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POSITIVE SOLUTIONS TO A SYSTEM OF DIFFERENTIAL EQUATIONS MODELING A COMPETITIVE INTERACTIVE SYSTEM WITH NONLOGISTIC GROWTH RATES

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Abstract. In this paper we investigate the existence of positive solutions to a system of differential equations which model a competitive system. Instead of assuming the usual logistic growth model for each population we allow the growth rate to increase for positive populations. This leads to models which exhibit the Allee effect.

1. Introduction. In the study of interacting populations one usually starts with a model where the growth rates for each population is strictly decreasing. For example, in the Lotka-Volterra model for two competitive populations u, v one starts with the system

$$\frac{du}{dt} = u(a - bu - cv)$$
$$\frac{dv}{dt} = v(d - eu - fv),$$

where $a, b, c, d, e, f \geq 0$. In these equations the functions $f(u) = a - bu$ and $g(v) = d - fv$ represent the growth rates for the two species u and v. We can generalize these equations to include the possibility of spatial dependence and diffusion; see [3–17] and the references therein. Let $\Omega \subset \mathbb{R}^n$ be an open set. The Kolmogorov form of the Lotka-Volterra equations is

$$u_t - \Delta u = uM(u, v)$$
$$v_t - \Delta v = vN(u, v).$$

For these equations to model a logistic system we require M_u, $N_v < 0$. The elliptic system

$$-\Delta u = uM(u, v)$$
$$-\Delta v = vN(u, v)$$

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are assumed to be the steady-state system for the parabolic system.

In this paper we will investigate a one-dimensional elliptic system which models a competitive interaction between two populations where we will allow for the possibility of nonlogistic growth rates. In particular, we will study the following system.

$$\begin{aligned} -\frac{d^2u}{dx^2} &= uM(u,v) \text{ in } (-L,L) \\ -\frac{d^2v}{dx^2} &= vN(u,v) \text{ in } (-L,L) \end{aligned} \qquad u(\pm L) = v(\pm L) = 0.$$

In the above equations $M(u,0)$ and $N(0,v)$ are assumed to represent the corresponding growth rates for the two species u and v. For ease of notation let us set $f(x) := M(x,0)$, $g(x) := N(0,x)$. For the rest of the paper we will assume that f and g are continuous and one of the following three types.

Type I: (i) $f(0) > 0$. (ii) $f'(x) < 0$ for $x > 0$. (iii) There exists $c > 0$ such that $f(x) < 0$ for $x > c$.

Type II: (i) $f(0) > 0$. (ii) There exists $m > 0$ such that $f'(x) > 0$, $x < m$, $f'(x) < 0$, $x > m$. (iii) There exists $c > m$ such that $f(x) < 0$, $x > c$.

Type III: (i) $f(0) < 0$. (ii) There exist constants $0 < a < b < \infty$ such that $f(a) = f(b) = 0$. (iii) There exists a constant $a < m < b$ such that $f'(x) > 0$, $x < m$ and $f'(x) < 0$, $x > m$.

In the third section we will investigate the existence of positive solutions to the single equation

$$-\frac{d^2u}{dx^2} = uf(u), \quad \text{in } (-L,L) \quad u(\pm L) = 0. \tag{1.1}$$

Type I functions are those which are logistic in nature. Under certain conditions on L equation (1.1) will exhibit a stable positive solution. When f is either type II or type III, the equation may exhibit two positive solutions $0 < u_1 < u_2$, where u_2 is stable and u_1 is unstable. Models with these two types can exhibit the Allee effect. That is, if the population is not large enough it will tend to zero. For more information concerning models of this type see [17].

The methods that we use are motivated by those in [18] and [7]. In the fourth section we will use the results from the third section to establish some conditions where a pair of equations modeling a competitive system will have positive solutions.

2. Preparations. In this section we present some theorems concerning degree theory and fixed-point index for compact operators. These methods will be the primary technique used in exhibiting the existence of a positive solution to a system of elliptic partial differential equations.

We use the notation that was introduced by Dancer ([10]) and extended by Li ([14]). Let E be a real Banach space and $W \subset E$ a closed convex set. W is called a wedge provided $\alpha W \subset W$ for all $\alpha \geq 0$. A wedge W is said to be a cone if

$W \cap \{-W\} = \{0\}$. Let $y \in W$ and define

$$W_y := \{x \in E : y + \nu x \in W \text{ for some } \nu > 0\}, \quad S_y := \{x \in \overline{W}_y : -x \in \overline{W}_y\}.$$

Then $\overline{W}_y$ is an wedge and S_y is a closed subspace of E. Let T be a compact linear operator on E which leaves $\overline{W}_y$ invariant. We say that T has property α on $\overline{W}_y$ if there is a $t \in (0,1)$ and a $w \in \overline{W}_y \backslash S_y$ such that $(I - tT)w \in S_y$. Let A be a compact map with fixed point y in a wedge W. Let $L = A'(y)$ be the Fréchet derivative of A at y. Then L is invariant on $\overline{W}_y$. We now state the main theorem concerning the fixed-point index of A at y.

Theorem 2.1. *Assume that $I - L$ is invertible on E.*

(i) *If L has property α in $\overline{W}_y$, then index$(A, y) = 0$.*
(ii) *If L does not have property α on $\overline{W}_y$ then index$(A, y) =$ index$_E(L, 0) = \pm 1$.*

In many cases where $I - L$ is not invertible on E but is on $\overline{W}_y \backslash \{0\}$, the following results hold:

(iii) *If $I - L : \overline{W}_y \to \overline{W}_y$ is not surjective then index$_W(A, y) = 0$.*
(iv) *If L does not have property α on $\overline{W}_y$, then index$_W(A, y) = \pm 1$.*

The first two results are due to Dancer ([10]) and the last two are due to Li ([14]). For more properties of degree theory and the fixed-point index we refer the reader to Amann ([1]). In what follows, we will usually drop the subscripts concerning the space that we are working in. Usually it will be taken to be W.

Let E be a real Banach space with a cone W. Then we can form an ordered Banach space by $u, v \in E$, $u > v$ if and only if $u - v \in W$. Let T be a compact positive linear operator on an ordered Banach space E. We use the notation $r(T)$ to denote the spectral radius of T.

We have the following lemma.

Lemma 2.2 ([14])**.** *Let T be a compact positive linear operator on an ordered Banach space. Let $u > 0$ be a positive element. We have the following conclusions.*

(i) *If $Tu > u$, then $r(T) > 1$.*
(ii) *If $Tu < u$, then $r(T) < 1$.*
(iii) *If $Tu = u$, then $r(T) = 1$.*

Throughout this paper λ_1 will denote the principal eigenvalue for the equation

$$-\frac{d^2}{dx^2}u = \lambda u \quad \text{in } (-L, L), \qquad u(-L) = u(L) = 0.$$

We let $\lambda_1(\frac{d^2}{dx^2} + v)$ denote the principal eigenvalue of the operator $\frac{d^2}{dx^2} + vI$ with homogeneous Dirichlet boundary conditions. Let $v \in L^\infty((-L, L))$ and $P > max\{|v(x)| : x \in (-L, L)\}$. Then

$$Tu := (-\frac{d^2}{dx^2} + P)^{-1}(v + P)u$$

is a compact positive operator on $C_0((-L, L))$, with the usual cone.

Lemma 2.3.

(i) *If* $\lambda_1(\frac{d^2}{dx^2}+v) > 0$, *then* $r(T) > 1$.

(ii) *If* $\lambda_1(\frac{d^2}{dx^2}+v) < 0$, *then* $r(T) < 1$.

Proof. We prove the first part. The proof of part (ii) is similar. (i) Let $\mu := \lambda_1(\frac{d^2}{dx^2}+v)$ and ϕ be the corresponding positive eigenfunction. Then we have

$$\begin{aligned} &\frac{d^2}{dx^2}\phi + v\phi = \mu\phi > 0, \\ &-\frac{d^2}{dx^2}\phi + P\phi < v\phi + P\phi = (v+P)\phi, \\ &\phi < (-\frac{d^2}{dx^2}+P)^{-1}(v+P)\phi. \end{aligned}$$

Thus $T\phi > \phi$. Hence, by Lemma 2.2, $r(T) > 1$.

Lemma 2.4 [1]**.** *Let* $f : \bar{W}_\rho := cl\{u \in W : ||u|| < \rho\} \to W$ *be a compact map such that* $f(0) = 0$. *Suppose that* f *has a right derivative* $f'_+(0)$ *at zero such that 1 is not an eigenvalue of* $f'_+(0)$ *corresponding to a positive eigenfunction. Then there exists a constant* $\sigma_0 \in (0, \rho]$ *such that for every* $\sigma \in (0, \sigma_0]$, *index*$(f, W_\rho) = 0$ *if* $f'_+(0)$ *has a positive eigenfuction corresponding to an eigenvalue greater than one.*

3. A single equation. In this section we will investigate the existence of positive solutions to the equations

$$-\frac{d^2u}{dx^2} = uf(u) \quad \text{in } (-L, L), \quad u(\pm L) = 0, \tag{3.1}$$

where $f(x)$ is one of the three types described in the introduction.

In what follows we will consider L to be a parameter and will study the existence of positive solutions to equation (3.1) depending on the size of L.

We start with f being of type I. Let $L_0 = \frac{\pi}{2\sqrt{f(0)}}$. We have the well-known theorem due to Berestycki and Lions, [2].

Theorem 3.1. *For* $L \le L_0$, $u_0 \equiv 0$ *is the only nonnegative solution to* (3.1). *For* $L > L_0$ *there is a unique positive solution to* (3.1).

Note: For a fixed L we have $\lambda_1 = (\frac{\pi}{2L})^2$. If $L > L_0$, then it follows that $f(0) > \lambda_1$, and if $L < L_0$ then $f(0) < \lambda_1$.

To investigate the existence of solutions when f is of type II or III, we proceed as in [19, 18]. First, rewrite equation (3.1) as a system of first order equations:

$$\begin{aligned} u' &= v \\ v' &= -uf(u), \end{aligned} \tag{3.2}$$

where $' = \frac{d}{dx}$. Solutions for (3.1) can be thought of as integral curves for system (3.2) which start and end on the line $u = 0$, and take $2L$ units of time to complete the journey. Let F be any antiderivative of $xf(x)$ and set $A^2 = 2F(b)$. For $0 < p < A$, let $\alpha(p)$ denote the point where the solution (u, v) of system (3.2), starting at $(0, p)$, crosses the line $v = 0$. If we let $T(p)$ denote the length of time that a solution (u, v) takes to travel from $(0, p)$ to $\alpha(p)$ then solutions of (3.1) will satisfy $L = T(p)$.

System (3.2) is a Hamiltonian system with $H(u, v) := \frac{1}{2}v^2 + F(u)$. Let $0 < p < A$ and (u, v) be the solution passing through $(0, p)$. Then, $H(0, p) = H(\alpha(p), 0) = F(\alpha(p))$. Thus, $v^2 = 2(F(\alpha(p)) - F(u))$. Using (3.2) (and t as the independent variable), we obtain

$$\frac{du}{\sqrt{2}\sqrt{F(\alpha(p)) - F(u)}} = dt. \tag{3.3}$$

Integrating over the interval $[0, T(p)]$ we obtain

$$T(p) = \frac{1}{\sqrt{2}} \int_0^{\alpha(p)} \frac{1}{\sqrt{F(\alpha(p)) - F(u)}} \, du, \quad 0 < p < A. \tag{3.4}$$

Since $\alpha'(p) > 0$ we can consider T as being a function of α. Set $S(\alpha) = T(\alpha)\sqrt{2}$. We introduce some notation that will be used throughout this section.

$$\theta(x) = 2F(x) - x^2 f(x), \quad \Delta F = F(\alpha(p)) - F(u).$$

Theorem 3.2. *Assume that f is of type II with $f''(x) < 0$ for $x > d > 0$ where $d < m$. Then $T(p)$ has a single critical point which is a minimum.*

Proof. First notice that $T'(p) = S'(\alpha)\alpha'$. Thus, T has a critical point if and only if S has one. We will show that S has exactly one critical point. Making a change of variable, we obtain

$$S(\alpha) := \int_0^1 \frac{\alpha}{\sqrt{F(\alpha) - F(t\alpha)}} \, dt. \tag{3.5}$$

Differentiating S we get

$$S'(\alpha) = \int_0^1 \frac{\theta(\alpha) - \theta(t\alpha)}{2(\Delta F)^{\frac{3}{2}}} \, dt = \int_0^\alpha \frac{\theta(\alpha) - \theta(u)}{2(\Delta F)^{\frac{3}{2}}} \frac{du}{\alpha}. \tag{3.6}$$

Since for $0 < x < m$, $f'(x) > 0$, then $\theta'(x) < 0$ for $0 < x < m$. Hence for $0 < \alpha < m$, $S'(\alpha) < 0$. For $x > m$ $\theta'(x) > 0$, and $\theta(c) = 2F(c) - cf(c) = 2F(c) - c^2 f(c) = 2F(c) > 0$. Hence, there exists a δ with $m < \delta < c$ such that $\theta(\delta) = 0$ and $\theta(x) \geq 0$ for $\delta \leq x \leq c$. If $\delta < \alpha < c$ then $\theta(\alpha) - \theta(u) > 0$, for $0 < u < \alpha$. Hence $S'(\alpha) > 0$ for $\alpha > \delta$. Thus it follows that $S'(\alpha)$ has a zero between m and δ. To show that S

has only a single critical point which is a minimum, we shall show that for $\alpha > m$, $S'' > 0$. Differentiating S a second time we get

$$\begin{aligned} S''(\alpha) &= \\ &= \int_0^1 \frac{(\theta'(\alpha) - \theta'(t\alpha)t)2(\Delta F)^{\frac{3}{2}} - 3(\Delta F)^{\frac{1}{2}}(f(\alpha) - tf(t\alpha))(\theta(\alpha) - \theta(t\alpha))}{4(\Delta F)^3}\,dt \\ &= \int_0^1 \frac{(\alpha\theta'(\alpha) - \alpha\theta'(t\alpha)t)2(\Delta F)^3 - \frac{3}{2}(\alpha f(\alpha) - t\alpha f(t\alpha))(\theta(\alpha) - \theta(t\alpha))}{2(\Delta F)^{\frac{5}{2}}}\,\frac{dt}{\alpha} \\ &= \int_0^1 \frac{(\alpha\theta'(\alpha) - t\alpha\theta'(t\alpha))\Delta F - \frac{3}{2}(\alpha f(\alpha) - t\alpha f(t\alpha))(\theta(\alpha) - \theta(t\alpha))}{2(\Delta F)^{\frac{5}{2}}}\,\frac{dt}{\alpha}. \end{aligned}$$

Combining S' and S'', we get

$$S'' + \frac{3}{\alpha}S' = \int_0^1 \frac{\frac{3}{2}(\Delta\theta)^2 + (\Delta F)(\Delta(\tilde{\theta}')}{2\alpha^2(\Delta F)^{\frac{5}{2}}}\,\frac{dt}{\alpha},$$

where $\Delta(\tilde{\theta}') = \alpha\theta'(\alpha) - t\alpha\theta'(t\alpha)$. We now show the above expression is positive when $\alpha \geq m$. This will follow if we can show that $(\alpha\theta'(\alpha) - t\alpha\theta'(t\alpha)) \geq 0$ for $0 \leq t \leq 1$.

Consider $(x\theta'(x))' = -3x^2 f'(x) - x^3 f''(x)$. From the hypotheses on f, $f'(x)$, $f''(x) < 0$ when $x > m$. Hence $(x\theta'(x))' > 0$ when $x > m$. Thus $(x\theta'(x))$ is increasing for $x > m$, and $\alpha\theta'(\alpha) - t\alpha\theta'(t\alpha) \geq 0$ when $\alpha > m$ and $0 \leq t \leq 1$. Thus

$$S'' + \frac{3}{\alpha}S' > 0 \quad \text{for } \alpha > m.$$

If $S'(\alpha) = 0$ for $m < \alpha < \delta$, then $S''(\alpha) > 0$ and we have a minimum. This also shows there can be only one critical point, a minimum.

Proposition 3.3. *Assume that f satisfies the conditions in Theorem* 3.2. *Then*

$$\lim_{p \to 0^+} T(p) = L_0 < \infty.$$

Proof. To establish the limit we will use the relation $T(\alpha) = \sqrt{2}S(\alpha)$. We will use the alternative form of $S(\alpha)$,

$$S(\alpha) = \int_0^1 \frac{\alpha}{\sqrt{F(\alpha) - F(t\alpha)}}\,dt.$$

Set

$$H(\alpha, t) := \frac{\alpha}{\sqrt{F(\alpha) - F(t\alpha)}}. \tag{3.7}$$

Fix $0 < t < 1$, and consider the limit $\lim_{\alpha\to 0^+} H(\alpha, t) = \frac{0}{0}$. Differentiating both the numerator and denominator with respect to α we obtain

$$\frac{2\sqrt{F(\alpha) - F(t\alpha)}}{\alpha(f(\alpha) - t^2 f(t\alpha))}. \tag{3.8}$$

Using integration by parts we can write

$$F(\alpha) = \frac{1}{2}\alpha^2 f(\alpha) - \frac{1}{2}\int_0^\alpha t^2 f'(t)\, dt.$$

Hence,

$$F(\alpha) - F(t\alpha) = \frac{1}{2}\left\{\alpha^2(f(\alpha) - t^2 f(t\alpha)) - \int_{t\alpha}^\alpha s^2 f'(s)\, ds\right\}.$$

Substituting into (3.8) we obtain

$$\sqrt{2}\sqrt{\frac{\alpha^2(f(\alpha) - t^2 f(t\alpha))}{\alpha^2(f(\alpha) - t^2 f(t\alpha))^2} - \frac{\int_{t\alpha}^\alpha s^2 f'(s)\, ds}{\alpha^2(f(\alpha) - t^2 f(t\alpha))^2}}. \tag{3.9}$$

Using L'Hopital's method on $\frac{\int_{t\alpha}^\alpha s^2 f'(s)\, ds}{\alpha^2(f(\alpha) - t^2 f(t\alpha))^2}$ we can show that it tends to 0 as $\alpha \to 0^+$. We now obtain

$$\lim_{\alpha\to 0^+} H(\alpha, t) = \frac{\sqrt{2}}{\sqrt{f(0)}}\frac{1}{\sqrt{1 - t^2}}. \tag{3.10}$$

For small $\alpha > 0$ we have shown above that $H_\alpha(\alpha, t) < 0$ for $0 < t < 1$. Thus by the monotone convergence theorem we have

$$\lim_{\alpha\to 0^+} S(\alpha) = \lim_{\alpha\to 0^+}\int_0^1 H(\alpha, t)\, dt = \frac{\sqrt{2}}{\sqrt{f(0)}}\int_0^1 \frac{1}{\sqrt{1 - t^2}}\, dt = \frac{\sqrt{2}}{\sqrt{f(0)}}\frac{\pi}{2}. \tag{3.11}$$

Corollary 3.4. *Assume that f satisfies the conditions in Theorem* 3.2 *. Let L_f denote the minimum of $T(p)$. For $L < L_f$, $u_0 \equiv 0$ is the only nonnegative solution of* (3.1)*. For $L_f < L < L_0$ there are two positive solutions $0 < u_1 < u_2$. For $L > L_0$ there exists a unique positive solution u_2.*

Proof. In finding solutions to (3.2) we will assume that the initial conditions are at $x = 0$. To obtain solutions for (3.1) just translate the solution by the given L. Since we are looking for nonnegative solutions of (3.1) we will need $u'(0) = v(0) > 0$. Thus we will only consider the case when $p > 0$. Also, note that $u_0 \equiv 0$ is a nonnegative solution for (3.1) for any $L > 0$. Suppose that $0 < L < L_f$. Then for all $0 < p < A$

we have $T(p) \neq L$. Hence, for any solution (u, v) of system (3.2) starting at $(0, p)$ we will have $u(2L) \neq 0$. Hence u is not a solution of (3.1) .

Now consider the case where $L_f < L < L_0$. Then there are $0 < p_1 < p_2 < A$ such that $T(p_1) = T(p_2) = L$. Hence, there are two solutions (u_1, v_1) and (u_2, v_2) starting at $(0, p_1)$ and $(0, p_2)$, respectively. Since $p_1 < p_2$ we have that for small $\epsilon > 0$, $u_1(x) < u_2(x)$ for $0 < x < \epsilon$. To show that $u_1 < u_2$ we need only show that this holds for $0 < x < L$. Suppose that $u_1(x) = u_2(x)$ for some x. Then there is a smallest $0 < x_0 < L$ such that $u_1(x_0) = u_2(x_0)$. Since integral curves do not cross we have $v_1(x) < v_2(x)$ for $0 < x < L$. Hence $u_1'(x_0) < u_2'(x_0)$. There is a $\delta > 0$ such that $u_2(x) < u_1(x)$ for $x_0 - \delta < x < x_0$, contrary to $u_1(x) < u_2(x)$ for $0 < x < x_0$. Hence $u_1 < u_2$.

Next suppose that $L_0 < L$. Then there is one $0 < p < A$ such that $T(p) = L$. Hence, there is a solution (u_2, v_2) of (3.2) starting at $(0, p)$. Since $u_2'(0) = p > 0$ and that $T(p) = L$ we have that $u_2(0) = u_2(2L) = 0$ and $u_2(x) > 0$ for $0 < x < 2L$.

Theorem 3.5. *Assume that f is of type III and that there exists a constant $a < d < m$ such that $f''(x) < 0$ for $x > d$. If $F(b) > 0$ then $T(p)$ has a unique critical point which is a minimum.*

Proof. As in the proof of Theorem 3.2 we have $S'(\alpha) < 0$ for $0 < \alpha < m$. Assume that $F(b) > 0$, then there is a δ such that $m < \delta < b$ and $\theta(\delta) = 0,\ \theta(x) > 0,\ x > \delta$. Then for $\alpha > \delta,\ S'(\alpha) > 0$. Thus S has at least one critical point in $m < \alpha < \delta$. Using the same arguments as as in the proof of Theorem 3.2 we can show that T has one critical point, which is a minimum.

Note: From the phase diagram for functions f of type III it is clear that

$$\lim_{p \to 0^+} T(p) = \infty.$$

Hence, we have the following corollary.

Corollary 3.6. *Assume that f satisfies the conditions in Theorem 3.5 . For $L < L_f$, $u_0 \equiv 0$ is the only nonnegative solution of (3.1) . For $L > L_f$ there exist two positive solutions $0 < u_1 < u_2$.*

Proof. The proof is similar to the proof for Corollary 3.4 with the obvious changes.

We next consider the stability of the solutions to (3.1). For the proofs of the following propositions, we refer the reader to [6] and Chapter 24 in [19].

Theorem 3.7 ([19])**.** *Suppose that $T'(p) \neq 0$. Then the corresponding solution $u(x)$ is nondegenerate in the strong sense that 0 is not in the spectrum of the linearized operator about u.*

Proposition 3.8. *Assume that f is of type I. For $L < L_0$, u_0 is stable. For $L > L_0$, u_0 is unstable while u_1 is stable.*

Proposition 3.9. *Assume that f is of type II and satisfies the conditions in Theorem* 3.2.

(i) *For $L < L_f$, u_0 is stable.*
(ii) *For $L_f < L < L_0$, u_0 and u_2 are stable while u_1 is unstable.*
(iii) *For $L_0 < L$, u_0 is unstable and u_2 is stable.*

Proposition 3.10. *Assume that f is of type III and satisfies the conditions given in Theorem* 3.5. *Then*

(i) *For $L < L_f$, u_0 is stable.*
(ii) *For $L_f < L$, u_0 and u_2 are stable while u_1 is unstable.*

4. Main results. In this section we will investigate the existence of positive solutions to the system,

$$\begin{aligned} -\frac{d^2u}{dx^2} &= uM(u,v) \text{ in } (-L,L), \quad u(\pm L)=0 \\ -\frac{d^2v}{dx^2} &= vN(u,v) \text{ in } (-L,L), \quad v(\pm L)=0, \end{aligned} \tag{4.1}$$

where the functions $f(x) := M(x,0)$, $g(x) := N(0,x)$ will be one of the three types presented in the introduction. In particular, we assume that the functions M and N satisfy:

(i) M and N are C^1 in u and v.
(ii) $M(x,0)$ and $N(0,x)$ are one of the three types presented in Section 1.
(iii) There exist positive constants c_1 and c_2 such that $M(c_1,0)$, $N(0,c_2) < 0$.
(iv) M_v, $N_u < 0$ for all $u, v > 0$.

We introduce some notation that will be used throughout this section. Let $h(x)$ be a function of the type presented in the first section. Define L_h to be the smallest $L > 0$ such that

$$\frac{d^2u}{dx^2} + uh(u) = 0 \text{ in } (-L,L),\ u(\pm L) = 0$$

has a nontrivial solution. If h is of type II or III then L_h is the critical value of $T(p)$. If h is of type I then $L_h = \frac{\pi}{2\sqrt{h(0)}}$. For functions of type I we have $L_h = L_0$.

$$L_{0,h} := \begin{cases} \frac{\pi}{2\sqrt{h(0)}} & \text{if } h \text{ is of type I or II} \\ +\infty & \text{if } h \text{ is of type III.} \end{cases}$$

Define the following sets.

$$\begin{aligned} K &:= \{u \in C_0((-L,L)) : 0 \le u(x) \text{for } x \in (-L,L)\}, \\ W &:= K \oplus K, \\ D &:= \{(u,v) \in C_0((-L,L)) \oplus C_0((-L,L)) : u \le c_1+1,\ v \le c_2+1\}, \\ D' &:= (intD) \cap W, \\ P_r &:= \{(u,v) \in K : u < r,\ v < r\} \quad \text{where } r > 0. \end{aligned}$$

Theorem 4.1. *Let* (u,v) *be a nonnegative solution of* (4.1). *Then* $u < c_1$ *and* $v < c_2$.

Proof. Let $(\bar{u},\bar{v})$ be a nonnegative solution of (4.1). Using the property $M_v < 0$ we have

$$0 = \frac{d^2}{dx^2}\bar{u} + \bar{u}M(\bar{u},\bar{v}) \leq \frac{d^2}{dx^2}\bar{u} + \bar{u}M(\bar{u},0).$$

Hence, $\bar{u}$ is a lower solution for

$$\frac{d^2u}{dx^2} + uM(u,0) = 0 \quad \text{in } (-L,L) \quad u(-L) = u(L) = 0. \tag{4.2}$$

By letting k be a large enough positive constant we have a positive upper solution, k, for (4.2). Using the method of upper and lower solutions, we know that there is a solution w of (4.2) with $\bar{u} \leq w \leq k$. Since $M(x,0)$ is one of the three types studied in Section 3, we know that the maximal solution of (4.2) is u_2. Hence $\bar{u} \leq u_2$. It is easy to show that $u_2 < c_1$. Thus, $0 \leq \bar{u} \leq c_1$. The proof that $0 \leq \bar{v} \leq c_2$ is done in the same way.

Theorem 4.2 ([11]). *Let* $M(x,0)$ *and* $N(0,x)$ *be of type I. If* $\max\{L_{0,f}, L_{0,g}\} < L$ *then there is a positive solution to* (4.1) *provided both* $\lambda_1(\frac{d^2}{dx^2} + M(0,v_2))$ *and* $\lambda_1(\frac{d^2}{dx^2} + N(u_2,0))$ *have the same sign.*

Proof. Notice that if $L > \max\{L_{0,f}, L_{0,g}\}$ then we have $M(0,0), N(0,0) > \lambda_1$. The remaining conditions are just those in Theorem 4.1 in [11]. $\square$

In proving the remaining results, the primary technique will be the method of fixed-point index. Let $P > 0$ be a constant such that

$$P > \max\{|M(x,y)|, |N(x,y)| : 0 \leq x \leq c_1,\ 0 \leq y \leq c_2\}.$$

Set $\rho = \max\{c_1, c_2\} + 1$. Define an operator $A : \overline{P_\rho} \to P$ by

$$A(u,v) := (-\frac{d^2}{dx^2} + P)^{-1}(u(M(u,v)+P), v(N(u,v)+P)).$$

Then A is a compact operator and (u,v) is a solution of (4.1) if and only if it is a fixed point of A. We will also be interested in the solutions of the equations

$$\frac{d^2}{dx^2}u + uM(u,0) = 0 \ \text{ in } (-L,L), \quad u(-L) = u(L) = 0 \tag{4.3}$$

$$\frac{d^2}{dx^2}v + vN(0,v) = 0 \ \text{ in } (-L,L), \quad v(-L) = v(L) = 0, \tag{4.4}$$

which produce trivial solutions of (4.1).

Lemma 4.3. *index* $(A, P_\rho) = 1$.

Proof. Define a new operator

$$A_\lambda(u,v) := (-\frac{d^2}{dx^2} + P)^{-1}(\lambda u M(u,v) + Pu, \lambda v N(u,v) + Pv),$$

for $\lambda \in [0,1]$. If $(\bar{u}, \bar{v})$ is a fixed point of A_λ we can show that $\bar{u} < c_1$ and $\bar{v} < c_2$. It is easy to see that $(0,0)$ is the only fixed point of $A_0(u,v) = (-\frac{d^2}{dx^2} + P)^{-1}(Pu, Pv)$. It follows that index $(A_0, P_\rho) = \text{index}(A_0, (0,0)) = 1$. Using homotopy invariance for the fixed-point index we conclude $\text{index}(A, P_\rho) = 1$.

Lemma 4.4. *If $M(0,0) > \lambda_1$ or $N(0,0) > \lambda_1$ and $M(0,0), N(0,0) \neq \lambda_1$, then index* $(A, (0,0)) = 0$.

Proof. Without loss of generality, suppose $M(0,0) > \lambda_1$. Set $B := A'(0,0)$. We first show that 1 is not an eigenvalue of B with a positive eigenfunction. Suppose otherwise. Then there exists a pair $(\phi, \psi) > (0,0)$ such that $B(\phi,\psi)^T = (\phi,\psi)^T$. That is,

$$-\frac{d^2}{dx^2}\phi = M(0,0)\phi, \quad -\frac{d^2}{dx^2}\psi = N(0,0)\psi.$$

Hence, both $M(0,0)$ and $N(0,0)$ are eigenvalues of $-\frac{d^2}{dx^2}$. Since $M(0,0) > \lambda_1$ then it follows that $\phi \equiv 0$. If $N(0,0) > \lambda_1$, then similarly $\psi \equiv 0$. Hence, $(\phi,\psi) \equiv 0$ contrary to the assumption. On the other hand, if $\psi > 0$ then we must have $N(0,0) = \lambda_1$, contrary to the hypotheses. Hence 1 is not an eigenvalue of B with positive eigenfunctions. Since $M(0,0) > \lambda_1$ then it follows that $\lambda_1(\frac{d^2}{dx^2} + M(0,0)) > 0$. Set $T := (-\frac{d^2}{dx^2} + P)^{-1}(M(0,0) + P)$. Then T is a positive compact linear operator. By Lemma 2.3 we have that $r(T) > 1$. By the Krein-Rutman theorem we have that $\mu := r(T)$ is an eigenvalue of T with positive eigenfunction, say ϕ. Consider $B(\phi, 0)$. Then

$$\begin{aligned} B(\phi,0) &= (-\frac{d^2}{dx^2} + P)^{-1}\begin{pmatrix} M(0,0)+P & 0 \\ 0 & N(0,0)+P \end{pmatrix}\begin{pmatrix}\phi \\ 0\end{pmatrix} \\ &= \begin{pmatrix} T\phi \\ 0 \end{pmatrix} = \mu\begin{pmatrix}\phi \\ 0\end{pmatrix}. \end{aligned}$$

Thus B has a positive eigenfunction corresponding to an eigenvalue greater than 1. Hence, by Lemma 2.3, $\text{index}(A, (0,0)) = 0$.

Lemma 4.5. *If $\lambda_1(\frac{d^2}{dx^2} + M(0,\bar{v})) > 0$ and $\bar{v}$ is a nondegenerate solution of* (4.4)*, then index* $(A, (0,\bar{v})) = 0$.

Proof. Set $B := A'(0,\bar{v})$, $y = (0,\bar{v})$. Then $\overline{W}_y = K \oplus C_0((-L,L))$, $S_y = \{0\} \oplus C_0((-L,L))$. Suppose that there are functions (ϕ,ψ) such that $B\binom{\phi}{\psi} = \binom{\phi}{\psi}$. Then

$$\frac{d^2}{dx^2}\phi + M(0,\bar{v})\phi = 0$$
$$\frac{d^2}{dx^2}\psi + (N(0,\bar{v}) + \bar{v}N_v(0,\bar{v}))\psi = -\bar{v}N_u(0,\bar{v})\phi.$$

Since $\lambda_1(\frac{d^2}{dx^2} + M(0,\bar{v})) > 0$ if $\phi \not\equiv 0$ then 0 is not the principal eigenvalue of $\frac{d^2}{dx^2} + M(0,\bar{v})I$. Thus $T_1 := (-\frac{d^2}{dx^2} + P)^{-1}(M(0,\bar{v}) + P)$ is invertible on K. That is, if $\phi \in K$ and $T_1\phi = \phi$ then $\phi \equiv 0$. By restricting the first component of (ϕ,ψ) to be from K, the second equation becomes

$$\frac{d^2}{dx^2}\psi + (N(0,\bar{v}) + \bar{v}N_v(0,\bar{v}))\psi = 0.$$

Since $\bar{v}$ is a nondegenerate solution of (4.4), it follows that $\psi \equiv 0$. Hence, $I - B$ is invertible on $\overline{W}_y$.

Using $\lambda_1(\frac{d^2}{dx^2} + M(0,\bar{v})) > 0$ and Lemma 2.3 we obtain $r(T_1) > 1$. Let $t = \frac{1}{r(T_1)}$ and ϕ be the positive eigenfunction of T_1 corresponding to $r(T_1)$; this is guaranteed to exist by the Krein-Rutman theorem. Let ξ be any function in $C_0((-L,L))$. Then we have

$$(I - tB)\begin{pmatrix}\phi\\ \xi\end{pmatrix} = $$
$$= \begin{pmatrix}\phi - tT_1\phi \\ \xi - t(-\frac{d^2}{dx^2} + P)^{-1}[(\bar{v}N_u(0,\bar{v}))\phi + (N(0,\bar{v}) + \bar{v}N_v(0,\bar{v}) + P)\xi]\end{pmatrix}.$$

The first component is 0, and the second component is a function in $C_0((-L,L))$. Hence $(I - tB)(\phi,\xi) \in S_y$, and B has property α on $\overline{W}_y$. By Theorem 2.1, $\operatorname{index}(A,(0,\bar{v})) = 0$.

Lemma 4.6. *If* $\lambda_1(\frac{d^2}{dx^2} + N(u,0)) < 0$ *and u is a stable nondegenerate solution of* (4.4)*, then* $\operatorname{index}(A,(u,0)) = 1$.

Proof. Set

$$B := A'(\bar{u},0) = (-\frac{d^2}{dx^2} + P)^{-1}\begin{pmatrix} M(\bar{u},0) + \bar{u}M_u(\bar{u},0) + P & \bar{u}M_v(\bar{u},0) \\ 0 & N(\bar{u},0) + P\end{pmatrix}.$$

Suppose that there are functions ϕ, ψ such that $B\binom{\phi}{\psi} = \binom{\phi}{\psi}$. That is,

$$\frac{d^2}{dx^2}\phi + (M(\bar{u},0) + \bar{u}M_u(\bar{u},0))\phi = \bar{u}M_v(\bar{u},0)\psi \tag{4.5}$$
$$\frac{d^2}{dx^2}\psi + N(\bar{u},0)\psi = 0. \tag{4.6}$$

The second equation implies that 0 is an eigenvalue for the operator $\frac{d^2}{dx^2}+N(\bar{u},0)I$, contrary to $\lambda_1(\frac{d^2}{dx^2}+N(\bar{u},0))<0$. Thus, $\psi=0$. The first equation becomes

$$\frac{d^2}{dx^2}\phi+(M(\bar{u},0)+\bar{u}M_u(\bar{u},0))\phi=0.$$

Since $\bar{u}$ is a nondegenerate solution of (4.3), $\phi\equiv 0$. Thus $I-B$ is invertible on E. We now show that B does not satisfy property α. In this case we have $\overline{W}_y=C_0((-L,L))\oplus K$ (where $y=(\bar{u},0)$), $S_y=C_0((-L,L))\oplus\{0\}$, and $\overline{W}_y\backslash S_y=C_0((-L,L))\oplus\{K\backslash\{0\}\}$. Suppose that there is a $t>0$ and a pair of functions $(\phi,\psi)\in\overline{W}_y\backslash S_y$ such that $(I-tB)\binom{\phi}{\psi}\in S_y$. That is,

$$\begin{aligned}&\phi-t(-\frac{d^2}{dx^2}+P)^{-1}[(M(\bar{u},0)+\bar{u}M_u(\bar{u},0)+P)\phi+(\bar{u}M_v(\bar{u},0))\psi]\\&\quad\in C_0((-L,L))\\&\psi-t(-\frac{d^2}{dx^2}+P)^{-1}[(N(\bar{u},0)+P)\psi]=0.\end{aligned}$$

The first equation is true regardless of our choice of ϕ, ψ. Set $T:=(-\frac{d^2}{dx^2}+P)^{-1}(N(\bar{u},0)+P)$. Then T is a compact positive operator and the second equation reduces to $T\psi=\frac{1}{t}\psi>\psi$. From Lemma 2.2, $r(T)>1$. However, from $\lambda_1(\frac{d^2}{dx^2}+N(\bar{u},0))<0$ and Lemma 2.3 we obtain $r(T)<1$. Hence, B does not have property α on $\overline{W}_y$. From Theorem 2.1, $\text{index}_W(A,(\bar{u},0))=\text{index}_E(B,(0,0))=\pm1$. Suppose that $\lambda>1$ is an eigenvalue of B with eigenvector (ϕ,ψ). Then we have

$$\begin{aligned}&(-\frac{d^2}{dx^2}+P)^{-1}[(M(\bar{u},0)+\bar{u}M_u(\bar{u},0)+P)\phi+(\bar{u}M_v(\bar{u},0))\psi]=\lambda\phi\\&(-\frac{d^2}{dx^2}+P)^{-1}[(N(\bar{u},0)+P)\psi]=\lambda\psi.\end{aligned}$$

The second equation is just $T\psi=\lambda\psi$, which implies that $r(T)>1$ contrary to $r(T)<1$. Thus $\psi\equiv 0$. The first equation reduces to

$$(-\frac{d^2}{dx^2}+P)^{-1}(M(\bar{u},0)+\bar{u}M_u(\bar{u},0)+P)\phi=\lambda\phi.$$

Let $T:=(-\frac{d^2}{dx^2}+P)^{-1}(M(\bar{u},0)+\bar{u}M_u(\bar{u},0)+P)$. If $\lambda>1$, then $r(T)>1$. Since $\bar{u}$ is a stable solution for (4.3), then $\lambda_1(-\frac{d^2}{dx^2}+(M(\bar{u},0)+\bar{u}M_u(\bar{u},0)))<0$. From Lemma 2.3 we obtain $r(T)<1$. Thus, $\phi\equiv 0$, and B has no eigenvalues greater than 1. Hence $\text{index}_W(A,(\bar{u},0))=\text{index}_E(B,(0,0))=1$.

Theorem 4.7. *Let $M(x,0)$ be of type I and $N(0,x)$ of type II.*

(i) *If* $\max\{L_f, L_g\} < L < L_{0,g}$, *and* $\lambda_1(\frac{d^2}{dx^2} + M(0,v_2)) < 0$ *and* $\lambda_1(\frac{d^2}{dx^2} + M(0,v_1)) > 0$ *then there is a positive solution of* (4.1).

(ii) *Let* $\max\{L_{0,f}, L_{0,g}\} < L$. *If* $\lambda_1(\frac{d^2}{dx^2} + M(0,v_2))$ *and* $\lambda_1(\frac{d^2}{dx^2} + N(u_2,0))$ *are both positive or negative then there is a positive solution to* (4.1).

Proof. (i) For a fixed L with $\max\{L_f, L_g\} < L < L_{0,g}$ we have $L_f < L_{0,g}$, $M(0,0) > \lambda_1$ and $N(0,0) < \lambda_1$. Then by Theorem 2.2 we have a unique positive solution u_2 of (4.3). By Corollary 2.5 we have two positive solutions $0 < v_1 < v_2$ of (4.4). By the assumptions on L we have the following trivial solutions to (4.1): $(0,0)$, $(u_2,0)$, $(0,v_1)$ and $(0,v_2)$. To show that there is a solution with both components positive we will use the theory of fixed-point index for the operator A. Notice that since $N_u < 0$ and $N(0,0) < \lambda_1$ we have $\lambda_1(\frac{d^2}{dx^2} + N(u_2,0)) < 0$.
$(0,0)$: In this case we have $M(0,0) > \lambda_1$. By Lemma 4.4 we have index$\,(A,(0,0)) = 0$.
$(u_2,0)$: Since u_2 is a stable solution for (4.3) and $\lambda_1(\frac{d^2}{dx^2} + N(u_2,0)) < 0$ we can use Lemma 4.6 to obtain index$\,(A,(u_2,0)) = 1$.
$(0,v_1)$: We can use Lemma 4.5 to obtain index$\,(A,(0,v_1)) = 0$.
$(0,v_2)$: This case is similar to $(u_2,0)$. Hence index$\,(A,(0,v_2)) = 1$.

Thus, we have

$$\text{index}(A,P_\rho) = 1, \quad \text{index}\,(A,(0,0)) = 0, \quad \text{index}\,(A,(u_2,0)) = 1,$$
$$\text{index}\,(A,(0,v_1)) = 0, \quad \text{index}\,(A,(0,v_2)) = 1.$$

Using the excision property, there must be another fixed point of A in W with both components positive. Hence, there is a positive solution for (4.1).

(ii) In this case we have $M(0,0) > \lambda_1$, $N(0,0) > \lambda_1$ and three trivial solutions $(0,0)$, $(u_2,0)$, $(0,v_2)$. We consider the case where $\lambda_1(\frac{d^2}{dx^2} + M(0,v_2))$ and $\lambda_1(\frac{d^2}{dx^2} + N(u_2,0))$ are both negative. The case where both are positive is proven in a similar manner.
$(0,0)$: Since $M(0,0) > \lambda_1$, then by Lemma 4.4, index$\,(A,(0,0)) = 0$.
$(u_2,0)$: In this case u_2 is a stable solution for (4.3) and $\lambda_1(\frac{d^2}{dx^2} + N(u_2,0)) < 0$; we can use Lemma 4.6. Whence, index$\,(A,(u_2,0)) = 1$.
$(0,v_2)$: This is the same as for $(u_2,0)$. Hence index$\,(A,(0,v_2)) = 1$.

Using the excision property as above we can show that there must be a positive solution for (4.1).

Theorem 4.8. *Let $M(x,0)$ and $N(0,x)$ be functions of types I and III, respectively. Let $L > \max\{L_f, L_g\}$. If $\lambda_1(\frac{d^2}{dx^2} + M(0,v_2)) < 0$ and $\lambda_1(\frac{d^2}{dx^2} + M(0,v_1)) > 0$, then there is a positive solution to* (4.1).

Proof. By the assumption on L there is a unique positive solution u_2 of

$$\frac{d^2}{dx^2}u + uM(u,0) = 0 \text{ in } (-L,L), \quad u(-L) = u(L) = 0,$$

and two positive solutions $0 < v_1 < v_2$ of

$$\frac{d^2}{dx^2}v + vN(0,v) = 0 \text{ in } (-L,L), \quad v(-L) = v(L) = 0.$$

In equation (4.3) we have that u_2 is stable and that in equation (4.4) 0 and v_2 are stable while v_1 is unstable. Using the fact that $N_u < 0$ we have $\lambda_1(\frac{d^2}{dx^2} + N(u_2,0)) < 0$.

We have the following trivial fixed points of A: $(0,0)$, $(u_2,0)$, $(0,v_1)$, and $(0,v_2)$. As above can show that $\text{index}(A,D',W) = 1$. We now calculate the fixed-point index for each of the above trivial solutions.
$(0,0)$: Since $M(0,0) > \lambda_1$ then from Lemma 4.4 we have $\text{index}(A,(0,0)) = 0$.
$(u_2,0)$: Here we have u_2 is a stable solution of (4.3) and $\lambda_1(\frac{d^2}{dx^2} + N(u_2,0)) < 0$. From Lemma 4.6, $\text{index}\,(A,(u_2,0)) = 1$.
$(0,v_1)$: In this case we have that v_1 is a nondegenerate solution of (4.4) and from the hypothesis $\lambda_1(\frac{d^2}{dx^2} + M(0,v_1)) > 0$. Using Lemma 4.5 we have $\text{index}(A,(0,v_1)) = 0$.
$(0,v_2)$: Since v_2 is a stable solution of (4.4) and $\lambda_1(\frac{d^2}{dx^2} + M(0,v_2)) < 0$ we can use Lemma 4.6 (with the obvious changes), to conclude $\text{index}\,(A,(0,v_2)) = 1$.

Thus we have

$$\begin{aligned} &\text{index}\,(A,(0,0)) = 0, \quad \text{index}\,(A,(u_2,0)) = 1, \\ &\text{index}\,(A,(0,v_1)) = 0, \quad \text{index}\,(A,(0,v_2)) = 1. \end{aligned}$$

Using the excision property for index and that $\text{index}\,(A,P_\rho) = 1$ we have that there exists a positive solution (u,v) of system (4.1).

Theorem 4.9. *Let $M(x,0)$ and $N(0,x)$ be of type II. Assume that $L > \max\{L_f, L_g\}$, so that there exist positive solutions to both equations* (4.3) *and* (4.4)*.*

- (i) *Let $\min\{L_{0,f}, L_{0,g}\} < L < \max\{L_{0,f}, L_{0,g}\}$. If $\lambda_1(\frac{d^2}{dx^2} + N(u_2,0)) < 0$ and $\lambda_1(\frac{d^2}{dx^2} + N(u_1,0)) > 0$ then there is a positive solution to* (4.1)*.*
- (ii) *Let $\max\{L_{0,f}, L_{0,g}\} < L$. There exists a positive solution of* (4.1) *provided $\lambda_1(\frac{d^2}{dx^2} + M(0,v_2))$ and $\lambda_1(\frac{d^2}{dx^2} + N(u_2,0))$ have the same sign.*

Proof. (i) Assume that we have the situation where $L_{0,g} < L < L_{0,f}$. Defining A as above, we have the trivial solutions $(0,0)$, $(u_1,0)$, $(u_2,0)$, $(0,v_2)$, where 0 and u_2 are stable solutions of (4.3) while u_1 is unstable and v_2 is stable for (4.4) while 0 is unstable. We have that $M(0,0) < \lambda_1$ and $N(0,0) > \lambda_1$. From this it follows that $\lambda_1(\frac{d^2}{dx^2} + M(0,v_2)) < 0$. We now calculate the fixed-point index for each trivial solution.
$(0,0)$: Since $N(0,0) > \lambda_1$ we obtain from Lemma 4.4 $\text{index}\,(A,(0,0)) = 1$.
$(u_1,0)$: Since $\lambda_1(\frac{d^2}{dx^2} + N(u_1,0)) > 0$ then by Lemma 4.5 $\text{index}\,(A,(u_1,0)) = 0$.

$(u_2, 0)$: In this case we have that u_2 is a stable solution for (4.3) and $\lambda_1(\frac{d^2}{dx^2} + N(u_2, 0)) < 0$; then Lemma 4.6 produces index $(A, (u_2, 0)) = 1$.
$(0, v_2)$: Since v_2 is a stable solution for (4.4) and $\lambda_1(\frac{d^2}{dx^2} + M(0, v_2) < 0$ then by Lemma 4.6, index $(A, (0, v_2)) = 1$.

Using the excision property we see that there must be a positive solution to (4.1).

(ii) In this case we have $M(0, 0)$, $N(0, 0) > \lambda_1$; 0, u_2 are the nonnegative solutions of (4.3); and 0, v_2 are the solutions of (4.4). Thus, the fixed points we need to consider are $(0, 0)$, $(u_2, 0)$, and $(0, v_2)$. We establish the theorem for the case where $\lambda_1(\frac{d^2}{dx^2} + M(0, v_2))$ and $\lambda_1(\frac{d^2}{dx^2} + N(u_2, 0))$ are both positive. The proof for when they are both negative is similar and left to the reader.
$(0, 0)$: Since $M(0, 0) > \lambda_1$, we have by Lemma 4.4 index $(A, (0, 0)) = 0$.
$(u_2, 0)$: Since u_2 is a nondegenerate solution of (4.3) and $\lambda_1(\frac{d^2}{dx^2} + N(u_2, 0)) > 0$, we obtain from Lemma 4.5 index $(A, (u_2, 0)) = 0$.
$(0, v_2)$: This is the same as $(u_2, 0)$. Hence, index $(A, (0, v_2)) = 0$.

$$\begin{aligned} &\text{index}\,(A, P_\rho) = 1, \quad \text{index}\,(A, (0,0)) = 0, \\ &\text{index}\,(A, (u_2, 0)) = 0, \quad \text{index}\,(A, (0, v_2)) = 0. \end{aligned}$$

Using the excision property for index we obtain that there must be a positive solution for (4.1).

Theorem 4.10. *Let $M(x, 0)$ be of type II and $N(0, x)$ of type III. Assume $L > \max\{L_g, L_{0,f}\}$. Then there is a positive solution to* (4.1) *provided* $\lambda_1(\frac{d^2}{dx^2} + N(u_2, 0)) < 0$.

Proof. In this case we have $M(0, 0) > \lambda_1$ and $N(0, 0) < \lambda_1$. Since $L > L_g$, then there are two positive solutions for (4.4). Hence we have the following trivial fixed points for A: $(0, 0)$, $(u_2, 0)$, $(0, v_1)$, $(0, v_2)$. As in the preceding proofs we use the fixed-point index to establish he existence of a fixed point with positive components. The details are left to the reader.

Remark. In the cases where L is such that there might be two positive solutions to the equations (4.3) and (4.4) fixed-point index does not work to establish the existence of a positive solution for (4.1). The problem lies in calculating the index for the unstable solutions. Here we can show that the index is ± 1. In the case where they are both 1, we can establish a positive solution. When they are both -1 then we obtain that the sum of the indices is the same as index(A, P_ρ).

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