Differential and Integral Equations Volume 12, Number 3, May 1999, 395–418.

COMPLEX MULTIPLICATIVE PERTURBATIONS OF ELLIPTIC OPERATORS: HEAT KERNEL BOUNDS AND HOLOMORPHIC FUNCTIONAL CALCULUS

Xuan Thinh Duong
Department of Mathematics, Macquarie University, NSW 2109, Australia

El Maati Ouhabaz
Equipe d'Analyse et de Mathématiques Appliquées, Université de Marne-La-Vallée
Cité Descartles, 77454 Marne-la-Vallee Cedex 2, France

(Submitted by: Herbert Amann)

Abstract. We study heat kernel bounds, regularity on space variables and the holomorphic functional calculus on L^p for operators of type bA where b is a complex bounded function and A is a second-order elliptic operator.

1. Introduction. Consider a triple (X, d, μ) consisting of a topological space X equipped with a metric d and a measure μ. Suppose that A is a linear operator on $L^2(X) := L^2(X, \mu)$ such that $-A$ generates an analytic semigroup e^{-tA}. Let $b : X \to \mathbb{C}$ be a bounded measurable function such that its real part $\Re b$ is bounded below by a positive constant. Under certain additional assumptions on A and b, the operator $-bA$ with domain $D(bA) = D(A)$ is a generator of an analytic semigroup e^{-tbA} on $L^2(X)$. The aims of this paper are the following.

1) Suppose that the semigroup e^{-tA} has a kernel $p_t(x, y)$ which is positive and satisfies an upper bound of gaussian type, namely

$$0 \leq p_t(x, y) \leq Ct^{-\alpha} e^{-c\frac{d^2(x,y)}{t}} \text{ for a.e. } x, y \in X \text{ and all } t > 0, \tag{1.1}$$

we show that the semigroup e^{-tbA} has a kernel $k_t(x, y)$ which satisfies the same upper bound. That is

$$|k_t(x, y)| \leq Ct^{-\alpha} e^{-c\frac{d^2(x,y)}{t}} \text{ for a.e. } x, y \in X \text{ and all } t > 0 \tag{1.2}$$

Received for publication October 1997.
AMS Subject Classifications: 47A63, 47F05, 58G11.

with some new constants C and c.

2) Consider the case $X = \mathbb{R}^n$ and A is a second order uniformly elliptic operator whose coefficients are real and Hölder continuous. We study the regularity in space variables of the heat kernel $k_t(x,y)$. It is shown that $k_t(.,y)$ is $C^{1,\eta}(\mathbb{R}^n)$ for all $\eta \in (0,1)$. Moreover, the space derivative $\nabla_x k_t(x,y)$ has a bound and Hölder bound which satisfy (1.2) (with $\alpha + \frac{1}{2}$ in place of α for $\nabla_x k_t(x,y)$ and $\alpha + \frac{1}{2} + \frac{\eta}{2}$ for its Hölder bound).

If the adjoint operator A^* has the same properties as A, then the "regularized" kernel $k_t(x,y)b(y)$ of $e^{-tbA}b$ has the same regularity properties in the $y-$variable.

We also show that all these estimates hold for $\frac{\partial}{\partial t} k_t(x,y)$ and $\nabla_x \frac{\partial}{\partial t} k_t(x,y)$ with the additional term $\frac{1}{t}$ and $\frac{1}{t^{3/2}}$, respectively.

3) If X and A are as in 2) with the additional assumption that A is of divergence form with some smoothness on its coefficients then bA is shown to have a bounded holomorphic functional calculus on $L^p(X), 1 < p < \infty$.

These results are in the spirit of those obtained recently by McIntosh and Nahmod [22] who studied the operator $-b\Delta$ on $L^2(\mathbb{R}^n)$ (here Δ is the Laplacian) and obtained polynomial decay on the heat kernel as well as on its space derivatives and also Hölder bounds. They also show that $-b\Delta$ has a bounded holomorphic functional calculus on $L^p(\mathbb{R}^n)$. The results in the present paper extend those in [22] in two directions. Firstly, improve the polynomial decay to exponential one and secondly, extend the results to a large class of elliptic operators with less regularity on their coefficients acting on domains more general than $\mathbb{R}^n$.

Our approach to prove the gaussian upper bound (1.2) is based on the idea of writing the resolvent of $-bA$ for $\lambda > 0$ as a resolvent of a "Schrödinger" operator with a complex potential and then use the fact that this resolvent is pointwise bounded by the resolvent of this operator with only the real part of the potential. This, of course, is valid for all powers of these resolvents. From this and (1.1) we deduce that for a large enough integer m, the power $(\lambda I + bA)^{-m}$ has an exponential decay. The rest of the proof consists of extending this decay to all λ in a sector of angle $> \frac{\pi}{2}$. This is achieved by applying the Phragmen-Lindelöf principle to a suitable function. This idea has been used by Davies [10] to extend heat kernel bounds for $t > 0$ to complex t. Once the exponential decay is proved for $(\lambda I + bA)^{-m}$ for all λ in this sector, then (1.2) is deduced easily by a standard contour integral. This approach is flexible enough and works for more general bounds (not necessarily of gaussian type) but as we are concerned with applications to uniformly elliptic operators on $\mathbb{R}^n$ or domains of $\mathbb{R}^n$ we only present the proof to the bounds of gaussian type. Note that the approach in [22] is different and is based on Sobolev embedding to obtain the uniform bound $t^{-n/2}$ and a commutator estimate to obtain the polynomial decay.

To prove the results in 2), we use the same strategy as in [22] which relies on Sobolev embedding. However, by taking the advantage of having gaussian bounds for $k_t(x,y)$ we obtain gaussian bounds for $\nabla_x k_t(x,y)$ and for its Hölder bound.

The bounds on $\frac{\partial}{\partial t}k_t(x,y)$ and $\nabla_x \frac{\partial}{\partial t}k_t(x,y)$ are shown by using the resolvent estimates obtained in the proof of the results in 1).

The boundedness of the H_∞ functional calculus on L^2 is equivalent to quadratic estimates for bA and its adjoint (see [21], [1] and [5]). In order to prove these quadratic estimates we use the estimates shown for the time derivative $\frac{\partial}{\partial t}k_t(x,y)$ and then conclude by applying a result of Semmes [28]. This shows that the operator $f(bA)$ where f is a bounded holomorphic function on a sector which contains the spectrum of bA, is a Calderón-Zygmund operator, hence it is bounded on L^p, $1 < p < \infty$. Note that in the case where b is real, the boundedness of H_∞ functional calculus on L^2 is known by different method which does not assume the Hölder continuity of the kernel, the boundedness of H_∞ functional calculus on L^p then follows from a result of Duong and Robinson [13]. See Remark 4.1. We note that the idea of using Semmes' theorem to show boundedness of holomorphic functional calculi on L^2 was first used in [22]. Note also that elliptic operators which possess some smoothness on the coefficients were known to have bounded $H_\infty-$ functional calculus on L^p spaces. Recently, Amann, Hieber and Simonett [14] proved the bounded $H_\infty-$ functional calculus under the Hölder continuity assumption of the coefficients. Duong and Simonett [14] extend this to the case of uniformly continuous coefficients. In the present paper, as in [22] there is no smoothness assumptions on the function b.

Notation. The norm in $L^p(X)$ is denoted by $\|.\|_p$ and the pairing between $L^p(X)$ and its dual $L^q(X)$ by $(.,.)$. For any real-valued measurable functions f and g, the inequality $f \le g$ is always understood in the $\mu-$a.e. sense. For $\theta \in [0,\pi], \Sigma(\theta)$ denotes the sector $\{z \in \mathbb{C}, z \neq 0, |\arg z| < \theta\}$. For $z \in \mathbb{C}, \Re z$ and $\Im z$ denote respectively the real and imaginary parts of z.

$W^{m,p}(\mathbb{R}^n)$ is the classical Sobolev spaces and $C^{1,\eta}(\mathbb{R}^n)$ is the space of bounded C^1-functions such that their (first-order) derivatives are bounded and uniformly Hölder continuous of order η on $\mathbb{R}^n$.

Acknowledgments. This work began while the second author was visiting

Macquarie University. He would like to thank Alan McIntosh for the invitation. Both authors thank Alan McIntosh for explaining the results in [22] and for helpful discussions.

2. Heat kernel bounds. Let (X, d, μ) denote a metric space X equipped with a measure μ. Consider an operator A on $L^2(X)$ with domain $D(A)$ such that $-A$ generates a bounded analytic semigroup e^{-tA}. Let $b : X \to \mathbb{C}$ be a measurable function which satisfies

$$b \in L^\infty(X), \text{and } \Re b \geq \delta > 0, \text{ where } \delta \text{ is a constant }. \tag{2.1}$$

Note that the analyticity assumption on e^{-tA} is equivalent to the existence of $\nu \in (0, \frac{\pi}{2})$ such that the sector $\Sigma(\nu + \frac{\pi}{2})$ is a subset of the resolvent set $\rho(-A)$ of $-A$ and

$$\sup_{\lambda \in \Sigma(\nu + \frac{\pi}{2})} \|\lambda(\lambda I + A)^{-1}\|_{\mathcal{L}(L^2(X))} := M < \infty. \tag{2.2}$$

Our aim in this section is to study heat kernel bounds of the multiplicative perturbation bA. We first need to ensure the existence of the semigroup e^{-tbA}.

Proposition 2.1. 1) *Suppose that A satisfies the assumption* (2.2) *and let $b \in L^\infty(X)$. If there exists a constant $k > 0$ such that either $\|1 - kb\|_\infty < \frac{1}{M+1}$ or $\|1 - kb^{-1}\|_\infty < \frac{1}{M}$ then $-bA$ with domain $D(A)$ generates a bounded analytic semigroup on $L^2(X)$.*

2) *If A is sectorial, that is $(Au, u) \in \Sigma(\frac{\pi}{2} - \nu)$ for all $u \in D(A)$ and if b satisfies* (2.1) *and $|arg b| \leq w$* for a.e. *$x \in X$ with some $w < \nu$ then $-bA$ (defined on $D(A)$) generates a bounded analytic semigroup on $L^2(X)$.*

Proof. 1) Let $\lambda \in \Sigma(\nu + \frac{\pi}{2})$ and write that

$$\lambda I + kbA = (I - (I - kb)A(\lambda I + A)^{-1})(\lambda I + A).$$

The norm of the operator $(I - kb)A(\lambda I + A)^{-1}$ is bounded by

$$\|1 - kb\|_\infty \|A(\lambda I + A)^{-1}\|_{\mathcal{L}(L^2(X))},$$

hence less than 1 if $\|1 - kb\|_\infty < \frac{1}{M+1}$. This implies that $\lambda \in \rho(-bA)$ and

$$\sup_{\lambda \in \Sigma(\nu + \frac{\pi}{2})} \|\lambda(\lambda I + bA)^{-1}\|_{\mathcal{L}(L^2(X)} < \infty.$$

Therefore, $-kbA$ (and hence $-bA$) generates a bounded analytic semigroup.

In the case $\|1-kb^{-1}\|_\infty < \frac{1}{M}$, write

$$\lambda I + k^{-1}bA = k^{-1}b(I-(-kb^{-1}+I)\lambda(\lambda I+A)^{-1})(\lambda I+A)$$

and argument is the same as above.

2) Let $u \in D(A)$ and $\lambda \in \Sigma(\frac{\pi}{2}+\nu-w)$. We have

$$\|(\lambda b^{-1}+A)u\|_2\|u\|_2 \geq |(\lambda(b^{-1}u,u)+(Au,u))| = |(b^{-1}u,u)||\lambda + \frac{(Au,u)}{(b^{-1}u,u)}|$$
$$\geq c\|u\|_2^2|\lambda + \frac{(Au,u)}{(b^{-1}u,u)}|.$$

The last inequality (in which c is a positive constant) follows from (2.1).

Note that $(b^{-1}u,u) \in \Sigma(w)$ for all $u \in L^2(X)$. Hence,

$$\|(\lambda b^{-1}+A)u\|_2 \geq c\|u\|_2 \mathrm{dist}(-\lambda, \Sigma(\frac{\pi}{2}-\nu+w)).$$

Thus, $(\lambda b^{-1}+A)$ is one-one. Note that the adjoint A^* satisfies the same sectoriality assumption as A in 2). This can be seen from the fact that A sectorial implies that A is the operator associated with a sectorial sesquilinear form. This form has the same numerical range as its adjoint form and the latter contains the numerical range of A^*, see [20, Chap. IV] for details. By the same argument, we then obtain that the adjoint of $(\lambda b^{-1}+A)$ is one-one and hence $(\lambda b^{-1}+A)$ is invertible. It then follows from the above estimate that

$$\|(\lambda b^{-1}+A)^{-1}\| \leq \frac{c}{\mathrm{dist}(-\lambda, \Sigma(\frac{\pi}{2}-\nu+w))}.$$

Since $(\lambda I+bA)^{-1} = (\lambda b^{-1}+A)^{-1}b^{-1}$, we obtain the desired result. □

Both statements in this proposition remain valid if one replaces the multiplication operator $Bf = bf$ by a more general bounded operator $B \in \mathcal{L}(L^2(X))$ which satisfies the assumptions like for b above. A similar result to that of 2) was proved in [5].

Note that if $(Au,u) \geq 0$ for all $u \in D(A)$ and b satisfies (2.1) then $-bA$ generates a bounded analytic semigroup. This is a particular case of 2) because (2.1) implies that

$$|\Im(bu,u)| \leq \|b\|_\infty(u,u) \leq \frac{\|b\|_\infty}{\delta}\Re(bu,u)$$

for all $u \in L^2(X)$.

Suppose now that for all $t > 0$, e^{-tA} has a kernel $p_t(x,y)$ which satisfies (1.1) with some positive constants C, c and α. Suppose in addition that e^{-tA} defines a bounded operator from $L^1(X)$ into $L^2(X)$ and from $L^2(X)$ into $L^\infty(X)$ which satisfies for some constant M

$$\|e^{-tA}\|_{1,2} + \|e^{-tA}\|_{2,\infty} \leq Mt^{\frac{-\alpha}{2}}. \tag{2.3}$$

Note that in general, this assumption is satisfied as a consequence of (1.1). In fact, if e^{-tA} is uniformly bounded on $L^1(X)$ and $L^\infty(X)$, then (2.3) follows from a standard interpolation argument. On the other hand, if α is such that

$$\int_X t^{-\alpha} e^{-c\frac{d^2(x,y)}{t}} d\mu(x) \leq k \text{ for a.e. } y \in X$$

for some constant k independent of t, then the upper bound (1.1) implies that e^{-tA} is uniformly bounded on $L^1(X)$ and $L^\infty(X)$. It is easy to see that the above condition is satisfied in particular if we assume that the volume of balls of center x and radius $\sqrt{t}$ is proportional to $t^{\frac{\alpha}{2}}$ (see for example [13], Proposition 2.1).

The main result of this section is the following.

Theorem 2.2. *Suppose that A satisfies* (2.2), (2.3), *b satisfies* (2.1) *and the operator $-bA$ generates a bounded analytic semigroup e^{-tbA} on $L^2(X)$. If the kernel $p_t(x,y)$ of e^{-tA} satisfies* (1.1), *then the kernel $k_t(x,y)$ of e^{-tbA} satisfies a similar upper bound, that is for some constants C and c,*

$$|k_t(x,y)| \leq Ct^{-\alpha} e^{-c\frac{d^2(x,y)}{t}} \text{ for a.e. } x, y \in X \text{ and all } t > 0.$$

Proof. The proof is divided into four steps.

Step 1. We first prove that there exists a constant c_0 such that for all $f \in L^2(X)$ and all $\lambda > 0$,

$$|(\lambda I + bA)^{-1} f| \leq \frac{1}{\delta} (\lambda c_0 I + A)^{-1} |f|. \tag{2.4}$$

Note that the positivity of $p_t(x,y)$ implies that the resolvent $(\lambda c_0 I + A)^{-1}$ is a positivity preserving operator for $\lambda > 0$. This is also the case for the operators $(sI+\lambda\Re(\frac{1}{b})+A)^{-1}$ for all $s,\lambda>0$ as a consequence of the Trotter product formula (see [26])

$$e^{-t(\lambda\Re(\frac{1}{b})+A)}f=\lim_{n\to\infty}(e^{-\frac{t}{n}\lambda\Re(\frac{1}{b})}e^{-\frac{t}{n}A})^n f,\quad \forall f\in L^2(X)$$

(in [26], the formula is given for contraction semigroups but it applies in this situation since both semigroups $e^{-t\Re(\frac{1}{b})}$ and e^{-tA} are contractions for the equivalent norm $\|f\|_* := \sup_{t\geq 0}\|e^{-tA}|f|\|_2$).

Note that by this formula we have the pointwise estimate

$$e^{-t(\lambda\Re(\frac{1}{b})+A)}|f|\leq e^{-tA}|f|,\quad \forall t>0,\ \text{and}\quad f\in L^2(X)$$

from which it follows that

$$(sI+\lambda\Re(\frac{1}{b})+A)^{-1}=\int_0^\infty e^{-st}e^{-t(\lambda\Re(\frac{1}{b})+A)}dt$$

exists for all $s,\lambda>0$.

Using again the Trotter product formula for $e^{-t(\frac{\lambda}{b}+A)}$, we deduce that

$$|e^{-t(\frac{\lambda}{b}+A)}f|\leq e^{-t(\lambda\Re(\frac{1}{b})+A)}|f|$$

which implies

$$|(sI+\frac{\lambda}{b}+A)^{-1}b^{-1}f|\leq\frac{1}{\delta}(sI+\lambda\Re(\frac{1}{b})+A)^{-1}|f|.$$

Since $\Re(\frac{1}{b})\geq\frac{\delta}{\|b\|_\infty^2}:=c_0$, it follows from the Trotter product formula that

$$|(sI+\frac{\lambda}{b}+A)^{-1}b^{-1}f|\leq\frac{1}{\delta}(sI+\lambda c_0 I+A)^{-1}|f|.$$

Since $(\frac{\lambda}{b}+A)^{-1}$ and $(\lambda c_0 I+A)^{-1}$ exist we conclude from this inequality that

$$|(\frac{\lambda}{b}+A)^{-1}b^{-1}f|\leq\frac{1}{\delta}(\lambda c_0 I+A)^{-1}|f|$$

which is the desired inequality.

Step 2. For all $\lambda>0$ and integers $m>\alpha+1$, the operator $(\lambda+bA)^{-m}$ has a kernel $R_{\lambda,m}(x,y)$ which satisfies

$$|R_{\lambda,m}(x,y)|\leq C_1\lambda^{-m+\alpha}e^{-c_1 d(x,y)\sqrt{\lambda}}\ \text{for a.e.}\quad x,y\in X, \tag{2.5}$$

where C_1, c_1 are two positive constants.

From (2.4) we obtain that for all $f \in L^2(X)$

$$|(\lambda I + bA)^{-m} f| \leq \frac{1}{\delta^m} (\lambda c_0 I + A)^{-m} |f|. \tag{2.6}$$

Since

$$(\lambda c_0 I + A)^{-m} f = \frac{1}{m!} \int_0^\infty t^{m-1} e^{-\lambda c_0 t} e^{-tA} f dt,$$

the operator $(\lambda c_0 + A)^{-m}$ has a kernel given by

$$\frac{1}{m!} \int_0^\infty t^{m-1} e^{-\lambda c_0 t} p_t(x, y) dt.$$

It is now straightforward that the assumption (1.1) implies that this kernel satisfies the estimate (2.5). From the domination property (2.6) it follows that the kernel $R_{\lambda,m}(x, y)$ satisfies (2.5). Note that the existence of this kernel is guaranteed by (2.6) and the fact that $(\lambda c_0 I + A)^{-m}$ is a bounded operator from $L^2(X)$ into $L^\infty(X)$.

Step 3. There exists $\theta \in (0, \frac{\pi}{2})$ such that for all $\lambda \in \Sigma(\theta + \frac{\pi}{2})$, the operator $(\lambda I + bA)^{-2m}$ with $m > \alpha + 1$ has a kernel $R_{\lambda,2m}(x, y)$ which satisfies the estimate

$$|R_{\lambda,2m}(x, y)| \leq C_1 |\lambda|^{-2m+\alpha} e^{-c_1 d(x,y) \sqrt{|\lambda|}} \text{ for } \; a.e. x, y \in X \tag{2.7}$$

with some constants C_1, c_1 depending on θ.

By assumptions, the semigroup e^{-tbA} is bounded, analytic. Then there exists $\nu > 0$ such that $\Sigma(\nu + \frac{\pi}{2}) \subset \rho(-bA)$ and the estimate (2.2) holds for bA. Let $m > \alpha + 1$ and $\lambda \in \Sigma(\nu + \frac{\pi}{2})$. By the resolvent equation we have

$$(\lambda I + bA)^{-2m} = (|\lambda| I + bA)^{-m} [I + (|\lambda| - \lambda)(\lambda I + bA)^{-1}]^{2m} (|\lambda| I + bA)^{-m}.$$

The property (2.2) applied to bA shows that the term $[I + (|\lambda| - \lambda)(\lambda I + bA)^{-1}]^{2m}$ is bounded on $L^2(X)$ with a uniform bound in the sector $\Sigma(\nu + \frac{\pi}{2})$. Assumption (2.3), inequality (2.6) and the formula

$$(|\lambda| I + A)^{-m} = \frac{1}{m!} \int_0^\infty t^{m-1} e^{-|\lambda| t} e^{-tA} dt$$

imply that $(|\lambda|+bA)^{-m}$ is a bounded operator from $L^2(X)$ into $L^\infty(X)$ and also from $L^1(X)$ into $L^2(X)$ with both norms controlled by $|\lambda|^{-m+\frac{\alpha}{2}}$. We then obtain that $(\lambda+bA)^{-2m}$ is a bounded operator from $L^1(X)$ into $L^\infty(X)$ with its norm controlled by $|\lambda|^{-2m+\alpha}$ (times a constant C_ν depending on ν). This implies that $(\lambda I + bA)^{-2m}$ has a kernel $R_{\lambda,2m}(x,y)$ which satisfies

$$|R_{\lambda,2m}(x,y)| \leq C_\nu |\lambda|^{-2m+\alpha} \text{ for } a.e.\ x,y \in X. \tag{2.8}$$

We now extend the estimate (2.5) to complex λ, using analytic continuation as in [10], Theorem 3.4.8.

Fix $0 < \nu_1 < \nu$. Define on $\Sigma(\nu + \frac{\pi}{2})$ the function

$$f(z) = R_{z,2m}(x,y) z^{-\alpha+2m} \exp\{ic_1 d(x,y)\sqrt{z}\frac{e^{-i(\frac{\nu_1}{2}+\frac{\pi}{4})}}{\sin(\frac{\nu_1}{2}+\frac{\pi}{4})}\},$$

where x, y are fixed in X and c_1 is the constant appearing in the estimate (2.5) with m replaced by $2m$.

For $\arg z \in (-\nu-\frac{\pi}{2}, \nu+\frac{\pi}{2})$, the function $z^\alpha = \exp(\alpha \log z)$ is analytic in the sector $\Sigma(\nu+\frac{\pi}{2})$. The analyticity of $z \to (z+bA)^{-2m}$ implies the analyticity of $z \to R_{z,2m}(x,y)$. Therefore, f is analytic in $\Sigma(\nu+\frac{\pi}{2})$.

If $z = \lambda > 0$, then $|f(\lambda)| \leq C_1$ as a consequence of (2.5) (with m replaced by $2m$). If $z = |\lambda| e^{i(\frac{\pi}{2}+\nu_1)}$, then $|f(z)| \leq C_{\nu_1}$ for some constant C_{ν_1} as a consequence of (2.8). Therefore, for $z = |z|e^{i\arg z}$,

$$\arg z \in [0, \frac{\pi}{2}+\nu_1], |f(z)| \leq C_{\nu_1} \exp(c_1 d(x,y)\frac{\sqrt{|z|}}{\sin(\frac{\nu_1}{2}+\frac{\pi}{4})}).$$

We now apply the Phragmen-Lindelöf theorem on the set $\{z, z \in \Sigma(\frac{\pi}{2} + \nu_1), \arg z \geq 0\}$ and deduce that $|f(z)|$ is bounded in this set (with some constant C_2 depending on ν_1). It follows that

$$|R_{\lambda,2m}(x,y)| \leq C_2 |\lambda|^{-2m+\alpha} \exp\{-c_1 d(x,y)\sqrt{|\lambda|}\frac{\sin(\frac{\nu_1}{2}+\frac{\pi}{4}-\frac{\arg\lambda}{2})}{\sin(\frac{\nu_1}{2}+\frac{\pi}{4})}\}$$

for all $\lambda \in \{z, z \in \Sigma(\frac{\pi}{2}+\nu_1), \arg z \geq 0\}$.

We now fix $0 < \theta < \nu_1$. For some constants $C_1(\theta), c_1(\theta)$, we have

$$|R_{\lambda,2m}(x,y)| \leq C_1(\theta)|\lambda|^{-2m+\alpha} e^{-c_1(\theta)d(x,y)\sqrt{|\lambda|}}$$

for all $\lambda \in \Sigma(\theta + \frac{\pi}{2})$ with $\arg\lambda \geq 0$. A similar argument extends this estimate to all $\lambda \in \Sigma(\frac{\pi}{2} + \theta)$.

Step 4. We now prove the heat kernel bounds using the bounds on the resolvent kernels in Step 3. The argument is well known but we give a proof for the sake of completeness.

Let $\alpha_0 \in (\frac{\pi}{2}, \frac{\pi}{2} + \theta), R > 0$ and represent the semigroup by the contour integral formula

$$e^{-tbA} = \frac{(2m-1)!}{2\pi i t^{2m-1}} \int_\Gamma e^{\lambda t} (\lambda + bA)^{-2m} d\lambda,$$

where $\Gamma = \{re^{-i\alpha_0}, r \geq R\} \cup \{Re^{i\phi}, |\phi| \leq \alpha_0\} \cup \{re^{i\alpha_0}, r \geq R\}$.

$$k_t(x,y) = \frac{(2m-1)!}{2\pi i t^{2m-1}} \int_\Gamma e^{\lambda t} R_{\lambda,2m}(x,y) d\lambda.$$

We now use similar estimates to those of Step 3 and conclude that for some constants C, c_θ and K,

$$\begin{aligned}
&|k_t(x,y)| \leq Ct^{-2m+1} \int_R^\infty e^{rt\cos\alpha_0} r^{-2m+\alpha} e^{-c_\theta d(x,y)\sqrt{R}} dr \\
&+ Ct^{-2m+1} \int_{\alpha_0}^{\alpha_0} e^{tR\cos\phi} R^{-2m+\alpha+1} e^{-c_\theta d(x,y)\sqrt{R}} d\phi \\
&\leq Ct^{-2m+1} \int_{tR}^\infty e^{u\cos\alpha_0} u^{-2m+\alpha} t^{2m-\alpha} e^{-c_\theta d(x,y)\sqrt{R}} t^{-1} du \\
&+ Ct^{-2m+1} R^{-2m+\alpha+1} e^{Rt - c_\theta d(x,y)\sqrt{R}} \\
&\leq KR^{-2m+\alpha} t^{-2m} e^{-c_\theta d(x,y)\sqrt{R}} + KR^{-2m+\alpha+1} t^{-2m+\alpha} e^{Rt - c_\theta d(x,y)\sqrt{R}}.
\end{aligned}$$

By choosing $R = \max\, \{t^{-1}, \frac{c_\theta^2 d^2(x,y)}{4t^2}\}$, we obtain

$$|k_t(x,y)| \leq Ct^{-\alpha} e^{-c\frac{d^2(x,y)}{t}}$$

for some positive constants C and c. $\square$

Remarks. 1) The Theorem 2.2 is still true for perturbations of the kind bAc where b and c are two bounded complex-valued functions satisfying the same assumptions as b in the theorem. Indeed, write $(\lambda + bAc)^{-1} = c^{-1}(c^{-1}b^{-1}\lambda + A)^{-1}b^{-1}$ for $\lambda > 0$, then use the same argument as above.

2) If we assume instead of (1.1) that $p_t(x,y)$ satisfies

$$0 \leq p_t(x,y) \leq Ct^{-\alpha} e^{-c\frac{d^2(x,y)}{t}} e^{wt}$$

with some constant w then the conclusion in Theorem 2.2 is

$$|k_t(x,y)| \leq C' t^{-\alpha} e^{-c'\frac{d^2(x,y)}{t}} e^{\frac{w}{c_0}t},$$

where c_0 is the constant in Step 1 of the above proof. This can proved as above by estimating the resolvent of $\frac{w}{c_0}I + bA$ and obtain for $\lambda > 0$

$$|(\lambda I + \frac{w}{c_0} I + bA)^{-1} f| \leq \frac{1}{\delta}(\lambda c_0 I + w + A)^{-1}|f|.$$

The heat kernel of $w + A$ satisfies (1.1) and the rest of the proof is exactly the same.

3) The theorem is stated for bounds of gaussian type but one can adapt the above proof to obtain the same result for more general bounds such as bounds of polynomial decay.

4) Suppose now that $\Im b = 0$ then for $\mu-$a.e. $x, y \in X$ and all $t > 0$ we have $k_t(x,y) \geq 0$. This can be seen from the fact that $(\lambda I + bA)^{-1} = (\lambda b^{-1} I + A)^{-1} b^{-1}$, $\lambda > 0$, is a positivity-preserving operator as explained in the above proof. Moreover, if the kernel $p_t(x,y)$ satisfies

$$\int_X p_t(x,y) d\mu(y) \leq 1 \text{ for } \mu - \text{a.e. } x \in X,$$

then the kernel $k_t(x,y)$ satisfies the same inequality. For a proof see [24]. Note that the above inequality is equivalent to the fact that the semigroup is $L^\infty-$ contractive.

5) Suppose again that $\Im b = 0$. As explained in the proof of the theorem, we can write for all $f \in L^2(X)$ and all $\lambda > 0$

$$(\lambda I + bA)^{-1}|f| \geq \frac{1}{\|b\|_\infty}(\lambda \delta^{-1} I + A)^{-1}|f|.$$

This suggests that if $p_t(x,y)$ has a lower gaussian bound then $k_t(x,y)$ may have a similar lower bound.

We now give some applications of Theorem 2.2 to differential operators.

Let $X = \Omega$ be an open set of $\mathbb{R}^n$ and $\mu = dx$ be the Lebesgue measure and d the Euclidean distance of $\mathbb{R}^n$. Let $b : \Omega \to \mathbb{C}$ be a measurable function satisfying (2.1).

1) We consider uniformly elliptic operators of the form

$$A = -\sum_{k,j=1}^{n} \frac{\partial}{\partial x_k}(a_{kj}\frac{\partial}{\partial x_j}) - \sum_{k=1}^{n} a_k \frac{\partial}{\partial x_k} + \frac{\partial}{\partial x_k}(b_k.) + a_0.$$

The operator A can be defined by sesquilinear form with Dirichlet or Neumann boundary conditions as usual. The coefficients are assumed to be real, bounded, and the matrix (a_{kj}) satisfies the usual ellipticity assumption. By adding a suitable constant to A, we obtain a sectorial operator. The fact that the coefficients of A are assumed to be real implies that e^{-tA} is positivity-preserving in both cases of Dirichlet and Neumann boundary conditions (see for example [23]). Moreover, for some constant w, the heat kernel of $(A+w)$ has the gaussian bound (1.1) with $\alpha = \frac{n}{2}$ (see [10] for the symmetric case, also [27], [4]. The domain Ω is assumed to have the extension property when A is subject to Neumann boundary conditions). If $b : \Omega \to \mathbb{C}$, satisfies (2.1) and $-bA$ generates a bounded analytic semigroup, then for some constant w' the heat kernel of $(bA + w')$ has the gaussian bound.

2) Consider a non-divergence uniformly elliptic operator given by

$$A = -\sum_{k,j=1}^{n} a_{kj} \frac{\partial^2}{\partial x_k \partial x_j} + \sum_{k=1}^{n} a_k \frac{\partial}{\partial x_k} + a_0$$

with real, Hölder continuous coefficients on $L^2(\mathbb{R}^n)$. It is known that the kernel of e^{-tA} has a gaussian bound for all $t > 0$ (after adding a suitable constant to a_0 if necessary). See for example [16], Chapter 9. Then $-bA$ generates an analytic semigroup on $L^2(\mathbb{R}^n)$ whenever b satisfies one of the conditions in Proposition 2.1 or if b is uniformly continuous ([2]). It follows from Theorem 2.2 that the heat kernel of $-bA + w$ has a gaussian bound (w is a positive constant). This is interesting because for non-divergence form operators with uniformly continuous coefficients, it is well known that gaussian bounds on heat kernels are not true in general [7], [8].

3) Let X be a Riemannian manifold, μ be the Riemann measure and $A = \Delta$ be the Laplace-Beltrami operator on X. Upper bounds of type (1.1) for the kernel of $e^{t\Delta}$ are known under various geometric conditions (see for

example [10], [17], [29]). If function $b : X \to \mathbb{C}$ satisfies (2.1) then we obtain from Theorem 2.2 a similar upper bound on the heat kernel of $b\Delta$.

We finish this section by mentioning some consequences of heat kernel bounds.

For simplicity, suppose that we are in the one of the situations 1) or 2) above. We then obtain a gaussian bound on the heat kernel of $-bA$. As a consequence of this, we have:

a) The semigroup e^{-tbA} extends to an analytic semigroup on $L^p(\Omega)$ for $1 \leq p < \infty$ and the sector of analyticity in $L^p(\Omega)$ is the same as in $L^2(\Omega)$. This was proved in [25] for self-adjoint operators and extended to the non-self-adjoint case in [13], [18], [4]. A proof of this fact follows easily from the estimate (2.7).

b) Denote by $\rho_p^+(bA)$ the right component of the resolvent set of the operator $-bA$ as an operator on $L^p(\Omega)$. It follows from [3] that $\rho_p^+(bA)$ is independent of p, $1 \leq p < \infty$.

c) Under suitable conditions on the space X, the operator bA has maximal regularity property [19], [9].

3. Regularity on space variables of the heat kernel. In this section we study regularity properties on space variables of the heat kernel $k_t(x, y)$ of the perturbed operator bA. We will consider the case where A is a second-order elliptic operator having Hölder continuous coefficients. The method which we apply is using the same strategy as in [22], but we obtain estimates for more general operators.

Consider the uniformly elliptic operator given by

$$A = -\sum_{k,j=1}^{n} a_{kj} \frac{\partial^2}{\partial x_k \partial x_j} + \sum_{k=1}^{n} a_k \frac{\partial}{\partial x_k} + a_0$$

acting on $L^2(\mathbb{R}^n)$. We assume that the coefficients are real, bounded and (a_{kj}) satisfies the usual ellipticity assumption. Moreover, we assume

$$\text{For } 1 \leq k, j \leq n,\ a_{kj} \text{ is Hölder continuous on } \mathbb{R}^n. \tag{3.1}$$

Under this assumption, the operator A considered on $L^p(\mathbb{R}^n), 1 < p < \infty$ with domain $W^{2,p}(\mathbb{R}^n)$ generates a positivity-preserving and analytic semigroup e^{-tA} on $L^p(\mathbb{R}^n)$ (see [16], [2]). Moreover, the kernel $p_t(x, y)$ of e^{-tA} satisfies the gaussian bound

$$p_t(x, y) \leq C' t^{\frac{-n}{2}} e^{-c' \frac{|x-y|^2}{t}} e^{w_0' t} \text{ for all } t > 0, \tag{3.2}$$

where C', c' and w'_0 are positive constants ([16, Chapter 9]).

Let $b : \mathbb{R}^n \to \mathbb{C}$ be a measurable function which satisfies the assumption (2.1). Through this section we assume that $-bA$ with domain $D(bA) = D(A)$ generates an analytic semigroup on $L^2(\mathbb{R}^n)$. It follows from Section 2 that e^{-tbA} is given by a kernel $k_t(x, y)$ satisfying

$$|k_t(x,y)| \le Ct^{\frac{-n}{2}} e^{-c\frac{|x-y|^2}{t}} e^{w_0 t} \text{ for all } t > 0 \tag{3.3}$$

with some positive constants C, c, w_0.

As explained before, it then follows that for all $p \in [1, \infty)$ the semigroup e^{-tbA} acts as an analytic semigroup on $L^p(\mathbb{R}^n)$. Moreover, we see from $(\lambda I + bA)^{-1} = (\lambda b^{-1} + A)^{-1} b^{-1}$ that the domain of bA as an operator on $L^p(\mathbb{R}^n)$ is $W^{2,p}(\mathbb{R}^n), 1 < p < \infty$.

We now study the regularity in the $x-$variable of the heat kernel $k_t(x, y)$. Our aim is to prove the following theorem.

Theorem 3.1. *Under the assumptions* (3.1), (3.2) *on* A, *and* (2.1) *on* b, *the function* $k_t(., y) \in C^{1,\eta}(\mathbb{R}^n)$ *for all* $\eta \in (0, 1)$ *and a.e.* $y \in \mathbb{R}^n$. *More explicitly, the following estimates are satisfied*

$$|k_t(x,y) - k_t(x',y)| \le C_1 |x - x'|^\eta t^{\frac{-\eta}{2}} t^{\frac{-n}{2}} e^{-c_1\frac{|x-y|^2}{t}} e^{wt} \tag{3.4}$$

$$|\nabla_x k_t(x,y)| \le C_1 t^{\frac{-1}{2}} t^{\frac{-n}{2}} e^{-c_1\frac{|x-y|^2}{t}} e^{wt} \tag{3.5}$$

$$|\nabla_x k_t(x,y) - \nabla_x k_t(x',y)| \le C_1 |x - x'|^\eta t^{\frac{-1-\eta}{2}} t^{\frac{-n}{2}} e^{-c_1\frac{|x-y|^2}{t}} e^{wt} \tag{3.6}$$

where C_1, c_1 *and* w *are positive constants and for* $|x - x'| \le \frac{1}{2}|x - y|$ *in* (3.4) *and* (3.6).

Proof. As mentioned in the end of the last section, the estimate (3.3) implies that the semigroup e^{-tbA} acts as an analytic semigroup on $L^p(\mathbb{R}^n)$, $1 \le p < \infty$. Let us now denote by $-bA_p$ the corresponding generator on $L^p(\mathbb{R}^n)$. In particular, $e^{-tbA_1} L^1(\mathbb{R}^n) \subset D(bA_1)$ for all $t > 0$. Let $f \in L^1(\mathbb{R}^n)$ and $t > 0$. Using (3.3), there exist constants C and w so that

$$\begin{aligned} &\Big\|\frac{e^{-sbA_p} e^{-tbA_1} f - e^{-tbA_1} f}{s} - bA_1 e^{-tbA_1} f\Big\|_p \\ &\le Ce^{wt} t^{\frac{-n}{2}(1-\frac{1}{p})} \Big\|\frac{e^{-sbA_1} e^{-\frac{t}{2}bA_1} f - e^{-\frac{t}{2}bA_1} f}{s} - bA_1 e^{-\frac{t}{2}bA_1} f\Big\|_1. \end{aligned}$$

Thus, $e^{-tbA_1}f \in D(bA_p) = W^{2,p}(\mathbb{R}^n)$ for all $p \in (1,\infty)$. For $\eta \in (0,1)$, choose $p > \frac{n}{1-\eta}$. The Sobolev imbedding theorem implies that e^{-tbA} defines a bounded operator from $L^1(\mathbb{R}^n)$ into $C^{1,\eta}(\mathbb{R}^n)$, hence the kernel $k_t(.,y) \in C^{1,\eta}(\mathbb{R}^n)$ for a.e. $y \in \mathbb{R}^n$ and all $t > 0$.

In order to show (3.4), we first show that for some constants C, w

$$\|\nabla_x k_t(x,y)\|_\infty \leq C t^{\frac{-1}{2}} t^{\frac{-n}{2}} e^{wt}. \tag{3.7}$$

This inequality is a consequence of (3.5) but we give a different proof which could be of independent interest.

The Gagliardo-Nirenberg inequality ([16]) asserts that for all $p > n$ there exists a constant c such that

$$\|f\|_\infty \leq c\|f\|_p^{1-\frac{n}{p}} \|\nabla f\|_p^{\frac{n}{p}} \text{ for all } f \in W^{1,p}(\mathbb{R}^n).$$

Apply this inequality to $\nabla e^{-tbA}f$ where $t > 0$ and $f \in L^1(\mathbb{R}^n)$ are fixed, we obtain

$$\|\nabla e^{-tbA}f\|_\infty \leq c\|\nabla e^{-tbA}f\|_p^{1-\frac{n}{p}} \|\Delta e^{-tbA}f\|_p^{\frac{n}{p}}. \tag{3.8}$$

The domains $D(\Delta) = D(bA) = W^{2,p}(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ implies that the graph norms of both operators are equivalent. Hence for some constant K,

$$\|\Delta e^{-tbA}f\|_p \leq K(\|bAe^{-tbA}f\|_p + \|e^{-tbA}f\|_p).$$

The analyticity of the semigroup e^{-tbA} on $L^p(\mathbb{R}^n)$ and (3.3) imply that

$$\|\Delta e^{-tbA}f\|_p \leq K_1 t^{-1} t^{\frac{-n}{2}(1-\frac{1}{p})} \|f\|_1 \text{ for } \quad 0 < t \leq 1. \tag{3.9}$$

For the same reasons as explained above, for $0 < t \leq 1$, the norm of e^{-tbA} as an operator from $L^p(\mathbb{R}^n)$ into $W^{2,p}(\mathbb{R}^n)$ is dominated by $\frac{1}{t}$ and is bounded independently of t as an operator from $L^p(\mathbb{R}^n)$ into itself. By interpolation, its norm as an operator from $L^p(\mathbb{R}^n)$ into $W^{1,p}(\mathbb{R}^n)$ is dominated by $\frac{1}{\sqrt{t}}$. In particular,

$$\|\nabla e^{-tbA}f\|_p \leq K_2 t^{\frac{-1}{2}} t^{\frac{-n}{2}(1-\frac{1}{p})} \|f\|_1 \text{ for } \quad 0 < t \leq 1. \tag{3.10}$$

Now we combine (3.10), (3.9) and (3.8) to deduce (3.7) for $0 < t \leq 1$. A simple iteration argument gives (3.7) for all $t > 0$.

In order to prove (3.4) for $\eta \in (0,1)$, we have

$$\begin{aligned}|k_t(x,y) - k_t(x',y)| &= |k_t(x,y) - k_t(x',y)|^\eta |k_t(x,y) - k_t(x',y)|^{1-\eta} \\ &\le |x-x'|^\eta \|\nabla_x k_t(.,y)\|_\infty^\eta (|k_t(x,y)| + |k_t(x',y)|)^{1-\eta}\end{aligned}$$

(3.4) now follows from (3.3) and (3.7).

To prove (3.5), we remind the reader that $\nabla_x k_t(.,y) \in C^\eta(\mathbb{R}^n)$ for a.e. $y \in \mathbb{R}^n$ and all $t > 0$. To obtain an explicit estimate, we use Sobolev imbeddings again. Let $p > n$ and $\eta = 1 - \frac{n}{p}$. Then for all $f \in L^1(\mathbb{R}^n)$,

$$\|\nabla e^{-tbA} f\|_{C^\eta} \le C \|\nabla e^{-tbA} f\|_{W^{1,p}}.$$

As in the above proof, we obtain

$$\|\nabla e^{-tbA} f\|_{C^\eta} \le C t^{-1} t^{-\frac{n}{2}(1-\frac{1}{p})} \|f\|_1$$

for $0 < t \le 1$. Thus,

$$\|\nabla_x k_t(.,y)\|_{C^\eta} \le C t^{\frac{-1-\eta}{2}} t^{\frac{-n}{2}} e^{wt} \tag{3.11}$$

for some constants C, w and all $t > 0$.

We can now deduce (3.5) from (3.11) by the same argument as in [22] as follows. Fix $x, y \in \mathbb{R}^n,\ t > 0$ and let $M = \Re(\frac{\partial}{\partial x_j} k_t(x,y))$. Assume that $M > 0$. Let e_j be the unit vector in the j'th direction.

$$|\Re(\frac{\partial}{\partial x_j} k_t(x + he_j, y)) - \Re(\frac{\partial}{\partial x_j} k_t(x,y))| \le C t^{\frac{-1-\eta}{2}} t^{\frac{-n}{2}} e^{wt} h^\eta = c(t) h^\eta$$

for all $h > 0$. Hence,

$$\Re(\frac{\partial}{\partial x_j} k_t(x + he_j, y)) \ge \frac{M}{2}$$

whenever $h^\eta c(t) \le \frac{M}{2}$.

Apply now the upper bound (3.3) to obtain

$$\begin{aligned}2C t^{\frac{-n}{2}} e^{-c\frac{|x-y|^2}{t}} e^{w_0 t} &\ge \Re(k_t(x+he_j,y)) - \Re(k_t(x,y)) \\ &= \int_0^h \Re(\frac{\partial}{\partial x_j} k_t(x + se_j, y))\, ds \ge \frac{hM}{2}\,.\end{aligned}$$

Consequently,

$$(\frac{M}{2}c(t)^{-1})^{\frac{1}{\eta}}\frac{M}{2} \leq 2Ct^{\frac{-n}{2}}e^{-c\frac{|x-y|^2}{t}}e^{wt}$$

which implies that

$$M \leq Ct^{\frac{-1}{2}}t^{\frac{-n}{2}}e^{-c\frac{|x-y|^2}{t}}e^{wt}$$

with some new constants C, c and w. The case $M < 0$ and the imaginary part $\Im(\frac{\partial}{\partial x_j}k_t(x,y))$ are treated in the same way. This proves (3.5).

Finally, we prove (3.6). Let $0 < \eta < \eta' < 1$, $\alpha = \frac{\eta}{\eta'}$ and write

$$\begin{aligned}&|\nabla_x k_t(x,y) - \nabla_x k_t(x',y)| \\ &\qquad = |\nabla_x k_t(x,y) - \nabla_x k_t(x',y)|^{\alpha}|\nabla_x k_t(x,y) - \nabla_x k_t(x',y)|^{1-\alpha}.\end{aligned}$$

Apply (3.11) with η' in place of η and apply (3.5) to obtain

$$\begin{aligned}&|\nabla_x k_t(x,y) - \nabla_x k_t(x',y)| \\ &\leq C|x-x'|^{\eta}(t^{\frac{-1-\eta'}{2}}t^{\frac{-n}{2}})^{\alpha}e^{wt}(t^{-\frac{1}{2}-\frac{n}{2}})^{1-\alpha}[e^{-c\frac{|x-y|^2}{t}} + e^{-c\frac{|x'-y|^2}{t}}]^{1-\alpha}.\end{aligned}$$

This clearly implies (3.6). □

We can also show that the time derivative $\frac{\partial}{\partial t}k_t(x,y)$ satisfies the same estimates as in the above theorem. More precisely, we have

Theorem 3.2. *Under the same assumptions as in Theorem* 3.1, *the function* $\frac{\partial}{\partial t}k_t(.,y) \in C^{1,\eta}(\mathbb{R}^n)$ *for all* $\eta \in (0,1)$ *and a.e.* $y \in \mathbb{R}^n$ *and satisfies the following estimates*

$$|\frac{\partial}{\partial t}k_t(x,y)| \leq Ct^{\frac{-n}{2}-1}e^{-c\frac{|x-y|^2}{t}}e^{w_0 t} \text{ for all } t > 0 \tag{3.12}$$

$$|\frac{\partial}{\partial t}k_t(x,y) - \frac{\partial}{\partial t}k_t(x',y)| \leq C_1|x-x'|^{\eta}t^{\frac{-\eta}{2}-1}t^{\frac{-n}{2}}e^{-c_1\frac{|x-y|^2}{t}}e^{wt} \tag{3.13}$$

$$|\nabla_x\frac{\partial}{\partial t}k_t(x,y)| \leq C_1 t^{\frac{-3}{2}}t^{\frac{-n}{2}}e^{-c_1\frac{|x-y|^2}{t}}e^{wt} \tag{3.14}$$

$$|\nabla_x\frac{\partial}{\partial t}k_t(x,y) - \nabla_x\frac{\partial}{\partial t}k_t(x',y)| \leq C_1|x-x'|^{\eta}t^{\frac{-1-\eta}{2}-1}t^{\frac{-n}{2}}e^{-c_1\frac{|x-y|^2}{t}}e^{wt}. \tag{3.15}$$

Here C_1, c_1, w *are positive constants* (w_0 *as in* (3.3)) *and* $|x-x'| \leq \frac{1}{2}|x-y|$ *in* (3.13) *and* (3.15).

Proof. Note that $\frac{\partial}{\partial t}k_t(x,y)$ is the kernel of

$$\frac{\partial}{\partial t}e^{-tbA} = -bAe^{-tbA}.$$

We write

$$bAe^{-tbA} = e^{-\frac{t}{2}bA}bAe^{-\frac{t}{2}bA}$$

and observe as in the above proof that bAe^{-tbA} is bounded from $L^1(\mathbb{R}^n)$ into $W^{2,p}(\mathbb{R}^n)$ for all $p \in (1,\infty)$. This implies that $\frac{\partial}{\partial t}k_t(.,y) \in C^{1,\eta}(\mathbb{R}^n)$ for all $\eta \in (0,1)$ and a.e. $y \in \mathbb{R}^n$. As explained in Section 2, the semigroup e^{-tbA} extends analytically in $L^1(\mathbb{R}^n)$, and this implies that $bAe^{-\frac{t}{2}bA}$ is a bounded operator on $L^1(\mathbb{R}^n)$ with its norm controlled by $\frac{1}{t}$. Using this and (3.7) we obtain

$$\|\nabla e^{-\frac{t}{2}bA}bAe^{-\frac{t}{2}bA}f\|_\infty \le Ct^{\frac{-3}{2}}t^{\frac{-n}{2}}e^{wt}\|f\|_1$$

for all $t > 0$ and $f \in L^1(\mathbb{R}^n)$. This means that

$$|\nabla_x \frac{\partial}{\partial t}k_t(x,y)| \le Ct^{\frac{-3}{2}}t^{\frac{-n}{2}}e^{wt}\,. \tag{3.16}$$

Once we show (3.12) then (3.13) will follow from (3.16) as in the proof of (3.4).

We now prove (3.12). In [11], upper bounds of time derivatives of positive symmetric heat kernels are studied and one may extend the arguments given there to more general kernels which are neither symmetric nor positive. Here we give another proof which goes over to a more general setting.

Let $m > \frac{n}{2} + 1$ be an integer. Using the same notation as in Section 2, the operator $bA(\lambda I + bA)^{-2m-1}$ has a kernel given by $R_{\lambda,2m}(x,y) - \lambda R_{\lambda,2m+1}(x,y)$. This kernel satisfies the estimate (2.7). By writing

$$tbAe^{-tbA} = \frac{(2m)!}{2\pi i t^{2m-1}} \int_\Gamma e^{\lambda t} bA(\lambda + bA)^{-2m-1} d\lambda$$

with an appropriate contour Γ we obtain as in the proof of Theorem 2.2 that $tbAe^{-tbA}$ has a kernel which satisfies (3.3). This gives the desired estimate (3.12). The proofs of (3.14) and (3.15) are based on (3.12) and (3.13) and are exactly as the same as those of (3.5) and (3.6).

Remark 3.1. In addition to the assumptions of the above theorem, suppose that the adjoint A^* of A satisfies the same properties as A, i.e., A^* defined as an operator on $L^p(\mathbb{R}^n)$, $1 < p < \infty$ has domain $W^{2,p}(\mathbb{R}^n)$. In this case the "regularized" kernel $k_t(x,y)b(y)$ of $e^{-tbA}b$ has the same properties in the $y-$variable as in the above theorem. That is, $k_t(x,.)b(.) \in C^{1,\eta}(\mathbb{R}^n)$ for all $x \in \mathbb{R}^n$ and the estimates (3.4)–(3.6) and (3.12 –(3.15) hold for $\nabla_y(k_t(x,y)b(y))$ and $\nabla_y \frac{\partial}{\partial t} k_t(x,y)b(y)$. To see this, observe that $e^{-tbA}b$ is the adjoint of $\overline{b}e^{-tA^*\overline{b}}$. Write

$$\overline{b}e^{-tA^*\overline{b}} = \overline{b}(I + A^*\overline{b})^{-1}(I + A^*\overline{b})e^{-tA^*\overline{b}}.$$

Now, in one hand, gaussian bounds are of course satisfied for the kernel of $e^{-tA^*\overline{b}}$ by Theorem 2.2 and this implies the boundedness on $L^p(\mathbb{R}^n)$, $1 \le p < \infty$ of the operator $(I + A^*\overline{b})e^{-tA^*\overline{b}}$ (for all $t > 0$). In the other hand, $\overline{b}(I + A^*\overline{b})^{-1} = (\overline{b}^{-1} + A^*)^{-1}$ which then maps $L^p(\mathbb{R}^n)$ into $D(A^*) = W^{2,p}(\mathbb{R}^n)$ for all $p \in (1,\infty)$. It follows that $\overline{b}e^{-tA^*\overline{b}}$ is bounded form $L^1(\mathbb{R}^n)$ into $W^{2,p}(\mathbb{R}^n)$ for all $p \in (1,\infty)$. The assertions of Theorem 3.1 are then satisfied for the kernel of $\overline{b}e^{-tA^*\overline{b}}$ with respect to the $x-$variable which means that the kernel $e^{-tbA}b$ satisfies these assertions with respect to the $y-$variable. This shows also that the kernel of $\frac{\partial}{\partial t}e^{-tbA}b$ satisfies the assertions of Theorem 3.2 with respect to the $y-$ variable.

Remark 3.2. Instead of $\mathbb{R}^n$, if we consider a domain Ω of class C^2 and the operator A on $L^2(\Omega)$ with Dirichlet boundary conditions then we have the same conclusions as in the above theorems. This is because the domain $D(A) = W_0^{2,p}(\Omega)$ for $p \in (1,\infty)$ in this case, and the same proof works.

Remark 3.3. Let us finally mention that if we start with a uniformly elliptic operator in divergence form

$$A = -\sum_{k,j}^{n} \frac{\partial}{\partial x_k}(a_{kj}(x)\frac{\partial}{\partial x_j})$$

with real coefficients belonging to $W^{1,\infty}(\mathbb{R}^n)$ and a function b which satisfies (2.1) so that $-bA$ is a generator of an analytic semigroup e^{-tbA} on $L^2(\mathbb{R}^n)$, then the kernel $k_t(x,y)$ (respectively $k_t(x,y)b(y)$) satisfies the assertions of Theorems 3.1 and 3.2 with respect to the $x-$variable (respectively, with respect to the $y-$variable).

4. The holomorphic functional calculus. We consider in this section the perturbed operator bA where A is a divergence-form elliptic operator with some smoothness on the coefficients. Our aim is to show that the operator bA has a bounded holomorphic functional calculus on $L^p(\mathbb{R}^n), 1 < p < \infty$.

Let A be the uniformly elliptic operator given by

$$Au = -\sum_{k,j}^{n} \frac{\partial}{\partial x_k}(a_{kj}(x)\frac{\partial}{\partial x_j}).$$

We suppose that the coefficients (a_{kj}) are real, bounded and satisfy the usual ellipticity assumption.

Let $b : \mathbb{R}^n \to \mathbb{C}$ be a measurable function and satisfies (2.1). Since the operator A is sectorial, $-bA$ generates a bounded analytic semigroup (Proposition 2.1) if $|\arg b| \leq w$ for some w less than the angle of analyticity of A. This is also true if we assume that b is uniformly continuous ([2]). From now on, we assume that b satisfies (2.1) and $-bA$ generates a bounded analytic semigroup e^{-tbA} on $L^2(\mathbb{R}^n)$.

Because of this assumption, bA satisfies (2.2), i.e.

$$\sup_{\lambda \notin \Sigma(\nu_0)} \|\lambda(bA - \lambda I)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} < \infty \tag{4.1}$$

for some $\nu_0 \in [0, \frac{\pi}{2})$. This condition is needed in order to study the holomorphic functional calculus in a sector of angle $\leq \frac{\pi}{2}$.

Following [21], [1] we say that a given operator B in a Banach space E in which B satisfies the above resolvent estimates has a bounded $H_\infty(\Sigma(\mu))-$functional calculus if for $\mu > \nu_0$ there exists a constant c_μ such that for all $f \in H_\infty(\Sigma(\mu)), f(B)$ is a bounded operator on E and

$$\|f(B)\|_{\mathcal{L}(E)} \leq c_\mu \|f\|_\infty. \tag{4.2}$$

Here $H_\infty(\Sigma(\mu))$ is the algebra of bounded holomorphic functions in the sector $\Sigma(\mu)$. For the definition of $f(B)$ and several characterizations of bounded holomorphic functional calculus, we refer to [21] and [1]. This functional calculus includes as a particular case the boundedness of purely imaginary powers of B.

Because of Theorem 2.2 and the fact that the heat kernel of A has a gaussian bound, the semigroup e^{-tbA} acts on $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$. The operator bA is then well defined on $L^p(\mathbb{R}^n)$ as the (minus) generator of the corresponding semigroup in $L^p(\mathbb{R}^n)$. As we mentioned at the end of Section 2, the gaussian bound of the heat kernel of $-bA$ implies that $-bA$ satisfies (4.1) in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ (with the same ν_0). We can then ask whether bA has a bounded $H_\infty-$calculus in $L^p(\mathbb{R}^n)$.

Theorem 4.1. 1) *Suppose that b is real-valued. Then bA has a bounded H_∞−functional calculus in $L^p(\mathbb{R}^n)$ for $1 < p < \infty$.*

2) *Suppose that $a_{kj} \in W^{1,\infty}(\mathbb{R}^n)$, $1 \le k, j \le n$. Then the operator $bA + w$ has a bounded H_∞−functional calculus in $L^p(\mathbb{R}^n)$, $1 < p < \infty$, where w is the constant in the estimate* (3.4)*.*

Proof. The operator A and the function b satisfy the assumptions of Theorem 2.2. Hence by the estimate (2.7) in the proof of this theorem and Theorem 4.3 of [13], it suffices to show that bA has a bounded H_∞−calculus in $L^2(\mathbb{R}^n)$. Note that we can also use the estimates of Theorem 3.1 to obtain that $f(A)$ (for f in a smaller class of $H_\infty(\Sigma(\mu))$) is a Calderón-Zygmund operator which then implies the H_∞−calculus in $L^p(\mathbb{R}^n)$, $1 < p < \infty$ without using the above mentioned theorem from [13].

1) Suppose that b is real-valued. The operator bA is similar to $\sqrt{b}A\sqrt{b}$. Since b is real-valued, the latter is accretive, that is

$$\Re(\sqrt{b}A\sqrt{b}u, u) = \Re(A\sqrt{b}u, \sqrt{b}u) \ge 0, \forall u \in D(A\sqrt{b}).$$

It is known (see [15, 1]) that accretive operators have bounded H_∞−calculus. It then follows by similarity that bA has a bounded H_∞−calculus in $L^2(\mathbb{R}^n)$ and hence in $L^p(\mathbb{R}^n)$, $1 < p < \infty$.

2) As in [22], the idea is to show quadratic estimates for $bA + w$ which we explain briefly. Let $-B$ be the generator of a bounded analytic semigroup e^{-tB} on a Hilbert space H. This is equivalent to write that for some $\nu_0 \in (0, \frac{\pi}{2})$

$$\sup_{\lambda \notin \Sigma(\nu_0)} \|\lambda(B - \lambda I)^{-1}\|_{\mathcal{L}(H)} < \infty.$$

Fix $\mu > \nu_0$ and let

$$\Psi(\Sigma(\mu)) = \{f \in H_\infty(\Sigma(\mu)) : |f(z)| \le C|z|^s(1 + |z|^{2s})^{-1} \text{ for some } C, s > 0\}.$$

For a given $\psi \in \Psi(\Sigma(\mu))$, let $\psi_t(z) = \psi(tz)$ and define $\psi_t(tB)$ by

$$\psi_t(B)u = \frac{1}{2\pi i}\int_\Gamma \psi(\lambda)(tB - \lambda I)^{-1}u d\lambda,$$

where Γ is a contour in $\mathbb{C}$ which consists of two rays given by $\{z \in \mathbb{C} : |\arg z| = \theta\}$ with $\nu_0 < \theta < \mu$. We say that the operator B satisfies a quadratic estimate with respect to $\psi \in \Psi(\Sigma(\mu))$ if

$$\{\int_0^\infty \|\psi_t(B)u\|_2^2 \frac{dt}{t}\}^{\frac{1}{2}} \le C\|u\|_2 \text{ for all } u \in L^2(\mathbb{R}^n),$$

where C is some positive constant. If B and its adjoint B^* satisfy quadratic estimates with respect to ψ and ψ^* which are positive on $(0, \infty)$ then B has a bounded H_∞−calculus in H (see [21] and [5]).

In order to show this for our operator $bA + w$ we choose $\psi(z) = ze^{-z}$, thus, $\psi(t(bA + w)) = t(bA + w)e^{-t(bA+w)}$. The quadratic estimate consists of showing that

$$\{\int_0^\infty \|t(bA + w)e^{-t(bA+w)}bu\|_2^2 \frac{dt}{t}\}^{\frac{1}{2}} \le C\|u\|_2 \tag{4.3}$$

for all $u \in L^2(\mathbb{R}^2)$ (recall that b satisfies (2.1)). Consider first the term $\int_0^\infty \|twe^{-t(bA+w)}bu\|_2^2 \frac{dt}{t}$. This term is clearly dominated by $C(w)\|u\|_2^2$ since the semigroup e^{-tbA} is uniformly bounded on $L^2(\mathbb{R}^n)$. Consider now the term $\int_0^\infty \|tbAe^{-t(bA+w)}bu\|_2^2 \frac{dt}{t}$. The operator $tbAe^{-t(bA+w)}b$ is given by the kernel

$$\Theta_t(x, y) = -e^{-wt}t\frac{\partial}{\partial t}k_t(x, y)b(y).$$

Note that by Theorem 3.2 and Remark 3.1, $\Theta_t(x, y)$ satisfies a gaussian bound and Hölder estimates in x and in y with an exponential decay in $|x - y|^2$. Moreover, it is well known that the semigroup e^{-tA} is conservative, that is

$$e^{-tA}1 = 1, \quad \text{for all } t \ge 0$$

this can be written as $A1 = 0$ (here A denotes the corresponding realization of A on L^∞. It is defined as the adjoint of the generator in L^1 of e^{-tA^*}). We then have

$$\int_{\mathbb{R}^n} \Theta_t(x, y)\frac{1}{b}(y)dy = te^{-tw}e^{-tbA}bA1 = 0.$$

By [28] this implies the boundedness on $L^2(\mathbb{R}^n)$ of the desired operator

$$\{\int_0^\infty \|te^{-tw}bAe^{-tbA}bu\|_2^2 \frac{dt}{t}\}^{\frac{1}{2}}.$$

The quadratic estimates are shown in the similar way for the adjoint. In fact,

$$tA^*\overline{b}e^{-tA^*\overline{b}}\frac{1}{\overline{b}} = te^{-tA^*\overline{b}}A^*1 = 0$$

and apply [28] as above.

Remark 4.1. The proof of part 1) is true for more general operator A. More precisely, assume that (X, d, μ) is a space of homogeneous type in the sense that d is a quasi-metric (X, d, μ) has the doubling volume property. Also assume that:

(i) A generates an analytic semigroup e^{-tA} on $L^2(X)$ with kernel $p_t(x, y)$ which satisfies (1.1).

(ii) A is accretive

(iii) The function b is real-valued, satisfies (2.1), and $-bA$ generates an analytic semigroup on $L^2(X)$.

Then we conclude that bA has a bounded $H_\infty-$functional calculus in $L^p(X)$ for $1 < p < \infty$.

Note also that by [12], Theorem 6, we have the same conclusion when the operator A is defined in $L^p(\Omega, d, \mu)$ where Ω is any measurable subset of X.

REFERENCES

[1] D. Albrecht, X.T. Duong and A. McIntosh, *Operator theory and harmonic analysis,* Workshop in Analysis and Geometry 1995, Part III, Proc. of the CMA, ANU, Canberra (1996), 77–136.

[2] H. Amann, M. Hieber and G. Simonett, *Bounded $H^\infty-$calculus for elliptic operators,* Differential and Integral Equations, 7 (1994), 613–653.

[3] W. Arendt, *Gaussian estimates and interpolation of the spectrum in L^p*, Differential and Integral Equations, 7 (1994), 1153–1168.

[4] W. Arendt and A.F.M. ter Elst, *Gaussian estimates for second-order elliptic operators with boundary conditions,* J. Op. Theory, to appear.

[5] P. Auscher, A. McIntosh, and A. Nahmod, *Holomorphic functional calculi of operators, quadratic estimates and interpolation,* Preprint 1995.

[6] P. Auscher, A. McIntosh, and Ph. Tchamitchian, *Heat kernels of second-order complex elliptic operators and applications,* J. F. Anal, to appear.

[7] P. Bauman, *Equivalence of the Green's functions for diffusion operators in $\mathbb{R}^n$: a counter example,* Proc. Amer. Math. Soc, 91 (1984), 64–68.

[8] P. Bauman, *Positive solutions of elliptic equations in nondivergence form and their adjoints,* Ark. Mat., 22 (1984), 153–173.

[9] T. Coulhon and X. Duong, *Maximal regularity and kernel bounds: observations on a theorem by Hieber and Prüss,* Preprint 1997.

[10] E.B. Davies, "Heat Kernels and Spectral Theory," Cambridge Univ. Press 1989.

[11] E.B. Davies, *Non-gaussian aspects of heat kernel behaviour,* Preprint 1996.

[12] X.T. Duong and A. McIntosh, *Singular integral operators on irregular domains,* Preprint 1996.

[13] X.T. Duong and D.W. Robinson, *Semigroup kernels, Poisson bounds and holomorphic functional calculus,* J. Funct. Analysis, 142 (1996), 89–128.

[14] X.T. Duong and G. Simonett, *$H^\infty-$calculus for operators with non-smooth coefficients,* Differential and Integral Equations, to appear.

[15] C. Foias and B. Nagy, "Harmonic Analysis of Operators on Hilbert Space," Akademiai Kiado, Budapest 1970.

[16] A. Friedman, "Partial Differential Equations of Parabolic Type," Prentice Hall, New Jersey 1964.

[17] A. Grigory'an, *Gaussian upper bounds for the heat kernel on arbitrary manifolds,* Preprint 1996.

[18] M. Hieber, *Gaussian estimates and holomorphy of semigroups on L^p spaces,* J. London Math. Soc., 54 (1996), 148–160.

[19] M. Hieber and J. Prüss, *Heat kernels and maximal L^p-L^q estimates for parabolic evolution equations,* Preprint 1996.

[20] T. Kato, "Perturbation Theory for Linear Operators," 2^{nd} edition, Springer-Verlag 1976.

[21] A. McIntosh, *Operators which have an H^∞ functional calculus,* "Miniconference on Operator Theory and Partial Differential Equations," Proceedings of the Centre for Mathematical Analysis, ANU, Canberra, 14 (1986), 210–231.

[22] A. McIntosh and A. Nahmod, *Heat kernel estimates and functional calculi of $-b\Delta$,* Preprint 1996.

[23] E.M. Ouhabaz, *$L^\infty-$contractivity of semigroups generated by sectorial forms,* J. London Math. Soc., 46 (1992), 529–542.

[24] E.M. Ouhabaz, *Semi-groupes sous-Markoviens engendrés par des opérateurs matriciels,* Math. Ann., 296 (1993), 667–676.

[25] E.M. Ouhabaz, *Gaussian estimates and holomorphy of semigroups,* Proc. Amer. Math. Soc., 123 (1995), 1465–1474.

[26] M. Reed and B. Simon, "Methods of Modern Mathematical Physics," Vol. 2, Academic Press, 1978.

[27] D.W. Robinson, "Elliptic Operators on Lie Groups," Oxford Univ. Press 1991.

[28] S. Semmes, *Square function estimates and the T(b) theorem,* Proc. Amer. Math. Soc., 110 (1990), 721–726.

[29] N.Th. Varopoulos, L. Saloff-Coste, and Th. Coulhon, "Analysis and Geometry on Groups," Cambridge Univ. Press 1992.