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LOCAL REGULARITY OF NON-RESONANT NONLINEAR WAVE EQUATIONS

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Dedicated to Professor Rentaro Agemi on his sixtieth birthday

(Submitted by: Sergiu Klainerman)

Abstract. We study the problem of minimal regularity required to ensure local well-posedness for systems of nonlinear wave equations with different propagation speeds in three space dimensions

$$(\partial_t^2 - C_1^2\Delta)u = F(u, v, \partial u, \partial v),$$
$$(\partial_t^2 - C_2^2\Delta)v = G(u, v, \partial u, \partial v).$$

We prove that if $C_2 > C_1$ and F, G have the form $\partial u \cdot v$, then the problem is well-posed in H^1. Our proof is based on the same type of space-time estimates as those of Klainerman and Machedon.

1. Introduction. In this paper we study the problem of the minimal regularity required on the initial conditions to ensure well-posedness (local existence, uniqueness and continuous dependence on the data) for systems of nonlinear wave equations with quadratic nonlinearities. Specifically, we consider the Cauchy problem for the systems in three space dimensions :

$$\begin{cases} \Box_1 u \equiv (\partial_t^2 - C_1^2\Delta)u = F(u, v, \partial u, \partial v), & t > 0, \ \ x \in R^3, \\ \Box_2 v \equiv (\partial_t^2 - C_2^2\Delta)v = G(u, v, \partial u, \partial v), & t > 0, \ \ x \in R^3, \\ u(0, x) = u_0(x), \quad \partial_t u(0, x) = u_1(x), & x \in R^3, \\ v(0, x) = v_0(x), \quad \partial_t v(0, x) = v_1(x), & x \in R^3, \end{cases} \tag{1.1}$$

where u, v are scalar functions of $t > 0$ and $x \in R^3$, the C_j, $j = 1, 2$ are positive constants, and $\partial = (\partial_t, \nabla) = (\partial_0, \partial_1, \partial_2, \partial_3)$. The constants C_1 and C_2 are the propagation speeds. The functions F and G are assumed to be quadratic and

$$F(u, v, p, q), \ G(u, v, p, q) \in C^\infty(R \times R \times R^4 \times R^4).$$

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First, let us consider the case where the nonlinear terms F and G are independent of u and v; i.e., F and G have the form

$$B_{\alpha\beta}(u,v) = \partial_\alpha u \partial_\beta v, \quad \alpha,\beta = 0,1,2,3. \tag{1.2}$$

The proof of the classical local existence theorem relies on the energy estimates and the Sobolev imbedding theorem. In R^{1+n}, this requires the initial conditions

$$u_0, v_0 \in H^s(R^n) \quad \text{and} \quad u_1, v_1 \in H^{s-1}(R^n) \tag{1.3}$$

with $s > n/2 + 1$ for (1.2). Recently, Klainerman and Machedon [4] proved that if $C_1 = C_2 = 1$, and F, G have the null forms,

$$Q_0(u,v) = \partial_t u \partial_t v - \nabla u \cdot \nabla v, \tag{1.4a}$$

$$Q_{\alpha\beta}(u,v) = \partial_\alpha u \partial_\beta v - \partial_\beta u \partial_\alpha v, \quad 0 \le \alpha < \beta \le 3, \tag{1.4b}$$

then the Cauchy problem is well-posed for $u_0, v_0 \in H^2(R^3)$, $u_1, v_1 \in H^1(R^3)$. This remarkable result is based on new space-time estimates for the null forms (1.4a), (1.4b). Moreover, in [5] they improved minimal regularity assumptions for well-posedness to be $u_0, v_0 \in H^s$, $u_1, v_1 \in H^{s-1}$ with $s > 3/2$ if $C_1 = C_2 = 1$, and both F and G have the null form Q_0 in (1.4a).

For general quadratically nonlinear terms of the form (1.2) in three space dimensions, the lower bound for the Sobolev exponent s in (1.3) can be lowered to $s > 2$. This result‡ of Ponce and Sideris [11] was proved by using a space-time estimate called the Strichartz estimate. In particular, in the spherically symmetric case, the Cauchy problem is well-posed for $u_0, v_0 \in H^2(R^3)$, $u_1, v_1 \in H^1(R^3)$ and for general nonlinear terms of the form (1.2)§. This was shown by Klainerman and Machedon [4].

On the other hand, in [8, 9] Lindblad gave sharp counterexamples so that the Cauchy problem for the scalar wave equation

$$\Box u \equiv (\partial_t^2 - \Delta)u = f(u, \partial_t u, Du)$$

is ill-posed for the following cases : $f = (\partial_t u)^2$ and $s < 2$; $f = (Du)^2$ and $s = 2$ (where $D = \partial_1 - \partial_t$) ; $f = u\partial_t u$ and $s < 1$; $f = uDu$ and $s = 1$; and $f = u^2$ and $s \le 0$.

One aim of this paper is to show that the same type of estimates as those of Klainerman and Machedon [4] hold for $C_2 > C_1$ and for the forms $B_{\alpha\beta}(u,v)$ in (1.2) and

$$B_\alpha(u,v) = \partial_\alpha u \cdot v, \quad \alpha = 0,1,2,3. \tag{1.5}$$

‡The result generalizes to system (1.1) with $C_1 \ne C_2$.

§This result is true not only for quadratically nonlinear terms of the form (1.2) but also for all nonlinear terms of the form $F(u, v, \partial u, \partial v)$ which are quadratic in $\partial u, \partial v$.

Theorem 1. *Consider the solutions u, v of a system of inhomogeneous wave equations*

$$\begin{cases} \Box_1 u \equiv (\partial_t^2 - C_1^2\Delta)u = F(t,x), & t>0,\ \ x\in R^3, \\ \Box_2 v \equiv (\partial_t^2 - C_2^2\Delta)v = G(t,x), & t>0,\ \ x\in R^3, \\ u(0,x)=u_0(x), \quad \partial_t u(0,x)=u_1(x), & x\in R^3, \\ v(0,x)=v_0(x), \quad \partial_t v(0,x)=v_1(x), & x\in R^3. \end{cases} \tag{1.6}$$

Suppose $C_2 > C_1$.

(i) *Let $u_0, v_0 \in H^1(R^3)$, $u_1, v_1 \in L^2(R^3)$ and $F, G \in L^1(R_+, L^2(R^3))$. Then, for the form* (1.5)*, we have, for all $T>0$,*

$$\int_0^T\int_{R^3}|B_\alpha(u,v)|^2dx\,dt \le C\big(\|\nabla u_0\|_{L^2(R^3)}+\|u_1\|_{L^2(R^3)} + \int_0^T\|F(t)\|_{L^2(R^3)}\,dt\big)^2\big(\|\nabla v_0\|_{L^2(R^3)}+\|v_1\|_{L^2(R^3)}+\int_0^T\|G(t)\|_{L^2(R^3)}\,dt\big)^2.$$

(ii) *Let $v_0\in H^2(R^3)$, $u_0, v_1\in H^1(R^3)$, $u_1\in L^2(R^3)$, $F\in L^1(R_+,L^2(R^3))$ and $G\in L^1(R_+,H^1(R^3))$. Then, for the form* (1.2)*, we have*

$$\int_0^T\int_{R^3}|B_{\alpha\beta}(u,v)|^2dx\,dt \le C\big(\|\nabla u_0\|_{L^2(R^3)}+\|u_1\|_{L^2(R^3)} + \int_0^T\|F(t)\|_{L^2(R^3)}dt\big)^2\big(\|\nabla^2 v_0\|_{L^2(R^3)}+\|\nabla v_1\|_{L^2(R^3)} + \int_0^T\|\nabla G(t)\|_{L^2(R^3)}dt\big)^2 \quad \textit{for all } T>0.$$

Similar estimates hold for other dimensions $1+n$. Let $\dot H^s(R^n)$ denote the homogeneous Sobolev space and $\omega_0 = (-\Delta)^{1/2}$. The following corollary follows easily from the proof of Theorem 1.

Corollary 1. *Let u, v be the solutions of* (1.6) *in R^{1+n}. Suppose $C_2 > C_1$.*

(i) *Let $u_0\in\dot H^1(R^n)$, $u_1\in L^2(R^n)$, $v_0\in\dot H^{(n-1)/2}(R^n)$, $v_1\in\dot H^{(n-3)/2}(R^n)$, $F\in L^1(R_+,L^2(R^n))$ and $G\in L^1(R_+,\dot H^{(n-3)/2}(R^n))$. Then, for B_α in* (1.5)*, we have*

$$\int_0^T\int_{R^n}|B_\alpha(u,v)|^2dx\,dt \le C\big(\|\nabla u_0\|_{L^2(R^n)}+\|u_1\|_{L^2(R^n)} + \int_0^T\|F(t)\|_{L^2(R^n)}dt\big)^2\big(\|\omega_0^{(n-1)/2}v_0\|_{L^2(R^n)}+\|\omega_0^{(n-3)/2}v_1\|_{L^2(R^n)} + \int_0^T\|\omega_0^{(n-3)/2}G(t)\|_{L^2(R^n)}\,dt\big)^2 \quad \textit{for all } T>0.$$

(ii) *Let* $u_0 \in \dot{H}^1(R^n)$, $u_1 \in L^2(R^n), v_0 \in \dot{H}^{(n+1)/2}(R^n), v_1 \in \dot{H}^{(n-1)/2}(R^n)$, $F \in L^1(R_+, L^2(R^n))$ *and* $G \in L^1(R_+, \dot{H}^{(n-1)/2}(R^n))$. *Then, for* $B_{\alpha\beta}$ *in* (1.2), *we have*

$$\int_0^T \int_{R^n} |B_{\alpha\beta}(u,v)|^2 dx\, dt \le C\big(\|\nabla u_0\|_{L^2(R^n)} + \|u_1\|_{L^2(R^n)} + \int_0^T \|F(t)\|_{L^2(R^n)} dt\big)^2 \big(\|\omega_0^{(n+1)/2} v_0\|_{L^2(R^n)} + \|\omega_0^{(n-1)/2} v_1\|_{L^2(R^n)} + \int_0^T \|\omega_0^{(n-1)/2} G(t)\|_{L^2(R^n)}\, dt\big)^2 \quad \text{for all } T > 0.$$

Even if $C_2 > C_1$, however, it seems impossible to construct local solutions for the form (1.2) and arbitrary initial data $u_0, v_0 \in H^2(R^3)$, $u_1, v_1 \in H^1(R^3)$ except spherically symmetric case. If we could show that

$$\int_0^T \int_{R^3} |\partial B_{\alpha\beta}(u,v)|^2 dx\, dt \le C\big(\|\nabla^2 u_0\|_{L^2(R^3)} + \|\nabla u_1\|_{L^2(R^3)} + \int_0^T \|\nabla F(t)\|_{L^2(R^3)} dt\big)^2 \times \big(\|\nabla^2 v_0\|_{L^2(R^3)} + \|\nabla v_1\|_{L^2(R^3)} + \int_0^T \|\nabla G(t)\|_{L^2(R^3)} dt\big)^2 \tag{1.7}$$

for all $T > 0$ and arbitrary $u_0, v_0 \in H^2(R^3)$, $u_1, v_1 \in H^1(R^3)$, $F, G \in L^1(R_+, H^1(R^3))$, then the well-posedness would be true for $u_0, v_0 \in H^2$, $u_1, v_1 \in H^1$. Klainerman and Machedon [4] showed the above inequality to be true if we replace $B_{\alpha\beta}(u,v)$ by Q_0 or $Q_{\alpha\beta}$ in (1.4) and set $C_1 = C_2 = 1$. This result is based on the symmetry between u and v[¶]. As mentioned above, by using this space-time estimate they proved that the Cauchy problem for the null forms (1.4) is well-posed for $u_0, v_0 \in H^2$, $u_1, v_1 \in H^1$ and $C_1 = C_2 = 1$.

But in our problem, since the propagation speeds C_1 and C_2 are different, the symmetry between u and v fails, and (1.7) thus seems wrong for $B_{\alpha\beta}$ and for general initial data.

Next let us consider the case where F and G have the form $\partial_\alpha u \cdot v$ or $u \partial_\alpha v$. In this case, the classical local existence theorem requires the initial

[¶]The following estimate holds for the solutions u, v of (1.1) with $C_1 = C_2 = 1, F = G = 0, u_0 = v_0 = 0$ and for $Q = Q_0$ or $Q_{\alpha\beta}$:

$$\|Q(u,v)\|_{L^2(R^{1+3})} \le C \min\{\|u_1\|_{L^2(R^3)} \|\nabla v_1\|_{L^2(R^3)}, \|\nabla u_1\|_{L^2(R^3)} \|v_1\|_{L^2(R^3)}\}.$$

conditions (1.3) with $s > n/2$ in R^{1+n}. If we suppose $C_1 = C_2$ and non-spherically symmetric case, then it seems impossible for us to construct local solutions for $u_0, v_0 \in H^1,\ u_1, v_1 \in L^2$ from the results of Lindblad [8,9] as above. However, in the case $C_2 > C_1$ the problem is well-posed for $u_0, v_0 \in H^1,\ u_1, v_1 \in L^2$ if F and G have the form (1.5). The following is the main result of this paper.

Theorem 2. *Assume the form* (1.5) *for the nonlinear terms F and G in* (1.1) *with $C_2 > C_1$. Then, for arbitrary initial data $u_0, v_0 \in H^1(R^3)$, $u_1, v_1 \in L^2(R^3)$ there exists a $T > 0$ depending only on C_1, C_2 and $\|u_0\|_{H^1(R^3)} + \|v_0\|_{H^1(R^3)} + \|u_1\|_{L^2(R^3)} + \|v_1\|_{L^2(R^3)}$ such that* (1.1) *has unique local solutions $(u(t), v(t))$ on $[0,T]$ satisfying*

$$u,\ v \in \bigcap_{j=0}^{1} C^j([0,T]; H^{1-j})$$

and

$$\int_0^T \|\partial_\alpha u \cdot v(t)\|^2_{L^2(R^3)} dt < \infty, \qquad \alpha = 0, 1, 2, 3.$$

Remark 1. Notice the asymmetry in (1.5). Let $B^*_\alpha(u,v) = u\partial_\alpha v, \quad \alpha = 0, 1, 2, 3$. From the proof of Theorem 1, it is conjectured that even in the case $C_2 > C_1$ the result of Theorem 2 is wrong for general initial conditions if the nonlinear terms F and G have this form.

We remark that in the case where F and G in (1.1) with $C_2 > C_1$ have the form $B_Z(u,v) = uv$, the Cauchy problem is well-posed for $u_0 \in L^2(R^3)$, $u_1 \in \dot{H}^{-1}(R^3)$, $v_0 \in H^1(R^3)$ and $v_1 \in L^2(R^3)$. This is proved by using the following corollary of Theorem 1.

Corollary 2. *Let u, v be the solutions of* (1.6)*. Suppose $C_2 > C_1$. Then, for the form $B_Z(u,v) = uv$, we have*

$$\begin{aligned}
&\int_0^T \int_{R^3} |B_Z(u,v)|^2 dx\, dt \\
&\le C\big(\|u_0\|_{L^2(R^3)} + \|\omega_0^{-1} u_1\|_{L^2(R^3)} + \int_0^T \|\omega_0^{-1} F(t)\|_{L^2(R^3)} dt\big)^2 \\
&\qquad \times \big(\|\nabla v_0\|_{L^2(R^3)} + \|v_1\|_{L^2(R^3)} + \int_0^T \|G(t)\|_{L^2(R^3)}\, dt\big)^2
\end{aligned}$$

for all $T > 0$ and $v_0 \in H^1(R^3)$, $u_0, v_1 \in L^2(R^3)$, $u_1 \in \dot{H}^{-1}(R^3)$, $F \in L^1(R_+, \dot{H}^{-1}(R^3))$, $G \in L^1(R_+, L^2(R^3))$.

Systems of wave equations with different propagation speeds appear in the theory of asymmetric elasticity and in crystal optics (see, e.g., [1], [2] and [10]). For example, the equations of classical elastodynamics are

$$\rho(x)\partial_t^2 u_i = \big(C_{ijkl}(x)u_{k,l} + \frac{\partial W}{\partial(u_{i,j})}\big)_{,j},$$

where $u_{i,j} = \partial_j u_i$, $\rho(x)$ is the mass density, $C_{ijkl}(x)$ are the elasticities, $W = W(\nabla u)$ is the nonlinear portion of the strain energy, and u is the displacement of a body (see, e.g., [7]).

Another motivation is that the problem of minimizing regularity conditions for (1.1) with $C_2 > C_1$ is related to the global existence problem. Let us recall the global existence result for nonlinear wave equations satisfying the null conditions.

It is well-known that in three space dimensions the scalar wave equations with quadratic nonlinearity have no global solutions in general even for small initial data. However, the system of wave equations with the same propagation speeds whose nonlinear terms satisfy the null conditions has global solutions for small initial data (see Klainerman [3]). This result is due to good decay estimates. The nonlinear terms satisfying the null conditions provide us a good decay property, which one can show by using the generators of the Lorentz group. Also, the Cauchy problem for the system with the nonlinear terms satisfying the null conditions is known to be well-posed under lower regularity assumptions by [4, 5].

On the other hand, the quadratic nonlinear terms of the coupled system (1.1) with $C_1 \neq C_2$ have also a good decay property, because the propagation speeds are different. Using this property, Kovalyov [6] proved the existence of global solutions of the system (1.1) with $C_1 \neq C_2$, F, $G = \partial_\alpha u \partial_\beta v$ for small initial data. The following natural question arises : can we improve the regularity requirements on the initial data to ensure well-posedness for systems of quadratic nonlinear wave equations with *distinct* propagation speeds? Our study began with this question. Although we have succeeded in reducing the regularity assumptions for some nonlinearity as above, we do not have any result which indicates a relation between decay estimates and minimal regularity assumptions.

Our plan in the present paper is as follows. In Section 2 we prove Theorem 1 following the method of [12]. In Section 3 we describe the proof of Theorem 2 by the standard iteration argument.

We conclude this section by giving several notations. For $\xi, \eta \in R^3$ we put $\xi \cdot \eta = \sum_{j=1}^3 \xi_j \eta_j$. Let $\mathcal{S}$ be the Schwartz space on R^3. For $f \in \mathcal{S}(R^3)$ we define the Fourier transform $\hat{f}(\xi)$ of f with respect to the space variables by

$$\hat{f}(\xi) = \int_{R^3} e^{-ix\cdot\xi} f(x)\, dx.$$

For a multi-index $\lambda = (\lambda_0, \lambda_1, \cdots, \lambda_n)$, we put $\partial^\lambda = \partial_t^{\lambda_0}\partial_1^{\lambda_1}\cdots\partial_n^{\lambda_n}$. In the course of calculations below, the various constants are simply denoted by C.

2. Proof of Theorem 1. We follow the method of [12]. We first prove part (i). we may assume that $\alpha = 0$ and $u_1 = v_1 = F = G = 0$ since the estimates for the more general case follow from the proof in this special case and the Duhamel Principle. We write $u = \frac{u_+ + u_-}{2}$, $v = \frac{v_+ + v_-}{2}$, where

$$u_\pm(t,x) = (2\pi)^{-3}\int_{R^3} e^{\pm iC_1 t|\xi|}\hat{u}_0(\xi)e^{ix\cdot\xi}\, d\xi,$$
$$v_\pm(t,x) = (2\pi)^{-3}\int_{R^3} e^{\pm iC_2 t|\xi|}\hat{v}_0(\xi)e^{ix\cdot\xi}\, d\xi.$$

Hence, $B_0(u,v)$ is expressed by

$$B_0(u,v) = \tfrac{1}{4}\{B_0(u_+,v_+) + B_0(u_+,v_-) + B_0(u_-,v_+) + B_0(u_-,v_-)\}.$$

We first estimate $B_0(u_+,v_+)$. Using the Fourier transform and making the change of variables $\xi \to \xi + \eta$, we get

$$B_0(u_+,v_+)(t,x) = (2\pi)^{-6}\int_{R^3}\int_{R^3} e^{i\cdot x(\xi+\eta)} iC_1|\xi| e^{iC_1 t|\xi| + iC_2 t|\eta|}\hat{u}_0(\xi)\hat{v}_0(\eta)d\xi\, d\eta.$$

Using spherical coordinates $\eta = \rho\omega$ with $|\omega| = 1$, we obtain

$$\|B_0(u_+,v_+)\|_{L^2(R^{1+3})} \le C\int_{S^2}\|T_\omega^{++}(u_0,v_0)\|_{L^2(R^{1+3})}\, d\omega, \tag{2.1}$$

where

$$T_\omega^{++}(u_0,v_0)(t,x) = \int_{R^3}\int_R e^{i\cdot x(\xi+\rho\omega)+iC_1 t|\xi|+iC_2 t\rho}|\xi|\hat{u}_0(\xi)\hat{v}_0(\rho\omega)\rho^2\, d\rho\, d\xi.$$

For each fixed ω, we change variables $(\tau,\gamma) = (C_1|\xi| + C_2\rho, \xi + \rho\omega)$. Note that the Jacobian is bounded below by some constant since

$$J \equiv \frac{\partial(\tau,\gamma)}{\partial(\rho,\xi)} = \left|C_2 - C_1\omega\cdot\frac{\xi}{|\xi|}\right| \ge C_2 - C_1 > 0.$$

Then we have

$$T_\omega^{++}(u_0,v_0)(t,x) = \int_{R^3}\int_R e^{i\cdot x\gamma+it\tau}|\xi|\hat{u}_0(\xi)\hat{v}_0(\rho\omega)\rho^2 J^{-1}\, d\tau\, d\gamma.$$

Hence, applying the Plancherel theorem, we obtain

$$\begin{aligned}\|T_\omega^{++}(u_0,v_0)\|^2_{L^2(R^{1+3})} &\le C\int_{R^3}\int_R \rho^4|\xi|^2|\hat{u}_0(\xi)|^2|\hat{v}_0(\rho\omega)|^2 J^{-2}d\tau\, d\gamma \\ &\le C\int_{R^3}\int_R \big||\xi|\hat{u}_0(\xi)\big|^2|\rho\hat{v}_0(\rho\omega)|^2\rho^2 d\rho\, d\xi. \end{aligned} \tag{2.2}$$

Combining (2.1) and (2.2) gives the desired estimate

$$\|B_0(u_+,v_+)\|^2_{L^2(R^{1+3})} \le C\|\nabla u_0\|^2_{L^2(R^3)}\|\nabla v_0\|^2_{L^2(R^3)}.$$

We next estimate $B_0(u_+,v_-)$. As before, we have

$$\|B_0(u_+,v_-)\|_{L^2(R^{1+3})} \le C\int_{S^2}\|T_\omega^{++}(u_0,v_0)\|_{L^2(R^{1+3})}\, d\omega,$$

where

$$T_\omega^{+-}(u_0,v_0)(t,x) = \int_{R^3}\int_R e^{i\cdot x(\xi+\rho\omega)+it(C_1|\xi|-C_2\rho)}|\xi|\hat{u}_0(\xi)\hat{v}_0(\rho\omega)\rho^2\, d\rho\, d\xi.$$

We make the change of variables $(\tau,\gamma) = (C_1|\xi| - C_2\rho, \xi+\rho\omega)$. Then the Jacobian satisfies

$$J = \frac{\partial(\tau,\gamma)}{\partial(\rho,\xi)} = \Big| - C_2 - C_1\omega\cdot\frac{\xi}{|\xi|}\Big| \ge C_2 - C_1 > 0.$$

Therefore, in the same way as before we obtain the desired estimate. Finally, we can similarly obtain the desired results for $\|B_0(u_-,v_+)\|_{L^2(R^{1+3})}$ and $\|B_0(u_-,v_-)\|_{L^2(R^{1+3})}$.

Part (ii) follows from the same procedure as above. In fact, only an extra factor $|\eta|$ appears in the integrand. This completes the proof of Theorem 1.

3. Existence and uniqueness. In this section we describe the proof of Theorem 2. We consider the Cauchy problem for a system of the type

$$\begin{cases} \Box_1 u = B_\alpha(u,v) = \partial_\alpha u\cdot v, & t>0,\ \ x\in R^3, \\ \Box_2 v = B_\alpha(u,v) = \partial_\alpha u\cdot v, & t>0,\ \ x\in R^3, \\ u(0,x) = f_0(x), \quad \partial_t u(0,x) = f_1(x), & x\in R^3, \\ v(0,x) = g_0(x), \quad \partial_t v(0,x) = g_1(x), & x\in R^3, \end{cases} \tag{3.1}$$

where $C_2 > C_1 > 0$, $f_0, g_0 \in H^1(R^3)$, $f_1, g_1 \in L^2(R^3)$. The proof is based on the result of Theorem 1.

Existence: We first prove the existence part of the theorem. The following proposition holds.

Proposition 3.1. *There exist positive constants ε and C depending only on*

$$\|f_0\|_{H^1(R^3)}, \|f_1\|_{L^2(R^3)}, \|g_0\|_{H^1(R^3)}, \|g_1\|_{L^2(R^3)}$$

and C_1, C_2 such that the problem (3.1) has solutions $(u(t), v(t))$ on $[0, \varepsilon]$ satisfying

$$\begin{aligned} u, v \in & \bigcap_{j=0}^{1} C^j([0,\varepsilon]; H^{1-j}(R^3)), \\ E_\varepsilon(u,v) \equiv & \sup_{0\le t\le\varepsilon} E(u,v)(t) \qquad (3.2a) \\ \equiv & \sup_{0\le t\le\varepsilon} \sum_{0\le|\alpha|\le1} (\|\partial^\alpha u(t)\|_{L^2(R^3)} + \|\partial^\alpha v(t)\|_{L^2(R^3)}) \le C, \end{aligned}$$

and

$$\int_0^\varepsilon \int_{R^3} |B_\alpha(u,v)|^2 dx\, dt \le C. \qquad (3.2b)$$

Proof. We use an iteration argument to prove the proposition. Let us define sequences of functions $\{u_n\}, \{v_n\}$ by

$$\begin{cases} \Box_1 u_n = B_\alpha(u_{n-1}, v_{n-1}),\ t > 0, & x \in R^3, \\ \Box_2 v_n = B_\alpha(u_{n-1}, v_{n-1}),\ t > 0, & x \in R^3, \\ u_n(0,x) = f_0(x), \quad \partial_t u_n(0,x) = f_1(x), & x \in R^3, \\ v_n(0,x) = g_0(x), \quad \partial_t v_n(0,x) = g_1(x), & x \in R^3 \end{cases} \qquad (3.3)$$

for $n \ge 1$ and $u_0 = v_0 = 0$. It suffices to show that there exist $\varepsilon \le 1$ and $M \ge 1$ such that

$$E(u_n, v_n)(t) \le \big(1 + \frac{1}{C_1} + \frac{1}{C_2}\big) M \quad \text{for } 0 \le t \le \varepsilon \text{ and all } n, \qquad (3.4a)$$

$$E(u_n - u_{n-1}, v_n - v_{n-1})(t) \le \frac{M}{2^{n-1}} \quad \text{for } 0 \le t \le \varepsilon \text{ and all } n. \qquad (3.4b)$$

To prove (3.4) we first show the following inequalities by induction on n :

$$\int_0^\varepsilon \|B_\alpha(u_n, v_n - v_{n-1})(s)\|^2_{L^2(R^3)} ds \le \frac{C_0 M^4}{2^{2(n-1)}}, \qquad (3.5)$$

$$\int_0^\varepsilon \|B_\alpha(u_n - u_{n-1}, v_n)(s)\|^2_{L^2(R^3)} ds \le \frac{C_0 M^4}{2^{2(n-1)}}, \qquad (3.6)$$

$$\int_0^\varepsilon \|B_\alpha(u_n, v_n)(s)\|^2_{L^2(R^3)}\, ds \le C_0 M^4, \qquad (3.7)$$

where C_0 is the constant in the estimate of Theorem 1(i).

Let $M \geq 2(\|f_0\|_{H^1} + \|f_1\|_{L^2} + \|g_0\|_{H^1} + \|g_1\|_{L^2})$ and $M \geq 1$. For $n = 1$, (3.5)–(3.7) hold by Theorem 1.

Assume that (3.5)–(3.7) are true for some $n \geq 1$. Then, for $n+1$, we have from Theorem 1,

$$\begin{aligned}
&\int_0^\varepsilon \|B_\alpha(u_{n+1}, v_{n+1})(s)\|^2_{L^2(R^3)}\, ds \\
&\quad \leq C_0\big(\frac{M}{2} + \int_0^\varepsilon \|B_\alpha(u_n, v_n)(s)\|_{L^2(R^3)}\, ds\big)^4 \\
&\quad \leq C_0\big(\frac{M}{2} + \varepsilon^{1/2}\big(\int_0^\varepsilon \|B_\alpha(u_n, v_n)(s)\|^2_{L^2(R^3)}\, ds\big)^{1/2}\big)^4 \\
&\quad \leq C_0 M^4\big(\frac{1}{2} + \varepsilon^{1/2} C_0^{1/2} M\big)^4,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^\varepsilon \|B_\alpha(u_{n+1}, v_{n+1} - v_n)(s)\|^2_{L^2(R^3)} ds \\
&\leq C_0\big(\frac{M}{2} + \int_0^\varepsilon \|B_\alpha(u_n, v_n)(s)\|_{L^2(R^3)} ds\big)^2 \\
&\quad \times \big(\int_0^\varepsilon \|B_\alpha(u_n, v_n) - B_\alpha(u_{n-1}, v_{n-1})\|_{L^2(R^3)} ds\big)^2 \\
&\leq C_0\big(\frac{M}{2} + \varepsilon^{1/2}\big(\int_0^\varepsilon \|B_\alpha(u_n, v_n)(s)\|^2_{L^2(R^3)} ds\big)^{1/2}\big)^2 \\
&\quad \times \big(\varepsilon^{1/2}\big(\int_0^\varepsilon \|B_\alpha(u_n, v_n - v_{n-1})(s)\|^2_{L^2(R^3)} ds\big)^{1/2} \\
&\quad + \varepsilon^{1/2}\big(\int_0^\varepsilon \|B_\alpha(u_n - u_{n-1}, v_{n-1})(s)\|^2_{L^2(R^3)} ds\big)^{1/2}\big)^2 \\
&\leq C_0\big(\frac{M}{2} + \varepsilon^{1/2} C_0^{1/2} M^2\big)^2 4\varepsilon C_0 \frac{M^4}{2^{2(n-1)}}.
\end{aligned}$$

If we choose ε so small that

$$\varepsilon^{1/2} C_0^{1/2} M \leq \frac{1}{2}, \tag{3.8}$$

$$\big(\frac{M}{2} + \varepsilon^{1/2} C_0^{1/2} M^2\big)^2 4\varepsilon C_0 \leq \frac{1}{4}, \tag{3.9}$$

then we obtain

$$\begin{aligned}
&\int_0^\varepsilon \|B_\alpha(u_{n+1}, v_{n+1})(s)\|^2_{L^2(R^3)} ds \leq C_0 M^4, \\
&\int_0^\varepsilon \|B_\alpha(u_{n+1}, v_{n+1} - v_n)(s)\|^2_{L^2(R^3)} ds \leq \frac{C_0 M^4}{2^{2n}}.
\end{aligned}$$

Therefore, (3.5) and (3.7) hold for $n \geq 1$. In the same way, inequality (3.6) is proved.

Now we are in a position to prove (3.4). We rewrite (3.3) as the following integral equations.

$$\begin{cases} u_n(t) = \cos C_1\omega_0 t \cdot f_0 + \dfrac{\sin C_1\omega_0 t}{C_1\omega_0} f_1 \\ \qquad\qquad + \displaystyle\int_0^t \frac{\sin C_1\omega_0(t-s)}{C_1\omega_0} B_\alpha(u_{n-1}, v_{n-1})(s)ds, \\ v_n(t) = \cos C_2\omega_0 t \cdot g_0 + \dfrac{\sin C_2\omega_0 t}{C_2\omega_0} g_1 \\ \qquad\qquad + \displaystyle\int_0^t \frac{\sin C_2\omega_0(t-s)}{C_2\omega_0} B_\alpha(u_{n-1}, v_{n-1})(s)ds, \end{cases} \tag{3.10}$$

where $\omega_0 = (-\Delta)^{1/2}$. From the Plancherel theorem, we have

$$\begin{aligned} &\|u_n(t)\|_{H^1(R^3)} + \|v_n(t)\|_{H^1(R^3)} \\ &\quad \leq \|f_0\|_{H^1(R^3)} + \big(t + \frac{1}{C_1}\big)\|f_1\|_{L^2(R^3)} \\ &\qquad + \|g_0\|_{H^1(R^3)} + \big(t + \frac{1}{C_2}\big)\|g_1\|_{L^2(R^3)} \\ &\qquad + 2\int_0^t (t-s)\|B_\alpha(u_{n-1}, v_{n-1})(s)\|_{L^2(R^3)}\, ds \\ &\qquad + \big(\frac{1}{C_1} + \frac{1}{C_2}\big)\int_0^t \|B_\alpha(u_{n-1}, v_{n-1})(s)\|_{L^2(R^3)}\, ds. \end{aligned} \tag{3.11}$$

When $n = 1$, from (3.11) we have

$$E(u_1, v_1)(t) \leq \big(1 + \frac{1}{C_1} + \frac{1}{C_2} + \varepsilon\big)\frac{M}{2} \equiv (C_{12} + \varepsilon)\frac{M}{2},$$

where $C_{12} = 1 + 1/C_1 + 1/C_2$. When $n \geq 2$, by (3.5)–(3.7) we have for $0 \leq t \leq \varepsilon$,

$$\begin{aligned} &E(u_n, v_n)(t) \\ &\leq (C_{12} + \varepsilon)\frac{M}{2} + \big(2\varepsilon + \frac{1}{C_1} + \frac{1}{C_2}\big)\varepsilon^{1/2}\Big(\int_0^\varepsilon \|B_\alpha(u_{n-1}, v_{n-1})(s)\|^2_{L^2(R^3)} ds\Big)^{1/2} \\ &\leq (C_{12} + \varepsilon)\frac{M}{2} + \big(2\varepsilon + \frac{1}{C_1} + \frac{1}{C_2}\big)\varepsilon^{1/2} C_0^{1/2} M^2 \\ &= M\Big(\frac{C_{12} + \varepsilon}{2} + \big(2\varepsilon + \frac{1}{C_1} + \frac{1}{C_2}\big)\varepsilon^{1/2} C_0^{1/2} M\Big) \end{aligned}$$

and

$$
\begin{aligned}
&E(u_n-u_{n-1},v_n-v_{n-1})(t)\\
&\le \big(2\varepsilon+\frac{1}{C_1}+\frac{1}{C_2}\big)\big(\int_0^\varepsilon \|B_\alpha(u_{n-1},v_{n-1})-B_\alpha(u_{n-2},v_{n-2})(s)\|_{L^2(R^3)}ds\big)\\
&\le \big(2\varepsilon+\frac{1}{C_1}+\frac{1}{C_2}\big)\times\big(\int_0^\varepsilon (\|B_\alpha(u_{n-1},v_{n-1}-v_{n-2})(s)\|_{L^2(R^3)}\\
&\quad+\|B_\alpha(u_{n-1}-u_{n-2},v_{n-2})(s)\|_{L^2(R^3)})ds\big)\\
&\le \big(2\varepsilon+\frac{1}{C_1}+\frac{1}{C_2}\big)\varepsilon^{1/2}\times\big\{\big(\int_0^\varepsilon \|B_\alpha(u_{n-1},v_{n-1}-v_{n-2})(s)\|^2_{L^2(R^3)}ds\big)^{1/2}\\
&\quad+\big(\int_0^\varepsilon \|B_\alpha(u_{n-1}-u_{n-2},v_{n-2})(s)\|^2_{L^2(R^3)}ds\big)^{1/2}\big\}\\
&\le \big(2\varepsilon+\frac{1}{C_1}+\frac{1}{C_2}\big)\varepsilon^{1/2}2\frac{C_0^{1/2}M^2}{2^{n-2}}.
\end{aligned}
$$

If we choose ε so small that

$$\varepsilon+\frac{1}{2}\le C_{12}\quad\text{and}\quad \big(2\varepsilon+\frac{1}{C_1}+\frac{1}{C_2}\big)\varepsilon^{1/2}2C_0^{1/2}M\le\frac{1}{2},$$

then we obtain (3.4). Inequalities (3.4) imply that there exist $(u,v)\in(\cap_{j=0}^1 C^j([0,\varepsilon];\ H^{1-j}))\times(\cap_{j=0}^1 C^j([0,\varepsilon];H^{1-j}))$ such that (u_n,v_n) converge to (u,v) as $n\to\infty$. Clearly, (u,v) satisfy the integral equations (3.10). This completes the proof of Proposition 3.1.

Uniqueness: We next prove the uniqueness part of the theorem.

Proposition 3.2. *Let (u,v) and (u',v') be the solutions of* (3.1) *satisfying* (3.2a) *and* (3.2b) *with u,u',v,v' and having the same initial data $f_0,g_0\in H^1$, $f_1,g_1\in L^2$. Then, $u=u'$ and $v=v'$.*

Proof. In view of (3.10) and the Plancherel theorem, we have

$$
\begin{aligned}
&E(u-u',v-v')(t)\\
&\le 2\int_0^t (t-s)(\|B_\alpha(u,v-v')(s)\|_{L^2(R^3)}+\|B_\alpha(u-u',v')(s)\|_{L^2(R^3)})ds\\
&+\big(\frac{1}{C_1}+\frac{1}{C_2}\big)\int_0^t(\|B_\alpha(u,v-v')(s)\|_{L^2(R^3)}+\|B_\alpha(u-u',v')(s)\|_{L^2(R^3)})ds.
\end{aligned}
\tag{3.12}
$$

Furthermore, Theorem 1 yields

$$\begin{aligned}
&\int_0^t (\|B_\alpha(u,v-v')(s)\|_{L^2(R^3)} + \|B_\alpha(u-u',v')(s)\|_{L^2(R^3)})ds \\
&\quad \le t^{1/2}\big\{\big(\int_0^t \|B_\alpha(u,v-v')(s)\|^2_{L^2(R^3)}ds\big)^{1/2} \\
&\qquad + \big(\int_0^t \|B_\alpha(u-u',v')(s)\|^2_{L^2(R^3)}ds\big)^{1/2}\big\} \\
&\quad \le Ct^{1/2}\big(\|f_0\|_{H^1} + \|f_1\|_{L^2} + \int_0^t \|B_\alpha(u,v)\|_{L^2(R^3)}ds\big) \\
&\qquad \times \big(\int_0^t (\|B_\alpha(u,v-v')(s)\|_{L^2(R^3)} + \|B_\alpha(u-u',v')(s)\|_{L^2(R^3)})ds\big) \\
&\qquad + Ct^{1/2}\big(\|g_0\|_{H^1} + \|g_1\|_{L^2} + \int_0^t \|B_\alpha(u',v')\|_{L^2(R^3)}ds\big) \\
&\qquad \times \big(\int_0^t (\|B_\alpha(u,v-v')(s)\|_{L^2(R^3)} + \|B_\alpha(u-u',v')(s)\|_{L^2(R^3)})ds\big) \\
&\quad \le Ct^{1/2}\big(\int_0^t (\|B_\alpha(u,v-v')(s)\|_{L^2(R^3)} + \|B_\alpha(u-u',v')(s)\|_{L^2(R^3)})ds\big)
\end{aligned}$$

for $0 \le t \le \varepsilon$. Here, since ε in Proposition 3.1 is sufficiently small, we obtain

$$\int_0^t (\|B_\alpha(u,v-v')(s)\|_{L^2(R^3)} + \|B_\alpha(u-u',v')(s)\|_{L^2(R^3)})ds \equiv 0$$

for all $0 \le t \le \varepsilon$. Thus, (3.12) implies

$$E(u-u',v-v')(t) \equiv 0 \quad \text{for all} \quad 0 \le t \le \varepsilon.$$

This completes the proof. □

Therefore, we have completed the proof of Theorem 2.

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