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ANNULUS ARGUMENTS IN THE STABILITY THEORY FOR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Dedicated to Professor Junji Kato on his sixtieth birthday

Abstract. An annulus argument is a method of proof which can detect that a curve in $\mathbb{R}^n$ crosses an annulus around the origin infinitely many times. In this paper we give annulus arguments not requiring the boundedness of the derivatives of the functions involved. Using these results we establish Lyapunov type theorems for the attractivity, asymptotic stability, and partial stability properties of the zero solution of nonautonomous functional differential equations whose right hand sides are not bounded with respect to the time. We apply these results to the scalar equation

$$x'(t) = -c(t)x(t) + b(t)x(t-h) \qquad (c(t) \geq 0),$$

the scalar equation with several delays

$$x'(t) = -c(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t-h_i) \qquad (c(t) \geq 0),$$

as well as to the system

$$x'(t) = B(t)x(t-h) - C(t)x(t),$$

where $B(t)$ and $C(t)$ are continuous matrix functions.

1. Introduction. An annulus argument is a method of proof for the existence of a limit which can detect that a trajectory $t \mapsto \psi(t)$, $\psi : \mathbb{R}_+ := [0, \infty) \to \mathbb{R}^n$ crosses the annulus $\varepsilon_1 \leq |x| \leq \varepsilon_2$ ($0 < \varepsilon_1 < \varepsilon_2$; $x \in \mathbb{R}^n$) infinitely many times. To recall its first abstract formulation from the monograph of N. Rouche, P. Habets, and M. Laloy, we need two notations. For a continuous function $a : \mathbb{R}_+ \to \mathbb{R}$ we denote by $D^+a(t)$ the Dini derivative

$$D^+a(t) := \limsup_{\tau \to 0+} \frac{a(t+\tau) - a(t)}{\tau}.$$

In the sequel we call a *wedge* any continuous strictly increasing function $W : \mathbb{R}_+ \to \mathbb{R}_+$ vanishing at zero.

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Lemma A [23, Basic Lemma 4.3, Ch. VIII]**.** *Let* $a, b : \mathbb{R}_+ \to \mathbb{R}$ *be two continuous functions satisfying the following conditions*:

(i) *a is nonincreasing and bounded from below;*
(ii) $b(t) \geq 0$ *for all* $t \in \mathbb{R}_+$;
(iii) *for every* $\varepsilon > 0$ *there are a wedge* W *and a constant* $A > 0$ *such that* $b(t) \geq \varepsilon$ *implies*

$$D^+a(t) \leq -W(b(t)),$$
$$D^+b(t) \leq A \quad (\textit{or } D^+b(t) \geq -A).$$

Then $\lim_{t\to\infty} b(t) = 0$.

The basic idea of the proof is that, by the second inequality in (iii), the point $b(t)$ must stay in any stripe $0 < \varepsilon_1 \leq x \leq \varepsilon_2$ longer than $A/(\varepsilon_2 - \varepsilon_1)$ while crossing it. Therefore, if $b(t)$ remained above ε_2 forever or crossed the stripe infinitely many times, then $a(t) \to -\infty$, which would be a contradiction.

This reasoning, which was called "annulus argument" (if $a(t) = |\psi(t)|$, $\psi : \mathbb{R}_+ \to \mathbb{R}^n$, then $0 < \varepsilon_1 \leq |x| \leq \varepsilon_2$, $x \in \mathbb{R}^n$ is an annulus) by T.A. Burton [2, Def. 6.1.5], appeared in the literature in 1940 in the well known Marachkov's theorem [22] on the asymptotic stability of the equilibria of ordinary differential equations (see [25, Sect. 14]). Here we cite its generalization to functional differential equations given by T. Yoshizawa (see also [8, Th. 5.2.1]).

Consider the functional differential equation

$$x'(t) = f(t, x_t), \tag{1.1}$$

where we use the following standard notations (see, e.g., [8]). For a fixed $h > 0$ we denote by C the space of the continuous functions $\varphi : [-h, 0] \to \mathbb{R}^n$ with the norm $\|\varphi\| := \max_{-h\leq s\leq 0} |\varphi(s)|$. If $x : [t_0 - h, t_0 + T) \to \mathbb{R}^n$ ($t_0 \in \mathbb{R}_+,\ T > 0$) is a continuous function and $t_0 \leq t < t_0 + T$, then $x_t \in C$ denotes the segment of x at t defined by $x_t(s) := x(t + s)$ for $s \in [-h, 0]$. Suppose that $f : \mathbb{R}_+ \times C \to \mathbb{R}^n$ is continuous and maps bounded sets into bounded sets. Moreover, we suppose $f(t, 0) \equiv 0$, i.e., (1.1) admits the zero solution. For $\varphi \in C,\ t_0 \in \mathbb{R}_+$ let $x(t; t_0, \varphi)$ denote the solution of (1.1) satisfying the initial condition $x_{t_0}(\cdot; t_0, \varphi) = \varphi$.

To establish stability properties we will use *Lyapunov functionals*, which are continuous functionals $V : \mathbb{R}_+ \times C \to \mathbb{R}_+$. The *derivative of* V *with respect to* (1.1) is defined by

$$V'(t, \varphi) := \limsup_{\tau\to 0+} \frac{1}{\tau}\big[V(t + \tau, x_{t+\tau}(\cdot; t, \varphi)) - V(t, \varphi)\big] = D^+V(t, x_t(\cdot)). \tag{1.2}$$

Theorem B [25, Thm. 33.3], [8, Thm. 5.2.1]**.** *Let* f *map* $\mathbb{R}\times$ (*bounded sets in* C) *into bounded sets in* $\mathbb{R}^n$. *Suppose that there are a Lyapunov functional* V *and wedges* W_1, W_2, W_3 *such that*

(i) $W_1(|\varphi(0)|) \leq V(t, \varphi) \leq W_2(\|\varphi\|)$;
(ii) $V'(t, \varphi) \leq W_3(|\varphi(0)|)$ *for all* $(t, \varphi) \in \mathbb{R}_+ \times C$.

Then the zero solution of (1.1) *is uniformly asymptotically stable.*

(For those not familiar with stability definitions, see Definition 4.1.) The proof is based upon the annulus argument. The boundedness condition on the right hand side f is supposed to guarantee $D^+\|x(t)\| \le A$ (see condition (iii) in Lemma A). As is known [12, 20, 21] this condition cannot be dropped from Theorem B. However, numerous papers have been devoted to the problem of weakening or substituting this condition (see [1–6, 9–21, 24–27] and the references therein). S. Busenberg and K. Cooke [6] considered the case of $f(t,\varphi) = -g(t,\varphi(0)) + h(t,\varphi)$ assuming $x^T Dg(t,x) \ge 0$ with some positive definite symmetric matrix D for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. They could drop the boundedness condition on the "ordinary part" g and replaced the boundedness of the "functional part" h with a weaker integral condition. Using their general results, they established sufficient conditions for the uniform asymptotic stability of the zero solution of the scalar equation

$$x'(t) = -c(t)x(t) + b(t)x(t-h) \qquad (c(t) \ge 0), \tag{1.3}$$

of the scalar equation with several delays

$$x'(t) = -c(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t-h_i) \qquad (c(t) \ge 0), \tag{1.4}$$

where $c, b_i : \mathbb{R}_+ \to \mathbb{R}$ are continuous, and h, h_i are positive constants $(i = 1, 2, \dots, n)$, as well as of the system

$$x'(t) = B(t)x(t-h) - C(t)x(t), \tag{1.5}$$

where $B(t)$ and $C(t)$ are continuous $n \times n$ matrix functions. They proved, among others, the following.

Theorem C [6, Thm. 3]**.** *Suppose that*

(i) $2ac(t) - b^2(t+h) - a^2 \ge q$ *for some* $a > 0$, $q > 0$;
(ii) $\int_t^{t+h} b^2(s)ds$ *is bounded, and given* $\eta > 0$ *there exists* $\tau > 0$ *such that*

$$\int_t^{t+\tau} |b(s)|\, ds < \eta \qquad \text{for } t \ge 0.$$

Then the zero solution of (1.3) *is uniformly asymptotically stable.*

The purpose of this paper is to weaken further the conditions of Theorem B. This weakening, especially those allowing real "nonautonomous" (i.e., explicitly depending on t) conditions, can result in loss of uniformity of asymptotic stability. The following is a simplified corollary to our results and is stated here to focus the paper.

Theorem 1.1. *Suppose that there are a Lyapunov functional V and wedges $W_1, \dots,$ W_5 satisfying the following conditions*:

(i) $W_1(|\varphi(0)|) \le V(t,\varphi)$;

(ii) *there is a locally integrable $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$V'(t,\varphi) \le -\eta(t) W_3(|\varphi(0)|);$$

(iii) *there is a locally integrable function $M : \mathbb{R}_+ \to \mathbb{R}$ such that*

$$V'(t,\varphi) \le -[\varphi^T(0) D f(t,\varphi)]_+ + M(t) W_4(\|\varphi\|)$$

with some symmetric positive definite matrix D, where $[a]_+ := \max\{a; 0\}$ denotes the positive part of real number a;

(iv)

$$\int_s^t \eta(r)\,dr \ge W_5\Big(\int_s^t M(r)\,dr\Big) \quad \textit{for all } s \le t.$$

Then the zero solution of (1.1) *is asymptotically stable.*

If, in addition,

(v) $V(t,\varphi) \le W_2(\|\varphi\|)$,

then the asymptotic stability is uniform.

Theorem 1.1 generalizes and improves Busenberg's and Cooke's result for (1.1). This can be illuminated by the fact that Theorem 1.1 makes it possible to drop condition (ii) from Theorem C (see Theorem 5.1). Moreover, we can handle also the case when D in (iii) is only positive semidefinite, which yields partial stability properties, i.e., stability properties with respect to a part of the components of vector x (see Definition 4.1). Very recently T.A. Burton and G. Makay [5], T.A. Burton [3], and K. Kobayashi [13] established stability criteria by annulus arguments; we go into details and discuss them later.

The paper is organized as follows. In Section 2 we prove lemmas, which are descendants of Lemma A and give the abstract formulation of our new annulus argument. Section 3 is devoted to Lyapunov type theorems generalizing Theorem B. In Section 4 we apply our results to equations (1.3)–(1.5), which are important model equations as well as test equations [15] in the stability theory for functional differential equations.

2. Lemmas. The first lemma is almost trivial; the proof is left to the reader.

Lemma 2.1. *Suppose that $a, b : \mathbb{R}_+ \to \mathbb{R}_+$ are locally absolutely continuous and a is nonincreasing. If for every $\varepsilon_1, \varepsilon_2$ $(0 < \varepsilon_1 < \varepsilon_2)$ there is a function $m \in L_1(\mathbb{R}_+; \mathbb{R}_+)$ such that $\varepsilon_1 \le b(t) \le \varepsilon_2$ implies either*

$$\begin{aligned} &a'(t) \le b'(t) + m(t) \quad \textit{for all } t \in \mathbb{R}_+, \qquad \textit{or} \\ &a'(t) \le -b'(t) + m(t) \ \textit{for all } t \in \mathbb{R}_+, \end{aligned}$$

then the finite or infinite $\lim_{t\to\infty} b(t)$ *exists.*

In the following lemma we will not suppose that m is integrable. For a locally integrable function $M : \mathbb{R}_+ \to \mathbb{R}_+$ and numbers $t \in \mathbb{R}_+$, $\varepsilon > 0$ we introduce the notations

$$\Gamma_M(t,\varepsilon) := \sup\Big\{\tau > 0 : \int_{t-\tau}^{t} M(s)\,ds \le \varepsilon\Big\},$$
$$\Delta_M(t,\varepsilon) := \sup\Big\{\tau > 0 : \int_{t}^{t+\tau} M(s)\,ds \le \varepsilon\Big\}.$$

Lemma 2.2. *Suppose that* $a_1, a_2, b : \mathbb{R}_+ \to \mathbb{R}_+$ *are locally absolutely continuous and* a_1, a_2 *are nonincreasing. Suppose, in addition, that for every* $\varepsilon_1, \varepsilon_2$ $(0 < \varepsilon_1 < \varepsilon_2)$ *there are locally integrable functions* $\alpha, M : \mathbb{R}_+ \to \mathbb{R}_+$ *and* $\delta > 0$ *such that*

$$\int_{t-\Gamma_M(t,(\varepsilon_2-\varepsilon_1)/2)}^{t} \alpha(s)\,ds \ge \delta \quad \textit{for all } t \in \mathbb{R}_+, \tag{2.1}$$

and the inequalities $t - \Gamma_M(t, \varepsilon_2 - \varepsilon_1) \le s \le t$, $\varepsilon_1 \le b(s) \le \varepsilon_2$ *imply*

$$a_1'(s) \le -\alpha(s), \qquad a_2'(s) \le -b'(s) + M(s). \tag{2.2}$$

Then the finite or infinite $\lim_{t\to\infty} b(t)$ *exists. If, in addition,* $\int_0^\infty \alpha = \infty$, *then either* $\lim_{t\to\infty} b(t) = 0$ *or* $\lim_{t\to\infty} b(t) = \infty$.

Proof. Suppose the contrary of the first statement, i.e.,

$$0 < \limsup_{t\to\infty} b(t) \ne \liminf_{t\to\infty} b(t) < \infty.$$

Then there are $\varepsilon_1, \varepsilon_2$ $(0 < \varepsilon_1 < \varepsilon_2)$ and sequences $\{r_i\}$, $\{s_i\}$ such that $r_i < s_i < r_{i+1}$; $b(r_i) = \varepsilon_1$, $b(s_i) = \varepsilon_2$, and $\varepsilon_1 \le b(t) \le \varepsilon_2$ for $t \in [r_i, s_i]$, $i = 1, 2, \dots$. If

$$s_i - r_i \le \Gamma_M(s_i, (\varepsilon_2 - \varepsilon_1)/2), \tag{2.3}$$

then by the second inequality in (2.2) we have

$$\begin{aligned} a_2(s_i) - a_2(r_i) &\le \int_{r_i}^{s_i} a_2'(u)\,du \le -\int_{r_i}^{s_i} b'(u)\,du + \int_{r_i}^{s_i} M(u)\,du \\ &\le -(\varepsilon_2 - \varepsilon_1) + \frac{\varepsilon_2 - \varepsilon_1}{2} = -\frac{\varepsilon_2 - \varepsilon_1}{2} < 0. \end{aligned}$$

If

$$s_i - r_i > \Gamma_M(s_i, (\varepsilon_2 - \varepsilon_1)/2), \tag{2.4}$$

then (2.1) and (2.2) imply

$$a_1(s_i) - a_1(r_i) \le -\int_{s_i-\Gamma_M(s_i,(\varepsilon_2-\varepsilon_1)/2)}^{s_i} \alpha(u)\,du \le -\delta.$$

At least one of (2.3)-(2.4) is satisfied for infinitely many i's, which means that $a_1(t) + a_2(t) \to -\infty$, and this is a contradiction. The second statement in the lemma is obvious.

Remark 2.3. The statement of Lemma 2.2 remains true if we replace (2.1)–(2.2) by

$$\int_t^{t+\Delta_M(t,(\varepsilon_2-\varepsilon_1)/2)} \alpha(s)\,ds \geq \delta \quad \text{for all } t \in \mathbb{R}_+, \tag{2.1$'$}$$

and the inequalities $t \leq s \leq t + \Delta_M(t, \varepsilon_2 - \varepsilon_1)$, $\varepsilon_1 \leq b(s) \leq \varepsilon_2$ imply

$$a_1'(s) \leq -\alpha(s), \qquad a_2'(s) \leq b'(s) + M(s). \tag{2.2$'$}$$

In the applications we will have to handle also the cases when (2.1) or (2.1$'$) is satisfied only along a sequence $\{t_i'\}$.

Lemma 2.4. *Suppose that* $a_1,\ a_2,\ b : \mathbb{R}_+ \to \mathbb{R}_+$ *are locally absolutely continuous, and there are sequences* $\{t_i'\}$, $\{t_i''\}$, $t_i' < t_i'' \leq t_{i+1}'$, $i = 1, 2, \ldots$ *such that the following conditions are satisfied*:

(i) a_1, a_2 *are nonincreasing on* $\mathbb{R}_+$;

(ii) *for every* $\varepsilon > 0$ *there are locally integrable functions* $\alpha, M : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $t \in \cup_{i=1}^{\infty}[t_i', t_i'']$, $b(t) \geq \varepsilon$ *imply*

$$a_1'(t) \leq -\alpha(t), \qquad a_2'(t) \leq -b'(t) + M(t); \tag{2.5}$$

(iii)

$$\sum_{i=1}^{\infty} \int_{\max\{t_i';\, t_i''-\Gamma_M(t_i'',\varepsilon)\}}^{t_i''} \alpha(t)\,dt = \infty.$$

Then $\liminf_{i\to\infty} b(t_i'') = 0$.

Proof. Suppose that the statement is not true. Then we can suppose without any loss of generality that there is an $\varepsilon > 0$ such that $b(t_i'') \geq 3\varepsilon$, $i = 1, 2, \ldots$. Then for any i either

(a) $b(t) \geq \varepsilon$ for $t \in [t_i', t_i'']$ or (b) there is a number $s_i \in [t_i', t_i'')$ with $b(s_i) = \varepsilon$ and $b(t) \geq \varepsilon$ for $t \in [s_i, t_i'']$.

In case (a) we have

$$a_1(t_i'') - a_1(t_i') \leq -\int_{t_i'}^{t_i''} \alpha(t)\,dt.$$

In case (b) either (b/1) $\Gamma_M(t_i'', \varepsilon) \geq t_i'' - s_i$ and

$$a_2(t_i'') - a_2(s_i) \leq -\int_{s_i}^{t_i''} (b'(t) + M(t))\,dt \leq -2\varepsilon + \varepsilon = -\varepsilon < 0,$$

or (b/2) $\Gamma_M(t_i'', \varepsilon) < t_i'' - s_i$ and we have

$$a_1(t_i'') - a_1(t_i'' - \Gamma_M(t_i'', \varepsilon)) \leq -\int_{t_i'' - \Gamma_M(t_i'', \varepsilon)}^{t_i''} \alpha(t)\, dt.$$

If case (b/1) occurs infinitely many times, then $\lim_{t\to\infty} a_2(t) = \infty$, which is a contradiction. Otherwise, there is a natural number I such that for any $i > I$ either (a) or (b/2) is satisfied. Then from condition (iii) we obtain

$$\sum_{i=I}^{\infty} \big(a_1(t_i'') - a_1(t_i')\big) \leq -\sum_{i=I}^{\infty} \int_{\max\{t_i';\, t_i'' - \Gamma_M(t_i'', \varepsilon)\}}^{t_i''} \alpha(t)\, dt = -\infty,$$

i.e., $\lim_{t\to\infty} a_1(t) = -\infty$, which is a contradiction again. The lemma is proved.

Lemma 2.5. *Suppose that conditions* (i)–(ii) *in Lemma* 2.4 *are satisfied, and for every* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that*

(iii)
$$\int_{\max\{t_i';\, t_i'' - \Gamma_M(t_i'', \varepsilon)\}}^{t_i''} \alpha(t)\, dt \geq \delta, \quad i = 1, 2, 3, \dots .$$

Then $\lim_{i\to\infty} b(t_i'') = 0$. *Furthermore, for every* $\varepsilon > 0$ *there is a natural number* N, *depending only on the magnitude of* $a_1(t_1')$, $a_2(t_1')$ *and* ε *such that* $b(t_1'') \geq 3\varepsilon$, $b(t_2'') \geq 3\varepsilon, \dots, b(t_N'') \geq 3\varepsilon$ *imply* $b(t_i'') < 3\varepsilon$ *for all* $i > N$.

Proof. If the first statement is not true then there are a subsequence $\{t_{i_k}'\}$ and $\varepsilon > 0$ such that $b(t_{i_k}') \geq \varepsilon,\ k = 1, 2, \dots$, and we get contradictions in the same way as in the proof of Lemma 2.4.

It can be seen from the estimates of the proof of Lemma 2.4 that if $b(t_i'') \geq 3\varepsilon$ for some i, then

$$a_1(t_i'') - a_1(t_i') + a_2(t_i'') - a_2(t_i') \leq -\min\{\varepsilon, \delta\} < 0,$$

hence the statement follows. The lemma is proved.

Remark 2.6. It can be seen from the proof that the second inequality of (ii) in Lemma 2.4 can be replaced by

$$a_2'(t) \leq b'(t) + M(t),$$

provided that, instead of (iii), we require

$$\sum_{i=1}^{\infty} \int_{t_i'}^{\min\{t_i'';\, t_i' + \Delta_M(t_i', \varepsilon)\}} \alpha(t)\, dt = \infty$$

and state $\liminf_{i\to\infty} b(t_i') = 0$.

Finally, we conclude this session by proving that Lemma A is a corollary of our Lemma 2.2. By assuming $D^+b(t) \le A$, all the conditions of Lemma 2.2 are satisfied if we choose $a_1(t) \equiv a_2(t) := a(t)$, $\alpha(t) := W(\varepsilon)$, $M(t) := A$. In fact, $D^+a(t) \le -D^+b(t) + A$ and we have $\Gamma_M(t,\varepsilon) \equiv \varepsilon/A$; therefore, condition (2.1) is obviously satisfied. By Lemma 2.2 we have either $\lim_{t\to\infty} b(t) = 0$ or $\lim_{t\to\infty} b(t) = \infty$. The second case is excluded by the conditions $D_+a(t) \le W(b(t))$ and $a(t) \ge 0$, which completes the proof of Lemma A.

If $D^+b(t) \ge -A$ is assumed, then we put $D^+a(t) \le D^+b(t) + A$ and take into account Remarks 2.3.

3. Lyapunov-type theorems for equation (1.1). Roughly speaking, the stability of the zero solution of (1.1) means that the graph $t \mapsto (t, x_t)$ of each solution x is close to the line $\varphi = 0$ in $\mathbb{R}_+ \times C$, provided that its initial point is close to this line. To establish this property one has to guarantee that $|x(t)|$, i.e., $|x_t(0)|$ is small for all t. For this reason the functional $|\varphi(0)|$ appears in the conditions of the theorems, e.g., of Theorem B, on the stability of the zero solution. In the stability theory and its application other invariant sets different from the line $\varphi = 0$ can play a role. In these cases, instead of $|\varphi(0)|$, a functional $F(t,\varphi)$ is involved in the theorems, which is called a "generalized" [25, p. 116] or "pseudo" [2, p. 297] Lyapunov functional.

Theorem 3.1. *Suppose that there are Lyapunov functionals* $V, F : \mathbb{R}_+ \times C \to \mathbb{R}_+$ *satisfying the following conditions*:

(i) $V'(t,\varphi) \le 0$ *for all* $(t,\varphi) \in \mathbb{R}_+ \times C$;

(ii) *for every* $\epsilon_1,\ \epsilon_2$ $(0 < \epsilon_1 < \epsilon_2)$ *there is a function* $m \in L_1(\mathbb{R}_+;\mathbb{R}_+)$ *such that* $\epsilon_1 \le F(t,\varphi) \le \epsilon_2$ *implies either*

$$V'(t,\varphi) \le F'(t,\varphi) + m(t) \quad \text{for all } t \in \mathbb{R}_+,$$

or

$$V'(t,\varphi) \le -F'(t,\varphi) + m(t) \quad \text{for all } t \in \mathbb{R}_+.$$

Then for every solution x *of* (1.1) *defined on* $[t_0 - h, \infty)$, *the finite or infinite* $\lim_{t\to\infty} F(t, x_t)$ *exists.*

Proof. For any solution x define $a(t) := V(t, x_t)$, $b(t) := F(t, x_t)$ and apply Lemma 2.1.

Theorem 3.2. *Suppose that there are Lyapunov functionals* $V_1, V_2, F : \mathbb{R}_+ \times C \to \mathbb{R}_+$ *satisfying the following conditions*:

(i) $V_2'(t,\varphi) \le 0$, *and there are a locally integrable* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ *and a wedge* W *such that*

$$V_1'(t,\varphi) \le -\eta(t) W(F(t,\varphi)) \quad \text{for all } t, \varphi;$$

(ii) *for every* ϵ_1, ϵ_2 $(0 < \epsilon_1 < \epsilon_2)$ *there is a locally integrable* $M : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $\epsilon_1 \le F(t,\varphi) \le \epsilon_2$ *implies*

$$V_2'(t,\varphi) \le -F'(t,\varphi) + M(t);$$

(iii) *for every* $\epsilon > 0$ *there is* $\delta > 0$ *such that*

$$\int_{t-\Gamma_M(t,\varepsilon)}^{t} \eta(s)\,ds \geq \delta \quad \text{for all } t \text{ large enough.}$$

Then for every solution x *of* (1.1) *defined on* $[t_0 - h, \infty)$ *the finite or infinite* $\lim_{t\to\infty} F(t, x_t)$ *exists. If, in addition,* $\int_0^\infty \eta = \infty$, *then for every solution* x *defined on* $[t_0 - h, \infty)$,

$$\lim_{t\to\infty} F(t, x_t) = 0. \tag{3.1}$$

Proof. For any solution x, define $a_1(t) := V_1(t, x_t), a_2(t) := V_2(t, x_t)$ and $b(t) := F(t, x_t)$. Then it is easy to see that all the conditions of Lemma 2.2 are satisfied. □

So far we did not suppose that the Lyapunov functionals V_1, V_2 admit any upper bound, or that they are positive definite in any sense. If we have such information then we can weaken condition (iii) in Theorem 3.2.

Theorem 3.3.. *Suppose that there are Lyapunov functionals* $V_1, V_2, V_3, F : \mathbb{R}_+ \times C \to \mathbb{R}_+$, *wedges* W_1, W_2, W_3, *and a sequence* $\{t_i\},\ t_{i+1} > t_i + h,\ i = 1, 2, \dots$ *which satisfy the following conditions*:

(i) *there is a locally integrable* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$V_1'(t,\varphi) \leq -\eta(t) W_1(F(t,\varphi)), \quad V_2'(t,\varphi) \leq 0, \quad V_3'(t,\varphi) \leq 0 \quad \text{for all } t, \varphi;$$

(ii) *for every* ϵ_1, ϵ_2 $(0 < \epsilon_1 < \epsilon_2)$ *there is a locally integrable* $M : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$t_i - h - \Gamma_M(t_i - h, \varepsilon_1) \leq t \leq t_i, \qquad \epsilon_1 \leq F(t,\varphi) \leq \epsilon_2$$

together imply

$$V_2'(t,\varphi) \leq -F'(t,\varphi) + M(t);$$

(iii) *for every* $\epsilon > 0$ *and for every sequence* $\{s_i\}$ $(s_i \in [t_i - h, t_i],\ i = 1, 2, \dots)$ *it is satisfied that*

$$\sum_{i=1}^{\infty} \int_{\max\{t_{i-1};\, s_i - \Gamma_M(s_i,\varepsilon)\}}^{s_i} \eta(t)\,dt = \infty;$$

(iv) $V_3(t, 0) \equiv 0$ *and* $W_2(F(t,\varphi)) \leq V_3(t,\varphi)$ *for all* t, φ;

(v) $V_3(t_i, y_{t_i}) \leq W_3\big(\sup_{t_i - h \leq s \leq t_i} F(s, y_s)\big)$ *for every continuous and bounded function* $y : \mathbb{R}_+ \to \mathbb{R}^n$, *and for all* $i = 1, 2, \dots$

Then for every solution x *of* (1.1) *starting from a sufficiently small neighborhood of the origin,* $\lim_{t\to\infty} F(t, x_t) = 0$.

Proof. For $t_0 \in \mathbb{R}_+$ and $\epsilon_2 > 0$ given, take $\varphi \in C$ such that $V_3(t_0, \varphi) < W_2(\epsilon_2)$. Then $F(t, x_t(\cdot; t_0, \varphi)) < \epsilon_2$ for all $t \geq t_0$; in other words, $x = 0$ is "stable with respect to F". We show that (3.1) holds.

Introduce the notations $b(t) := F(t, x_t(\cdot; t_0, \varphi))$, $\bar{b}_i := \sup_{t_i - h \le s \le t_i} b(t)$. By conditions (iv)-(v), if $\liminf_{i\to\infty} \bar{b}_i = 0$, then $\lim_{t\to\infty} V_3(t, x_t) = 0$, and (3.1) holds. Therefore, to complete the proof it is enough to show $\liminf_{i\to\infty} \bar{b}_i = 0$.

Suppose the contrary. Then, without loss of generality, we can assume $\bar{b}_i \ge \epsilon_0$ for all $i = 1, 2, \dots$ with some $\epsilon_0 > 0$. It means that for every i there is a $t_i'' \in [t_i - h, t_i]$ such that $b(t_i'') \ge \epsilon_0$. On the other hand, conditions (i)- (iii) make it possible to apply Lemma 2.4 to the functions $a_1(t) :\equiv V_1(t, x_t)$, $a_2(t) := V_2(t, x_t)$, $b(t) := F(t, x_t)$ by choosing $t_i' := t_{i-1}$, $\alpha(t) := -\eta(t) W_1(\epsilon)$. Then we obtain $\liminf_{i\to\infty} b(t_i'') = 0$, which contradicts the definition of t_i''. The proof is complete.

Remark 3.4. Theorem 3.2 and 3.3 have duals for the case when conditions (ii) contain the inequality

$$V_2'(t, \varphi) \le F'(t, \varphi) + M(t).$$

In this case only conditions (iii) have to be modified according to Remark 2.3.

4. Corollaries for D-stability. We formulate the corollaries of the theorems of the previous section for the case when $F(t, \varphi)$ is a positive semidefinite quadratic form of $\varphi(0)$.

Let D be an $n \times n$ symmetric positive semidefinite matrix and, for a column vector $x \in \mathbb{R}^n$, let $|x|_D := (x^T D x)^{1/2}$, where x^T denotes the transpose of x. For $\varphi \in C$ define

$$||\varphi||_D := \max_{-h \le s \le 0} \big(\varphi^T(s) D \varphi(s)\big)^{1/2} = \max_{-h \le s \le 0} |\varphi(s)|_D.$$

Definition 4.1. The zero solution of (1.1) is said to be

(a) *D-stable* if for every $\epsilon > 0$ and $t_0 \in \mathbb{R}_+$ there is a $\delta = \delta(\epsilon, t_0) > 0$ such that $||\varphi|| < \delta$ implies $|x(t; t_0, \varphi)|_D < \epsilon$ for all $t \ge t_0$;

(b) *uniformly D-stable* if it is D-stable with $\delta = \delta(\epsilon)$ independent of t_0;

(c) *asymptotically D-stable* if it is D-stable and for every $t_0 \in \mathbb{R}_+$ there is a $\sigma = \sigma(t_0) > 0$ such that $||\varphi|| < \sigma$ implies $\lim_{t\to\infty} |x(t; t_0, \varphi)|_D = 0$.

(d) *uniformly asymptotically D-stable* if it is uniformly D-stable, it is asymptotically D-stable with $\sigma > 0$ independent of t_0, and for every $\eta > 0$ there is a $T = T(\eta)$ such that $||\varphi|| < \sigma$ and $t \ge t_0 + T$ imply $|x(t; t_0, \varphi)|_D < \eta$.

If D is a projection, i.e., $Dx = D(x_1, x_2, \dots, x_n)^T = (x_1, x_2, \dots, x_m, 0, \dots, 0)^T$, $m \le n$, then the D-stability properties are known in the literature as stability properties with respect to a part of variables or *partial stability properties* [18]. If D is the unit matrix, i.e., $m = n$, then one obtains the standard stability properties [8, Ch. 5].

Theorem 4.2. *Suppose that there is a Lyapunov functional* $V : \mathbb{R}_+ \times C \to \mathbb{R}_+$ *which satisfies the following conditions for all* t, φ :

(i) V *is positive D-definite, i.e.,*

$$W_1(|\varphi(0)|_D) \le V(t, \varphi)$$

with some wedge W_1*;*

(ii) *there are a locally integrable* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ *with* $\int_0^\infty \eta = \infty$ *and a wedge* W_2 *such that*

$$V'(t,\varphi) \le -\eta(t) W_2\big(|\varphi(0)|_D\big);$$

(iii) *there are a locally integrable* $M : \mathbb{R}_+ \to \mathbb{R}_+$ *and a wedge* W_3 *such that*

$$V'(t,\varphi) \le -\varphi(0)^T D f(t,\varphi) + M(t) W_3(||\varphi||_D);$$

(iv) *either* $\int_0^\infty M < \infty$ *or*

$$\int_s^t \eta(r)\,dr \ge W_4\Big(\int_s^t M(r)\,dr\Big)$$

for large s,t $(t - \Gamma_M(t,1) \le s \le t)$ *with some wedge* W_4*.*

Then the zero solution of (1.1) *is asymptotically* D*-stable.*

If, in addition, there is a wedge W_5 *with* $V(t,\varphi) \le W_5(\|\varphi\|_D)$*, and*

$$\lim_{S\to\infty} \int_T^{T+S} \eta(t)\,dt = \infty$$

uniformly with respect to $T \ge 0$*, then the asymptotic stability is uniform.*

Proof. To show D-stability, for ϵ, t_0 choose $\delta(\epsilon,t_0) > 0$ so that $||\varphi|| < \delta(\epsilon,t_0)$ implies $V(t_0,\varphi) < W_1(\epsilon)$. For fixed t_0,φ with $||\varphi|| < \sigma(t_0) := \delta(1,t_0)$ we show

$$\lim_{t\to\infty} \big|x(t;t_0,\varphi)\big|_D = 0. \tag{4.1}$$

To this end we apply Theorems 3.1 and 3.2 putting $F(t,\varphi) = \varphi(0)^T D\varphi(0)/2 = |\varphi(0)|_D^2/2$. According to equation (1.1) we have $F'(t,\varphi) = \varphi(0)^T D f(t,\varphi)$. From condition (iii) we obtain

$$V'(t,\varphi) \le -F'(t,\varphi) + M(t) W_3(1). \tag{4.2}$$

If $\int_0^\infty M < \infty$, then Theorem 3.1 guarantees the existence of the limit involved in (4.1). If it differs from zero then condition (ii) implies $\lim_{t\to\infty} V(t,x_t) = -\infty$, a contradiction.

To the case $\int_0^\infty M = \infty$ we apply Theorem 3.2 with $V_1 := V_2 := V$. By condition (iv), for any $\epsilon > 0$ we have

$$\int_{t-\Gamma_{MW_3(1)}(t,\epsilon)}^t \eta(r)\,dr \ge W_4\Big(\frac{\epsilon}{W_3(1)}\Big),$$

for large t, which shows that conditions (ii)–(iii) in Theorem 3.2 are satisfied. The proof of the asymptotic stability is complete. The uniformity will be proved after Theorem 4.4. □

Condition (iv) in Theorem 4.2 is very restrictive: the integral of η has to dominate the integral of M over every interval $[s,t]$. If we have an upper bound for V at least along a sequence, then this condition can be considerably weakened. Theorem 3.3 yields

Theorem 4.3. *Suppose that conditions* (i)–(iii) *in Theorem* 4.2 *are satisfied. Suppose, in addition, that*

(iv) *there are a sequence* $\{t_i\}, t_{i+1} \geq t_i + h, i = 1, 2, \dots$ *and a wedge* W_5 *such that* $V(t_i, \varphi) \leq W_5(||\varphi||_D)$ *for* $i = 1, 2, \dots$;

(v) *for every* $\epsilon > 0$ *and* $s_i \in [t_i - h, t_i]$,

$$\sum_{i=1}^{\infty} \int_{\max\{t_{i-1};\, s_i - \Gamma_M(s_i,\epsilon)\}}^{s_i} \eta(r)\, dr = \infty.$$

Then the zero solution of (1.1) *is asymptotically* D*-stable.*

We conclude this section by results on uniform asymptotic D-stability.

Theorem 4.4. *Suppose that there are Lyapunov functionals* $V_1, V_2 : \mathbb{R}_+ \times C \to \mathbb{R}_+$ *and wedges* $W_1, W_2, \dots, W_7$ *satisfying the following conditions for all* $(t, \varphi) \in \mathbb{R}_+ \times C$:

(i) V_1, V_2 *are positive* D*-definite and* V_1 *admits an upper bound with* $||\varphi||$, *i.e.,*

$$W_1(|\varphi(0)|_D) \leq V_1(t, \varphi) \leq W_2(||\varphi||), \qquad W_3(|\varphi(0)|_D) \leq V_2(t, \varphi);$$

(ii) *there is a locally integrable* $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$V_1'(t, \varphi) \leq -\eta(t) W_4(|\varphi(0)|_D), \qquad V_2'(t, \varphi) \leq 0;$$

(iii) *there is a locally integrable* $M : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$V_2'(t, \varphi) \leq -\varphi(0)^T D f(t, \varphi) + M(t) W_5(||\varphi||_D);$$

(iv) *there are a sequence* $\{t_i\}$ *and constants* k, K *such that*

$$t_i + h < t_{i+1} \leq t_i + K, \quad \int_{t_i}^{t_{i+1}-h} \eta(t)\, dt \geq k, \quad V_1(t_i, \varphi) + V_2(t_i, \varphi) \leq W_6(||\varphi||_D)$$

for $i = 1, 2, \dots$ *and* $\varphi \in C$;

(v) *if* $t \in [t_i - h, t_i]$ *and* $\max\{t_{i-1};\, t - \Gamma_M(t, 1)\} \leq s \leq t$, *then*

$$\int_s^t \eta(r)\, dr \geq W_7\Big(\int_s^t M(r)\, dr\Big)$$

for all $i = 1, 2, \dots$

Then the zero solution of (1.1) *is uniformly asymptotically* D*-stable.*

Proof. For $\epsilon > 0$ given, define $\delta(\epsilon) := W_2^{-1}\big(W_1(\epsilon)\big)$. For any solution $x(t) = x(t; t_0, \varphi)$ with $||\varphi|| < \delta(\epsilon)$ we have

$$\begin{aligned} W_1\big(|x(t)|_D\big) &\leq V_1(t, x_t) \leq V_1(t_0, x_{t_0}) = V_1(t_0, \varphi) \\ &\leq W_2(\|\varphi\|) \leq W_2(\delta(\varepsilon)) = W_1(\epsilon), \end{aligned}$$

i.e., $|x(t)|_D < \epsilon$ for all $t \geq t_0$, which means that the zero solution is uniformly D-stable.

To complete the proof take $\eta > 0$ and define $\gamma(\eta) := W_6^{-1}\big(W_3(\eta)\big)$. It is enough to find a $T(\eta)$ such that for every t_0 and φ with $||\varphi|| \leq \sigma := \delta(1)$ there is an I such that $t_I - t_0 < T(\eta)$ and $||x_{t_I}(\cdot; t_0, \varphi)||_D < \gamma(\eta)$. In fact, then for $t \geq t_0 + T(\eta) > t_I$ we have

$$W_3(|x(t)|_D) \leq V_2(t, x_t) \leq V_2(t_I, x_{t_I}) \leq W_6(||x_{t_I}||_D) \leq W_6(\gamma(\eta)) = W_3(\eta),$$

i.e., $|x(t)|_D \leq \eta$ for all $t \geq t_0 + T(\eta)$, which is to be proved.

Now we prove that such a $T(\eta)$ exists. For any $t_0 \in \mathbb{R}_+$ there is a natural number i_* such that $t_0 + 2h \leq t_{i_*} \leq t_0 + 3K$. For every $i > i_*$ let t_i'' denote any point of $[t_i - h, t_i]$ for which $|x(t_i'')|_D = ||x_{t_i}||_D$ holds. Apply now Lemma 2.5 to the functions $a_1(t) := V_1(t, x_t)$, $a_2(t) := V_2(t, x_t)$, $b(t) := |x(t)|_D^2/2$ putting $t_i' := t_{i-1}$. Lemma 2.5 guarantees the existence of a natural number N such that $|x(t_i'')|_D \geq \gamma(\eta)$, $i = i_* + 1, i_* + 2, \ldots, i_* + N$ imply $|x(t_{i_*+N+1})|_D < \gamma(\eta)$. Moreover, N depends only on $\gamma(\eta)$ and the magnitude of $V_1(t_{i_*}, x_{t_{i_*}}), V_2(t_{i_*}, x_{t_{i_*}})$. Since $V_1(t_{i_*}, x_{t_{i_*}}) + V_2(t_{i_*}, x_{t_{i_*}}) \leq W_6(1)$, N depends only on η ($N = N(\eta)$). Now we can choose $T(\eta) := (N(\eta) + 4)K$, which completes the proof of Theorem 4.3.

Proof of the uniformity in Theorem 4.2. By the additional condition there is an S_1 such that $\int_T^{T+S_1} \eta \geq 1$ for every $T \geq 0$. Now define $t_i := i(S_1 + h)$, and apply Theorem 4.4 putting $V_1 \equiv V_2 \equiv V$.

Remark 4.5. Theorem 1.1 in the Introduction is a simple corollary of Theorem 4.2.

Finally, analyze the condition

$$\int_s^t \eta(r)\, dr \geq W\Big(\int_s^t M(r)\, dr\Big) \text{ for large } s \leq t \tag{4.3}$$

involved in Theorems 4.2 and 4.4. It is strongly connected with the well known property of integral positivity [4, 9, 24,26–27] of η. A measurable function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is called *integrally positive* if

$$\liminf_{t\to\infty} \int_t^{t+\delta} \eta(s)\, ds > 0$$

for every $\delta > 0$.

Lemma 4.6. *If $\int_0^t M(s)\, ds$ is uniformly continuous on $\mathbb{R}_+$, then the integral positivity of η suffices to show* (4.3).

Proof. If $\int_0^t M$ is uniformly continuous, then for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that $0 \leq t - s \leq \delta$ implies $\int_s^t M < \epsilon$. Suppose that η is integrally positive. Then there exists a $\gamma = \gamma(\delta) > 0$ with $\int_t^{t+\delta} \eta \geq \gamma(\delta)$ for t large enough. It is easy to see that (4.3) is satisfied by the choice $W(\epsilon) := \gamma(\delta(\epsilon))$.

In this way we get the following corollary of Theorem 4.4 for the standard uniform asymptotic stability.

Corollary 4.7. *Suppose that there are a Lyapunov functional $V : \mathbb{R}_+ \times C \to \mathbb{R}_+$ and wedges W_1, W_2, W_3, W_4 satisfying the following conditions for all $(t, \varphi) \in \mathbb{R}_+ \times C$:*

(i) *V is positive definite and admits an upper bound with $||\varphi||$, i.e.,*

$$W_1(|\varphi(0)|) \leq V(t, \varphi) \leq W_2(||\varphi||);$$

(ii) *there is a locally integrable $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$V'(t, \varphi) \leq -\eta(t) W_3(|\varphi(0)|);$$

(iii) *there is a locally integrable $M : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$V'(t, \varphi) \leq -\varphi(0)^T f(t, \varphi) + M(t) W_4(||\varphi||);$$

(iv) *there are a sequence $\{t_i\}$ and constants K, $\gamma > 0$ such that $t_i + h + \gamma \leq t_{i+1} \leq t_i + K$, $i = 1, 2, \ldots$, and $\int_0^t M(s)\, ds$ is uniformly continuous on the set $H := \cup_{i=1}^{\infty} [t_i - h - \gamma, t_i]$;*

(v) *for every $\delta > 0$,*

$$\liminf_{t\to\infty,\ [t,t+\delta]\subset H} \int_t^{t+\delta} \eta(s)\, ds > 0.$$

Then the zero solution of (1.1) *is uniformly asymptotically stable.*

Obviously, Theorem B is a special case of Corollary 4.7.

Remark 4.8. The conditions (iii) in Theorems 4.2-4.4 and Corollary 4.7 can be replaced by requiring the existence of a locally integrable $M : \mathbb{R}_+ \to \mathbb{R}_+$ and a wedge W such that

(iii′) $$\varphi(0)^T D f(t, \varphi) \leq M(t) W(\|\varphi\|_D) \quad \text{for all } t, \varphi \tag{4.4}$$

(in the case of Corollary 4.7, $D =$ unit matrix). In fact, since $V'(t, \varphi) \leq 0$ the new condition (iii′) implies conditions (iii) in all the above mentioned theorems and the corollary.

It is obvious that the converse is not true. However, it is not obvious that conditions (iii) can be met essentially easier in practice than (iii′). This fact will be illustrated by Theorem 5.1 (see Remark 5.2).

5. Applications.

5.1. Scalar equation with one delay. Consider the equation

$$x'(t) = -c(t)x(t) + b(t)x(t - h) \tag{5.1}$$

where $b, c : \mathbb{R}_+ \to \mathbb{R}$ are continuous, $c(t) \geq 0$ for $t \geq 0$, and the positive constant h denotes the time lag. This is an important model equation; it describes a process in which there is an instantaneous and a delayed feedback. We investigate the conditions of the asymptotic constancy of the solutions and the asymptotic stability of the zero solution.

For the case of constant coefficients

$$x'(t) = -c_0 x(t) + b_0 x(t-h) \qquad (b_0, c_0 \in \mathbb{R}), \tag{5.2}$$

it is known [8, p.136] that the subdomain $|b_0| < c_0$ of the stability region is the exact region of the asymptotic stability where the fact of the asymptotic stability is independent of the size of the delay h (see Figure 1). This means that if we associate the point (b_0, c_0) of the (b, c)-plain to equation (5.2), then the points of the domain $|b| < c$ corresponds to those equations having an asymptotically stable zero solution for all $h > 0$. If we want to construct an analogous theory for the nonautonomous equation (5.1), then it seems natural to associate the curve $t \mapsto (b(t), c(t))$ on the (b, c)-plain to equation (5.1). Then to give a criterion for the asymptotic stability for (5.1) is nothing else but to characterize the curves on the plain corresponding to those equations (5.1) having an asymptotically stable zero solution. The simplest way to characterize them is to locate them. The first such locations had to use bounded regions on the plain (b, c) (see, e.g., those of [8, pp. 136, 154] which can be seen on Figures 2, 3) because the first Lyapunov-type theorems for the asymptotic stability required boundedness with respect to t of the right hand side of (1.1) (see Theorem B). The locations of this type cannot be sharp; e.g., the curve $c(t) \equiv (t+1)^2$, $b(t) \equiv 0$ corresponds to the ordinary differential equation $x' = -(t+1)^2 x$ having an asymptotically stable zero solution; nevertheless, it cannot be located by a bounded domain. As was mentioned in the Introduction, Busenberg and Cooke [6] could drop the boundedness of c but b continued to be bounded at least "in mean". However, it is a natural conjecture that the zero solution can be asymptotically stable also for "very" unbounded b's provided that the ordinary part $-c(t)x(t)$, which drives the solution to zero, strongly dominates the delayed part $b(t)x(t-h)$ (e.g., $c(t) \equiv (t+1)^2,\ b(t) \equiv t$).

Our purpose in this section is to establish criteria for the asymptotic stability of the zero solution of (5.1) and its generalizations (1.4), (1.5) working also in the cases of unbounded coefficients of the delayed parts. It will be pointed out that nonuniform asymptotic stability can be guaranteed without any boundedness conditions on b and c. However, to establish *uniform* asymptotic stability we also need $\int_{t-h}^{t} |b(s)|\, ds$ (sometimes $\int_{t-h}^{t} b^2(s)\, ds$) bounded.

The crucial point of the stability theory for the nonautonomous equation (5.1) is to find the region of asymptotic stability corresponding to the domain $|b| < c$ of the autonomous case. One might think that, analogously to the point (b, c) of the autonomous case, the nonautonomous equation (5.1) could be represented by the curve $t \mapsto \big(b(t),\, c(t)\big)$ on the half-plane $\mathbb{R} \times \mathbb{R}_+$ (see Figures 2-3). Finer dominating conditions, descendants of $|b| < c$ of the autonomous case, show, however, that it is the curve $t \mapsto \big(b(t), b(t+h), c(t)\big)$ in the half space $\mathbb{R}^2 \times \mathbb{R}_+$ that is the relevant representation of (5.1) from the point of view of asymptotic stability. So the stability region must be found in the half-space. This can be illuminated by the following important result of Busenberg and Cooke.

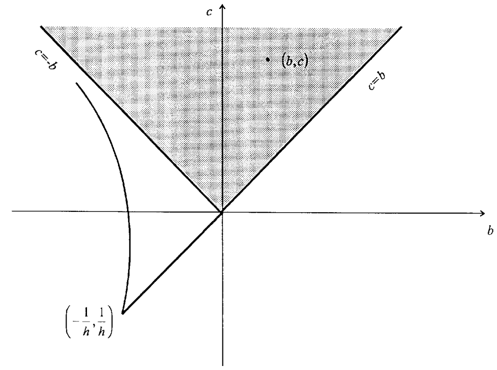

Figure 1

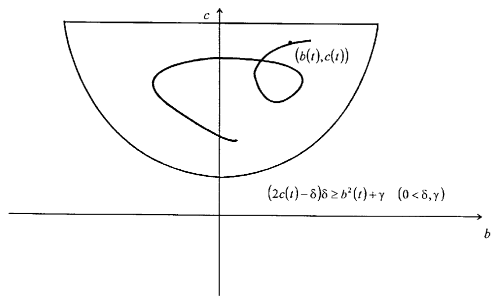

Figure 2

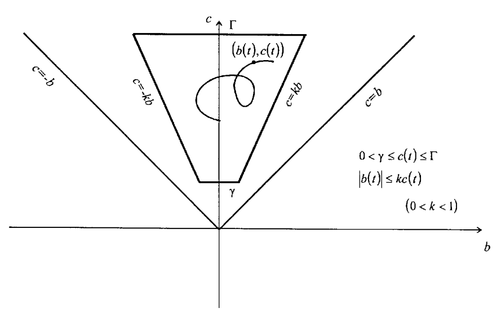

Figure 3

Theorem D [6, Theorem 2]**.** *Suppose that*

(i) *there exist* $a > 0$ *and* $q > 0$ *such that*

$$H_a(t) := 2c(t) - a|b(t)| - \frac{1}{a}|b(t+h)| \geq q \quad \text{for } t \geq 0; \tag{5.3}$$

(ii) *given* $\eta > 0$ *there exists* $\tau > 0$ *such that*

$$\int_t^{t+\tau} |b(s)|\, ds < \eta \quad \text{for } t \geq 0.$$

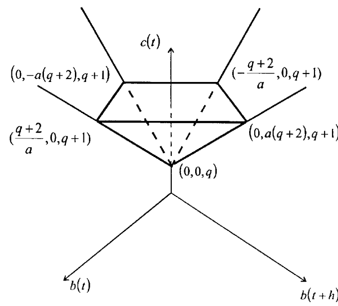

Figure 4

Then the zero solution of (5.1) *is uniformly asymptotically stable.*

Inequality (5.3) determines a polyhedral cone with 4 faces locating the curve $t \mapsto \big(b(t), b(t+h), c(t)\big)$ (see Figure 4).

The stability regions in Theorems C and D can be reached by the Lyapunov functional [6]

$$V(t,\varphi) := a\varphi^2(0) + \int_{-h}^{0} K(t+s)\varphi^2(s)\,ds \quad (t \in \mathbb{R}_+, \varphi \in C), \tag{5.4}$$

where $a > 0$ is a constant and $K : [-h,\infty) \to \mathbb{R}_+$ is a continuous function. Its derivative with respect to (5.1) reads

$$V'(t,\varphi) = [K(t) - 2ac(t)]\varphi^2(0) + 2ab(t)\varphi(0)\varphi(-h) - K(t-h)\varphi^2(-h). \tag{5.5}$$

Using the identity

$$a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{12}x_2^2 = \big(a_{11} - \frac{a_{12}^2}{a_{22}}\big)x_1^2 + a_{22}\big(x_2 + \frac{a_{12}}{a_{22}}x_1\big)^2 \tag{5.6}$$

we obtain the estimate

$$V'(t,\varphi) \le \big[K(t) - 2ac(t) + \frac{a^2b^2(t)}{K(t-h)}\big]\varphi^2(0). \tag{5.7}$$

Consider now the functional $F(\varphi) := \xi\varphi^2(0)$ $(0 < \xi = \text{const.},\ \varphi \in C)$, whose derivative with respect to (5.1) can be estimated

$$F'(t,\varphi) = \xi\big(-2c(t)\varphi^2(0) + 2b(t)\varphi(0)\varphi(-h)\big) \le 2\xi|b(t)||\varphi(0)||\varphi(-h)|. \tag{5.8}$$

Combining (5.5)–(5.7) with (5.8) we get

$$V'(t,\varphi) \le -F'(t,\varphi) + \big[K(t) - 2ac(t) + \frac{(a+\xi)^2b^2(t)}{K(t-h)}\big]\varphi^2(0). \tag{5.9}$$

Theorem 5.1. *If there are constants $a > 0$, $q > 0$ such that*

$$L_a(t) := 2ac(t) - b^2(t+h) - a^2 \geq q \quad \textit{for } t \geq 0,$$

then the zero solution of (1.1) *is asymptotically stable. If, in addition,* $\int_{t-h}^t b^2(s)\,ds$ *is bounded on* $\mathbb{R}_+$, *then the asymptotic stability is uniform.*

Proof. Consider the Lyapunov functional (5.4) with $K(t) := b^2(t+h)$. Then, by (5.9) we have the inequality

$$V'(t,\varphi) \leq -F'(t,\varphi) + \big[-L_a(t) + \xi(\xi+2a)\big]\varphi^2(0) \leq -F'(t,\varphi),$$

provided that ξ is chosen so that $\xi(\xi+2a) \leq q$. Since

$$V(t,\varphi) \leq \big(a + \int_{t-h}^t b^2(s)\,ds\big)||\varphi||,$$

we can apply Theorem 4.2 putting $\eta(t) \equiv q$, $M(t) \equiv 0$, and $D = \xi$. Theorem 5.1 is proved. □

Theorem 5.1 yields the same region of asymptotic stability for (5.1) as Theorem C but without any boundedness condition on b. Moreover, the asymptotic stability is uniform in this region under the only additional assumption that $\int_{t-h}^t b^2(s)\,ds$ is bounded; i.e., the uniform continuity of $\int_0^t |b(s)|\,ds$ can be dropped from Theorem C.

The following question arises after this theorem: Is an analogous result true for Theorem D? In other words: Can condition (5.3) alone imply asymptotic stability (nonuniform)? I do not know the answer to this question. What only follows from estimate (5.9) by Theorem 3.1 with $K(t) := |b(t+h)|$ is that condition $H_a(t) \geq q|b(t)|$ implies the asymptotic constancy of the solutions. The following theorem gives a little bit sharper result.

Theorem 5.3. (A) *Suppose that the following conditions are satisfied*:

(i) *there is a constant $a > 0$ such that*

$$H_a(t) := 2c(t) - a|b(t)| - \frac{1}{a}|b(t+h)| \geq 0 \quad \textit{for } t \geq 0;$$

(ii) *there is a wedge W such that*

$$\int_s^t H_a(r)\,dr \geq W\big(\int_s^t |b(r)|\,dr\big) \quad \textit{for } t \geq 0, t - \Gamma_{|b|}(1,t) \leq s \leq t.$$

Then the limit $x(\infty)$ *exists and is finite for every solution* x *of* (5.1).

(B) *If, in addition,* $\int_0^\infty H_a(t)\,dt = \infty$, *then the zero solution of* (5.1) *is asymptotically stable.*

(C) *If, in addition to conditions* (i), (ii), *the function* $t \to \int_{t-h}^t |b(s)|\,ds$ *is bounded on* $\mathbb{R}_+$, *and*

$$\lim_{S\to\infty}\int_T^{T+S}\big(2c(t)-\big(a+\frac{1}{a}\big)|b(t)|\big)\,dt=\infty \qquad \textit{uniformly with respect to } T\ge 0,$$

then the zero solution of (5.1) *is uniformly asymptotically stable.*

Proof. Consider the Lyapunov functional (5.4) with $K(t) := |b(t+h)|$. Then, by (5.7), we have $V'(t,\varphi) \le H_a(t)\varphi^2(0)$. Consequently, all the solutions are bounded on $\mathbb{R}_+$, and (5.9) yields

$$\begin{aligned} V'(t,\varphi) &\le -F'(t,\varphi)+\big[-aH_a(t)+\xi(\xi+2a)|b(t)|\big]^2\varphi(0) \\ &\le -F'(t,\varphi)+k_1|b(t)| \qquad (k_1=\text{const}). \end{aligned} \tag{5.10}$$

The assertions (A) and (B) follow from Theorems 3.2 and 4.2, respectively. Assertion (C) also needs Theorem 4.2 and the computation

$$\begin{aligned} \int_T^{T+S}\big(2c(t)-a|b(t)|-\frac{1}{a}|b(t+h)|\big)\,dt &= \int_T^{T+h}\big(2c(t)-a|b(t)|\big)\,dt \\ + \int_{T+h}^{T+S}\big(2c(t)-(a+\frac{1}{a})|b(s)|\big)\,ds &- \frac{1}{a}\int_{T+S}^{T+S+h}|b(u)|\,du. \end{aligned}$$

Since the last member is bounded uniformly with respect to $T, S \in \mathbb{R}_+$, Theorem 4.2 can be applied, which completes the proof.

It is easy to see that assertion (C) generalizes and improves Theorem D. However, the plain simple truth is that (ii) in Theorem 5.3 is not an easy to check condition. In the following theorem it will be pointed out that this condition can be dropped at the cost of strengthening condition (i).

In order to make the key condition of the theorem more plausible, let us start with the main idea of the proof. We want to get rid of the integral condition (ii) in Theorem 5.3, so we try to satisfy the conditions of Theorem 3.1 instead of Theorem 3.2. To this end, consider again the Lyapunov functional

$$V_a(t,\varphi) := a\varphi^2(0)+\int_{-h}^0 |b(t+h+s)|\varphi^2(s)\,ds \quad (0<a=\text{const}).$$

Obviously, for any $\xi>0$ we have $(V_{a+\xi}(t,\varphi))' \equiv (V_a(t,\varphi))' + (\xi\varphi^2(0))'$. Therefore, to have a pair $(V_a(t,\varphi))' \le 0$, $(V_a(t,\varphi))' \le -(\xi\varphi^2(0))'$ it is enough to find $0<a_1<a_2$ such that $(V_{a_1}(t,\varphi))' \le 0$, $(V_{a_2}(t,\varphi))' \le 0$ for all t,φ. On the other hand, by (5.5) we know that

$$V_a'(t,\varphi) = \big(|b(t+h)|-2ac(t)\big)\varphi^2(0)+2ab(t)\varphi(0)\varphi(-h)-|b(t)|\varphi^2(-h).$$

Consequently, for the existence of a_1, a_2 it is necessary and sufficient that the inequalities

$$\frac{-c(t)+\sqrt{c^2(t)-|b(t)||b(t+h)|}}{-2|b(t)|} \le a_1 < a_2 \le \frac{-c(t)-\sqrt{c^2(t)-|b(t)||b(t+h)|}}{-2|b(t)|}$$

hold for all t, φ. An easy computation shows that they hold if and only if

$$\begin{aligned} H_{a_1}(t) &\ge 0 \qquad \text{if} \qquad c(t) \ge \frac{a_1+a_2}{2}|b(t)|, \\ H_{a_2}(t) &\ge 0 \qquad \text{if} \qquad c(t) \le \frac{a_1+a_2}{2}|b(t)| \end{aligned} \tag{5.11}$$

for all t, φ.

Theorem 5.4. (A) *Suppose that there exist a_1, a_2 $(0 < a_1 < a_2)$ such that* (5.11) *is satisfied for all t, φ. Then the limit $x(\infty)$ exists and is finite for every solution x of* (5.1).

(B) *If, in addition, $\int_0^\infty H_{\frac{a_1+a_2}{2}}(t)\,dt = \infty$, then the zero solution of* (5.1) *is asymptotically stable.*

(C) *If, in addition to the condition in* (A), *the function $t \mapsto \int_{t-h}^t |b(s)|\,ds$ is bounded on $\mathbb{R}_+$, and*

$$\lim_{S\to\infty} \int_T^{T+S} \big(2c(t) - \big(\frac{a_1+a_2}{2} + \frac{2}{a_1+a_2}\big)|b(t)|\big)\,dt = \infty$$

uniformly with respect to $T \ge 0$, then the zero solution of (5.1) *is uniformly asymptotically stable.*

Proof. (A) As the computation before the theorem shows, (5.11) implies the pair $\big(V_{a_1}(t,\varphi)\big)' \le 0$, $\big(V_{a_1}(t,\varphi)\big)' \le -\big((a_2-a_1)\varphi^2(0)\big)'$, and Theorem 3.1 can be applied.

(B) and (C) can be deduced from Theorem 4.2 by putting $V := V_{\frac{a_1+a_2}{2}}$, $M(t) \equiv 0$, $D = (a_2 - a_1)$. The proof is complete. □

Condition (5.11) geometrically says that the curve $t \mapsto \big(b(t), b(t+h), c(t)\big)$ in the (x,y,z)-space is located by the domain

$$\begin{aligned} z &\ge \frac{a_1}{2}|x| + \frac{1}{2a_1}|y| \quad \text{if} \quad z \ge \frac{a_1+a_2}{2}x, \\ z &\ge \frac{a_2}{2}|x| + \frac{1}{2a_2}|y| \quad \text{if} \quad z \le \frac{a_1+a_2}{2}x, \end{aligned} \tag{5.12}$$

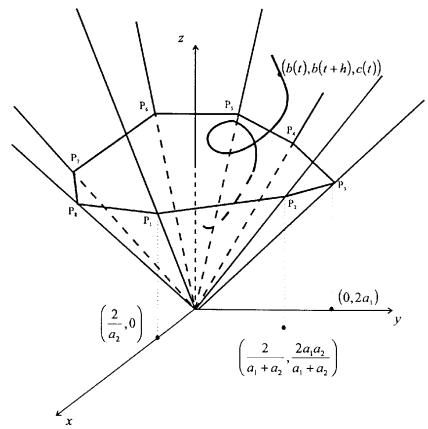

Figure 5

which is a polyhedral cone with 8 faces (see Figure 5).

As we already mentioned, we do not known whether condition (5.3) in Theorem D alone implies asymptotic stability. We do know, however, that condition (ii) in the same theorem can be required only on a sequence of intervals.

Theorem 5.5. *Suppose that condition* (i) *in Theorem* D *is satisfied, and*

(ii) *there are a sequence* $\{t_i\}_{i=1}^{\infty}$ *and a constant* $h_* > h$ *such that* $t_{i+1} \ge t_i + h_*$, $i = 1, 2, \dots,$ *and* $\int_0^t |b|$ *is uniformly continuous on the set* $U = \bigcup_{i=1}^{\infty} [t_i - h_*, t_i]$.

Then the zero solution of (5.1) *is asymptotically stable.*

If, in addition, $\int_{t-h}^{t} |b|$ *is bounded on* $\mathbb{R}_+$, *and there is a constant* C *such that* $t_{i+1} \le t_i + C, i = 1, 2, \dots,$ *then the asymptotic stability is uniform.*

Proof. We consider the Lyapunov functional (5.4) with $K(t) := |b(t+h)|$ again, whose derivative satisfies (5.10), and apply Theorem 4.3. By condition (ii), the sequence $\int_{t_i-h}^{t_i} |b|$, $i = 1, 2, \dots$ is bounded. On the other hand, given $\varepsilon > 0$, there exists a $\tau > 0$ such that $[t-\tau, t] \subset U$ implies $\int_{t-\tau}^{t} |b| < \varepsilon$. Therefore, $\Gamma_{k_1|b|}(t, \varepsilon) \ge \tau$, and for every $s_i \in [t_i - h, t_i]$ we have

$$\int_{\max\{t_{i-1};\, s_i - \Gamma_{k_1|b|}(s_i, \varepsilon)\}}^{s_i} aH_a(t)\, dt \ge aq \min\{\tau;\, h_* - h\} = \text{const.}, \quad i = 1, 2, \dots .$$

This means that all the conditions of Theorem 4.3 are satisfied. The uniformity is a consequence of Corollary 4.7, and this completes the proof.

Remark 5.6. Theorem 5.1 illuminates that condition (iii) in Theorem 4.2 is essentially better than (iii′) in Remark 4.8, i.e., (4.4). If we had applied Theorem 4.2 with condition (iii′) instead of (iii), then we would have had to use the estimate

$$|b(t)x(t-h)| \le |b(t)|\, \|x_t\| = M(t)\|x_t\| \quad (M(t) := |b(t)|)$$

and demand that $\int_s^t L_a(r)\,dr \geq W\big(\int_s^t |b(r)|\,dr\big)$ for $s \leq t$ with some wedge W, which is not satisfied automatically at all if $|b|$ is not bounded.

To conclude the subsection on equation (5.1) consider the Lyapunov functional

$$V_a(t,\varphi) := a|\varphi(0)| + \int_{-h}^{0} |b(t+s+h)||\varphi(s)|\,ds \quad (a>0), \tag{5.13}$$

recently used with $a=1$ for (5.1) by T. A. Burton and G. Makay [5]. The derivative of V_a with respect to (5.1) can be estimated

$$(V_a(t,\varphi))' \leq \big(ac(t) - |b(t+h)|\big)|\varphi(0)| - (1-a)|b(t)||\varphi(-h)|. \tag{5.14}$$

To illuminate their annulus argument, Burton and Makay proved

Theorem E [5]**.** *Suppose there are constants c_1, c_2, $c_3 > 0$ with*

(a) $c(t) - |b(t+h)| \geq c_1$*;*

(b) *there is a sequence $\{t_n\} \uparrow \infty$ and $K>0$ with $t_{n+1}-t_n \leq K$ and $\int_{t_n-h}^{t_n} |b(s+h)|\,ds \leq c_2$;*

(c) $c(t) + |b(t)| \leq c_3(t+1)\ln(t+2)$.

Then the zero solution of (5.1) *is asymptotically stable.*

T. Krisztin and the author [11] could replace (c) by $|b(t)| \leq c_3(t+1)\ln(t+2)$. They showed also that condition (c) cannot be dropped. Now we prove that a stronger form of condition (a) alone is sufficient.

Theorem 5.6. (A) *Suppose that there is a constant $\nu > 1$ such that*

$$c(t) - \nu|b(t+h)| \geq 0 \quad \text{for } t \geq 0. \tag{5.15}$$

Then the limit $x(\infty)$ exists and is finite for every solution x of (5.1).

(B) *If, in addition, $\int_0^\infty \big(c(t) - |b(t+h)|\big)\,dt = \infty$, then the zero solution of* (5.1) *is asymptotically stable.*

(C) *If, in addition, $\int_{t-h}^{t} |b|$ is bounded, and*

$$\lim_{S\to\infty} \int_T^{T+S} \big(c(t) - |b(t+h)|\big)\,dt = \infty \quad \text{uniformly with respect to } T \geq 0,$$

then the asymptotic stability is uniform.

Proof. Using definition (5.13), estimate (5.14), and condition (i) we obtain the pair

$$(V_{a=1}(t,\varphi))' \leq -\big(c(t) - |b(t+h)|\big)|\varphi(0)|,$$
$$(V_{a=1}(t,\varphi))' = (V_{a=1/\nu}(t,\varphi))' + ((1-\frac{1}{\nu})|\varphi(0)|)' \leq 0.$$

Now applying Theorem 3.1 to $V = V_{a=1/\nu}$, $F(t,\varphi) = (1-1/\nu)|\varphi(0)|$ we obtain (A). Assertion (B) follows from Theorem 3.2 with $V_1 := V_{a=1}$, $V_2 := V_{a=1/\nu}$, $M(t) \equiv 0$.

The proof of the uniformity in (C) is based on Lemma 2.5; it is very similar to that of Theorem 4.4, so it is omitted. □

If b is unbounded, then condition (5.15) may be essentially stronger than (a) in Theorem E. So it can be interesting that we can handle the case when (5.15) is satisfied only on a sequence of intervals.

Theorem 5.7. (A) *Suppose that*

(i) *there are a sequence* $\{t_i\}_{i=1}^{\infty}$ *and constants* $\gamma > 0$, $\nu > 0$ *such that* $t_{i+1} \geq t_i + h + \gamma$ *and*

$$c(t) - \nu|b(t+h)| \geq 0 \quad \text{for } t \in [t_i - h - \gamma, t_i],\ i = 1, 2, \dots;$$

(ii) $c(t) - |b(t+h)| \geq 0$ *for* $t \geq 0$*;*

(iii) *the sequence* $\left\{\int_{t_i-h}^{t_i} |b(t+h)|\,dt\right\}_{i=1}^{\infty}$ *is bounded;*

(iv) $\sum_{i=1}^{\infty} \int_{t_i-h-\gamma}^{t_i-h} \big(c(t) - |b(t+h)|\big)\,dt = \infty$.

Then the zero solution of (5.1) *is asymptotically stable.*

(B) *Suppose that, in addition to* (i)–(iii), *there are constants* $\delta > 0$, K *such that*

(iv′) $\int_{t_i-h-\gamma}^{t_i-h} \big(c(t) - |b(t+h)|\big)\,dt \geq \delta$ *for* $i = 1, 2, \dots;$

(v) $t_{i+1} \leq t_i + K$ *for* $i = 1, 2, \dots$.

Then the zero solution of (5.1) *is uniformly asymptotically stable.*

Proof. It can be done in the same spirit as the proof for Theorem 5.6 by using Theorem 3.3 and Lemma 2.5.

5.2. Scalar equation with several delays. Our technique can be applied also to the equations with several delays of the form

$$x'(t) = -c(t)x(t) + \sum_{i=1}^{n} b_i(t)x(t-h_i). \tag{5.16}$$

For a constant a, $0 < a < 1$, consider the Lyapunov functional

$$V_a(t,\varphi) := a\varphi^2(0) + \sum_{i=1}^{n} \int_{-h_i}^{0} |b_i(t+s+h_i)|\varphi^2(s)\,ds,$$

whose derivative with respect to (5.16) reads as follows:

$$\begin{aligned} V_a'(t,\varphi) =& \Big(-2ac(t) + \sum_{i=1}^{n} |b_i(t+h_i)|\Big)\varphi^2(0) \\ &+ 2a\sum_{i=1}^{n} b_i(t)\varphi(0)\varphi(-h_i) - \sum_{i=1}^{n} |b_i(t)|\varphi^2(-h_i). \end{aligned}$$

Using the inequality between the arithmetic and geometric means we obtain

$$\begin{aligned} V_a'(t,\varphi) \leq& -a\Big(\Big(2c(t) - \sum_{i=1}^{n} |b_i(t)| - \frac{1}{a}\sum_{i=1}^{n} |b_i(t+h_i)|\Big)\varphi^2(0) \\ &+ \sum_{i=1}^{n} (\frac{1}{a} - 1)|b_i(t)|\varphi^2(-h_i)\Big). \end{aligned} \tag{5.17}$$

Theorem 5.8. (A) *Suppose that there is a constant* $\nu > 1$ *such that*

$$c(t) \geq \sum_{i=1}^{n} \frac{|b_i(t)| + \nu |b_i(t+h_i)|}{2} \qquad \text{for } t \geq 0. \tag{5.18}$$

Then the limit $x(\infty)$ *exists and is finite for every solution* x *of* (5.16).

(B) *If, in addition,*

$$\int_0^\infty \big(\sum_{i=1}^{n} |b_i(t)|\big)\, dt = \infty \qquad \text{or} \qquad \int_0^\infty c(t)\, dt = \infty,$$

then the zero solution of (5.16) *is asymptotically stable.*

(C) *If, in addition to conditions in* (A), $\int_{t-h_i}^{t} |b_i(s)|\, ds$, $i = 1, 2, \dots n$ *are bounded for* $t \geq 0$, *and*

$$\lim_{S\to\infty} \int_T^{T+S} \big(c(t) - \sum_{i=1}^{n} |b_i(t)|\big)\, dt = \infty \qquad \text{uniformly with respect to } T \geq 0,$$

then the zero solution of (5.16) *is uniformly asymptotically stable.*

Proof. Consider the Lyapunov functionals $V_1 = V_{a=1}$ and $V_\nu = V_{a=\nu}$. Then, by 5.17, we have the pair

$$V_1'(t,\varphi) \leq -R(t)\varphi^2(0), \quad R(t) := 2c(t) - \sum_{i=1}^{n} \big(|b_i(t)| + |b_i(t+h_i)|\big) \geq 0,$$
$$V_1'(t,\varphi) = V_\nu'(t,\varphi) - ((\nu-1)\varphi^2(0))' \leq -F'(t,\varphi), \quad F(t,\varphi) := (\nu-1)\varphi^2(0).$$

To apply Theorem 4.1 it remains to estimate the integral of the coefficient R. It is obvious that $\int_0^\infty \big(\sum_{i=1}^\infty |b_i|\big) < \infty$ and $\int_0^\infty c = \infty$ imply $\int_0^\infty R = \infty$. On the other hand, by condition (5.18) we have

$$\begin{aligned}\int_0^S R(t)\, dt \geq & (\nu-1)\sum_{i=1}^{n}\int_0^S |b_i(t+h_i)|\, dt = (\nu-1)\int_0^S \big(\sum_i |b_i(t)|\big)\, dt \\ & - (\nu-1)\sum_{i=1}^{n}\int_0^{h_i} |b_i(t)|\, dt + (\nu-1)\sum_{i=1}^{n}\int_S^{S+h_i} |b_i(t)|\, dt;\end{aligned}$$

therefore, $\int_0^\infty \Big(\sum_{i=1}^\infty |b_i|\Big) = \infty$ implies $\int_0^\infty R = \infty$.

To deduce assertion (C) we need the identity

$$\begin{aligned}\int_T^{T+S} R(t)\, dt = & 2\int_T^{T+S} \big(c(t) - \sum_{i=1}^{n} |b_i(t)|\big)\, dt + \sum_{i=1}^{n}\int_T^{T+h_i} |b_i(t)|\, dt \\ & - \sum_{i=1}^{n}\int_{T+S}^{T+S+h_i} |b_i(t)|\, dt.\end{aligned}$$

An application of Theorem 4.2 completes the proof.

Corollary 5.9. *Suppose that b_i is periodic of period h_i, $i = 1, 2, \dots, n$. If there is a constant $\nu > 1$ such that*

$$c(t) - \nu \sum_{i=1}^{n} |b_i(t)| \geq 0 \quad \text{for } t \geq 0, \tag{5.19}$$

and $\int_0^\infty c(t)\,dt = \infty$, then the zero solution is asymptotically stable.

If, in addition,

$$\lim_{S\to\infty} \int_T^{T+S} c(t)\,dt = \infty \quad \text{uniformly with respect to } T \geq 0, \tag{5.20}$$

then the asymptotic stability is uniform.

Busenberg and Cooke [6, Corollary 3] also considered the periodic case. They could guarantee uniform asymptotic stability assuming

$$c(t) - \sum_{i=1}^{n} |b_i(t)| \geq q \quad \text{for } t \geq 0 \qquad (0 < q = \text{const}). \tag{5.21}$$

The function $\sum_{i=1}^{n} |b_i(t)|$ is bounded, so (5.21) implies (5.19); therefore, Busenberg's and Cooke's result is a consequence of Corollary 5.9. The converse is not true; for the trivial case $b_i(t) \equiv 0$ ($t \geq 0,\ i = 1, 2, \dots, n$) Corollary 5.9 gives the best possible condition (5.20) in contrast to (5.21) yielding $c(t) \geq q$. Besides, Corollary 5.9 can detect (nonuniform) asymptotic stability, too.

5.3. Systems. Consider the equation

$$x'(t) = -C(t)x(t) + B(t)x(t-h), \tag{5.22}$$

where $B, C : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ are continuous matrix functions, $x(t) \in \mathbb{R}^n$ is a column vector, and n is a natural number.

Theorem 5.9. *Let $D, E \in \mathbb{R}^{n\times n}$ be symmetric matrices, and suppose that $D \geq 0$; i.e., D is positive semidefinite.*

If there is a number $\beta \neq 0$ such that

$$C^T(t)D + DC(t) - B^T(t+h)B(t+h) - D^2 \geq 0,$$

$$C^T(t)(D+\beta E) + (D+\beta E)C(t) - B^T(t+h)B(t) - (D+\beta E)^2 \geq 0$$

for all $t \geq 0$ (the superscript ^{T} denotes the transpose of a matrix), then the finite or infinite $\lim_{t\to\infty} x^T(t)Ex(t)$ exists for every solution x of (5.22).

Proof. Consider the Lyapunov functional

$$V(t,\varphi) := \varphi^T(0)D\varphi(0) + \int_{-h}^{0} \varphi^T(s)B^T(t+s+h)B(t+s+h)\varphi(s)\,ds$$

defined for $t \geq 0,\ \varphi \in C\big([-h,0];\mathbb{R}^n\big)$, whose derivative with respect to (5.22) reads

$$\begin{aligned} V'(t,\varphi) = &- \varphi^T(0)\big(C^T(t)D + DC(t) - B^T(t+h)B(t+h)\big)\varphi(0) \\ &+ 2\varphi^T(0)DB(t)\varphi(-h) - \varphi^T(-h)B^T(t)B(t)\varphi(-h). \end{aligned}$$

Using the identity

$$2y^T DBz = -(Dy - Bz)^T(Dy - Bz) + y^T D^2 y + z^T B^T Bz \quad (y, z \in \mathbb{R}^n;\ D^T = D)$$

we obtain

$$\begin{aligned} V'(t,\varphi) = &- \varphi^T(0)\big(C^T(t)D + DC(t) - B^T(t+h)B(t+h) - D^2\big)\varphi(0) \\ &- \big(D\varphi(0) - B(t)\varphi(-h)\big)^T\big(D\varphi(0) - B(t)\varphi(-h)\big). \end{aligned} \tag{5.23}$$

By the conditions of the theorem we have the pair

$$V'(t,\varphi) \leq 0,\ V'(t,\varphi) \leq -\big(\beta\varphi^T(0)E\varphi(0)\big)'$$

and can apply Theorem 3.1 to complete the proof. □

The case $D = E$ is of special interest. The following theorem gives simple sufficient conditions for asymptotic D-stability.

Theorem 5.10. *Let $D \in \mathbb{R}^{n\times n}$ be a positive semidefinite symmetric matrix. Suppose that there is a number $\mu > 1$ such that*

$$C^T(t)D + DC(t) \geq B^T(t+h)B(t+h) + \mu D^2 \text{ for } t \geq 0. \tag{5.24}$$

Then the zero solution of (5.22) *is asymptotically D-stable.*

If, in addition, there is a wedge W such that

$$\int_{-h}^{0} \varphi^T(s)B^T(t+s)B(s)\varphi(s)\,ds \leq W\big(\max_{-h\leq s\leq 0} \varphi^T(s)D\varphi(s)\big) \quad \text{for } t \geq 0,$$

then the asymptotic D-stability is uniform.

Proof. By identity (5.23), condition (5.24) implies the pair $V'(t,\varphi) \leq -(\mu - 1)|D\varphi(0)|^2,\ V'(t,\varphi) \leq -\big((\mu-1)\varphi^T(0)D\varphi(0)\big)'$; therefore, the assertions follow from Theorem 4.2.

If D is positive definite, then Theorem 5.10 guarantees asymptotic stability without any boundedness condition on $B(t)$ and $C(t)$. Busenberg and Cooke [6, Theorem 7] deduced uniform asymptotic stability for this case from (5.24) and the extra condition that $\int_0^t \|B(s)\|\,ds$ is uniformly continuous on $\mathbb{R}_+$.

We conclude the paper with an application when the pseudo Lyapunov functional $F(\varphi)$ uses not only the single value $\varphi(0)$ but the whole function φ. It will handle the dual condition to (5.24) when the constant μ appears at the first member of the right hand side.

Theorem 5.11. *If there are a positive semidefinite matrix $D \in \mathbb{R}^{n\times n}$ and a constant $\mu > 1$ such that*

$$C^T(t)D + DC(t) \geq \mu B^T(t+h)B(t+h) + D^2 \quad \text{for } t \geq 0, \tag{5.25}$$

then for every solution x of (5.22) *the limit*

$$\lim_{t\to\infty} \int_{t-h}^{t} x^T(s)B^T(s+h)B(s+h)x(s)\,ds \tag{5.26}$$

exists and is finite.

Proof. We use the same technique as in Theorems 5.9 and 5.10. By identity (5.23), condition (5.25) implies the pair

$$V'(t,\varphi) \leq 0, \quad V'(t,\varphi) \leq -\Big((\mu-1)\int_{-h}^{0} \varphi^T(s)B^T(t+s+h)B(t+s+h)\varphi(s)\,ds\Big)'.$$

An application of Theorem 3.1 guarantees the existence of the finite or infinite limit (5.26). If it were equal to infinity, then $V(t,x_t) \to \infty$ as $t \to \infty$, which is a contradiction since $V(t,x_t)$ is nonincreasing. The proof is complete.

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