infty.$$

**Proof.** We start by showing that (2.11) implies

$$\lim_{t\to\infty}\frac{f'(t+z(t))}{\sqrt{F(t)}}=0 \tag{2.12}$$

for every function $z(t)$ satisfying $0\le z(t)\le l$, $l$ finite. Indeed, by (2.11) and de l'Hôpital's rule we get $f^2/F^{3/2}\to 0$ as $t\to\infty$, which is equivalent to $f/F^{3/4}\to 0$. This implies $F/f\to\infty$. As a consequence, since $F$ and $f$ are increasing, we have

$$1<\frac{F(t+l)}{F(t)}\le\frac{F(t+l)}{F(t+l)-l\,f(t+l)}\to 1.$$

The above result and the inequality

$$\frac{f'(t+z(t))}{\sqrt{F(t)}}\le\frac{f'(t+z(t))}{\sqrt{F(t+z(t))}}\sqrt{\frac{F(t+l)}{F(t)}}$$

complete the step.

Consider first the case when $v(r)-w(r)$ is not monotone as $r\to R$, so that $v-w$ attains infinitely many local minima near $R$. Denote by $\{r_i\}$ a sequence of minimum points such that $r_i\to R$ and $\lim_{i\to\infty}(v-w)(r_i)=\liminf_{r\to R}(v-w)(r)=:\alpha$. Assume $\alpha<\infty$. Clearly $(v-w)(r_i)<2\alpha=:l$ for large $i$. By using (2.3) and (2.4) at $r_i$ we find

$$\begin{aligned}0&\le v''(r_i)-w''(r_i)=f(v(r_i))-f(w(r_i))-\frac{N-1}{r_i}v'(r_i)\\&\le f'(w(r_i)+z_i)l-\frac{N-1}{r_i}w'(r_i)<f'(w(r_i)+z_i)l-\frac{N-1}{R}\sqrt{2F(w(r_i))},\end{aligned}$$

where $z_i$ is a suitable number satisfying $0\le z_i\le l$. The above inequality contradicts (2.12) as $r_i\to R$, hence $v(r_i)-w(r_i)$ cannot be bounded in this situation.

Now consider the case when $v(r)-w(r)$ is monotone near $R$. Arguing again by contradiction, suppose there exists some finite $l$ such that

$$v(r)-w(r)<l,\quad \forall r\in(0,R). \tag{2.13}$$

By using (2.4) as well as (2.13) one finds

$$r^{1-N}(r^{N-1}v')'=f(v)<f(w+l).$$

Equation (2.3) may be rewritten as

$$r^{1-N}(r^{N-1}w')' = f(w) + \frac{N-1}{r}w'.$$

Hence

$$\begin{aligned} r^{1-N}(r^{N-1}(v'-w'))' &< f(w+l) - f(w) - \frac{N-1}{r}w' \\ &\le f'(w(r)+z(r))l - \frac{N-1}{R}w', \end{aligned}$$

where $z(r)$ is a suitable function satisfying $0 \le z(r) \le l$. The last inequality, (2.3) and (2.12) imply that for some $r_1 < R$, one has

$$\big(r^{N-1}(v'-w')\big)' \le -r^{N-1}\frac{N-1}{2R}w' < -r_1^{N-1}\frac{N-1}{2R}w', \quad \forall r \in (r_1, R).$$

Integration on $(r_1, r)$ leads to

$$v'(r) - w'(r) < -\Big(\frac{r_1}{R}\Big)^{N-1}\frac{N-1}{2R}w(r) + c_1.$$

Integrating once more one finds

$$v(r) - w(r) < -\Big(\frac{r_1}{R}\Big)^{N-1}\frac{N-1}{2R}\int_{r_1}^{r} w(\rho)d\rho + c_2. \tag{2.14}$$

If $\psi(t)$ denotes the inverse function of $\phi(s)$ we have

$$\int_{r_1}^{R} w(\rho)\,d\rho = \int_{r_1}^{R} \phi(R-\rho)\,d\rho = -\int_a^\infty t\psi'(t)\,dt = \int_a^\infty \frac{tdt}{\sqrt{2F(t)}}, \tag{2.15}$$

where $a = \phi(R - r_1)$. Condition (2.11) implies that for $t$ larger than some $t_1$,

$$F''(t) < \sqrt{F(t)}.$$

Multiplying by $F'(t)$ and integrating on $(t_1, t)$ we find

$$\frac{F'(t)^2}{2} < \frac{2}{3}F(t)^{\frac{3}{2}} + \frac{F'(t_1)^2}{2} < F(t)^{\frac{3}{2}}, \quad \forall t > t_2.$$

Hence

$$F'(t) < \sqrt{2}F(t)^{\frac{3}{4}}, \quad \forall t > t_2.$$

Integration in the last inequality leads to

$$F(t) < \Big(\frac{\sqrt{2}}{4}t + F(t_2)^{\frac{1}{4}}\Big)^4 < t^4, \quad \forall t > t_3.$$

This inequality and (2.15) imply that

$$\int_{r_1}^{R} w(\rho)\,d\rho = \infty.$$

Since $v(r) - w(r) > 0$, the last result contradicts (2.14) as $r$ approaches $R$. The proof of the lemma has been completed. $\square$

Let us prove the main results of this section. Recall that $\delta(x)$ denotes the distance from $x$ to the boundary of $D$.

**Theorem 2.4.** *Let $D$ be a bounded smooth convex domain in $R^N$, $N > 1$, and let $R$ be the smallest radius of curvature of $\partial D$. If $f(t)$ satisfies* (2.1) *and* (2.5) *then any solution $u(x)$ to problem* (1.1) *satisfies, for $x \in D$ with $\delta(x) < R$,*

$$0 < u(x) - \phi(\delta(x)) \leq M,$$

*where $M$ is a constant depending on $N/R$.*

**Proof.** The proof uses the comparison principle for the equation in (1.1) ([9], Theorem 9.2). Let $x \in D$ with $\delta(x) < R$ and let $z_x \in \partial D$ be the point of $\partial D$ nearest to $x$. Consider the half space $\Omega$ containing $D$ and tangent to $\partial D$ at $z_x$. The functions $u$ and $\phi$ are both solutions to the equation in (1.1) in $D$ and satisfy $u(y) > \phi(\delta'(y))$ on $\partial D$, where $\delta'(y)$ denotes the distance from $y$ to the boundary of $\Omega$. Hence, by the comparison principle,

$$u(x) > \phi(\delta'(x)) = \phi(\delta(x)). \tag{2.16}$$

Let $B_x$ be the ball of radius $R$ and center $c_x$, contained in $D$ and tangent to $\partial D$ at $z_x$. Let $v(r)$ be a solution to problem (2.4). By the comparison principle we have

$$u(x) < v(|x - c_x|) = v(R - \delta(x)). \tag{2.17}$$

The theorem follows by (2.16), (2.17) and Lemma 2.1.

**Theorem 2.5.** *Let $D$ be a bounded smooth convex domain in $R^N$, $N > 1$, and let $d < \infty$ be the maximum radius of curvature of $\partial D$. If $f(t)$ satisfies* (2.1) *and* (2.8) *then any solution $u(x)$ to problem* (1.1) *satisfies*

$$u(x) - \phi(\delta(x)) \geq \sigma,$$

*where $\sigma$ is a positive constant depending on $(N-1)/d$.*

**Proof.** Let $x \in D$ and let $z_x \in \partial D$ be the point of $\partial D$ nearest to $x$. Consider the ball $B_x$ of radius $d$ and center $c_x$ containing $D$ and tangent to $\partial D$ at $z_x$. Let $v(r)$ be a solution to problem (1.4) with $R = d$. By the comparison principle we have

$$u(x) > v(|x - c_x|) = v(R - \delta(x)).$$

The theorem follows by Lemma 2.2.

**Theorem 2.6.** *Let $D$ be a bounded smooth convex domain in $R^N$, $N > 1$. If $f(t)$ satisfies* (2.1) *and* (2.11) *then any solution $u(x)$ to problem* (1.1) *satisfies*

$$\lim_{\delta(x) \to 0} [u(x) - \phi(\delta(x))] = \infty.$$

**Proof.** The proof is almost the same as that of Theorem 2.5 and uses Lemma 2.3 instead of Lemma 2.2.

**3. Convexity.** In this section $f(t)$ is supposed to satisfy condition (2.1) with $t_0 = 0$. The case $t_0 \neq 0$ finite can be reduced to the case $t_0 = 0$ by translation.

**Lemma 3.1.** *Let $f(t)$ satisfy* (2.1) *and* (1.9). *If $u(x)$ is the solution to problem* (1.1) *in a smooth domain $D$, then*

$$\lim_{\delta(x)\to 0} \frac{u(x)}{\phi(\delta(x))} = 1, \tag{3.1}$$

*where $\phi(s)$ is as in* (2.2).

**Proof.** We refer to [3]. □

If $v(x) = \log u(x)$, problem (1.1) becomes

$$\Delta v = f(e^v)e^{-v} - |\nabla v|^2 \quad \text{in} \quad D, \quad v(x) \to \infty \quad \text{as} \quad x \to \partial D, \tag{3.2}$$

and (3.1) reads as

$$\lim_{\delta(x)\to 0} [v(x) - \log\big(\phi(\delta(x))\big)] = 0. \tag{3.3}$$

For proving the convexity of $v(x)$, the domain $D$ is supposed to be convex (not necessarily strictly convex). Assume the following additional hypotheses on $f$:

$$\frac{f(t)}{t} \quad \text{is nondecreasing for all} \quad t > 0, \tag{3.4}$$

$$\frac{e^s}{f(e^s)} \quad \text{is convex for all} \quad s. \tag{3.5}$$

Assumptions (2.1), (1.9), (3.4) and (3.5) are fulfilled by $f(t) = t^p,\ p > 1$.

**Lemma 3.2.** *Let $f(t)$ satisfy* (2.1), (1.9), (3.4) *and* (3.5). *If $v(x)$ is the solution to problem* (3.2) *in a convex domain $D$, then the concavity function*

$$C(v; x, y) = 2v(z) - v(x) - v(y), \qquad z = (x + y)/2 \tag{3.6}$$

*cannot assume any local positive maximum in $D \times D$.*

**Proof.** It is well known [11, 13] that the result holds when $v(x)$ satisfies $\Delta v = b(v, \nabla v)$, where $b(v, p)$ is positive, nondecreasing and harmonic concave with respect to $v$ for every fixed vector $p$. We have

$$b(v, \nabla v) = f(e^v)e^{-v} - |\nabla v|^2 = u^{-2}[f(u)u - |\nabla u|^2], \tag{3.7}$$

where $u = e^v$. By using (3.4) we find $f(t) \le t f(u)/u$ for $0 < t < u$, hence

$$2F(u) = 2\int_0^u f(t)\,dt \le 2\,\frac{f(u)}{u}\int_0^u t\,dt = f(u)u. \tag{3.8}$$

Insertion of the last estimate into (3.7) leads to

$$b(v, \nabla v) \geq u^{-2}[2F(u) - |\nabla u|^2].$$

By a result of Payne and Philippin [18], the function $\Psi(x) = 2F(u) - |\nabla u|^2$ satisfies $\Psi(x) \geq 2F(u_{min})$ in $D$ whenever $u(x)$ solves the equation in (1.1) and is constant on $\partial D$. Since the solution $u(x)$ to problem (1.1) is the limit as $m \to \infty$ of the solutions $u_m(x)$ to the equation in (1.1) satisfying $u(x) = m$ on $\partial D$, the positivity of $\Psi(x)$ follows (cf. [2]).

By (3.4), the function $b(v, p)$ is nondecreasing with respect to $v$. Let us prove that assumption (3.5) implies that $b(v, p)$ is harmonic concave with respect to $v$. Let $g(s) = f(e^s)e^{-s}$ and let $G(s) = g(s) - c > 0$, for a nonnegative $c$. We have to prove that $G$ is harmonic concave, that is, $2(G')^2 - GG'' \geq 0$ [11]. Indeed, by (3.5) we have

$$2(g')^2 - gg'' \geq 0. \tag{3.9}$$

If $g'' > 0$, then $2(G')^2 - GG'' \geq 2(g')^2 - gg'' \geq 0$, where (3.9) has been used (cf. [10]). If $g'' \leq 0$ then $2(G')^2 - GG'' \geq 0$ trivially. The lemma is proved.

**Lemma 3.3.** *Under the assumptions* (2.1) *and* (1.9) *the function* $\log\big(\phi(s)\big)$ *is convex.*

**Proof.** By using Proposition A we find

$$\phi^2(s)\big(\log\big(\phi(s)\big)\big)'' = \phi''(s)\phi(s) - \phi'(s)^2 = f(\phi)\phi - 2F(\phi) \geq 0,$$

where the last inequality is proved as in (3.8). The lemma is proved.

**Theorem 3.4.** *Let* $f(t)$ *satisfy* (2.1), (1.9), (3.4) *and* (3.5). *If* $v(x)$ *is the solution to problem* (3.2) *in a convex domain* $D$ *then the concavity function* (3.6) *cannot be positive in* $D \times D$.

**Proof.** In view of Lemma 3.2, it suffices to prove that $C(v; x, y)$ cannot become positive near $\partial(D \times D)$. In other words, we must prove that $C(v; x_i, y_i)$ cannot become positive when $x_i \to x$, $y_i \to y$, where either $x \in D$, $y \in \partial D$ or $x, y \in \partial D$. The first case can be excluded because, when $x \in D$, $y \in \partial D$, then $z \in D$, hence $C(v; x, y) = -\infty$. The case $x, y \in \partial D$ and $z \in D$ is similar. Finally, consider the case $x, y, z \in \partial D$. We have

$$C(v; x_i, y_i) = C(w; x_i, y_i) + C(v - w; x_i, y_i).$$

Recall that the distance function $\delta(x)$ with $x \in D$ is concave when $D$ is convex ([4], Proposition 3.2). By using this fact as well as Lemma 3.3 one finds that the function $w(x) = \log\big(\phi(\delta(x))\big)$ is convex. Hence $C(w; x_i, y_i) \leq 0$ for all $i$. On the other hand, (3.3) implies that $C(v - w; x_i, y_i) \to 0$ as $i \to \infty$. Hence $\limsup_{i\to\infty} C(v; x_i, y_i) \leq 0$. The theorem is proved. $\square$

Of course, the nonpositivity of $C(v;x,y)$ in $D \times D$ is equivalent to the convexity of $v(x)$ or to the log-convexity of $u(x)$.

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