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ORBITAL STABILITY OF SOLITARY WAVES OF COUPLED KDV EQUATIONS

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Abstract. This paper is concerned with the orbital stability for solitary waves of a system of coupled KdV-equations. Although the coupled equations are not in Hamiltonian form, we can apply the abstract results of Grillakis et al. [4, 5] and use a detailed spectral analysis to obtain the stability of the solitary waves.

1. Introduction. In this paper, we consider the stability of the solitary waves of the following coupled KdV system:

$$\begin{cases} u_t + u_{xxx} + 6uu_x - 2bvv_x = 0, \\ v_t + v_{xxx} + 3uv_x = 0, \end{cases} \qquad x \in R. \tag{1.1}$$

The system (1.1) describes the interaction of two long waves with different dispersion relations [1, 2]. If there is no effect of one of the long waves on the other, the latter obeys the ordinary KdV equation, so (1.1) can be regarded as a natural extension of the single KdV equation

$$u_t + u_{xxx} + 6uu_x = 0. \tag{1.2}$$

The initial-value problem for (1.1) was investigated in [3], and the existence of solitary waves of (1.1) was obtained in [1].

Unlike most evolution equations such as (1.2), (1.1) can not be written as an abstract Hamiltonian system [4, 5] in the form

$$\frac{du}{dt} = JE'(u(t)), \tag{1.3}$$

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where J is a skew-symmetric linear operator and E is a nonlinear functional (the energy). Nevertheless, the abstract theory of Shatah et al. [4, 5] can be applied here, by the use of two appropriate invariants of motion, and by a detailed spectral analysis. We thereby obtain the stability of the solitary waves.

Remarks. Recently, many authors have been studying the problem of stability and instability of solitary waves for the single KdV-type equations like (1.2) (see [6–8]). Pego and Weinstein did a linearized analysis for a generalized KdV equation, and obtained asymptotic stability and the linearized exponential instability of the solitary-wave solutions. Their methods involve Evans's function techniques and are different from the approach taken here (see [9, 10]).

This paper is organized as follows: in Section 2, we state the result for the existence of solitary waves, in Section 3, we state the assumptions and the stability results (Theorem 2 and Theorem 3), in Section 4, we obtain an abstract stability result (Theorem 2), while in Section 5, we prove the stability of solitary-wave solutions of (1.1).

2. The existence of solitary waves. Consider the following system

$$\begin{cases} u_t + u_{xxx} + 6uu_x - 2bvv_x = 0, \\ v_t + v_{xxx} + 3uv_x = 0, \end{cases} \qquad x \in R. \tag{2.1}$$

Let

$$u(t,x) = \varphi_c(x-ct) \tag{2.2}$$
$$v(t,x) = \psi_c(x-ct) \tag{2.3}$$

be solitary-wave solutions of (2.1). Put (2.2)–(2.3) into (2.1) and supposing φ_c, φ_c'', ψ_c, ψ_c', $\psi_c''' \to 0$, as $x \to \infty$, we obtain

$$-\varphi_c'' - 3\varphi_c^2 + c\varphi_c + b\psi_c^2 = 0, \tag{2.4}$$
$$\psi_c''' - c\psi_c' + 3\varphi_c\psi_c' = 0. \tag{2.5}$$

Suppose that

$$\varphi_c = k\psi_c^2 \tag{2.6}$$

satisfies (2.4)–(2.5) with a constant $k \neq 0$ to be determined later. Then we have

$$-\varphi_c'' - 3\varphi_c^2 + c\varphi_c + \frac{b}{k}\varphi_c = 0, \tag{2.7}$$

$$\psi_c'' - c\psi_c + k\psi_c^3 = 0. \tag{2.8}$$

Suppose ψ_c has the form $c_1 \operatorname{sech} c_2 x$, where the constants c_1, c_2 will be determined later. Then we obtain

$$c_2^2 = c, \quad 2c_2^2 = kc_1^2, \quad 4c_2^2 = c + \frac{b}{k}. \tag{2.9}$$

It follows that $k = \frac{b}{3c}$, $c_2 = \sqrt{c}$, $c_1 = \sqrt{\frac{6}{b}}c$, where

$$\varphi_c(x) = 2c \operatorname{sech}^2(\sqrt{c}x), \qquad \psi_c(x) = c\sqrt{\tfrac{6}{b}} \operatorname{sech}(\sqrt{c}x). \tag{2.10}$$

In summary, we have the following result.

Theorem 1. *For any real positive constants b and c, there exist solitary-wave solutions of* (2.1) *in the form of* (2.2)–(2.3), *with φ_c, ψ_c, b, c satisfying* (2.10).

3. Main results. Write $\vec{u} = \begin{pmatrix} u \\ v \end{pmatrix}$. The function space in which we shall work is $X = L^2(R) \times H^1(R)$, with inner product

$$(\vec{f}, \vec{g}) = \int_R (f_1 g_1 + f_2 g_2 + f_{2x} g_{2x})\, dx, \quad \text{for} \quad \vec{f}, \vec{g} \in X. \tag{3.1}$$

The dual space of X is $X^* = L^2(R) \times H^{-1}(R)$, and there is a natural isomorphism $I : X \to X^*$ defined by

$$\langle I\vec{f}, \vec{g} \rangle = (\vec{f}, \vec{g}), \tag{3.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between X and X^*, which is to say,

$$\langle \vec{f}, \vec{g} \rangle = \int_R \Big(\sum_{i=1}^{2} f_i g_i\Big)\, dx. \tag{3.3}$$

By (3.1)–(3.3), it is obvious that $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 - \frac{\partial^2}{\partial x^2} \end{pmatrix}$ Because the stability in view here refers to perturbations of the solitary-wave profile itself, a study of the initial-value problem for (1.1) is necessary. The following lemma states that the initial-value problem for (1.1) is well posed in Hadamard's classical sense (see [3]).

Lemma 1. *Let $b > 0$ and $s \geq 3$. For each $\vec{u}_0 \in H^s(R) \times H^s(R)$, there exists $T_* = T_*(\|\vec{u}_0\|_{H^s}) > 0$ and a unique solution $\vec{u} \in C([0, T_*); H^s \times H^s)$ of* (1.1) *with $\vec{u}(0) = \vec{u}_0$. In addition, either $T_* = \infty$ or $\|\vec{u}(x,t)\|_X \to \infty$ as $t \to T_*$.*

Let T be the one-parameter group of unitary operators on X defined by

$$T(s)\vec{u}(\cdot) = \vec{u}(\cdot - s) \quad \text{for} \quad \vec{u}(\cdot) \in X, \quad s \in R. \tag{3.4}$$

Obviously, we have

$$T'(0) = \begin{pmatrix} -\frac{\partial}{\partial x} & 0 \\ 0 & -\frac{\partial}{\partial x} \end{pmatrix}.$$

It follows from Theorem 1 and (1.1) that there exist solitary waves $T(ct)$ $\vec{\varphi}_c(x)$ of (1.1), with $\vec{\varphi}_c(x)$ defined by

$$\vec{\varphi}_c(x) = \begin{pmatrix} \varphi_c(x) \\ \psi_c(x) \end{pmatrix}. \tag{3.5}$$

In this and the following section, we shall consider the orbital stability of solitary waves $T(ct)\vec{\varphi}_c(x)$ of (1.1). Note that the system (1.1) is invariant under $T(\cdot)$. Define orbital stability as follows:

Definition 1. The solitary wave $T(ct)\vec{\varphi}_c(x)$ is orbitally stable if for any $\epsilon > 0$, there exists a $\delta > 0$ with the property that if $\vec{u_0} \in X$, $\|\vec{u_0} - \vec{\varphi}_c\|_X < \delta$, and $\vec{u}(t)$ is a solution of (1.1) in some interval $[0, t_0)$ with $\vec{u}(0) = \vec{u}_0$, then $\vec{u}(t)$ can be continued to a solution in $0 \leq t < +\infty$, and

$$\sup_{0<t<+\infty} \inf_{s \in R} \|\vec{u}(t) - T(s)\vec{\varphi}_c\|_X < \epsilon. \tag{3.6}$$

Otherwise $T(ct)\vec{\varphi}_c(x)$ is called orbitally unstable.

Define

$$E(\vec{u}) = E(u,v) = \int_R b(v_x^2 - uv^2)\, dx, \tag{3.7}$$

$$V(\vec{u}) = V(u,v) = \int_R (\frac{3}{2}u^2 + bv^2)\, dx. \tag{3.8}$$

It is easy to verify that $E(\vec{u})$ and $V(\vec{u})$ are invariant under T, and formally conserved under the flow of (1.1). Namely, we have

$$E(T(s)\vec{u}) = E(\vec{u}), \text{ for any } s \in R, \tag{3.9}$$

$$V(T(s)\vec{u}) = V(\vec{u}), \text{ for any } s \in R, \tag{3.10}$$

and if $\vec{u}(t)$ is the flow generated by (1.1),

$$E(\vec{u}(t)) = E(\vec{u}(0)), \tag{3.11}$$

$$V(\vec{u}(t)) = V(\vec{u}(0)). \tag{3.12}$$

Note that equation (1.1) can not be written as a Hamiltonian system in the form

$$\frac{d\vec{u}}{dt} = JE'(\vec{u}), \tag{3.13}$$

where J is a skew-symmetric linear operator and E is a functional (the energy). However, by (2.6–2.8), we have

$$E'(\vec{\varphi}_c) + cV'(\vec{\varphi}_c) = 0, \tag{3.14}$$

where E' and V' are the Frechet derivatives of E and V, with

$$E'(\vec{u}) = \begin{pmatrix} -bv^2 \\ -2bv_{xx} - 2buv \end{pmatrix}, \quad V'(\vec{u}) = \begin{pmatrix} 3u \\ 2bv \end{pmatrix}.$$

Moreover, the linearized operator H_c of $E' + cV'$ around $\vec{\varphi}_c$ is

$$H_c = E''(\vec{\varphi}_c) + cV''(\vec{\varphi}_c) = \begin{pmatrix} 3c & -2b\psi_c \\ -2b\psi_c & 2b(-\frac{\partial^2}{\partial x^2} + c - \varphi_c) \end{pmatrix}. \tag{3.15}$$

Observe that $H_c : X \to X^*$ is self-adjoint in the sense that $H_c^* = H_c$, and that $I^{-1/2} H_c I^{-1/2}$ is a self-adjoint operator on $L^2(R) \times L^2(R)$, where

$$I^{-1/2} = \begin{pmatrix} 1 & 0 \\ 0 & (1 - \frac{\partial^2}{\partial x^2})^{-1/2} \end{pmatrix}.$$

The 'spectrum' of H_c consists of the real numbers λ such that $H_c - \lambda I$ is not invertible. We claim that $\lambda = 0$ belongs to the spectrum of H_c. By (3.4) (3.9) (3.10) and (3.14), it is easy to prove that

$$H_c T'(0)\vec{\varphi}_c(x) = 0. \tag{3.16}$$

Let

$$Z = \{k_1 T'(0)\vec{\varphi}_c(x) | k_1 \in R\}. \tag{3.17}$$

By (3.16) Z is contained in the kernel of H_c.

We continue by stating our assumptions on H_c. These are the following.

Assumption 1. (Spectral decomposition of H_c.)

The space X is decomposed as a direct sum

$$X = N + Z + P, \tag{3.18}$$

where Z is defined above, N is a one dimensional subspace such that

$$\langle H_c\vec{u}\,,\,\vec{u}\rangle < 0 \text{ for } 0 \neq \vec{u} \in N \tag{3.19}$$

and P is a closed subspace such that

$$\langle H_c\vec{u}\,,\,\vec{u}\rangle \geq \delta\|\vec{u}\|_X^2 \text{ for } \vec{u} \in P \tag{3.20}$$

with some constant $\delta > 0$, independent of $\vec{u}$.

Remarks. It may be helpful to introduce some general terminology. Let X be a vector space and $h : X \times X \to R$ be a symmetric bilinear form. A subspace N (or P) is called negative (or positive) if $h(\vec{u},\vec{u}) < 0$ (or > 0) for all $0 \neq \vec{u} \in N$ (or P). The kernel of h is $Z = \{\vec{u} \in X | h(\vec{u},\vec{v}) = 0, \ \ \forall\vec{v} \in X\}$. Then all maximal negative subspaces have the same dimension, which we call the negative index n(h). Similarly, there is a positive index p(h). In case $h(\vec{u},\vec{u}) = \langle H_c\vec{u},\vec{u}\rangle$, where X is a Hilbert space, we write $n(H_c)$ and $p(H_c)$ for these indices. We also write $Z(H_c)$ for the kernel of H_c (=kernel of h) and $z(H_c) = \dim Z(H_c)$. In our case (Assumption 1), we have $n(H_c) = \dim N = 1$, $z(H_c) = \dim Z(H_c) = 1$ and $p(H_c) = \dim P = \infty$. Thus $n(H_c) = 1$ is the number of negative eigenvalues of H_c, and H_c has a unique, negative, simple eigenvalue $-\lambda^2$, with $\lambda > 0$ and eigenfunction $\vec{\chi}_c$.

We define $d(c) : R \to R$ by

$$d(c) = E(\vec{\varphi}_c) + cV(\vec{\varphi}_c). \tag{3.21}$$

Let us now state our main results about the stability of solitary waves of (1.1).

Theorem 2. *Assume that there exist two functionals $E(\vec{u})$, $V(\vec{u})$ satisfying* (3.9–3.12) *and there exist the solitary wave solutions $T(ct)\ \vec{\varphi}_c(x)$ of* (1.1) *such that* (3.14) *and Assumption* 1 *hold. Then the solitary wave $T(ct)\ \vec{\varphi}_c(x)$ is orbitally stable if $d''(c) > 0$.*

Theorem 3. *Under the conditions of Theorem* 1*, the solitary waves* $T(ct)\vec{\varphi}_c(x)$ *of* (1.1) *are orbitally stable.*

4. The Proof of Theorem 2. To prove Theorem 2, we need some lemmas. The stability of the solitary-wave solutions of (1.1) is an immediate consequence of the fact that $d''(c) > 0$ implies that $\vec{\varphi}_c$ is a local minimum of E subject to the constancy of V. This is a general fact, not special to the equations under consideration in this paper. The proof is, in essence, a special case of the theory given by Grillakis et al. [4].

Lemma 2. *There exists* $\epsilon > 0$ *and a unique* C^1 *map* $\alpha : U_\epsilon \to R$*, such that for every* $\vec{u} \in U_\epsilon$ *and* $r \in R$*.*

(i) $\langle \vec{u}(\cdot + \alpha(\vec{u})), T'(0)\vec{\varphi}_c\rangle = 0$,
(ii) $\alpha(\vec{u}(\cdot + r)) = \alpha(\vec{u}) - r$,

where U_ϵ *is the "tube",*

$$U_\epsilon = \{\vec{u} \in X : \inf_{s\in R} \|\vec{u} - T(s)\vec{\varphi}_c\|_X < \epsilon\}.$$

Proof. Consider the functional

$$(\vec{u}, \alpha) \to \int_R \vec{u}(x + \alpha) \cdot T'(0)\vec{\varphi}_c(x)dx = 0,$$

define on pairs $\vec{u} \in L^2(R) \times L^2(R)$ and $\alpha \in R$. It's derivative with respect to α at $\alpha = 0$ and $\vec{u} = \vec{\varphi}_c$ is $-\langle T'(0)\vec{\varphi}_c(x), T'(0)\vec{\varphi}_c(x)\rangle$, which is non-zero. By the implicit function theorem, there is a unique C^1 functional $\alpha(\vec{u})$ satisfying (i) in a neighborhood of $\vec{\varphi}_c$. By translation invariance, $\alpha(\vec{u})$ can be uniquely extended to a tube of the form U_ϵ for $\epsilon > 0$ small enough. By (i),

$$\vec{u}(\cdot + \alpha(\vec{u})) = u(\cdot + r + (\alpha(\vec{u}) - r))$$

is orthogonal to $T'(0)\vec{\varphi}_c$. Therefore by the uniqueness of $\alpha(\vec{u})$, guaranteed by the conclusion of the implicit function theorem, $\alpha(\vec{u}) - r = \alpha(\vec{u}(\cdot + r))$ so that (ii) holds.

Lemma 3. *Let* $d''(c) > 0$*. If* $\vec{y} \in X$ *is orthogonal to both* $V'(\vec{\varphi}_c)$ *and* $T'(0)\vec{\varphi}_c$*, then* $\langle H_c\vec{y}, \vec{y}\rangle > 0$.

Proof. (3.14) and (3.21) imply that $d'(c) = V(\vec{\varphi}_c)$, and hence that

$$0 < d''(c) = \langle V'(\vec{\varphi}_c), \frac{d\vec{\varphi}_c}{dc}\rangle = -\langle H_c\frac{d\vec{\varphi}_c}{dc}, \frac{d\vec{\varphi}_c}{dc}\rangle.$$

Write

$$\frac{d\vec{\varphi}_c}{dc} = a_0\vec{\chi}_c + b_0 T'(0)\vec{\varphi}_c + \vec{p}_0,$$

where $\vec{p}_0$ is in the positive subspace of H_c. Recall that $H_c\vec{\chi}_c = -\lambda^2\vec{\chi}_c$ with $\lambda > 0$ and $H_c(T'(0)\vec{\varphi}_c) = 0$. It follows that

$$\langle H_c\vec{p}_0, \vec{p}_0\rangle < a_0^2\lambda^2.$$

Now suppose that $\langle \vec{y}, V'(\vec{\varphi}_c)\rangle = \langle \vec{y}, T'(0)\vec{\varphi}_c\rangle = 0$ and decompose $\vec{y}$ into the sum $a\vec{\chi}_c + \vec{p}$ with $\vec{p}$ in the positive subspace of H_c. Because

$$0 = -\langle V'(\vec{\varphi}_c), \vec{y}\rangle = \langle H_c\frac{d\vec{\varphi}_c}{dc}, \vec{y}\rangle = -a_0 a\lambda^2 + \langle H_c\vec{p}_0, \vec{p}\rangle,$$

it is inferred that

$$\begin{aligned}\langle H_c\vec{y}, \vec{y}\rangle &= -a^2\lambda^2 + \langle H_c\vec{p}, \vec{p}\rangle \\ &\geq -a^2\lambda^2 + \langle H_c\vec{p}, \vec{p}_0\rangle^2/\langle H_c\vec{p}_0, \vec{p}_0\rangle \\ &> -a^2\lambda^2 + (a_0 a\lambda^2)^2/a_0^2\lambda^2 = 0,\end{aligned}$$

as required.

Lemma 4. *Let $d''(c) > 0$. There exist constants $c_0 > 0$ and $\epsilon > 0$ such that*

$$E(\vec{u}) - E(\vec{\varphi}_c) \geq c_0\|\vec{u}(\cdot + \alpha(\vec{u})) - \vec{\varphi}_c\|_X^2 = c_0\|T(-\alpha(\vec{u}))\vec{u} - \vec{\varphi}_c\|_X^2,$$

for all $\vec{u} \in U_\epsilon$ which satisfy $V(\vec{u}) = V(\vec{\varphi}_c)$.

Proof. Write

$$\vec{u}(\cdot + \alpha(\vec{u})) - \vec{\varphi}_c = aV'(\vec{\varphi}_c) + \vec{y},$$

where

$$\langle V'(\vec{\varphi}_c), \vec{y}\rangle = 0$$

and a is a scalar. Then, by the translation invariance of V and Taylor's theorem,

$$\begin{aligned}&V(\vec{\varphi}_c) = V(\vec{u}) = V(\vec{u}(\cdot + \alpha(\vec{u}))) \\ &= V(\vec{\varphi}_c) + \langle V'(\vec{\varphi}_c), \vec{u}(\cdot + \alpha(\vec{u})) - \vec{\varphi}_c\rangle + O(\|\vec{u}(\cdot + \alpha(\vec{u})) - \vec{\varphi}_c\|_X^2).\end{aligned}$$

The middle term is precisely $a\|V'(\vec{\varphi}_c)\|_X^2$. Note that $V'(\vec{\varphi}_c) \in X$ so that

$$a = O(\|\vec{u}(\cdot + \alpha(\vec{u})) - \vec{\varphi}_c\|_X^2).$$

Writing $L = E + cV$, another Taylor expansion gives

$$L(\vec{u}) = L(\vec{u}(\cdot + \alpha(\vec{u})) = L(\vec{\varphi}_c) + \frac{1}{2}\langle H_c\vec{w}, \vec{w}\rangle + o(\|\vec{w}\|_X^2),$$

where $\vec{w} = \vec{u}(\cdot + \alpha(\vec{u})) - \vec{\varphi}_c = aV'(\vec{\varphi}_c) + \vec{y}$. This can be written as

$$\begin{aligned} L(\vec{u}) - L(\vec{\varphi}_c) = E(\vec{u}) - E(\vec{\varphi}_c) &= \frac{1}{2}\langle H_c\vec{w}, \vec{w}\rangle + o(\|\vec{w}\|_X^2) \\ &= \frac{1}{2}\langle H_c\vec{y}, \vec{y}\rangle + O(a^2) + O(a\|\vec{w}\|_X) + o(\|\vec{w}\|_X^2) \\ &= \frac{1}{2}\langle H_c\vec{y}, \vec{y}\rangle + o(\|\vec{w}\|_X^2). \end{aligned}$$

But $\vec{y}$ is orthogonal to both $V'(\vec{\varphi}_c)$ and $T'(0)\vec{\varphi}_c$. Therefore by Lemma 3

$$E(\vec{u}) - E(\vec{\varphi}_c) \geq 2c_0\|\vec{y}\|_X^2 + o(\|\vec{w}\|_X^2),$$

for some positive constant c_0. Because

$$\|\vec{y}\|_X = \|\vec{w} - aV'(\vec{\varphi}_c)\|_X \geq \|\vec{w}\|_X - O(\|\vec{w}\|_X^2),$$

for $\|\vec{w}\|$ small, it follows that

$$E(\vec{u}) - E(\vec{\varphi}_c) \geq c_0\|\vec{w}\|_X^2.$$

The proof of Lemma 4 is now complete.

Proof of Theorem 2. If $\vec{\varphi}_c$ is unstable, there exists a sequence of initial data $\vec{u}_n(0)$ and $\epsilon > 0$ such that

$$\|\vec{u}_n(0) - \vec{\varphi}_c\|_X \to 0, \text{ as } n \to \infty,$$

but

$$\sup_{t>0} \inf_{s} \|\vec{u}_n(t) - T(s)\vec{\varphi}_c\|_X \geq \epsilon,$$

where $\vec{u}_n(t)$ is the unique solution of (1.1) with initial data $\vec{u}_n(0)$. Let $t_n > 0$ be the first time that

$$\inf_s \|\vec{u}_n(t_n) - T(s)\vec{\varphi}_c\|_X = \epsilon. \tag{4.1}$$

Such a value exists by continuity. Because E and V are continuous on X and translation invariant, it follows that

$$E(\vec{u}_n(\cdot, t_n)) = E(\vec{u}_n(0)) \to E(\vec{\varphi}_c)$$

and

$$V(\vec{u}_n(\cdot, t_n)) = V(\vec{u}_n(0)) \to V(\vec{\varphi}_c).$$

Next choose $\vec{w}_n \in U_\epsilon$ so that $V(\vec{w}_n) = V(\vec{\varphi}_c)$ and $\|\vec{w}_n - \vec{u}_n(\cdot, t_n)\| \to 0$. By Lemma 4,

$$\begin{aligned} 0 \leftarrow E(\vec{w}_n) - E(\vec{\varphi}_c) &\geq c_0\|T(-\alpha(\vec{w}_n))\vec{w}_n - \vec{\varphi}_c\|_X^2 \\ &= c_0\|\vec{w}_n - T(\alpha(\vec{w}_n))\vec{\varphi}_c\|_X^2. \end{aligned}$$

Hence $\|\vec{u}_n(t_n) - T(\alpha(\vec{w}_n))\vec{\varphi}_c\|_X \to 0$, which contradicts (4.1). This means the orbit of $\vec{\varphi}_c$ is stable, and thus Theorem 2 is established.

Remark 1. Along the lines of the proof of Theorem 2, replacing s by $\alpha(\vec{u}_n(t))$, and because of $\|T(\alpha(\vec{u}_n(t_n)))\vec{\varphi}_c - T(\alpha(\vec{w}_n))\vec{\varphi}_c\|_X \to 0$, as $n \to \infty$, we can also deduce the following stability result.

Definition 2. The solitary wave $T(ct)\vec{\varphi}_c(x)$ is orbitally stable if for all $\epsilon > 0$, there exists $\delta > 0$ with the following property: If $\|\vec{u_0} - \vec{\varphi}_c\|_X < \delta$ and $\vec{u}(t)$ is a solution of (1.1) in some interval $[0, t_0)$ with $\vec{u}(0) = \vec{u}_0$, then $\vec{u}(t)$ can be continued to a solution in $0 \leq t < +\infty$, and

$$\sup_{0<t<+\infty} \|\vec{u}(t) - T(\alpha(\vec{u}))\vec{\varphi}_c\|_X < \epsilon, \tag{4.2}$$

with $\alpha(\vec{u})$ defined by Lemma 2. Otherwise $T(ct)\vec{\varphi}_c(x)$ is called orbitally unstable.

Remark 2. Recently, Bona and Soyeur have proven that in quite general situations, $\frac{d\alpha(\vec{u})}{dt} = c + o(1)$, where c is the speed of the unperturbed solitary wave (see [12]). This general fact also applies to the case considered here.

5. The Proof of Theorem 3. In virtue of (3.4)–(3.21), we can apply Theorem 2 to (1.1). To prove Theorem 3, it is sufficient to prove that under the condition (2.11), Assumption 1 holds and $d''(c) > 0$.

First we prove that Assumption 1 holds and $n(H_c) = 1$.

For any $\vec{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in X$, by (3.15),

$$\begin{aligned} \langle H_c(\vec{\varphi}_c)\vec{y}, \vec{y}\rangle &= \langle L_1 y_2, y_2\rangle - 4b\int_R (\psi_c y_1 y_2)dx + 3c\langle y_1, y_1\rangle \\ &= \langle L_2 y_2, y_2\rangle + \frac{b}{k}\int_R (y_1 - 2k\psi_c y_2)^2 dx, \end{aligned} \tag{5.1}$$

with

$$L_1 = 2b(-\frac{\partial^2}{\partial x^2} + c - \varphi_c), \tag{5.2}$$

$$L_2 = 2b(-\frac{\partial^2}{\partial x^2} + c - 3\varphi_c). \tag{5.3}$$

Since $b > 0$, and $c > 0$, note that

$$L_1 = -2b\frac{\partial^2}{\partial x^2} + 2bc + M_1(x), \tag{5.4}$$

with

$$M_1(x) \to 0 \text{ as } |x| \to +\infty. \tag{5.5}$$

Thus, by Weyl's theorem on the essential spectrum (see [11]), we have

$$\sigma_{\text{ess}}(L_2) = [2bc, +\infty). \tag{5.6}$$

Let $\sigma_1 = 2bc$. Then (2.11) assures that $\sigma_1 > 0$. Using (2.10), (3.5) and recalling that $H_c T'(0)\vec{\varphi}_c(x) = 0$, we have

$$L_2\psi_{cx} = 0. \tag{5.7}$$

By (2.10) and (5.7), we see that ψ_{cx} has a simple zero at $x = 0$, so Sturm-Liouville theory implies that 0 is the second eigenvalue of L_2, and L_2 has exactly one, strictly negative eigenvalue $-\sigma_-^2$, with an eigenfunction χ_1.

In virtue of (5.3)–(5.7), as in [8], we have the following lemma.

Lemma 5. *For any real functions* $y_2 \in H^1(R)$ *satisfying*

$$\langle y_2, \chi_1 \rangle = \langle y_2, \psi_{cx} \rangle = 0, \tag{5.8}$$

there exists a positive number $\delta_1 > 0$ *such that*

$$\langle L_2 y_2, y_2 \rangle \geq \delta_1 \|y_2\|_{H^1}^2. \tag{5.9}$$

Choose

$$y_1^- = \frac{2b}{3c}\psi_c \chi_1, \quad y_2^- = \chi_1, \quad \vec{y}_- = \begin{pmatrix} y_1^- \\ y_2^- \end{pmatrix}, \tag{5.10}$$

then

$$\langle H_c \vec{y}_-, \vec{y}_- \rangle = -\sigma_-^2 \langle \chi_1, \chi_1 \rangle < 0. \tag{5.11}$$

Also notice that the vector

$$\vec{y}_0 = \begin{pmatrix} \varphi_{cx} \\ \psi_{cx} \end{pmatrix}. \tag{5.12}$$

is in the kernel of H_c. Let

$$Z = \{k_1 \vec{y}_0 / k_1 \in R\}, \tag{5.13}$$

$$P = \{\vec{p} \in X / \vec{p} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}, \quad \langle p_2, \chi_1 \rangle = \langle p_2, \psi_{cx} \rangle = 0\}, \tag{5.14}$$

$$N = \{k_2 \vec{y}_- / \, k_2 \in R\}. \tag{5.15}$$

Obviously (3.19) holds. For any $\vec{u} \in X$, $\vec{u} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, choose $a = \langle y_2, \chi_1 \rangle$ and $b_1 = (\langle y_2, \psi_{cx} \rangle / \langle \psi_{cx}, \psi_{cx} \rangle)$. Then $\vec{u}$ can be uniquely represented in the form

$$\vec{u} = a\vec{y}_- + b_1 \vec{y}_0 + \vec{p}, \tag{5.16}$$

with $\vec{p} \in P$, which implies (3.18).

For the subspace P, it remains to prove (3.20).

Remarks. It is not necessary that the decomposition (5.16) is orthogonal, this is a general fact. Indeed, in the present case, if $n(H_c) \geq 2$, then there exists a $\vec{y} \in X$, where $\langle \vec{y}, \vec{y_-} \rangle = 0$, $\vec{y} \neq 0$, such that

$$\langle H_c(\vec{y} + l\vec{y_-}), \vec{y} + l\vec{y_-} \rangle < 0, \text{ for any } l \in R.$$

By (5.16), $\vec{y}$ can be uniquely represented as

$$\vec{y} = a\vec{y}_- + b_1\vec{y}_0 + \vec{p},$$

where,

$$\langle H_c(\vec{y} - a\vec{y_-}), \vec{y} - a\vec{y_-}\rangle = \langle H_c(b_1\vec{y}_0 + \vec{p}), b_1\vec{y}_0 + \vec{p}\rangle = \langle H_c\vec{p}, \vec{p}\rangle \geq 0.$$

This is a contradiction. Hence, $n(H_c) = 1$. In the same way, we can prove $z(H_c) = 1$.

Lemma 6. *For any $\vec{p} \in P$ defined by* (5.14)*, there exists a constant $\delta > 0$ such that*

$$\langle H_c\vec{p}, \vec{p}\rangle \geq \delta\|\vec{p}\|_X \tag{5.17}$$

with δ independent of $\vec{p}$.

Proof. For any $\vec{p} \in P$, by (5.14) and Lemma 5, we have

$$\langle H_c\vec{p}, \vec{p}\rangle \geq \delta_1\|p_2\|^2_{H^1} + 3c\int_R (p_1 - \frac{2b}{3c}\psi_c p_2)^2 dx. \tag{5.18}$$

(1) If

$$\|p_1\|^2_{L^2} \geq \frac{16b^2M}{9c^2}\|p_2\|^2_{L^2}, \quad M = |\psi_c|^2_\infty, \tag{5.19}$$

then

$$3c\int_R (p_1 - \frac{2b}{3c}\psi_c p_2)^2 dx \geq 3c(\frac{1}{2}\|p_1\|^2_{L^2} - \frac{4b^2M}{9c^2}|\|p_2\|^2_{L^2}) \geq \frac{3c}{4}\|p_1\|^2_{L^2}, \tag{5.20}$$

(2) If

$$\|p_1\|^2_{L^2} \leq \frac{16b^2M}{9c^2}\|p_2\|^2_{L^2}, \tag{5.21}$$

then

$$\delta_1\|p_2\|^2_{H^1} \geq \frac{\delta_1}{2}\|p_2\|^2_{H^1} + \frac{9\delta_1 c^2}{32b^2M}\|p_1\|^2_{L^2}. \tag{5.22}$$

Thus, for any $\vec{y} \in P$, we have

$$\langle H_c\vec{y}, \vec{y}\rangle \geq \delta\|\vec{y}\|^2_X, \tag{5.23}$$

where $\delta > 0$ is independent of $\vec{y}$. Thus under the condition of (2.11). Assumption 1 holds, and so $n(H_c) = 1$.

In the following, we shall verify that $d''(c) > 0$ under the conditions of Theorem 1. Note that (3.14) and (3.21) imply

$$\begin{aligned} d'(c) = V(\vec{\varphi}_c) &= \int_R (\frac{3}{2}\varphi_c^2(x) + b\psi_c^2(x))dx \\ &= 6c^2 \int_R sech^4(\sqrt{c}x)dx + 3c \int_R sech^2(\sqrt{c}x)dx \\ &= 6c^{3/2} \int_R sech^4(x)dx + 3c^{1/2} \int_R sech^2(x)dx. \end{aligned}$$

It follows that

$$d''(c) = 9\sqrt{c} \int_R sech^4(x)dx + \frac{3}{2\sqrt{c}} \int_R sech^2(x)dx > 0.$$

This completes the proof of Theorem 3.

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