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OSCILLATORY SOLUTIONS AND ROTATORY SOLUTIONS FOR A PERIODICALLY FORCED LIÉNARD SYSTEM

Jitsuro Sugie[1]
Department of Mathematics, Faculty of Science, Shinshu University, Matsumoto 390, Japan

Tadayuki Hara[2]
Department of Mathematical Sciences, University of Osaka Prefecture, Sakai 593, Japan

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Dedicated to Professor Junji Kato on the occasion of his 60th birthday

Abstract. We give a necessary and sufficient condition for all solutions of a periodically forced Liénard system to be oscillatory. This paper also deals with the question when all trajectories keep on rotating around the origin. Several examples and global phase portraits are given to illustrate our results.

1. Introduction. We consider the generalized Liénard equation with forcing term

$$\ddot{x} + f(x)\dot{x} + g(x) = p(t)$$

or an equivalent system

$$\begin{aligned} \dot{x} &= y - F(x) + P(t) \\ \dot{y} &= -g(x), \end{aligned} \tag{1.1}$$

where

$$F(x) = \int_0^x f(\xi)\, d\xi \quad \text{and} \quad P(t) = \int_0^t p(s)\, ds.$$

The theory of oscillations of (1.1) has very wide application in applied science and engineering. A great deal of effort has been made to study the phenomena of oscillations. One of the main subjects is whether or not for every solution $\big(x(t), y(t)\big)$ of (1.1), $x(t)$ has arbitrarily large zeros, that is, $x(t)$ is oscillatory. However, little attention has been given to the problem of whether $y(t)$ is oscillatory or not.

It is well-known that system (1.1) is also equivalent to the system

$$\begin{aligned} \dot{u} &= v \\ \dot{v} &= -f(v)u - g(u) + p(t). \end{aligned} \tag{1.2}$$

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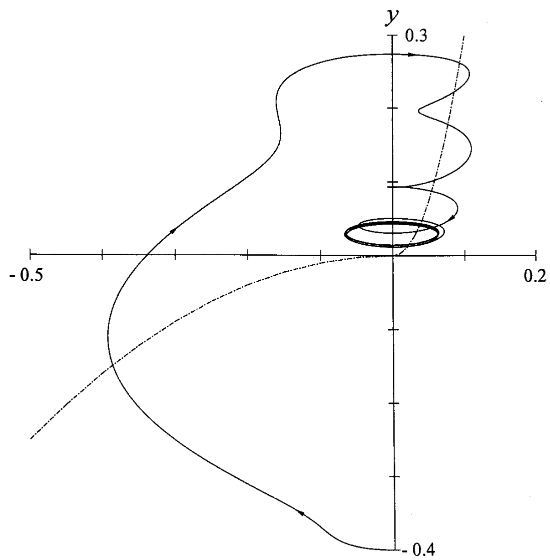

Figure 1. Trajectory of system (1.1) with

$$F(x) = \begin{cases} 30x^2 & (x \geq 0) \\ -x^2 & (x < 0), \end{cases} \qquad g(x) = x \;\text{ and }\; P(t) = \frac{1}{4}\sin 4t.$$

Let $\big(u(t), v(t)\big)$ be a solution of (1.2). Then it is easy to see that if $u(t)$ is oscillatory, then $v(t)$ is necessarily oscillatory.

But, system (1.1) does not have the same relation between $x(t)$ and $y(t)$. We explain the situation by an example. Let

$$F(x) = \begin{cases} 30x^2 & (x \geq 0) \\ -x^2 & (x < 0), \end{cases} \qquad g(x) = x \qquad \text{and} \qquad P(t) = \frac{1}{4}\sin 4t.$$

Then, as shown in Figure 1, the positive semitrajectory initiating at $(t_0, x_0, y_0) = (0, 0, -0.4)$ crosses the y-axis infinitely many times and eventually stays in the upper half-plane. Hence, $x(t)$ is oscillatory, but $y(t) > 0$ for large t. The broken curve in Figure 1 indicates the graph $y = F(x)$.

We would like to clear up the reason why such a strange phenomenon occurs.

Likewise, the trajectory in Figure 1 does not rotate around the origin. Note that even if both $x(t)$ and $y(t)$ are oscillatory, the corresponding positive semitrajectory of (1.1) or (1.2) does not always rotate around the origin. For example, consider system (1.1) with $F(x) = g(x) = x$ and $P(t) = -1 + \cos t$. Then system (1.1) has a solution $\big(x(t), y(t)\big) = (\cos t, 1 - \sin t)$, whose trajectory is $x^2 + (y-1)^2 = 1$. It is clear that $x(t)$ and $y(t)$ have an infinite number of zeros. However, the trajectory does not go around the origin.

The plan of this paper is the following. In Section 2, we consider the case that $P(t) = \frac{E}{\omega}\sin \omega t$, where $E > 0$ and $\omega > 0$, and give a necessary and sufficient condition under which all solutions of (1.1) are oscillatory in usual sense (Theorem 2.1). To see this, we introduce an important concept in the oscillation problem for (1.1). The proof of Theorem 2.1 will consist of a series of Propositions. In Theorem 2.2 we also clarify a difference between the oscillations of (1.1) and the unforced Liénard system

$$\begin{aligned}\dot{x} &= y - F(x)\\ \dot{y} &= -g(x).\end{aligned}$$

In Section 3, we present a sufficient condition for all positive semitrajectories of (1.1) with $P(t) = \frac{E}{\omega}\sin\omega t$ to keep on rotating around the origin. Finally, in Section 4, we illustrate Theorems 2.1 and 3.1 by some numerical examples. We also observe that the condition in Theorem 3.1 is reasonable in some sense.

We have not yet found the real reason why the trajectory of (1.1) does not go around the origin in Figure 1. The sign of $P(t)$ seems to play some role, but the situation is not so easy even if $P(t) = \frac{E}{\omega}\sin\omega t$. The magnitude of E and ω actually affects the behavior of the trajectory of (1.1) (see Observations 4.1–4.3 in Section 4). Our question is yet to be settled.

2. Oscillation of solutions. In this section we consider the periodically forced Liénard system

$$\begin{aligned}\dot{x} &= y - F(x) + \frac{E}{\omega}\sin\omega t\\ \dot{y} &= -g(x),\end{aligned} \tag{2.1}$$

where E and ω are positive constants; $F(x)$ and $g(x)$ satisfy suitable smoothness conditions for the uniqueness of solutions of the initial value problem and

$$xg(x) > 0 \quad \text{if} \quad x \neq 0. \tag{2.2}$$

We assume that all solutions of (2.1) are continuable in the future (for the continuation problem, we refer to [6], [7], [9]). But we do not necessarily assume that

$$xF(x) > 0 \quad \text{for} \ \ |x| \ \ \text{large}. \tag{2.3}$$

The case that $xF(x) \to -\infty$ as $|x| \to \infty$ may be admitted.

As mentioned in the introduction, system (2.1) comes from the classical Liénard equation. Hence, $F(0)$ is zero by definition. However, we intend to investigate system (2.1) in general, and so $F(x)$ may not vanish at zero.

Definition 2.1. A solution $(x(t), y(t))$ of (2.1) is said to be *oscillatory* if there exists a sequence $\{t_n\}$ tending to ∞ such that $x(t_n) = 0$.

Burton and Townsend ([1, Theorem 3.1]) gave a necessary and sufficient condition for the oscillation of all solutions of a forced Liénard system which is slightly more general than (2.1). Graef ([2, Theorem 4.11]) improved this result. He discussed the oscillation problem under the rather restricted assumption, namely, (2.3).

We will give another necessary and sufficient condition for all solutions of (2.1) to be oscillatory without requiring (2.3). To this end, we need the following concept concerning the unforced Liénard system:

$$\begin{aligned} \dot{x} &= y - F(x) \\ \dot{y} &= -g(x). \end{aligned} \tag{2.4}$$

Definition 2.2. System (2.4) has *property* (X^+) *in the right half-plane* (respectively *left half-plane*) if, for every point P in the region $\{(x,y) : x > 0 \text{ and } y > F(x)\}$ (respectively $\{(x,y) : x < 0 \text{ and } y < F(x)\}$), the positive semitrajectory starting at P crosses the curve $y = F(x)$.

Our main result on oscillation of solutions of (2.1) is stated as follows:

Theorem 2.1. *Assume* (2.2) *and* $F(0) = 0$. *Then all solutions of* (2.1) *are oscillatory if and only if system* (2.4) *has property* (X^+) *in the right and left half-plane.*

It is useful to tell about property (X^+) before proving Theorem 2.1. In the last decade, several articles ([3], [5], [8], [10]–[12]) have been devoted to the study of property (X^+). The authors gave the following necessary and sufficient condition for system (2.4) to have property (X^+) in a recent paper ([4]).

Proposition 2.1. *Assume* (2.2). *Then system* (2.4) *fails to have property* (X^+) *in the right half-plane if and only if there exist constants* $b \geq 0$ *and* $c \in \mathbf{R}$, *and a continuous function* $\psi(x)$ *such that*

$$F(x) < \psi(x) \quad \textit{and} \quad \int_b^x \frac{g(\xi)}{\psi(\xi) - F(\xi)}\, d\xi \leq c - \psi(x) \quad \textit{for } x \geq b.$$

Turning our attention to the left half-plane, we can obtain the following result which is analogous to Proposition 2.1.

Proposition 2.2. *Assume* (2.2). *Then system* (2.4) *fails to have property* (X^+) *in the left half-plane if and only if there exist constants* $b \geq 0$ *and* $c \in \mathbf{R}$, *and a continuous function* $\psi(x)$ *such that*

$$F(x) > \psi(x) \quad \textit{and} \quad \int_{-b}^x \frac{g(\xi)}{\psi(\xi) - F(\xi)}\, d\xi \geq c - \psi(x) \quad \textit{for } x \leq -b.$$

Conditions in Propositions 2.1 and 2.2 are represented implicitly. Some explicit conditions for property (X^+) are also given in [4, Theorem 4.1 and Theorems 5.1–5.4]. For example, in case $g(x) = x$, we have the following:

- **(i)** if $F(x) \geq -2x + \frac{x}{\log x}$ for x large, then system (2.4) has property (X^+) in the right half-plane;
- **(ii)** if $F(x) \leq -2x + \frac{x}{4(\log x)^2}$ for x large, then system (2.4) fails to have property (X^+) in the right half-plane.

To prove Theorem 2.1, we first prepare Lemma 2.1 below. Consider the following system:

$$\begin{aligned} \dot{x} &= y - \widetilde{F}(x) \\ \dot{y} &= -g(x), \end{aligned} \tag{2.5}$$

where $\widetilde{F}(x) = F(x) - M$ and M is a constant. Then we have:

Lemma 2.1. *Assume* (2.2)*. Then system* (2.4) *has property* (X^+) *in the right half-plane* (*respectively, left half-plane*) *if and only if system* (2.5) *has property* (X^+) *in the right half-plane* (*respectively, left half-plane*)*.*

Proof. We consider only the case $M > 0$ because the other case is similar. Suppose that system (2.4) fails to have property (X^+) in the right half-plane. Then, by Proposition 2.1 there exist constants $b \geq 0$ and $c \in \mathbf{R}$, and a continuous function $\psi(x)$ such that

$$F(x) < \psi(x) \quad \text{and} \quad \int_b^x \frac{g(\xi)}{\psi(\xi) - F(\xi)}\, d\xi \leq c - \psi(x) \quad \text{for } x \geq b.$$

We therefore have $\widetilde{F}(x) < F(x) < \psi(x)$ and

$$\int_b^x \frac{g(\xi)}{\psi(\xi) - \widetilde{F}(\xi)}\, d\xi < \int_b^x \frac{g(\xi)}{\psi(\xi) - F(\xi)}\, d\xi \leq c - \psi(x)$$

for $x \geq b$. Hence, Proposition 2.1 shows that system (2.5) fails to have property (X^+) in the right half-plane.

For the converse, suppose that system (2.5) fails to have property (X^+) in the right half-plane. Then, by using Proposition 2.1, there exist constants $\widetilde{b} \geq 0$ and $\widetilde{c} \in \mathbf{R}$, and a continuous function $\widetilde{\psi}(x)$ such that

$$\widetilde{F}(x) < \widetilde{\psi}(x) \quad \text{and} \quad \int_{\widetilde{b}}^x \frac{g(\xi)}{\widetilde{\psi}(\xi) - \widetilde{F}(\xi)}\, d\xi \leq \widetilde{c} - \widetilde{\psi}(x) \quad \text{for } x \geq \widetilde{b}.$$

Let $b = \widetilde{b}$, $c = \widetilde{c} + M$ and $\psi(x) = \widetilde{\psi}(x) + M$. Then we get for $x \geq b$

$$F(x) = \widetilde{F}(x) + M < \widetilde{\psi}(x) + M = \psi(x)$$

and

$$\int_b^x \frac{g(\xi)}{\psi(\xi) - F(\xi)}\, d\xi = \int_{\widetilde{b}}^x \frac{g(\xi)}{\widetilde{\psi}(\xi) - \widetilde{F}(\xi)}\, d\xi \leq \widetilde{c} - \widetilde{\psi}(x) = c - \psi(x).$$

Thus, it follows from Proposition 2.1 that system (2.4) fails to have property (X^+) in the right half-plane.

For property (X^+) in the left half-plane, by means of Proposition 2.2, we can use the same argument in the right half-plane. Thus, the proof is complete.

The proof of Theorem 2.1 will consist of Propositions 2.3–2.5 below.

Proposition 2.3. *Assume* (2.2). *If all solutions of* (2.1) *are oscillatory, then system* (2.4) *has property* (X^+) *in the right and left half-plane.*

Proof. Suppose that system (2.4) fails to have property (X^+) in the right half-plane. Consider an auxiliary system

$$\begin{aligned} \dot{x} &= y - F(x) - \frac{E}{\omega} \\ \dot{y} &= -g(x). \end{aligned} \tag{2.6}$$

Then Lemma 2.1 shows that system (2.6) also fails to have property (X^+) in the right half-plane. To be more exact, there exists a point P in the region $\{(x,y)\colon x > 0 \text{ and } y > F(x)+\frac{E}{\omega}\}$ such that the positive semitrajectory of (2.6) starting at P does not cross the curve $y = F(x)+\frac{E}{\omega}$. Compare the slope of this positive semitrajectory with that of the positive semitrajectory of (2.1) starting at the same point P. Then we can see that the latter as well as the former runs to infinity without intersecting the curve $y = F(x) + \frac{E}{\omega}$. This means that system (2.1) has a solution which is not oscillatory.

Similarly, if system (2.4) fails to have property (X^+) in the left half-plane, then from Lemma 2.1 and a comparison of the slope of trajectories of system (2.1) and

$$\begin{aligned} \dot{x} &= y - F(x) + \frac{E}{\omega} \\ \dot{y} &= -g(x), \end{aligned} \tag{2.7}$$

we conclude that there exists a nonoscillatory solution of (2.1). The proof is complete.

Proposition 2.4. *Assume* (2.2). *If system* (2.4) *has property* (X^+) *in the right and left half-plane, then all unbounded solutions of* (2.1) *are oscillatory.*

Proof. Let $\big(x(t), y(t)\big)$ be an unbounded solution of (2.1) which is not oscillatory. Then there exists a $T > 0$ such that either $x(t) > 0$ or $x(t) < 0$ for $t \geq T$. We prove only the case $x(t) > 0$ for $t \geq T$ since the proof of the other case is carried out in the same way. It follows from (2.2) that

$$\dot{y}(t) = -g\big(x(t)\big) < 0 \quad \text{for} \;\; t \geq T.$$

Consequently, we have $y(t) \leq y(T)$ for $t \geq T$. Let $Q = \big(x(T), y(T)\big)$ and denote by γ^+ the positive semitrajectory of (2.7) starting at the point Q.

Claim 1. The positive semitrajectory γ^+ meets the negative y-axis.

Define the regions R_L and R_U as follows:

$$\begin{aligned} R_L &= \big\{(x,y)\colon x > 0 \;\text{ and }\; y \leq F(x) - E/\omega\big\}; \\ R_U &= \big\{(x,y)\colon x > 0 \;\text{ and }\; y > F(x) - E/\omega\big\}. \end{aligned}$$

To prove the claim we consider two possible cases.

Case (**i**) $Q \in R_L$: We may regard γ^+ as a solution of

$$\frac{dy}{dx} = -\frac{g(x)}{y - F(x) + \frac{E}{\omega}}.$$

Hence γ^+ has no vertical asymptotes in R_L, and therefore, by (2.2) it must meet the negative y-axis.

Case (**ii**) $Q \in R_U$: By assumption and Lemma 2.1, system (2.7) has property (X^+) in the right half-plane. Hence γ^+ crosses the curve $y = F(x) - \frac{E}{\omega}$ in the right half-plane. The remainder of the proof is reduced to that of the case (i).

Consider the region S which is enclosed with γ^+, the y-axis and the line $y = y(T)$. From Claim 1 we conclude that S is bounded. Comparing the vector field of (2.1) with that of (2.7), we see that the positive semitrajectory of (2.1) which corresponds to the unbounded solution $\big(x(t), y(t)\big)$ remains in S for all future time. This is impossible and completes the proof.

Proposition 2.5. *Assume* (2.2) *and* $F(0) = 0$. *Then all bounded solutions of* (2.1) *are oscillatory.*

Proof. Let $\big(x(t), y(t)\big)$ be a bounded solution of (2.1) which is not oscillatory. Then there exist positive constants B and T such that for $t \geq T$

$$|x(t)| + |y(t)| \leq B \quad \text{and} \quad x(t) \neq 0.$$

We may assume without loss of generality that $x(t) > 0$ for $t \geq T$. Hence, by (2.2) we have

$$\dot{y}(t) = -g\big(x(t)\big) < 0 \quad \text{for } \ t \geq T,$$

and therefore, $y(t)$ approaches the limit $\bar{y}$ from above, because $y(t)$ is bounded.

Claim 2. There exists a sequence $\{s_n\}$ tending to ∞ such that $x(s_n) \to 0$ as $n \to \infty$.

Suppose that the claim is not true. Then there exists a $\lambda > 0$ such that $\lambda \leq x(t) \leq B$ for $t \geq T$. Let $K = \min\big\{g(x) : \lambda \leq x \leq B\big\}$. Then we get

$$\dot{y}(t) = -g\big(x(t)\big) \leq -K \quad \text{for } \ t \geq T,$$

and so

$$y(t) - y(T) \leq -K(t - T) \longrightarrow -\infty \quad \text{as } \ t \to \infty.$$

This contradicts the fact that $y(t)$ is bounded.

Claim 3. $x(t)$ tends to 0 as $t \to \infty$.

Suppose that the claim is false; that is,

$$\limsup_{t\to\infty} x(t) > 2\mu \quad \text{for some } \ \mu > 0.$$

Then, taking Claim 2 into account, we conclude that there exist two sequences $\{\tau_n\}$ and $\{\widetilde{\tau}_n\}$ with $\tau_1 > T, \tau_n < \widetilde{\tau}_n < \tau_{n+1}$ and $\tau_n \to \infty$ as $n \to \infty$ such that

$$x(\tau_n) = \mu, \quad x(\widetilde{\tau}_n) = 2\mu \quad \text{and} \quad \mu < x(t) < 2\mu \quad \text{for } \tau_n < t < \widetilde{\tau}_n.$$

Define the following constants.

$$L = \max\bigl\{|F(x)|\colon \mu \le x \le 2\mu\bigr\}, \quad M = \min\bigl\{g(x)\colon \mu \le x \le 2\mu\bigr\},$$

$$\hat{y} = \max\{\bar{y}, 0\} + 1 \quad \text{and} \quad N = \hat{y} + L + \frac{E}{\omega}.$$

Since $y(t)$ approaches $\bar{y}$ from above, we may assume that $y(t) < \hat{y}$ for $t \ge \tau_1$. Hence we have for $\tau_n \le t \le \widetilde{\tau}_n$

$$\dot{x}(t) = y(t) - F\bigl(x(t)\bigr) + \frac{E}{\omega}\sin\omega t < \hat{y} + L + \frac{E}{\omega} = N.$$

Integrating this inequality from τ_n to $\widetilde{\tau}_n$, we obtain

$$\widetilde{\tau}_n - \tau_n > \frac{1}{N}\bigl\{x(\widetilde{\tau}_n) - x(\tau_n)\bigr\} = \frac{\mu}{N},$$

and therefore,

$$y(\widetilde{\tau}_n) - y(\tau_n) = -\int_{\tau_n}^{\tilde{\tau}_n} g\bigl(x(s)\bigr)\, ds \le -M(\widetilde{\tau}_n - \tau_n) < -\frac{M\mu}{N}.$$

Hence we have

$$y(\widetilde{\tau}_n) - y(T) < \sum_{k=1}^{n}\bigl\{y(\widetilde{\tau}_k) - y(\tau_k)\bigr\} \longrightarrow -\infty \quad \text{as } n \to \infty.$$

This is also a contradiction to the boundedness of $y(t)$.

Since $F(x)$ is continuous and $F(0) = 0$, there exists a sequence $\{\delta_n\}$ tending to 0 such that

$$|y - F(x) - \bar{y}| < \frac{1}{n} \quad \text{for } (x, y) \in D_n, \tag{2.8}$$

where $D_n = \bigl\{(x, y)\colon x^2 + (y - \bar{y})^2 < \delta_n^2\bigr\}$. For reasons mentioned above, the solution $\bigl(x(t), y(t)\bigr)$ converges to the point $(0, \bar{y})$. Hence, there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\bigl(x(t_n), y(t_n)\bigr) \in \partial D_n, \quad \bigl(x(t), y(t)\bigr) \in D_n \quad \text{for } t > t_n. \tag{2.9}$$

There are two cases to consider.

Case (i): $\bar{y} > 0$. Let $m > 0$ be chosen so that $\bar{y} > \frac{1}{m}$. Then, by (2.8) and (2.9) we get

$$\dot{x}(t) = y(t) - F\big(x(t)\big) + \frac{E}{\omega}\sin\omega t > \bar{y} - \frac{1}{m} + \frac{E}{\omega}\sin\omega t$$

for $t > t_m$. Hence

$$x(t) - x(t_m) > \int_{t_m}^{t} \big(\bar{y} - \frac{1}{m} + \frac{E}{\omega}\sin\omega s\big)\, ds = \big(\bar{y} - \frac{1}{m}\big)(t - t_m) + \frac{E}{\omega^2}(\cos\omega t_m - \cos\omega t)$$

which tends to ∞ as $t \to \infty$. This is a contradiction to (2.9).

Case (ii): $\bar{y} \le 0$. Let $i > 0$ and $j > \frac{2\omega}{E}$ be any integers and define a sequence $\{I_{ij}\}$ of intervals by

$$I_{ij} = \Big\{ t \in [2(i-1)\pi/\omega, 2i\pi/\omega] : \tfrac{E}{\omega}\sin\omega t \le -\tfrac{1}{j} - \tfrac{E}{2\omega} \Big\}.$$

Since $\sin\omega t$ is a periodic function, the length of I_{ij} is independent of i, and so we can denote by d_j the length. By the definition of I_{ij}, we see that d_j approaches $\frac{2\pi}{3\omega}$ from below. Therefore, since $\delta_n \to 0$ as $n \to \infty$, there exists an $m > 0$ with

$$\frac{E}{2\omega} d_m > \delta_m. \tag{2.10}$$

Next, we choose an $l > 0$ such that $t_m < \frac{2(l-1)\pi}{\omega}$. Then it follows from (2.8) and (2.9) that

$$\dot{x}(t) = y(t) - F\big(x(t)\big) + \frac{E}{\omega}\sin\omega t < \bar{y} + \frac{1}{m} - \big(\frac{1}{m} + \frac{E}{2\omega}\big) \le -\frac{E}{2\omega}$$

for $t \in I_{lm}$. Hence, together with (2.10), we conclude that the decrease of $x(t)$ on I_{lm} is larger than δ_m which is the radius of D_m. This means that the solution $\big(x(t), y(t)\big)$ does not stay in D_m for $t \in I_{lm}$.

On the other hand, by (2.9) we have $\big(x(t), y(t)\big) \in D_m$ for $t > t_m$, which is a contradiction.

Thus, both cases (**i**) and (**ii**) are impossible. The proof is now complete.

Under the assumptions of Proposition 2.5, we cannot conclude that all bounded solutions of the unforced system (2.4) are oscillatory. Hence, if $E = 0$, then the statement of Theorem 2.1 is not true. In oscillation results for system (2.4), we need to assume some condition which guarantees that the origin is locally repulsive, such as $xF(x) < 0$ in a neighborhood of the origin (for details, see [2], [5], [8], [11]). The following theorem give us a material reason why there is no necessity for assuming such a condition in Theorem 2.1.

Theorem 2.2. *Suppose that* $F(0) = 0$. *Then system* (2.1) *has no solution* $(x(t), y(t))$ *converging to the origin as* $t \to \infty$.

Proof. Suppose that there exists a solution $\big(x(t), y(t)\big)$ of (2.1) such that

$$\big(x(t), y(t)\big) \longrightarrow (0,0) \quad \text{as} \;\; t \to \infty.$$

Then, taking $\bar{y} = 0$, we can use the same argument as in the proof of Theorem 2.1. Hence, we see the following in order of mention:

(**i**) there exists a sequence $\{\delta_n\}$ with $\delta_n \to 0$ as $n \to \infty$ and

$$|y - F(x)| < \frac{1}{n} \quad \text{on} \quad U_n \equiv \big\{(x,y)\colon x^2 + y^2 < \delta_n^2\big\};$$

(**ii**) there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$,

$$\big(x(t_n), y(t_n)\big) \in \partial U_n \quad \text{and} \quad \big(x(t), y(t)\big) \in U_n \quad \text{for} \;\; t > t_n;$$

(**iii**) there exist integers $m > \frac{2\omega}{E}$ and $l > 0$ with

$$\frac{E}{2\omega} d_m > 2\delta_m \quad \text{and} \quad t_m < \frac{2(l-1)\pi}{\omega}, \tag{2.11}$$

$$\dot{x}(t) < -\frac{E}{2\omega} \quad \text{for} \;\; t \in I_{lm}, \tag{2.12}$$

where d_m is the length of

$$I_{lm} = \big\{t \in [2(l-1)\pi/\omega, 2l\pi/\omega]\colon \tfrac{E}{\omega} \sin \omega t \le -\tfrac{1}{m} - \tfrac{E}{2\omega}\big\}.$$

From (2.11) and (2.12) we conclude that

$$\int_{I_{lm}} \dot{x}(t)\, dt < -\frac{E}{2\omega} d_m < -2\delta_m,$$

which means that the solution $\big(x(t), y(t)\big)$ does not stay in the neighborhood U_{lm} of the origin. This contradicts the fact (**ii**), and hence the proof is complete.

Remark 2.1. In Theorem 2.2, condition (2.2) is not necessary.

Consider again system (1.1) under the following conditions:

$$P(t) \;\; \text{is a periodic function}; \tag{2.13}$$

$$\int_0^\infty P(t)\, dt > -\infty; \tag{2.14}$$

$$\text{there exists a } t_1 \text{ such that } \; P(t_1) < 0. \tag{2.15}$$

Then we can modify Theorems 2.1 and 2.2 as follows:

Theorem 2.3. *Assume* (2.2), (2.13)–(2.15) *and* $F(0) = 0$. *Then all solutions of* (1.1) *are oscillatory if and only if system* (2.4) *has property* (X^+) *in the right and left half-plane.*

Theorem 2.4. *Assume* (2.13)–(2.15) *and suppose that* $F(0) = 0$. *Then system* (1.1) *has no solution* $\big(x(t), y(t)\big)$ *converging to the origin as* $t \to \infty$.

3. Rotation of solutions. As mentioned in Section 2, there is an essential difference between the oscillation for the forced system (2.1) and that for the unforced system (2.4). Under the assumptions that

- **(i)** $F(0) = 0$ and $xg(x) > 0$ if $x \neq 0$; and
- **(ii)** system (2.4) has property (X^+) in the right and left half-plane,

all solutions of (2.1) are oscillatory, but all solutions of (2.4) are not necessarily oscillatory. If, in addition to the assumptions above,

- **(iii)** the origin is locally repulsive for (2.4),

as $xF(x) < 0$ for $|x|$ sufficiently small, then all solutions of (2.4) are oscillatory; moreover, all trajectories of (2.4) keep on rotating around the origin clockwise.

It may seem that all trajectories of (2.1) rotate around the origin if the assumptions **(i)**–**(iii)** hold. However, this is not true (see Example 4.1). The question then arises: instead of **(iii)**, what kind of condition is necessary for all trajectories of (2.1) to go around the origin? In this section we give an answer to this question.

We here put the following definition.

Definition 3.1. A solution $\big(x(t), y(t)\big)$ of (2.1) is said to be *rotatory* if there exists a sequence $\{t_n\}$ tending monotonically to ∞ such that for each positive integer k

$$\begin{aligned}
x(t_n) &= 0 \quad \text{and} \quad y(t_n) > 0 \quad \text{if } n = 4k-3;\\
x(t_n) &> 0 \quad \text{and} \quad y(t_n) = 0 \quad \text{if } n = 4k-2;\\
x(t_n) &= 0 \quad \text{and} \quad y(t_n) < 0 \quad \text{if } n = 4k-1;\\
x(t_n) &< 0 \quad \text{and} \quad y(t_n) = 0 \quad \text{if } n = 4k.
\end{aligned}$$

Theorem 3.1. *Assume* (2.2) *and suppose that system* (2.4) *has property* (X^+) *in the right and left half-plane. If*

$$F(x) = \alpha x \quad \text{and} \quad g(x) = \beta x \quad \text{for } |x| \leq \frac{2\beta E}{\alpha\omega\widetilde{\omega}}, \tag{3.1}$$

where $\alpha > 0$, $\beta > 0$ *and* $\widetilde{\omega} = \min\{\beta, \omega\}$, *then all solutions of* (2.1) *are rotatory.*

To prove Theorem 3.1, we need Lemmas 3.1 and 3.2 below. We first construct a positive invariant set with respect to (2.1). Let

$$P_2 = \Big(\frac{2\beta E}{\alpha\omega\widetilde{\omega}}, \frac{\beta E}{\omega\widetilde{\omega}}\Big) \quad \text{and} \quad Q_2 = \Big(-\frac{2\beta E}{\alpha\omega\widetilde{\omega}}, -\frac{\beta E}{\omega\widetilde{\omega}}\Big),$$

and consider two trajectories of

$$\begin{aligned} \dot{x} &= y - \alpha x + \frac{\beta E}{\omega\widetilde{\omega}} \\ \dot{y} &= -\beta x \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \dot{x} &= y - \alpha x - \frac{\beta E}{\omega\widetilde{\omega}} \\ \dot{y} &= -\beta x \end{aligned} \tag{3.3}$$

passing through P_2 and Q_2, respectively. It is clear that

(**i**) the negative semitrajectory of (3.2) starting at P_2 meets the positive y-axis;
(**ii**) if $\alpha^2 \geq 4\beta$, then the positive semitrajectory of (3.2) starting at P_2 approaches $\big(0, -\frac{\beta E}{\omega\widetilde{\omega}}\big)$, and otherwise it meets the negative y-axis.

Let P_1 and P_3 be the intersecting points of the closure of the trajectory of (3.2) passing through P_2 with the positive y-axis and the negative y-axis, respectively. Similarly, we denote by Q_1 and Q_3 the intersecting points of the closure of the trajectory of (3.3) passing through Q_2 with the negative y-axis and the positive y-axis, respectively. Note that the points P_i and Q_i $(i = 1, 2, 3)$ are symmetric with respect to the origin.

Let Ω be the region which is enclosed by the arcs $P_1P_2P_3$ and $Q_1Q_2Q_3$, and the line segments P_1Q_3 and P_3Q_1. Then we have:

Lemma 3.1. *Assume* (3.1). *Then the region* Ω *is a positive invariant set of* (2.1).

Proof. Let $\big(x(t), y(t)\big)$ be the solution of (3.2) satisfying $\big(x(0), y(0)\big) = P_2$. Then there exist $t_1 < 0$ and $t_3 > 0$ with

$$\big(x(t_1), y(t_1)\big) = P_1 \quad \text{and} \quad \big(x(t_3), y(t_3)\big) = P_3.$$

We will show that

$$y(t_1) + y(t_3) > 0. \tag{3.4}$$

In case $\alpha^2 \geq 4\beta$, it is easy to see that (3.4) is satisfied. In fact,

$$y(t_1) > y(0) = \frac{\beta E}{\omega\widetilde{\omega}} \quad \text{and} \quad y(t_3) = -\frac{\beta E}{\omega\widetilde{\omega}}.$$

Suppose that $\alpha^2 < 4\beta$. Then the solution $\big(x(t), y(t)\big)$ is given by

$$x(t) = \frac{2\beta E}{\omega\widetilde{\omega}\sqrt{4\beta-\alpha^2}} \exp\big(-\frac{\alpha}{2}t\big)\Big\{\sin\frac{\sqrt{4\beta-\alpha^2}}{2}t + \frac{\sqrt{4\beta-\alpha^2}}{\alpha}\cos\frac{\sqrt{4\beta-\alpha^2}}{2}t\Big\}$$

and

$$y(t) = \frac{2\beta E}{\omega\widetilde{\omega}\sqrt{4\beta-\alpha^2}} \exp\big(-\frac{\alpha}{2}t\big)\Big\{\big(\alpha - \frac{2\beta}{\alpha}\big)\sin\frac{\sqrt{4\beta-\alpha^2}}{2}t + \sqrt{4\beta-\alpha^2}\cos\frac{\sqrt{4\beta-\alpha^2}}{2}t\Big\} - \frac{\beta E}{\omega\widetilde{\omega}}.$$

Since $x(t_1) = x(t_3) = 0$, we have

$$\sin \frac{\sqrt{4\beta - \alpha^2}}{2} t_i + \frac{\sqrt{4\beta - \alpha^2}}{\alpha} \cos \frac{\sqrt{4\beta - \alpha^2}}{2} t_i = 0 \qquad (i = 1 \text{ or } 3);$$

that is,

$$\sin\big(\frac{\sqrt{4\beta - \alpha^2}}{2} t_i + \theta\big) = 0 \qquad (i = 1 \text{ or } 3),$$

where $0 < \theta < \frac{\pi}{2}$, $\cos\theta = \sqrt{\alpha^2/4\beta}$ and $\sin\theta = \sqrt{1 - (\alpha^2/4\beta)}$. Hence we get

$$t_1 = -\frac{2\theta}{\sqrt{4\beta - \alpha^2}} \quad \text{and} \quad t_3 = \frac{2(\pi - \theta)}{\sqrt{4\beta - \alpha^2}}.$$

Taking notice that $\sin(-\theta) = -\sin\theta = -\sqrt{1 - (\alpha^2/4\beta)}$ and $\sin(\pi - \theta) = \sin\theta = \sqrt{1 - (\alpha^2/4\beta)}$, we obtain the following:

$$\begin{aligned} y(t_1) &= \frac{2\beta E}{\omega\widetilde{\omega}\sqrt{4\beta - \alpha^2}} \exp\big(-\frac{\alpha}{2} t_1\big)\big\{-\frac{2\beta}{\alpha} \sin \frac{\sqrt{4\beta - \alpha^2}}{2} t_1\big\} - \frac{\beta E}{\omega\widetilde{\omega}} \\ &= \frac{2\beta E}{\omega\widetilde{\omega}\sqrt{4\beta - \alpha^2}} \exp\big(\frac{\alpha\theta}{\sqrt{4\beta - \alpha^2}}\big) \left\{-\frac{2\beta}{\alpha}\sin(-\theta)\right\} - \frac{\beta E}{\omega\widetilde{\omega}} \\ &= \frac{2\beta\sqrt{\beta}E}{\alpha\omega\widetilde{\omega}} \exp\big(\frac{\alpha\theta}{\sqrt{4\beta - \alpha^2}}\big) - \frac{\beta E}{\omega\widetilde{\omega}}; \\ y(t_3) &= \frac{2\beta E}{\omega\widetilde{\omega}\sqrt{4\beta - \alpha^2}} \exp\big(-\frac{\alpha}{2} t_3\big)\big\{-\frac{2\beta}{\alpha} \sin \frac{\sqrt{4\beta - \alpha^2}}{2} t_3\big\} - \frac{\beta E}{\omega\widetilde{\omega}} \\ &= \frac{2\beta E}{\omega\widetilde{\omega}\sqrt{4\beta - \alpha^2}} \exp\big(-\frac{\alpha(\pi - \theta)}{\sqrt{4\beta - \alpha^2}}\big) \left\{-\frac{2\beta}{\alpha}\sin(\pi - \theta)\right\} - \frac{\beta E}{\omega\widetilde{\omega}} \\ &= -\frac{2\beta\sqrt{\beta}E}{\alpha\omega\widetilde{\omega}} \exp\big(\frac{\alpha(\theta - \pi)}{\sqrt{4\beta - \alpha^2}}\big) - \frac{\beta E}{\omega\widetilde{\omega}}. \end{aligned}$$

Since $0 < \theta < \frac{\pi}{2}$, we conclude that

$$\begin{aligned} y(t_1) + y(t_3) &= \frac{2\beta\sqrt{\beta}E}{\alpha\omega\widetilde{\omega}}\big\{\exp\big(\frac{\alpha\theta}{\sqrt{4\beta - \alpha^2}}\big) - \exp\big(\frac{\alpha(\theta - \pi)}{\sqrt{4\beta - \alpha^2}}\big) - \frac{\alpha}{\sqrt{\beta}}\big\} \\ &> \frac{2\beta\sqrt{\beta}E}{\alpha\omega\widetilde{\omega}}\big\{\exp\big(\frac{\alpha\theta}{\sqrt{4\beta - \alpha^2}}\big) - \exp\big(-\frac{\alpha\theta}{\sqrt{4\beta - \alpha^2}}\big) - \frac{\alpha}{\sqrt{\beta}}\big\}. \end{aligned}$$

For simplicity, let $\zeta = \frac{\alpha}{\sqrt{\beta}}$. Then $0 < \zeta < 2$ and

$$y(t_1) + y(t_3) > \frac{2\beta E}{\zeta\omega\widetilde{\omega}}\left\{\exp\big(\zeta h(\zeta)\big) - \exp\big(-\zeta h(\zeta)\big) - \zeta\right\},$$

where $h(\zeta) = \frac{1}{\sqrt{4-\zeta^2}}\mathrm{Tan}^{-1}\frac{\sqrt{4-\zeta^2}}{\zeta}$. Since

$$\frac{d}{d\zeta}h(\zeta) = \frac{1}{4-\zeta^2}\Big(\frac{\zeta}{\sqrt{4-\zeta^2}}\mathrm{Tan}^{-1}\frac{\sqrt{4-\zeta^2}}{\zeta} - 1\Big) < 0$$

and

$$h(\zeta) = \frac{1}{\zeta}\Big(\frac{\zeta}{\sqrt{4-\zeta^2}}\mathrm{Tan}^{-1}\frac{\sqrt{4-\zeta^2}}{\zeta}\Big) \longrightarrow \frac{1}{2} \quad \text{as } \zeta \to 2,$$

it follows that

$$h(\zeta) > \frac{1}{2} \quad \text{for } 0 < \zeta < 2.$$

Define $z = \zeta h(\zeta)$ and $k(z) = \exp(z) - \exp(-z) - 2z$ for $0 < z < 1$. Then we have

$$y(t_1) + y(t_3) > \frac{2\beta E}{\zeta\omega\tilde{\omega}}\big(\exp(z) - \exp(-z) - \zeta\big) > \frac{2\beta E}{\zeta\omega\tilde{\omega}}k(\zeta) > 0$$

because $\frac{d}{dz}k(z) > 0$ and $k(z) \to 0$ as $z \to 0$. Hence, (3.4) is satisfied.

Since the arcs $P_1P_2P_3$ and $Q_1Q_2Q_3$ are symmetric with respect to the origin, the inequality (3.4) means that P_1 and P_3 lie above Q_3 and Q_1, respectively.

Consider any positive semitrajectory of system (2.1) with (3.1) starting in Ω. Then, since $\frac{E}{\omega}\sin\omega t \le \frac{\beta E}{\omega\tilde{\omega}}$ for $t \in \mathbf{R}$, it traverses the line segments P_1Q_3 and P_3Q_1. Also, a simple comparison of the direction of the vector field shows that such a trajectory does not cross the arcs $P_1P_2P_3$ and $Q_1Q_2Q_3$. Thus, Ω is a positive invariant set with respect to (2.1). The lemma is proved.

Remark 3.1. The region Ω is contained in the strip $\big\{(x,y)\colon |x| \le \frac{2\beta E}{\alpha\omega\tilde{\omega}}\big\}$.

Remark 3.2. In case $\alpha^2 \ge 4\beta$, the positive semitrajectory of (3.2) starting at $R_1 = \big(0, \frac{\beta E}{\omega\tilde{\omega}}\big)$ approaches $R_2 = \big(0, -\frac{\beta E}{\omega\tilde{\omega}}\big)$ through the right half-plane and the positive semitrajectory of (3.3) starting at R_2 approaches R_1 through the left half-plane. Hence, the arcs R_1R_2 and R_2R_1 constitute another positive invariant set of (2.1). This set is included in Ω.

We next give a result on the asymptotic behavior of trajectories of (2.1).

Lemma 3.2. *Under the assumption* (3.1), *every positive semitrajectory of* (2.1) *starting at a point in* Ω *spirals toward an ellipse* $(\beta x)^2 + (\omega y)^2 = \frac{(\beta E)^2}{(\beta-\omega^2)^2+(\alpha\omega)^2}$ *in clockwise order.*

Proof. Consider a linear system

$$\begin{aligned} \dot{x} &= y - \alpha x - \frac{E}{\omega}\sin\omega t \\ \dot{y} &= -\beta x. \end{aligned} \tag{3.5}$$

Then it is easy to see that system (3.5) has a particular solution

$$x_p(t) = \sqrt{a^2+b^2}\cos(\delta - \omega t)$$
$$y_p(t) = \frac{\beta}{\omega}\sqrt{a^2+b^2}\sin(\delta - \omega t),$$

where

$$a = \frac{(\beta-\omega^2)E}{(\beta-\omega^2)^2+(\alpha\omega)^2}, \quad b = \frac{\alpha\omega E}{(\beta-\omega^2)^2+(\alpha\omega)^2},$$

$$\cos\delta = \frac{a}{\sqrt{a^2+b^2}} \quad \text{and} \quad \sin\delta = \frac{b}{\sqrt{a^2+b^2}}.$$

Note that the trajectory of $\big(x_p(t), y_p(t)\big)$ rotates in a clockwise direction on an ellipse $(\beta x)^2 + (\omega y)^2 = \frac{(\beta E)^2}{(\beta-\omega^2)^2+(\alpha\omega)^2}$.

The general solution $\big(\xi(t), \eta(t)\big)$ of (3.5) has the form

$$\xi(t) = x_h(t) + x_p(t) \quad \text{and} \quad \eta(t) = y_h(t) + y_p(t),$$

where $\big(x_h(t), y_h(t)\big)$ is the general solution of the corresponding homogeneous system

$$\dot{z} = Az; \qquad A = \begin{pmatrix} -\alpha & 1 \\ -\beta & 0 \end{pmatrix}, \qquad z = \begin{pmatrix} x \\ y \end{pmatrix}.$$

Since $\alpha > 0$ and $\beta > 0$, the eigenvalues of A have negative real parts. Hence,

$$\big(x_h(t), y_h(t)\big) \longrightarrow (0,0) \quad \text{as } t \to \infty,$$

and therefore, the phase portrait of (3.5) consists of clockwise spirals tending to the ellipse as t increases.

Let $\big(x(t), y(t)\big)$ be any solution of (2.1) leaving a point in Ω at $t = t_0$. Then it follows from Lemma 3.1 that the trajectory of $\big(x(t), y(t)\big)$ stays in Ω for all future time, and so we have

$$|x(t)| \le \frac{2\beta E}{\alpha\omega\widetilde{\omega}} \quad \text{for } t \ge t_0.$$

Hence, by (3.1) we may regard $\big(x(t), y(t)\big)$ as a solution of (3.5) satisfying

$$x(t) = x_h(t) + x_p(t) \quad \text{and} \quad y(t) = y_h(t) + y_p(t) \quad \text{for } t \ge t_0.$$

We therefore conclude that every trajectory of (2.1) starting in Ω spirals toward the elliptical orbit, in clockwise order, as $t \to \infty$. The proof is complete.

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\big(x(t), y(t)\big)$ be a solution of (2.1). If there exists a $t_1 > 0$ with $\big(x(t_1), y(t_1)\big) \in \Omega$, then by Lemmas 3.1 and 3.2 we see that $\big(x(t), y(t)\big)$

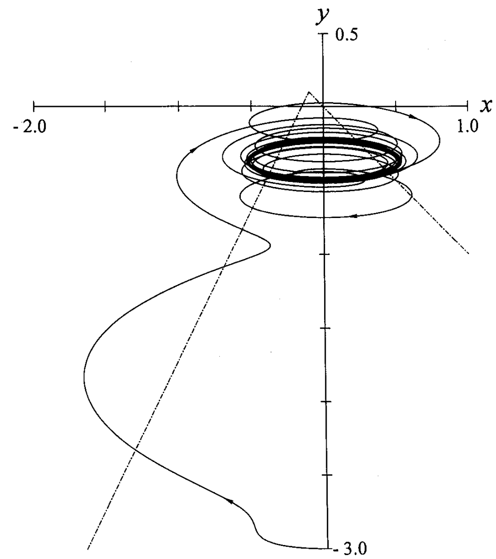

Figure 2. Trajectory of system (2.1) with $E = 8$, $\omega = 4$,

$$F(x) = \begin{cases} -x & (x \geq -0.1) \\ 2x + 0.3 & (x < -0.1), \end{cases} \qquad \text{and} \quad g(x) = x.$$

remains in Ω for $t \geq t_1$; and the trajectory of $\big(x(t), y(t)\big)$ spirals toward the ellipse $(\beta x)^2 + (\omega y)^2 = \frac{(\beta E)^2}{(\beta - \omega^2)^2 + (\alpha\omega)^2}$. Hence, $\big(x(t), y(t)\big)$ is rotatory.

Suppose that $\big(x(t), y(t)\big)$ does not go in Ω. Theorem 2.1 shows that $\big(x(t), y(t)\big)$ is oscillatory. Hence, taking account of the vector field of (2.1), we see that the trajectory of $\big(x(t), y(t)\big)$ keeps on going around Ω; that is, $\big(x(t), y(t)\big)$ is rotatory. This completes the proof.

4. Numerical examples and observations. In this section we give some global phase portraits for the forced Liénard system (2.1) to illustrate our results. To clarify the essence of Theorems 2.1 and 3.1, we restrict ourselves to the case that $F(x)$ is piecewise linear and $g(x) = x$. Then the unforced system (2.4) has property (X^+) in the right and left half-plane if and only if the gradient of $F(x)$ is greater than -2 for $|x|$ sufficiently large. In Figures 2–8 the broken lines represent the graph $y = F(x)$.

Example 4.1. Consider system (2.1) with $E = 8,\ \omega = 4$,

$$F(x) = \begin{cases} -x & (x \geq -0.1) \\ 2x + 0.3 & (x < -0.1) \end{cases} \qquad \text{and} \qquad g(x) = x.$$

Then system (2.4) has property (X^+) in the right and left half-plane, and therefore, by Theorem 2.1 all solutions of (2.1) are oscillatory.

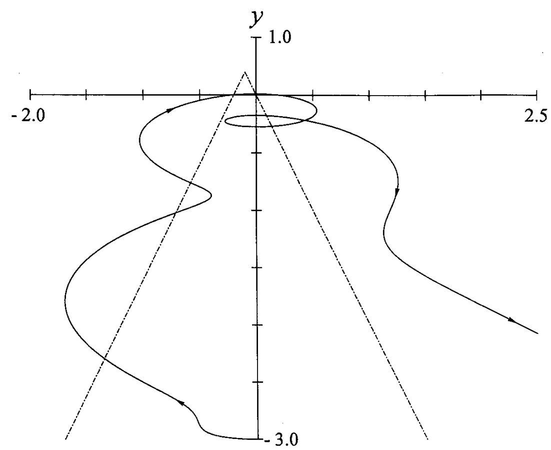

Figure 3. Trajectory of system (2.1) with $E = 8$, $\omega = 4$,

$$F(x) = \begin{cases} -2x & (x \geq -0.1) \\ 2x + 0.4 & (x < -0.1), \end{cases} \quad \text{and} \quad g(x) = x.$$

Figure 2 shows that the positive semitrajectory initiating at $(t_0, x_0, y_0) = (0, 0, -3)$ crosses the negative y-axis infinitely many times and coils around an oval in the lower half-plane. Hence the solution in Figure 2 is not rotatory though it is oscillatory.

Note that all solutions of (2.4) with $F(x)$ and $g(x)$ in Example 4.1 are rotatory because $xF(x) < 0$ for $|x|$ sufficiently small (the origin is locally repulsive for (2.4)).

Example 4.2. Consider system (2.1) with $E = 8,\ \omega = 4$,

$$F(x) = \begin{cases} -2x & (x \geq -0.1) \\ 2x + 0.4 & (x < -0.1) \end{cases} \quad \text{and} \quad g(x) = x.$$

Then system (2.4) fails to have property (X^+) in the right half-plane. Hence, by Theorem 2.1 there exists a nonoscillatory solution of (2.1).

From Figure 3 we see that the positive semitrajectory initiating at $(t_0, x_0, y_0) = (0, 0, -3)$ does not oscillate and goes to infinity through the right half-plane.

Example 4.3. Consider system (2.1) with $E = \omega = 4$,

$$F(x) = \begin{cases} -x + 3 & (x \geq 1) \\ 2x & (-1 \leq x < 1) \\ -x - 3 & (-3 \leq x < -1) \\ x + 3 & (x < -3) \end{cases} \quad \text{and} \quad g(x) = x.$$

Then the conditions of Theorem 3.1 hold, and so all solutions of (2.1) are rotatory. In fact, system (2.4) has property (X^+) in the right and left half-plane, and condition (3.1) is satisfied with $\alpha = 2$ and $\beta = \widetilde{\omega} = 1$.

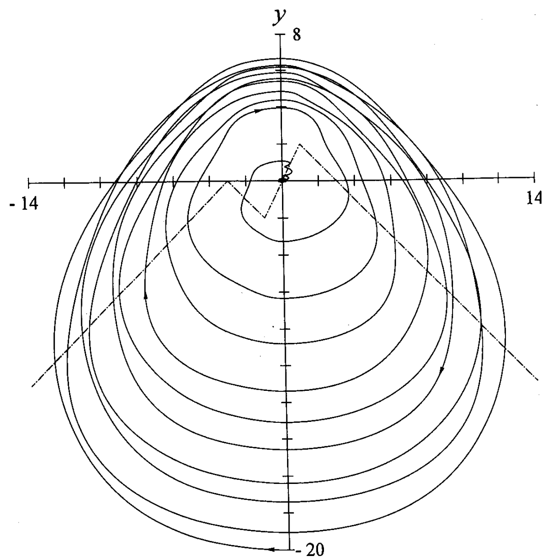

Figure 4. Trajectory of system (2.1) with $E = \omega = 4$,

$$F(x) = \begin{cases} -x+3 & (x \geq 1) \\ 2x & (-1 \leq x < 1) \\ -x-3 & (-3 \leq x < -1) \\ x+3 & (x < -3) \end{cases} \quad \text{and} \quad g(x) = x.$$

Let us turn our attention to the positive semitrajectory initiating at $(t_0, x_0, y_0) = (0, 0, -20)$. This trajectory spirals clockwise and then approaches the origin (see Figure 4). Ultimately, it winds around an ellipse surrounding the origin (see Figure 5).

In Theorem 3.1, condition (3.1) seems to be much too restricted. But, by three observations below, we can show that (3.1) is a reasonable condition for all solutions of (2.1) to be rotatory.

Observation 4.1. Consider system (2.1) with $E = \omega = 4$,

$$F(x) = \begin{cases} 2x & (x \geq -0.1) \\ -x-0.3 & (x < -0.1) \end{cases} \qquad \text{and} \qquad g(x) = x.$$

Then, as shown in Figure 6, the positive semitrajectory initiating at $(t_0, x_0, y_0) = (0, 0, -0.4)$ does not rotate around the origin. Since $\alpha = 2$ and $\beta = \widetilde{\omega} = 1$, we have

$$\frac{2\beta E}{\alpha\omega\widetilde{\omega}} = 1 > 0.1.$$

Hence, $F(x)$ does not satisfy condition (3.1).

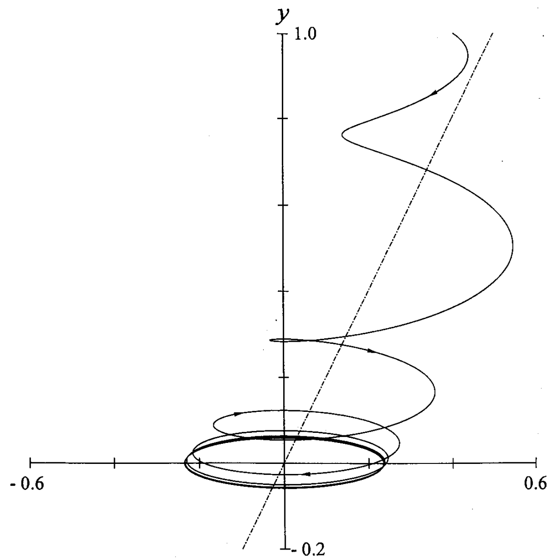

Figure 5. Trajectory of system (2.1) with $E = \omega = 4$,

$$F(x) = \begin{cases} -x+3 & (x \geq 1) \\ 2x & (-1 \leq x < 1) \\ -x-3 & (-3 \leq x < -1) \\ x+3 & (x < -3) \end{cases} \qquad \text{and} \quad g(x) = x.$$

Observation 4.2. Consider system (2.1) with $E = 10^5$, $\omega = 500$,

$$F(x) = \begin{cases} 2x & (x \geq 0) \\ \\ 1.99x & (x < 0) \end{cases} \qquad \text{and} \qquad g(x) = x.$$

Then $F(x)$ is nearly linear, but not linear. Of course, condition (3.1) is not satisfied. Figure 7 indicates that the solution passing through $(x_0, y_0) = (0, 1)$ at $t_0 = 0$ is not rotatory.

Observation 4.3. Consider system (2.1) with $E = 60$, $\omega = 10$,

$$F(x) = 3x \qquad \text{and} \qquad g(x) = \begin{cases} x + \frac{1}{2}x \sin \frac{10}{|x|} & (x \neq 0) \\ 0 & (x = 0). \end{cases}$$

Then $g(x)$ is nonlinear and so condition (3.1) is not satisfied. The positive semi-trajectory initiating at $(t_0, x_0, y_0) = (0, 1, 2)$ does not rotate around the origin (see Figure 8). From this observation, we conclude that even though

$$F(x) \text{ and } g(x) \text{ are odd functions,} \tag{4.1}$$

all solutions of (2.1) are not always rotatory.

Taking these observations into consideration, we finally present the following:

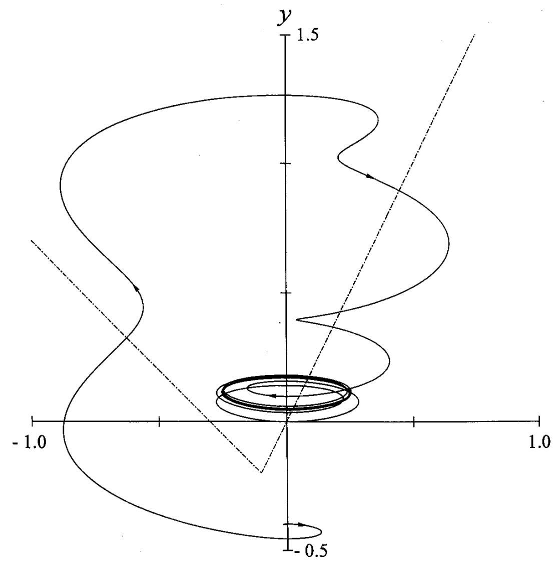

Figure 6. Trajectory of system (2.1) with $E = \omega = 4$,

$$F(x) = \begin{cases} 2x & (x \geq -0.1) \\ -x - 0.3 & (x < -0.1) \end{cases} \qquad \text{and} \quad g(x) = x.$$

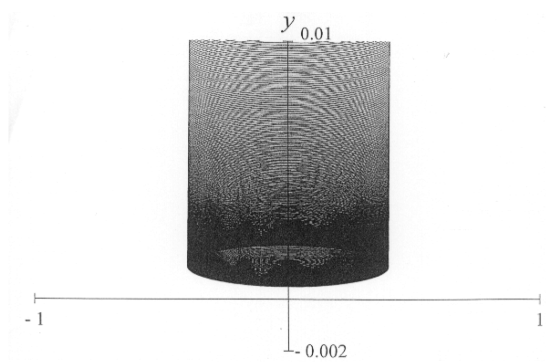

Figure 7. Trajectory of system (2.1) with $E = 10^5, \omega = 500$,

$$F(x) = \begin{cases} 2x & (x \geq 0) \\ 1.99x & (x < 0) \end{cases} \qquad \text{and} \quad g(x) = x.$$

Problem. *Suppose that system* (2.4) *has property* (X^+) *in the right and left half-plane. If, in addition to conditions* (2.2) *and* (4.1),

$$g(x) \text{ is monotone,}$$

then are all solutions of (2.1) *rotatory?*

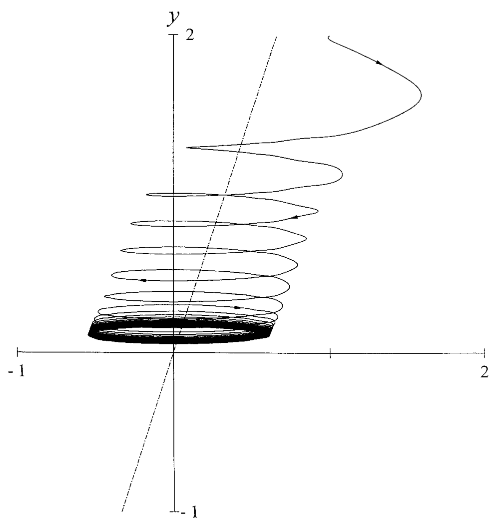

Figure 8. Trajectory of system (1.1) with $E = 60$, $\omega = 10$,

$$F(x) = 3x \quad \text{and} \quad g(x) = \begin{cases} x + \frac{1}{2}x\sin\frac{10}{|x|} & (x \neq 0) \\ 0 & (x = 0). \end{cases}$$

Many numerical examples suggest that the answer to this problem is affirmative. The problem is yet to be proved.

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