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EXISTENCE RESULTS FOR STEADY FLOW OF WEAKLY COMPRESSIBLE VISCOELASTIC FLUIDS WITH A DIFFERENTIAL CONSTITUTIVE LAW*

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Abstract. In this paper we study flows of viscoelastic weakly compressible fluids having a differential constitutive equation. In both cases, Jeffreys and Maxwell type constitutive equations, we establish a result of existence and uniqueness of solutions. We also show that, when the compressibility goes to zero, then the weakly compressible steady solution goes to the incompressible one.

1. Introduction. We consider Jeffreys' and Maxwell's models given by the following partial differential equations (see [3], [4], [8], [9]):

$$\begin{cases} \rho \mathrm{Re}\,(u.\nabla)u - (1-\epsilon)(\Delta u + \nabla \mathrm{div}\, u) + \nabla p = f + \mathrm{div}\,\tau, \\ \mathrm{div}\,(\rho u) = 0 \\ \mathrm{We}\,((u.\nabla)\tau + \mathbf{g}(\nabla u, \tau)) + \tau = 2\epsilon \mathbf{D}[u]. \end{cases} \tag{1}$$

The system is satisfied in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2$ or 3), and is completed by the homogeneous boundary condition

$$u|_{\partial\Omega} = 0. \tag{2}$$

The unknowns are u, p, and τ (respectively, the velocity, the pressure, and the extra-stress tensor). Re and We are the Reynolds and Weissenberg (positive) numbers, ϵ is a positive parameter satisfying $0 < \epsilon \leq 1$. We

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have also used the notation $\mathbf{g}(\nabla u, \tau) = \tau\mathbf{W} - \mathbf{W}\tau - a(\mathbf{D}\tau + \tau\mathbf{D})$, where $\mathbf{W} = \frac{1}{2}(\nabla u - \nabla u^T)$, $\mathbf{D} = \frac{1}{2}(\nabla u + \nabla u^T)$, and $a \in [-1, 1]$. In the case of incompressible fluids, i.e., $\operatorname{div} u = 0$, M. Renardy [9] proved the existence and the uniqueness of the slow stationary flows associated to an external force for both models, Jeffreys and Maxwell. C. Guillopé and J.C. Saut [3] proved the existence and the stability of slightly viscoelastic (i.e., We sufficiently small) steady flows close to Newtonian flows (i.e., $\epsilon = 0$), on the assumption of quite important viscosity (i.e., ϵ sufficiently small). In this paper, we shall establish the existence and the uniqueness of weakly compressible stationary flows in Jeffreys' and Maxwell's cases associated to an external small force. Then we shall establish the convergence of the weakly compressible problem to the incompressible problem.

To obtain a model describing a weakly compressible flow, we shall consider the approximation of the density (see [8]) under the assumption of isothermal compressibility

$$\rho = \big(1 + \frac{p - p_0}{\beta}\big), \quad \beta > 0, \tag{3}$$

then we shall make an approximation of the equation $(1)_2$ under the assumption of weak compressibility, $\frac{|p-p_0|}{\beta} << 1$. The weakly compressible flow obtained will be close to an incompressible one. As $(1)_2$ and (2) imply

$$\int_\Omega \operatorname{div}(\rho u) dx = 0,$$

equation $(1)_2$ is equivalent to

$$\operatorname{div}(\rho u) - \frac{1}{|\Omega|} \int_\Omega \operatorname{div}(\rho u) = 0, \tag{4}$$

where $|\Omega|$ denotes the measure of Ω. When we replace the mass density by expression (3) and make the hypothesis $\frac{|p-p_0|}{\beta} << 1$ we obtain

$$\frac{1}{\beta}(u.\nabla)p + \operatorname{div} u - \frac{1}{|\Omega|} \int_\Omega \operatorname{div} u - \frac{1}{\beta|\Omega|} \int_\Omega (u.\nabla)p = 0.$$

Therefore, we will study the following system of partial differential equations modelling the weakly compressible viscoelastic fluids of Jeffreys type (0 <

$\epsilon < 1$) or Maxwell type ($\epsilon = 1$),

$$\begin{cases} \mathrm{Re}\,(u.\nabla)u - (1-\epsilon)(\Delta u + \nabla \mathrm{div}\, u) + \nabla p = f + \mathrm{div}\, \tau, \\ \dfrac{1}{\beta}\Pi((u.\nabla)p) + \mathrm{div}\, u = 0, \\ \mathrm{We}\,((u.\nabla)\tau + \mathbf{g}(\nabla u, \tau)) + \tau = 2\epsilon \mathbf{D}[u], \end{cases} \tag{5}$$

$$u|_{\partial\Omega} = 0, \tag{6}$$

where we define the projection operator by

$$\Pi(f) = f - \frac{1}{|\Omega|}\int_\Omega f\, dx. \tag{7}$$

Remark 1. We have made the approximation of equation (4) rather than equation $(1)_2$ to keep the conservative character of the obtained equation. Indeed equation $(1)_2$ expresses the conservation of the total mass to a zero flux of fluid (i.e., $\int_\Omega \mathrm{div}\,(\rho u)\, dx = 0$), and equation $(5)_2$ takes this fact into account. If we directly make the approximation of equation $(1)_2$, we obtain

$$\frac{1}{\beta}(u.\nabla)p + \mathrm{div}\, u = 0,$$

which, integrated onto Ω , implies the condition of orthogonality

$$\int_\Omega (u.\nabla)p\, dx = 0.$$

This condition does not seem realistic in a bounded domain.

Our aims are to establish two theorems of existence and uniqueness of solutions to problem (5)–(6), and to show the convergence toward the incompressible problem when β tends to the infinity. The method which is used here results from the crossing of two methods. The first one is used by M. Renardy [9], C. Guillopé and J.C. Saut [3] in the case of incompressible viscoelastic flows. The second one is used by A. Novotný and M. Padula [7], A. Novotný [6] to prove the existence of slow flows for compressible Newtonian fluids in bounded or external domains.

Notation. Ω is a bounded domain in $\mathbb{R}^N$, $N = 2$ or 3, with a regular boundary $\partial\Omega$. Throughout this paper we will use the following spaces:

$\mathrm{L}^p(\Omega)$, $1 \le p \le \infty$, with norms $\|.\|_{\mathrm{L}^p}$; the Sobolev spaces $H^k(\Omega)$, $k = -1, 0, 1, ...$ with norms $\|.\|_k$ and inner product $((.,.))_k$; the Sobolev space $H_0^1(\Omega)$, and its dual $H^{-1}(\Omega)$; the vector spaces $\mathbb{L}^2(\Omega)$ and $\mathbb{H}^k(\Omega)$ of vector valued or tensor valued functions with components in $\mathrm{L}^2(\Omega)$ and $H^k(\Omega)$ respectively, their norms being denoted in the same way as above. To simplify the notation we will also indicate the $\mathbb{L}^2$-norm by $\|.\|$.

We shall show the following results:

Theorem 1. *Let $k = 1, 2, ...$ Let $\Omega \in C^{k+2}$ be a bounded domain in $\mathbb{R}^N (N = 2, 3)$, $f \in \mathbb{H}^k(\Omega)$ and $0 < \epsilon < 1$. Then there exist three constants $\gamma_0, \gamma_1, \beta_1$ such that for all $\|f\|_k \le \gamma_1$ and $\beta > \beta_1$, there exists a unique solution of* (5)–(6) *in the set $B_{\gamma_0}^{k+2} = \{(u, p, \tau) \in (\mathbb{H}^{k+2}(\Omega) \cap \mathbb{H}_0^1(\Omega)) \times (H^{k+1}(\Omega) \cap (L^2(\Omega)/\mathbb{R})) \times \mathbb{H}^{k+1}(\Omega), \|u\|_{k+2} + \|p\|_{k+1} + \|\tau\|_{k+1} \le \gamma_0\}$.*

Theorem 2. *Let $k = 1, 2, ...$ Let $\Omega \in C^{k+2}$ be a bounded domain in $\mathbb{R}^N (N = 2, 3)$, $f \in \mathbb{H}^{k+1}(\Omega)$ and $\epsilon = 1$. Then there exist three constants $\gamma'_0, \gamma'_1, \beta_2$ such that for all $\|f\|_{k+1} \le \gamma'_1$ and $\beta > \beta_2$, there exists a unique solution of* (5)–(6) *in $B_{\gamma'_0}^{k+2}$.*

Theorem 3 (convergence)**.** *Let $k = 1, 2, ...$ For all fixed ϵ, $0 < \epsilon < 1$, the solution $(u_\beta, p_\beta, \tau_\beta)$ of the weakly compressible Jeffreys problem* (5)–(6) *given by Theorem* 1 *converges to the solution of the incompressible one when $\beta \to +\infty$ in $\mathbb{H}^{k+1}(\Omega) \times (H^k(\Omega) \cap (L^2(\Omega)/\mathbb{R})) \times \mathbb{H}^k(\Omega)$. If $\epsilon = 1$ the solution $(u_\beta, p_\beta, \tau_\beta)$ of the weakly compressible Maxwell problem* (5)–(6) *given by Theorem* 2 *converges to the solution of the incompressible one when $\beta \to +\infty$ in $\mathbb{H}^{k+1}(\Omega) \times (H^k(\Omega) \cap (L^2(\Omega)/\mathbb{R})) \times \mathbb{H}^k(\Omega)$.*

2. Transformation of the problem (Jeffreys case $0 < \epsilon < 1$). We apply the operator div to equation $(5)_3$, and calculate :

$$\begin{aligned}\operatorname{div}\{(I + \mathrm{We}(u.\nabla))\tau\} &= 2\epsilon \operatorname{div} D[u] - \mathrm{We} \operatorname{div} \mathbf{g}(\nabla u, \tau)\\ &= \mathrm{We}\Big(\sum_{j=1}^N \sum_{k=1}^N \frac{\partial u_k}{\partial x_j} \frac{\partial \tau_{ij}}{\partial x_k}\Big)_{i=1,\ldots,N} + \mathrm{We}(u.\nabla)\operatorname{div}\tau + \operatorname{div}\tau,\end{aligned}$$

where

$$\begin{aligned}&\operatorname{div}\mathbf{g}(\nabla u, \tau) = \tau \operatorname{div}(W - aD) - (W + aD)\operatorname{div}\tau\\ &+ \Big(\sum_{j,k=1}^N \frac{\partial \tau_{ik}}{\partial x_j}(W_{kj} - aD_{kj})\Big)_{i=1,\ldots,N} - \Big(\sum_{j,k=1}^N \frac{\partial (W_{ik} + aD_{ik})}{\partial x_j}\tau_{kj}\Big)_{i=1,\ldots,N}.\end{aligned}$$

Therefore, equation $(5)_3$ becomes

$$\big((I+\mathrm{We}(u.\nabla)\big)\mathrm{div}\,\tau \;=\; \epsilon(\Delta u+\nabla\mathrm{div}\,u)+F(u,\tau), \tag{8}$$

where $F(u,\tau)$ admits the following expression

$$F(u,\tau) \;=\; \mathrm{We}\{l_1(\partial u,\partial\tau)+l_2(\partial^2 u,\tau), \tag{9}$$

with

$$l_1(\partial u,\partial\tau)=(W+aD)\mathrm{div}\,\tau-\big(\sum_{j,k=1}^{N}\frac{\partial\tau_{ik}}{\partial x_j}(W_{kj}-aD_{kj})+\frac{\partial u_k}{\partial x_j}\;\frac{\partial\tau_{ij}}{\partial x_k}\big)_{i=1,\ldots,N}, \tag{10}$$

$$l_2(\partial^2 u,\tau)=\big(\sum_{j,k=1}^{N}\;\frac{\partial(W_{ik}+aD_{ik})}{\partial x_j}\tau_{kj}\big)_{i=1,\ldots,N}-\tau\mathrm{div}(W-aD). \tag{11}$$

It is well known that the operator $(I+\mathrm{We}(u.\nabla))$ is invertible with continuous inverse from $\mathbb{H}^k$ onto $\mathbb{H}^k$. More precisely we have the following proposition.

Proposition 1. *Let Ω be a bounded domain of $\mathbb{R}^N$ $(N=2\,or\,3)$. Let $k=-1,0,...,$ we suppose that*

$$\begin{aligned}&\Omega\in\mathcal{C}^{k+1}\cap\mathcal{C}^3,\quad u\in\mathbb{H}^{k+1}\cap\mathbb{H}^3\cap\mathbb{H}^1_0,\\ &\mathrm{We}\big\|u\big\|_{k+1}<1\quad\textit{if } k\geq 2\quad\textit{and}\quad \mathrm{We}\big\|u\big\|_3<1\quad\textit{if } k=-1,0,1\;.\end{aligned} \tag{12}$$

Then the operator $(I+\mathrm{We}(u.\nabla))\,:\mathbb{H}^k\to\mathbb{H}^k$ is an isomorphism from its domain onto $\mathbb{H}^k$. It is also an isomorphism from its domain (as a subspace of $\mathbb{H}^k\cap\mathbb{H}^1_0$) onto $\mathbb{H}^k\cap\mathbb{H}^1_0$.

For a proof of this proposition, see Beirão da Veiga [1]. Estimates for the continuity of the inverse in the cases $k\geq 1$ is given by Renardy [9]. A full proof for the cases $k=1,2$ is given in Guillopé-Saut [3]. Proposition 1 implies that the equation (8) is equivalent to

$$\mathrm{div}\,\tau \;=\; (I+\mathrm{We}(u.\nabla))^{-1}[\epsilon(\Delta u+\nabla\mathrm{div}\,u)+F(u,\tau)]. \tag{13}$$

We now replace $\mathrm{div}\tau$ in $(5)_1$ by its expression, and obtain :

$$-L[u](\Delta u+\nabla\mathrm{div}\,u)=f+(I+\mathrm{We}(u.\nabla))^{-1}F(u,\tau)-\mathrm{Re}(u.\nabla)u-\nabla p \tag{14}$$

where $L[u]=(1-\epsilon)I+\epsilon(I+\mathrm{We}(u.\nabla))^{-1}$.

Proposition 2. *Under the assumptions of Proposition* 1*, the operator* $L[u]\ :\ \mathbb{H}^k \to \mathbb{H}^k$, $k \geq -1$, *is an isomorphism. In the same way for* $L[u]\ :\ \mathbb{H}^1_0 \to \mathbb{H}^1_0$.

Proof. It is easy to see that $L[u]\ : \mathbb{H}^k \to \mathbb{H}^k$ is continuous, which results from proposition 1. $L[u]$ is invertible, this result from the theorem of Lax-Milgram : in the case $k = 0$ we obtain

$$(L[u]v, v) = (1 - \epsilon)\|v\|^2 + \epsilon((I + \mathrm{We}(u.\nabla))^{-1}v, v);$$

the hypothesis $\mathrm{We}\|u\|_3 < 1$ implies $((I + \mathrm{We}(u.\nabla))^{-1}v, v) \geq 0$. In fact let $v \in \mathbb{L}^2(\Omega)$; then there exists a unique $w \in \mathbb{L}^2(\Omega)$ such that

$$w + \mathrm{We}(u.\nabla)w = v; \tag{15}$$

therefore, we have

$$\begin{aligned}(v, (I + \mathrm{We}(u.\nabla))^{-1}v) &= (w + \mathrm{We}(u.\nabla)w, w) \\ &= \|w\|^2 - \frac{1}{2}\mathrm{We}\int_\Omega \mathrm{div}u|w|^2 dx \geq \big(1 - \frac{1}{2}\mathrm{We}\|u\|_3\big)\|w\|^2.\end{aligned}$$

For the case $k > 0$, we proceed in a similar way, differentiating k times equation (15), (see also [1]). The case $k = -1$ is obtained by applying Proposition 1.

Let us show now that the restriction of $L[u]$ to $\mathbb{H}^1_0$ remains an isomorphism from $\mathbb{H}^1_0$ onto itself. We just need to show that it is defined from $\mathbb{H}^1_0 \to \mathbb{H}^1_0$. Let $v \in \mathbb{H}^1_0$, then there exists a unique vector $w \in \mathbb{H}^1$ such that $L[u]w = v$, i.e.,

$$(1 - \epsilon)w + \epsilon(I + \mathrm{We}(u.\nabla))^{-1}w = v.$$

We then deduce

$$(1 - \epsilon)w|_{\partial\Omega} + (1 - \epsilon)(u.\nabla)w|_{\partial\Omega} + \epsilon w|_{\partial\Omega} = v|_{\partial\Omega} + \mathrm{We}(u.\nabla)v|_{\partial\Omega},$$

which implies $w|_{\partial\Omega} = 0$. Conversely, if $w \in \mathbb{H}^1_0$ then we know that $L[u]w \in \mathbb{H}^1$. Besides $[(1 - \epsilon)w + \epsilon(I + \mathrm{We}(u.\nabla))^{-1}w]|_{\partial\Omega} = v|_{\partial\Omega}$, so that, $\epsilon(I + \mathrm{We}(u.\nabla))^{-1}w|_{\partial\Omega} = v|_{\partial\Omega}$. From Proposition 1, $(I + \mathrm{We}(u.\nabla))^{-1}w|_{\partial\Omega} = 0$ hence $v|_{\partial\Omega} = 0$. This shows Proposition 2.

We apply $L[u]^{-1}$ to (14). Thanks to Proposition 2, and after calculating the commutator of $L[u]^{-1}$ and ∇ (see Appendix A) we can write

$$\begin{aligned}&L[u]^{-1}\nabla p = \nabla(L[u]^{-1}p) - \\ &\quad L[u]^{-1}\{\epsilon \mathrm{We}(I + \mathrm{We}(u.\nabla))^{-1}(\nabla u)^T\nabla[(I + \mathrm{We}(u.\nabla))^{-1}L[u]^{-1}p]\}.\end{aligned} \tag{16}$$

Finally system (5) is transformed into the equivalent system

$$\begin{cases} -(\Delta u + \nabla \operatorname{div} u) + \nabla(L[u]^{-1}p) = F_1(u,p,\tau), \\ \frac{1}{\beta}\Pi((u.\nabla)p) + \operatorname{div} u = 0, \\ \mathrm{We}\{(u.\nabla)\tau + \mathbf{g}(\nabla u, \tau)\} + \tau = 2\epsilon D[u], \end{cases} \tag{17}$$

where

$$F_1(u,p,\tau) = L[u]^{-1}\{f + (I + \mathrm{We}(u.\nabla))^{-1}F(u,\tau) - \mathrm{Re}(u.\nabla u)\} \tag{18}$$
$$+ L[u]^{-1}\{\epsilon \mathrm{We}(I + \mathrm{We}(u.\nabla))^{-1}(\nabla u)^T \nabla[(I + \mathrm{We}(u.\nabla))^{-1}L[u]^{-1}p]\}.$$

Proposition 3. *The operator* Π *defined by* (7) *is continuous* $\mathbb{H}^k \to \mathbb{H}^k$, $k \geq 0$.

The demonstration is immediate. We only need to notice that

$$\|\Pi(f)\|_k \leq \|f\|_k + \|f\|_0 \leq 2\|f\|_k.$$

Remark 2. Under the assumptions of proposition 1, we see that $L[u]^{-1}1 = 1$. As a matter of fact $L[u]1 = (1-\epsilon)1 + \epsilon(I + \mathrm{We}(u.\nabla))^{-1}1$. However there exists a unique $v \in \mathbb{H}^k,\ k \geq 0$, such that $1 = v + \mathrm{We}(u.\nabla)v$. But $v = 1$ is a solution and according to the uniqueness, it is the only solution (by linearity it is true for all the constants, $L[u]^{-1}\lambda = \lambda$, $\forall \lambda \in \mathbb{R}$.)

We shall now explain the method of demonstration of Theorem 1. We first solve the following linear problem

$$\begin{cases} -(\Delta u + \nabla \operatorname{div} u) + \nabla(L[w]^{-1}p) = F, \\ \frac{1}{\beta}\Pi((w.\nabla)p) + \operatorname{div} u = 0, \\ \mathrm{We}\{(w.\nabla)\tau + \mathbf{g}(\nabla w, \tau)\} + \tau = 2\epsilon D[u], \end{cases} \tag{19}$$

$$u = 0 \ \text{ on } \ \partial\Omega. \tag{20}$$

Then, we look for a solution of problem (17)–(20) as a fixed point of the mapping $N : (w, \mu, \check{\tau}) \to (u, p, \tau)$, where (u, p, τ) is a solution of (19)–(20) corresponding to $F = F_1(w, \mu, \check{\tau})$ (see [10] and for the Newtonian case [6], [7] for a similar proof).

3. A linearized problem (Jeffreys case). In this section, we study problem (19)–(20) for $0 < \epsilon < 1$. We introduce the following space: for $w \in \mathbb{H}^{k+1} \cap \mathbb{H}^1_0$ if $k \geq 2$, or for $w \in \mathbb{H}^3 \cap \mathbb{H}^1_0$ if $k = -1, 0, 1$,

$$H^k_w(\Omega) = \{\mu;\ \mu \in H^k(\Omega);\ (w.\nabla)\mu \in H^k(\Omega)\}.$$

$H_w^k(\Omega)$ is a Banach space for the norm $\|\mu\|_{H_w^k(\Omega)} = \|\mu\|_k + \|(w.\nabla)\mu\|_k$. ($H_w^k(\Omega)$ is the domain for the operator $(I + \lambda(w.\nabla)),\ \lambda > 0$).

3.1. Some auxiliary problems. We first recall a few results of existence, uniqueness and regularity of solutions to the Neumann and the Dirichlet problems for the Laplacian operator and to the Stokes problem.

Consider the Neumann problem:

$$\begin{cases} -\Delta\varphi = h \text{ in } \Omega, \\ \nabla\varphi.\nu|_{\partial\Omega} = 0, \end{cases} \tag{21}$$

where ν is the outwards unit vector normal to $\partial\Omega$.

Lemma 1. *Let $k = 0, 1, ..., \Omega \in \mathcal{C}^{k+2}$ be a bounded domain of $\mathbb{R}^N$ $(N = 2, 3)$. Let $h \in \mathbb{H}^k(\Omega)$ such that $\int_\Omega h\,dx = 0$. Then there exists a unique solution $\varphi \in \mathbb{H}^{k+2}(\Omega) \cap \mathbb{L}^2(\Omega)/\mathbb{R}$ to problem* (21)*. Moreover,*

$$\|\varphi\|_{k+2} \le c\|h\|_k. \tag{22}$$

Consider the Dirichlet problem:

$$\begin{cases} -\Delta\theta = h \text{ in } \Omega, \\ \theta|_{\partial\Omega} = 0, \end{cases} \tag{23}$$

Lemma 2. *Let $k = -1, 0, 1, ..., \Omega \in \mathcal{C}^{k+2}$ ($\mathcal{C}^2$ if $k = -1$) be a regular bounded domain of $\mathbb{R}^N$ $(N = 2, 3)$. Let $h \in \mathbb{H}^k(\Omega)$. Then there exists a unique solution θ of problem* (23) *verifying the estimate*

$$\|\theta\|_{k+2} \le c\|h\|_k. \tag{24}$$

See [5] for the proofs of Lemma 1 and Lemma 2.

Consider the Stokes problem:

$$\begin{cases} -\Delta u + \nabla p = F \quad \text{ in } \ \Omega, \\ \operatorname{div} u = l \quad \text{ in } \ \Omega, \\ u|_{\partial\Omega} = \psi. \end{cases} \tag{25}$$

Lemma 3. *Let* $k = -1, 0, 1, ..., \Omega \in \mathcal{C}^{k+2}$ ($\mathcal{C}^2$ *if* $k = -1$) *be a bounded domain of* $\mathbb{R}^N$ ($N = 2, 3$). *Let* $F \in \mathbb{H}^k(\Omega)$, $\psi \in \mathbb{H}^{k+\frac{3}{2}}(\partial\Omega)$, $l \in \mathbb{H}^{k+1}(\Omega)$ *and* $\int_\Omega \psi.\nu \, d\sigma = \int_\Omega l \, dx$. *Then there exists a unique solution of problem* (25), $u \in \mathbb{H}^{k+2}(\Omega)$, $p \in H^{k+1}(\Omega) \cap L^2(\Omega)/\mathbb{R}$, *satisfying the estimate*

$$\|u\|_{k+2} + \|p\|_{k+1} \leq c(\|F\|_k + \|\psi\|_{k+\frac{3}{2}} + \|l\|_{k+1}). \tag{26}$$

The Stokes problem is classical. (See, for instance, [11]).

3.2. A "transport" problem. Let w be a vector verifying the following hypotheses (see also proposition 1)

$$w \in \mathbb{H}^{k+2}(\Omega) \cap \mathbb{H}^3(\Omega) \cap \mathbb{H}^1_0(\Omega) \quad k = -1, 0, ... \tag{27}$$

We want to solve the following problem:

$$L[w]^{-1}p + \frac{2}{\beta}(w.\nabla)p = p_1, \tag{28}$$

where $L[w] = (1-\epsilon)I + \epsilon(I + \mathrm{We}(w.\nabla))^{-1}$ is an isomorphism of $\mathbb{H}^k$, for all $k \geq -1$ (see Proposition 2).

Lemma 4. *Let* $k = -1, 0, 1, ..., \Omega \in \mathcal{C}^{k+2}$ ($\mathcal{C}^3$ *if* $k = -1, 0$) *be a bounded domain of* $\mathbb{R}^N$ ($N = 2, 3$), w *satisfying* (27) *and* $p_1 \in H^{k+1}(\Omega)$. *Then if* $c\|w\|_{k+2} < 1$ ($c\|w\|_3 < 1$ *if* $k = -1, 0$) *where* $c = c(\mathrm{We}, \Omega, \frac{1}{\beta})$ *and* $\beta > 2\epsilon(\frac{1}{\mathrm{We}} + c_1)$, *there exists a unique solution of problem* (28) *in* H^{k+1} *such that*

$$\|p\|_{k+1} \leq c^0 \|p_1\|_{k+1}, \tag{29}$$

and, if $k \geq 0$,

$$\|\Delta p\|_{k-1} \leq c^1 \big(\|\Delta p_1\|_{k-1} + (\|w\|_{k+2} + \frac{2\epsilon}{\beta})\|p\|_{k+1}\big). \tag{30}$$

The proof of this result is given in Sections 3.4 and 3.5. We now study the linear problem (19)–(20).

3.3. Resolution of a linearized problem. We want to solve the linear problem (19)–(20), where w, F are given. We seek u in the Helmholtz form, i.e., $u = v + \nabla\varphi$ where v and φ satisfy $\operatorname{div} v = 0$, $\nabla\varphi.\nu|_{\partial\Omega} = 0$, $v|_{\partial\Omega} = -\nabla\varphi|_{\partial\Omega}$. We define the mapping

$$\mathcal{L} : \mu \to p \tag{31}$$

in the following way.

i) For μ given, φ is the solution of $(19)_2$ with p replaced by μ, so that φ satisfies

$$\begin{cases} -\Delta\varphi = \frac{1}{\beta}\Pi((w.\nabla)\mu) \quad \text{ in } \Omega, \\ \nabla\varphi.\nu|_{\partial\Omega} = 0 \quad \text{ on } \partial\Omega. \end{cases} \tag{32}$$

ii) For φ known, we replace $u = v + \varphi$ in $(19)_1$, we take into account (32), so that we obtain the equation

$$-\Delta v + \nabla(L[w]^{-1}p + \frac{2}{\beta}(w.\nabla)\mu) = F. \tag{33}$$

Then we solve the Stokes problem,

$$\begin{cases} -\Delta v + \nabla p_1 = F \text{ in } \Omega, \\ \operatorname{div} v = 0 \text{ in } \Omega, \\ v|_{\partial\Omega} = -\nabla\varphi|_{\partial\Omega}. \end{cases} \tag{34}$$

iii) Equation (33) suggests to seek p as a solution of the following equation

$$L[w]^{-1}p + \frac{2}{\beta}(w.\nabla)p = p_1. \tag{35}$$

Indeed, it is now obvious that the solution to (19)–(20) is (u, p, τ) where p is obtained as a fixed point of the mapping (31) and $u = v + \nabla\varphi$ where v and $\nabla\varphi$ are solutions of problems (32) and (34). Finally, we solve the equation in τ. Indeed, let p be a fixed point of $\mathcal{L}$, i.e., $\mathcal{L}(p) = p$. Then by (32),

$$\begin{cases} -\Delta\varphi = \frac{1}{\beta}\Pi((w.\nabla)p) \quad \text{ in } \Omega, \\ \nabla\varphi.\nu|_{\partial\Omega} = 0. \end{cases} \tag{36}$$

By (35), we have $L[w]^{-1}p + \frac{2}{\beta}(w.\nabla)p = p_1$. We put the expression of p_1 into (34) and we get

$$\begin{cases} -\Delta v + \nabla(L[w]^{-1}p + \frac{2}{\beta}(w.\nabla)p) = F \quad \text{ in } \Omega, \\ \operatorname{div} v = 0 \quad \text{ in } \Omega, \\ v|_{\partial\Omega} = -\nabla\varphi|_{\partial\Omega}. \end{cases} \tag{37}$$

Equations (36), (37) and $(19)_3$ give (19)–(20).

We have the following result.

Theorem 4. *Let $k = 0, 1, ...,$ and*

$$\Omega \in \mathcal{C}^{k+2} \cap \mathcal{C}^3, \quad w \in \mathbb{H}^{k+2}(\Omega) \cap \mathbb{H}^3(\Omega) \cap \mathbb{H}^1_0(\Omega), \quad F \in \mathbb{H}^k.$$

Then if $c_g^1 \mathrm{We}\|w\|_{k+2} < 1, (c_g^1 \mathrm{We}\|w\|_3 < 1$ if $k = 0)$ and β is sufficiently large, there exists a unique solution $(u, p, \tau) \in \mathbb{H}^{k+2}(\Omega) \cap \mathbb{H}^1_0(\Omega) \times (H^{k+1}(\Omega) \cap L^2(\Omega)/\mathbb{R}) \times \mathbb{H}^{k+1}$ for problem (19)–(20), which satisfies the inequalities

$$\|u\|_{k+2} + \|p\|_{k+1} \leq c^1 \|F\|_k, \tag{38}$$

$$\|\tau\|_{k+1} \leq \epsilon c^2 \|F\|_k. \tag{39}$$

Proof of Theorem 4. (i) Let w verify the hypothesis of Theorem 4. The operator $\mathcal{L}$ defined by (31) maps H_w^{k+1} into itself if $\mathrm{We}\|w\|_{k+2} < 1$ ($\mathrm{We}\|w\|_3 < 1$ if $k = 0$), and $\beta > 2\epsilon(\frac{1}{\mathrm{We}} + c_1)$. This results from Lemmas 1, 3, and 4.

(ii) $\mathcal{L}$ is a contraction in H_w^{k+1}. In fact we deduce from Lemma 2, the estimate

$$\|\varphi\|_{k+3} \leq \frac{c_1^{'}}{\beta} \|\Pi(w.\nabla)\mu\|_{k+1} \leq 2\frac{c_1^{'}}{\beta} \|(w.\nabla)\mu\|_{k+1}. \tag{40}$$

We know that $(w.\nabla)\mu|_{\partial\Omega} = 0$ for $w|_{\partial\Omega} = 0$. The Laplace operator is an isomorphism from $\mathbb{H}^{k+1}(\Omega) \cap \mathbb{H}^1_0(\Omega)$ onto $\mathbb{H}^{k-1}(\Omega)$ (see Lemma 2). We deduce the inequality

$$\|\varphi\|_{k+3} \leq \frac{c_2^{'}}{\beta} \|\Delta[(w.\nabla)\mu]\|_{k-1}. \tag{41}$$

Now, we apply the divergence operator to equation $(34)_1$, which implies

$$-\mathrm{div}\Delta v + \mathrm{div}\nabla p = \mathrm{div} F.$$

Therefore, $\Delta p_1 = \mathrm{div} F$ and

$$\|\Delta p_1\|_{k-1} \leq \|F\|_k. \tag{42}$$

By using estimate (29) of Lemma 4,

$$\|p\|_{k+1} \leq c_3 \|p_1\|_{k+1}. \tag{43}$$

Applying Lemma 3 to problem (34), we infer

$$\|v\|_{k+2} + \|p_1\|_{k+1} \leq c_4(\|F\|_k + \|\varphi\|_{k+3}). \tag{44}$$

We use (41) to estimate $\|\varphi\|_{k+3}$ and we obtain

$$\|v\|_{k+2} + \|p_1\|_{k+1} \leq c_4^1\big(\|F\|_k + \|\Delta[(\frac{w}{\beta}.\nabla)\mu]\|_{k-1}\big). \tag{45}$$

Also from Lemma 4, inequality (30) implies the estimate:

$$2\|\Delta[(\frac{w}{\beta}.\nabla)p]\|_{k-1} \leq c_5\big(\|\Delta p_1\|_{k-1} + \big(\|w\|_{k+2} + \frac{2\epsilon}{\beta}\big)\|p\|_{k+1}\big). \tag{46}$$

Therefore, if we take into account the preceding inequalities and (46), there follows

$$\|(\frac{w}{\beta}.\nabla)p\|_{k+1} \leq c_5^{'}\big[\big(1+\big(\|w\|_{k+2}+\frac{2\epsilon}{\beta}\big)\big)\|F\|_k+\big(\|w\|_{k+2}+\frac{2\epsilon}{\beta}\big)\|(\frac{w}{\beta}.\nabla)\mu\|_{k+1}\big]. \tag{47}$$

Estimate (45) implies

$$\|p\|_{k+1} \leq c_4^{'}\big(\|F\|_k + \|(\frac{w}{\beta}.\nabla)\mu\|_{k+1}\big). \tag{48}$$

Let now μ, $\mu_1 \in H_w^{k+1}$, let $p = \mathcal{L}\mu$, $p_1 = \mathcal{L}(\mu_1)$, F being fixed. We easily check that, if $\tilde{\mu} = \mu - \mu_1$ and $\tilde{p} = p - p_1$, then $\mathcal{L}(\tilde{\mu}) = \tilde{p}$ for $F = 0$. Therefore $\tilde{p}$ satisfy estimates (47) and (48) with $F = 0$, i.e.,

$$\|\tilde{p}\|_{k+1} \leq c_4^{'}\|(\frac{w}{\beta}.\nabla)\tilde{\mu}\|_{k+1}, \tag{49}$$

$$\|(\frac{w}{\beta}.\nabla)\tilde{p}\|_{k+1} \leq c_5^{'}\big(\|w\|_{k+2} + \frac{2\epsilon}{\beta}\big)\|(\frac{w}{\beta}.\nabla)\tilde{\mu}\|_{k+1}. \tag{50}$$

From the combination (49)+$\alpha c_4^{'}$(50), we obtain

$$\begin{aligned}\|\tilde{p}\|_{k+1} + \alpha c_4^{'}\|(w.\nabla)\tilde{p}\|_{k+1} \leq \alpha c_4^{'}\big(\frac{1}{\alpha} + c_5^{'}\big(\|w\|_{k+2} + \frac{2\epsilon}{\beta}\big)\big)\|(w.\nabla)\tilde{\mu}\|_{k+1} \\ + \big(\frac{1}{\alpha} + c_5^{'}\big(\|w\|_{k+2} + \frac{2\epsilon}{\beta}\big)\big)\|\tilde{\mu}\|_{k+1}.\end{aligned}$$

Therefore, we see that if $\frac{1}{\alpha} + c_5^{'}\big(\|w\|_{k+2} + \frac{2\epsilon}{\beta}\big) < 1$, then $\mathcal{L}$ is a contraction in H_w^{k+1} for the norm $\|\mu\|_{k+1} + \alpha c_4^{'}\|(w.\nabla)\mu\|_{k+1}$. It is now easy to see that $\alpha \in \mathbb{R}_*^+$ is such that

$$\frac{1}{\alpha} + c_5^{'}\big(\|w\|_{k+2} + \frac{2\epsilon}{\beta}\big) < 1$$

as soon as

$$\|w\|_{k+2} < \big(\frac{1}{c_5^{'}} - \frac{2\epsilon}{\beta}\big), \quad \beta\ > 2c_5^{'}\epsilon. \tag{51}$$

Therefore, under hypothesis (51) $\mathcal{L}$ admits a unique fixed point p. We have then proved that (u, p) is a solution of $(19)_1$, $(19)_2$ and (20). Moreover as $u = v + \nabla\varphi$ is known and $u \in \mathbb{H}^{k+2}(\Omega) \cap \mathbb{H}_0^1(\Omega)$, we can then solve $(19)_3$ in τ. Proposition 1 cannot be directly applied, to obtain the existence of τ because of the term $\mathbf{g}(\nabla u, \tau)$. But a slight modification of the demonstration of Proposition 1 allows to conclude thanks to the linearity of $\mathbf{g}(\nabla u, \tau)$ in τ (see for example Beirão da Veiga [1] or Renardy [9]).

Let us now show inequalities (38) and (39). For (38), from $u = v + \nabla\varphi$, we obtain

$$\|u\|_{k+2} \le \|v\|_{k+2} + \|\varphi\|_{k+3}. \tag{52}$$

by (40)

$$\|\varphi\|_{k+3} \le c_2 \|\big(\frac{w}{\beta}.\nabla\big)p\|_{k+1}, \tag{53}$$

by (43) and (44)

$$\|u\|_{k+2} \le c_4\big(\|F\|_k + \|\big(\frac{w}{\beta}.\nabla\big)p\|_{k+1}\big), \tag{54}$$

$$\|p\|_{k+1} \le c_4\big(\|F\|_k + \|\big(\frac{w}{\beta}.\nabla\big)p\|_{k+1}\big). \tag{55}$$

For estimating τ, we apply Proposition 1 with $\epsilon D[u] - \mathrm{We}\,\mathbf{g}(\nabla w, \tau)$ as a second member and we obtain:

$$(1 - c_{\mathbf{g}}\mathrm{We}\|w\|_{k+2})\|\tau\|_{k+1} \le \epsilon c_3 \|v\|_{k+2},$$

so that if

$$c_{\mathbf{g}}\mathrm{We}\|w\|_{k+2} < 1, \tag{56}$$

we obtain the estimate $\|\tau\|_{k+1} \le \epsilon c^4 \|F\|_k$.

Now, from estimate (47), we deduce

$$\big(1 - c_5^{'}\big(\|w\|_{k+2} + \frac{2\epsilon}{\beta}\big)\big)\|(\frac{w}{\beta}.\nabla)p\|_{k+1} \le c_5^{'}\big(1 + \big(\|w\|_{k+2} + \frac{2\epsilon}{\beta}\big)\big)\|F\|_k$$

Therefore, thanks to (51), there exists a constant c_6 depending on ϵ, β, $c_5^{'}$ such that

$$\|(\frac{w}{\beta}.\nabla)p\|_{k+1} \le c_6\|F\|_k; \tag{57}$$

We now use (54), (55) and (57) to obtain

$$\|u\|_{k+2} + \|p\|_{k+1} \leq c^1 \|F\|_k, \tag{58}$$

$$\|\tau\|_{k+1} \leq \epsilon c^2 \|F\|_k, \tag{59}$$

which ends the proof of Theorem 4.

Remark 3. The constant $c^1_{\mathbf{g}}$ defined in Theorem 4 is obtained from (56) and (51) and given by

$$c^1_{\mathbf{g}} = \mathrm{Sup}\big(c_{\mathbf{g}}, \big(\frac{1}{c_5^{'}} - \frac{2\epsilon}{\beta}\big)^{-1}\big). \tag{60}$$

3.4. Strong solutions to the transport problem. Let us recall the problem given by (28)

$$L[w]^{-1}p + \frac{2}{\beta}(w.\nabla)p = p_1.$$

The aim of this section and the next one, is to prove Lemma 4, i.e., to study the existence of strong and of weak solutions to problem (28). We designate by strong solution a solution $p \in H^{k+1}$, $k \geq 0$, and by weak solution a solution (in a sense to be defined) $p \in H^{k+1}$, $k = -2, -1$.

Remark 4. We need to define weak solutions of problem (28), as Beirão da Veiga [1] did for the classical transport operator (Proposition 1), to establish inequality (30) in the case $k = 0, 1$ (i.e., to estimate $\|\Delta p\|_{-1}$ and $\|\Delta p\|$).

Let $k = 0, 1, ...$ We transform (28) by applying the operator $L[w]$ to it, so that we obtain the equivalent equation

$$p + \frac{2}{\beta}L[w]\big((w.\nabla)p\big) = p_1^{'}, \tag{61}$$

with $p_1^{'} = L[w]p_1$. If we take into account the expression of $L[w]$, equation (61) becomes

$$L_1[w]p + \frac{2\epsilon}{\beta}L_2[w]p = p_1^{'}, \tag{62}$$

with

$$L_1[w] = I + \frac{2(1-\epsilon)}{\beta}(w.\nabla), \qquad L_2[w] = (I + \mathrm{We}(w.\nabla))^{-1}(w.\nabla).$$

The operator $L_1[w]$ is of the same nature as the operator $(I + \mathrm{We}(w.\nabla))$ (see proposition 1). In particular it is invertible on H^k if $\frac{2(1-\epsilon)}{\beta}\|w\|_{k+2} < 1$ $\Big(\frac{2(1-\epsilon)}{\beta}\|w\|_3 < 1,\ \text{if } k = 0\Big)$, with continuous inverse from H^k onto H^k_w.

The operator $L_2[w]$ is defined and continuous from H^k, with values in H^k, if

$$\begin{cases} \mathrm{We}\|w\|_{k+2} < 1 & k = 1, ..., \\ \mathrm{We}\|w\|_3 < 1 & k = 0. \end{cases} \tag{63}$$

Indeed, we first check that $L_2[w]$ maps H^k into H^k . If $q \in H^k$, we set $L_2[w]q = g_1$ and

$$g = \frac{q}{\mathrm{We}} - (I + \mathrm{We}(w.\nabla))^{-1}\big(\frac{q}{\mathrm{We}}\big).$$

As $q \in H^k$ then $g \in H^k$. It is now easy to see that by the properties of the operator $(I + \mathrm{We}(w.\nabla))^{-1}$, we obtain $g = g_1$. In fact, we obtain

$$(I + \mathrm{We}(w.\nabla))g = -\frac{q}{\mathrm{We}} + (I + \mathrm{We}(w.\nabla))\big(\frac{q}{\mathrm{We}}\big).$$

Since $(I + \mathrm{We}(w.\nabla))^{-1}\ :\ H^{k-1} \to H^{k-1}_w$ is continuous

$$(I + \mathrm{We}(w.\nabla))g_1 = -\frac{q}{\mathrm{We}} + (I + \mathrm{We}(w.\nabla))\big(\frac{q}{\mathrm{We}}\big),$$

$(I + \mathrm{We}(w.\nabla))(g - g_1) = 0$ in H^{k-1}, Proposition 1 implies $g = g_1$.

Let us now show the continuity of $L_2[w]$. Let $q \in H^k$, $(k \geq 0)$, we have shown that $g = L_2[w]q = \frac{q}{\mathrm{We}} - (I + \mathrm{We}(w.\nabla))^{-1}\big(\frac{q}{\mathrm{We}}\big)$. Therefore $\|g\|_k \leq \frac{1}{\mathrm{We}}\|q\|_k + \frac{c_1}{\mathrm{We}}\|q\|_k$, which shows the continuity of $L_2[w]$: denote c_2 a constant depending on k, such that $\|L_2[w]\|_{\mathcal{L}(H^k)} \leq c_2$ for any w verifying conditions (27) and (63) (c_1, c_2 are independent of w according to the hypothesis of Proposition 1). We now are going to prove the existence and the uniqueness of the solution to (62). The Banach fixed point theorem is used. We suppose

$$\mathrm{Sup}\big(\frac{2(1-\epsilon)}{\beta}, \mathrm{We}\big)\|w\|_{k+2} < 1.$$

We write (62) in the equivalent form

$$p + \frac{2\epsilon}{\beta}L_1[w]^{-1}L_2[w]p = L_1[w]^{-1}p_1^{'}. \tag{64}$$

We define the mapping

$$\Phi \ : \ \begin{cases} H^{k+1} \to H^{k+1} \\ p \to L_1[w]^{-1}\big(p_1^{'} - \frac{2\epsilon}{\beta} L_2[w]p\big). \end{cases}$$

Φ is a contraction if $\beta > 2\epsilon c_1 c_2$. As a matter of fact, if p_1, $p_2 \in H^{k+1}$ then

$$\Phi(p_1) - \Phi(p_2) = \frac{2\epsilon}{\beta} L_1[w]^{-1} L_2[w](p_1 - p_2),$$

and

$$\|\Phi(p_1) - \Phi(p_2)\|_{k+1} \leq \frac{2\epsilon c_1 c_2}{\beta} \|(p_1 - p_2)\|_{k+1},$$

Therefore, if $\beta > 2\epsilon c_1 c_2$, then problem (64) admits a unique solution,

$$p = L_1[w]^{-1}(p_1^{'} - \frac{2\epsilon}{\beta} L_2[w]^{-1} p).$$

In particular,

$$\|p\|_{k+1} \leq c_1\big(\|p_1^{'}\|_{k+1} + \frac{2\epsilon}{\beta} c_2 \|p\|_{k+1}\big),$$

and, for $\beta > 2\epsilon c_1 c_2$, we have

$$\|p\|_{k+1} \leq \frac{c_1}{(1 - \frac{2\epsilon}{\beta} c_1 c_2)} \|p_1^{'}\|_{k+1}.$$

Let us recall that $p_1^{'} = L[w]p_1$, so that $\|p_1^{'}\|_{k+1} \leq ((1-\epsilon) + \epsilon c_1^{'})\|p_1\|_{k+1}$. Finally, we have shown inequality (29)

$$\|p\|_{k+1} \leq \frac{c_1((1-\epsilon) + \epsilon c_1^{'})}{(1 - \frac{2\epsilon}{\beta} c_1 c_2)} \|p_1\|_{k+1},$$

with $c^0 = \frac{c_1((1-\epsilon)+\epsilon c_1^{'})}{(1-\frac{2\epsilon}{\beta} c_1 c_2)}$. To prove the second inequality (30) of Lemma 4, we apply the operator $\Delta = \operatorname{Tr} \nabla^2$ to equation (28). Then after estimating the commutator of ∇^2 and $L[w]^{-1}$ (see Appendix A), we obtain

$$L[w]^{-1} \Delta p + \frac{2}{\beta}(w.\nabla))\Delta p = g(p_1, w, p),$$

where

$$g(p_1,w,p)=\Delta p_1-\frac{2}{\beta}\nabla w\nabla^2 p-\frac{2}{\beta}(w.\nabla)p-\mathrm{Tr}\,\nabla^2 L[w]p-2\mathrm{Tr}\,\nabla L[w]^{-1}\nabla p.$$

If $k=2,3,...$, then $g(p_1,w,p)\in H^{k-1}$ so that there exists a constant c^0 such that

$$\|\Delta p\|_{k-1}\le c^0\|g(p_1,w,p)\|_{k-1}.$$

The estimate of the left side term is given by

$$\|g(p_1,w,p)\|_{k-1}\le\|\Delta p_1\|_{k-1}+\frac{4}{\beta}\|w\|_{k+2}\|p\|_{k+1} \tag{65}$$
$$+\,\epsilon(1-\epsilon)\mathrm{We}^2c_4\|w\|^2_{k+2}\|p\|_{k+1}+2\epsilon\mathrm{We}c_4\|w\|_{k+2}\|p\|_{k+1}+c_4\|w\|_{k+2}\|p\|_{k+1}.$$

Under hypotheses (56) and (51), there exists a constant c^1 independent of $\|w\|_{k+2}$, and depending only on We, Ω, $\frac{1}{\beta}$, such that

$$\|\Delta p\|_{k-1}\le c^1\big(\|\Delta p_1\|_{k-1}+\big(\|w\|_{k+2}+\frac{2\epsilon}{\beta}\big)\|p\|_{k+1}\big).$$

Therefore, we have proved the part of Lemma 4 concerning strong solutions.

Let us notice that if we can define weak solutions to equation (28), i.e., solutions in H^{-1} (respectively in L^2) verifying $\|p\|_{-1}\le\|p_1\|_{-1}$ (respectively $\|p\|\le\|p_1\|$), then inequality (30) is true also for $k=0,1$ (same demonstration). We now define a solution in $H^{-1}(\Omega)$ (respectively in L^2) of (28).

3.5. Weak solutions to the transport problem. Let us recall equation (61)

$$p+\frac{2}{\beta}L[w]\big((w.\nabla)p\big)=p_1^{'}.$$

We are going to proceed as Beirão da Veiga [1] did for the transport operator defined in Proposition 1. The operator $L[w]$ is an isomorphism of H^1_0 (see Proposition 2), then we can define its adjoint operator $L^*[w]$: its domain is

$$D(L[w]^*)=\{v\in(H^1_0)^*,\,\exists c\ge 0\text{ such that }\big|<v,L[w]u>\big|\le c\|u\|_1,\forall\,u\in H^1_0\}$$
$$=H^{-1}.$$

Moreover, $L[w]^*$ is an isomorphism in H^{-1} and $(L[w]^*)^{-1}=(L[w]^{-1})^*$. In addition, $L[w]$ and $L[w]^*$, also by Proposition 2, have in L^2 properties similar to the ones mentioned above.

Case 1: Solution in L^2.

Definition 1. Let $p_1^{'} \in L^2$, $w \in \mathbb{H}^3 \cap \mathbb{H}_0^1$, $\Omega \in \mathcal{C}^3$. The function $p \in L^2$ is a weak solution to problem (61) if

$$\big(\varphi - \frac{2}{\beta}\sum_{i=1}^{n} D_i(w_i L[w]^*\varphi), p\big) = \big(\varphi, p_1^{'}\big)\ \forall \varphi \in H_0^1, \tag{66}$$

where $(.,.)$ designates the scalar product in L^2.

We need to prove that, under the assumptions of Lemma 4 ($k = -1$), equation (61) admits a unique solution defined by (66).

$p_1^{'} \in L^2(\Omega)$, therefore there exists a sequence $(p_{m1}^{'})$ of H_0^1 converging to $p_1^{'}$ in $L^2(\Omega)$. By the previous section the equation

$$p_m + \frac{2}{\beta} L[w]((w.\nabla)p_m) = p_{m1}^{'}$$

admits a unique solution $p_m \in H_0^1$ verifying the estimate

$$\big\|p_m\big\| \le c\big\|p_{m1}^{'}\big\|. \tag{67}$$

Indeed, to prove this estimate we write $p_m = L_1[w]^{-1}(p_{m1}^{'} - \frac{2\epsilon}{\beta}L_2[w]p_m)$, and

$$\big\|p_m\big\| \le \big\|L_1[w]^{-1}(p_{m1}^{'} - \frac{2\epsilon}{\beta}L_2[w]p_m)\big\| \le c_1\big\|p_{m1}^{'}\big\| + \frac{2c_1c_2\epsilon}{\beta}\big\|p_m\big\|.$$

We deduce $(1 - \frac{2c_1c_2\epsilon}{\beta})\big\|p_m\big\| \le c_1\big\|p_{m1}^{'}\big\|$. So let p_m and p_n be two solutions corresponding respectively to $p_{m1}^{'}$ and $p_{n1}^{'}$. From (67) it follows that $\big\|p_m - p_n\big\| \le c\big\|p_{m1}^{'} - p_{n1}^{'}\big\|$, which implies that the sequence (p_m) converges in L^2 to p. One easily verifies that p is the desired solution, and we can pass to the limit in inequality (67).

Case 2: Solution in H^{-1}.

Definition 2. Let $p_1^{'} \in H^{-1}$, $w \in \mathbb{H}^3 \cap \mathbb{H}_0^1$, $\Omega \in \mathcal{C}^3$. The function $p \in H^{-1}$ is a weak solution to problem (61) if

$$\big\langle\varphi - \frac{2}{\beta}\sum_{i=1}^{n} D_i(w_i L[w]^*\varphi), p\big\rangle = \big\langle\varphi, p_1^{'}\big\rangle\ \forall \varphi \in \mathcal{D}(\Omega), \tag{68}$$

where $\mathcal{D}(\Omega) = \{u,\ u \in \mathcal{C}^\infty(\Omega)\ \text{ with compact support in } \Omega\}$.

Proposition 4. *Let $p_1^{'} \in H^{-1}$, $w \in \mathbb{H}^3 \cap \mathbb{H}_0^1$, $\Omega \in \mathcal{C}^3$. Then equation* (61) *admits a weak solution in $H^{-1}(\Omega)$.*

Proof. We consider the following equation

$$\varphi - \frac{2}{\beta}(w.\nabla)(L[w]^*\varphi) - \frac{2}{\beta}\operatorname{div} w(L[w]^*\varphi) = g. \tag{69}$$

We admit for a moment the following result.

Lemma 5. *We suppose that $\Omega \in \mathcal{C}^3$, $g \in H_0^1$ and $w \in \mathbb{H}^3(\Omega) \cap \mathbb{H}_0^1$. Then there exists a constant $c = c\big(\mathrm{We}, \Omega, \frac{1}{\beta}\big)$ such that if $c\|w\|_3 < 1$ equation* (69) *admits a unique solution $\varphi \in H_0^1(\Omega)$. Moreover, the operator*

$$B \ : \ \begin{cases} H_0^1 \to H_0^1 \\ g \to \varphi, \end{cases}$$

where φ is the solution to (69)*, is continuous.*

Denote $A = B^{-1}$, A is closed in H_0^1 with domain $H_{0w}^1 = \{\varphi \in H_0^1;\ (w.\nabla)\varphi \in H_0^1\}$, because $L[w]^*$ is an isomorphism of H_0^1 (see point (ii) below). As $\mathcal{D}(\Omega) \subset H_{0w}^1$, H_{0w}^1 is dense in H_0^1. Denoted by A^* the adjoint of A. Since $A^{-1} = B \in \mathcal{L}(H_0^1)$ a well known result of functional analysis (see [2]) implies $(A^*)^{-1} = (A^{-1})^* = B^* \in \mathcal{L}(H^{-1})$ (with $\|B^*\| = \|B\|$). It follows that the equation $A^*p = p_1{}'$ has a unique solution $p = B^*p_1^{'}$ $\forall p_1^{'} \in H^{-1}$. This equation is equivalent to $\langle A\varphi, p\rangle = \langle \varphi, p_1{}'\rangle \ \ \forall\varphi \in H_{0w}^1$; i.e.,

$$\Big\langle \varphi - \frac{2}{\beta}\sum_{i=1}^{n} D_i(w_i L[w]^*\varphi), p\Big\rangle = \big\langle \varphi, p_1{}'\big\rangle \ \ \forall\varphi \in H_{0w}^1,$$

so that $p = B^*p_1{}'$ is a weak solution of equation (61) verifying

$$\|p\|_{-1} \leq \|B^*\|\, \|p_1^{'}\|_{-1} \leq \|B\|\, \|p_1^{'}\|_{-1}.$$

Proof of Lemma 5. We want to solve equation (69). We set $\phi = L[w]^*\varphi$, which we replace in equation (69) by its value, and we obtain:

$$L[w]^{*-1}\phi - \frac{2}{\beta}(w.\nabla)\phi - \frac{2}{\beta}(\operatorname{div} w)\phi = g, \tag{70}$$

where $L[w]^* = (1-\epsilon)I + \epsilon(I + \mathrm{We}(w.\nabla))^{-1*}$. We apply $L[w]^*$ to (70), and we obtain

$$\phi - \frac{2}{\beta}L[w]^*(w.\nabla)\phi - \frac{2}{\beta}L[w]^*(\operatorname{div} w)\phi = L[w]^*g, \tag{71}$$

so that

$$\phi - \frac{2(1-\epsilon)}{\beta}(w.\nabla)\phi - \frac{2\epsilon}{\beta}(I + \mathrm{We}(w.\nabla))^{-1*}(w.\nabla)\phi - \frac{2}{\beta}L[w]^*\operatorname{div} w\, \phi = L[w]^*g. \tag{72}$$

(i) The restriction of the operator $(I + \mathrm{We}(w.\nabla))^{-1*}$ to H_0^1 defines an continuous linear operator in H_0^1. As a matter of fact let $\varphi \in H_0^1$. We obtain

$$\left|\left\langle (I + \mathrm{We}(w.\nabla))^{-1*}\varphi, v\right\rangle_{H^{-1},H_0^1}\right| \leq c_1\|\varphi\|_1\|v\|_{-1} \qquad \forall v \in H_0^1. \tag{73}$$

As H_0^1 is dense in H^{-1}, Hahn-Banach's theorem allows to extend the above linear form to $v \in H^{-1}$, so that $(I + \mathrm{We}(w.\nabla))^{-1*}\varphi$ defines a continuous linear map on H^{-1}, that is to say $(I + \mathrm{We}(w.\nabla))^{-1*}\varphi \in H_0^1$ and, according to (73),

$$\|(I + \mathrm{We}(w.\nabla))^{-1*}\varphi\|_1 \leq c\|\varphi\|_1.$$

(ii) The restriction of $L[w]^*$ to H_0^1 is an isomorphism in H_0^1. In fact

$$\left(L[w]^*v, v\right)_1 = ((1-\epsilon)v, v)_1 + \epsilon(v, (I + \mathrm{We}(w.\nabla))^{-1}v)_1 \geq (1-\epsilon)\|v\|_1^2, \tag{74}$$

because $(v, (I + \mathrm{We}(w.\nabla))^{-1}v)_1 \geq 0$ (see Proposition 2 or [8]).

Now Lemma 5 follows from points (i) and (ii), and from the proof of the case $k = 1$, already studied in the previous section. Indeed, the following equation :

$$\phi - \frac{2(1-\epsilon)}{\beta}(w.\nabla)\phi - \frac{2\epsilon}{\beta}(I + \mathrm{We}(w.\nabla))^{-1*}(w.\nabla)\phi = \psi \tag{75}$$

admits a unique solution in H_0^k, this is proved in the same way as for equation (62). Now to obtain a solution of (72) we use the fixed point method on equation (75). To this end we denote $L_3[w] = -(I + \mathrm{We}(w.\nabla))^{-1*}(w.\nabla)$ and ϕ_i the solution of equation (75), with $\psi = \frac{2}{\beta}L[w]^*(\operatorname{div} w)\varphi_i + L[w]^*g \;\; \varphi_i \in H_0^1,\ i = 1,2$. It follows that $\phi_1 - \phi_2$ is solution of equation

$$L_1[-w](\phi_1 - \phi_2) + \frac{2\epsilon}{\beta}L_3[w](\phi_1 - \phi_2) = \frac{2}{\beta}L[w]^*(\operatorname{div} w)(\phi_1 - \phi_2),$$

with

$$\|\phi_1 - \phi_2\|_1 \leq \frac{2c}{\beta}\|L[w]^*(\operatorname{div} w)(\varphi_1 - \varphi_2)\|_1 \leq \frac{2c^{'}}{\beta}\|w\|_3\|\varphi_1 - \varphi_2\|_1,$$

which allows us to conclude by the Banach fixed point theorem if β sufficiently large.

Remark 5. We can directly solve equation (69) without using the case $k = 1$, like in Guillopé-Saut [3]. To this end, we have to consider the following regularized problem

$$\begin{cases} -\lambda\Delta(L[w]^*\varphi) + \varphi - \frac{2}{\beta}(w.\nabla)(L[w]^*\varphi) - \frac{2}{\beta}\operatorname{div} w(L[w]^*\varphi) = g \text{ in } \Omega, \\ L[w]^*\varphi|_{\partial\Omega} = 0 \end{cases} \tag{76}$$

4. Demonstration of Theorem 1. Let us recall the nonlinear problem (17)–(20)

$$\begin{cases} -(\Delta u + \nabla \operatorname{div} u) + \nabla(L[u]^{-1}p) = F_1(u, p, \tau), \\ \frac{1}{\beta}\Pi((u.\nabla)p) + \operatorname{div} u = 0, \\ \mathrm{We}\{(u.\nabla)\tau + \mathbf{g}(\nabla u, \tau)\} + \tau = 2\epsilon D[u], \end{cases}$$
$$u|_{\partial\Omega} = 0,$$

where the expression of $F_1(u, p, \tau)$ is given by equality (18). We seek a solution of this problem as a fixed point of the mapping $N : (w, \mu, \check{\tau}) \to (u, p, \tau)$, where (u, p, τ) is the solution of the linear problem (19)–(20) given by Theorem 4 and corresponding to $F = F_1(w, \mu, \check{\tau})$. We define the ball $B^{k+2}_{\gamma_0}$ by

$$B^{k+2}_{\gamma_0} = \Big\{(w, \mu, \check{\tau}), (w, \mu, \check{\tau}) \in \mathbb{H}^{k+2} \cap \mathbb{H}^1_0 \times H^{k+1} \times \mathbb{H}^{k+1}; \\ \|w\|_{k+2} + \|\mu\|_{k+1} + \|\check{\tau}\|_{k+1} \leq \gamma_0\Big\} \tag{77}$$

Lemma 6. *Let $k = 1, 2, \ldots,$ there exist two constants γ_0 and γ_1 such that if $\|f\|_k \leq \gamma_1$ then $N : B^{k+2}_{\gamma_0} \to B^{k+2}_{\gamma_0}$.*

Proof. Let us start by estimating the norm of $F_1(w, \mu, \check{\tau})$. From its expression and by using (10) and (11) it follows

$$\|F_1(w, \mu, \check{\tau})\|_k \leq c_3\|f\|_k + \|w\|_{k+2}\big(2c_3\mathrm{We}\|\check{\tau}\|_{k+1} \\ + \mathrm{Re}\|w\|_{k+2} + \epsilon c_4\|w\|_{k+2}\|\mu\|_{k+1}\big).$$

From Theorem 4 we get the following estimate

$$\begin{aligned}&\|u\|_{k+2}+\|p\|_{k+1}+\|\tau\|_{k+1}\\&\leq\big(c^1+\epsilon c^2\big)\big(\|f\|_k+\|w\|_{k+2}(2c_3\mathrm{We}\|\check{\tau}\|_{k+1}+\mathrm{Re}\|w\|_{k+2}+\epsilon c_4\|\mu\|_{k+1})\big).\end{aligned}$$

For $\|f\|_k\leq\gamma_1$, the previous estimate implies that if $(w,\mu,\check{\tau})$ and $(u,p,\tau)\in B_{\gamma_0}^{k+2}$, then

$$\begin{aligned}&(c^1+\epsilon c^2)\big(\gamma_1+\gamma_0^2(2c_3\mathrm{We}+\mathrm{Re}+\epsilon c_4)\big)\leq\gamma_0\\&(c^1+\epsilon c^2)\gamma_1\leq\gamma_0\big(1-\gamma_0(c^1+\epsilon c^2)(2c_3\mathrm{We}+\mathrm{Re}+\epsilon c_4)\big).\end{aligned}$$

One choose

$$\gamma_0\leq\frac{1}{c_\gamma}\quad\text{where}\quad c_\gamma=(c^1+\epsilon c^2)(2c_3\mathrm{We}+\mathrm{Re}+\epsilon c_4), \tag{78}$$

and

$$\gamma_1\leq\frac{(1-\gamma_0c_\gamma)\gamma_0}{(c^1+\epsilon c^2)}. \tag{79}$$

The choices (78) and (79) imply Lemma 6.

Remark 6. The constants c_3, c_4 are independent of w and β, and depend only on the domain Ω and on We.

Remark 7. The more Re and We increase, the more γ_0 and γ_1 decrease. At first sight γ_0 and γ_1 depend only on β through c^1 and c^4, but we will see in the Section 4, that they depend only on the lower bound of β.

Let us show now that N is a contraction of $B_{\gamma_0}^{k+2}$. Let $(w,\mu,\check{\tau})$ and $(w_1,\mu_1,\check{\tau}_1)\in B_{\gamma_0}^{k+2}$; we define

$$\begin{gathered}N(w,\mu,\check{\tau})=(u,p,\tau),\quad N(w_1,\mu_1,\check{\tau}_1)=(u_1,p_1,\tau_1),\quad F=F_1(w,\mu,\check{\tau}),\\F_1=F_1(w_1,\mu_1,\check{\tau}_1),\quad\tilde{F}=F-F_1,\ \tilde{p}=p-p_1,\ \tilde{u}=u-u_1,\ \tilde{\tau}=\tau-\tau_1,\\\tilde{\mu}=\mu-\mu_1,\ \tilde{w}=w-w_1,\ \tilde{\check{\tau}}=\check{\tau}-\check{\tau}_1.\end{gathered}$$

Then we easily check that $(\tilde{u},\tilde{p},\tilde{\tau})$ verifies the following system

$$\begin{cases}-(\Delta\tilde{u}+\nabla\mathrm{div}\,\tilde{u})+\nabla(L[w_1]^{-1}\tilde{p})=\tilde{F}-\nabla((L[w]^{-1}-L[w_1]^{-1})p),\\\frac{1}{\beta}\Pi((w_1.\nabla))\tilde{p}+\mathrm{div}\tilde{u}=-\frac{1}{\beta}\Pi((\tilde{w}.\nabla)p),\\\mathrm{We}\big((w_1.\nabla)\tilde{\tau}+\mathbf{g}(\nabla w_1,\tilde{\tau})\big)+\tilde{\tau}=2\epsilon D[\tilde{u}]-\mathrm{We}\big((\tilde{w}.\nabla)\tau+\mathbf{g}(\nabla\tilde{w},\tau)\big),\\\tilde{u}|_{\partial\Omega}=0.\end{cases} \tag{80}$$

For $w_1 \in \mathbb{H}^{k+2} \cap \mathbb{H}^1_0$, $k = 1, 2, ...$, Theorem 4 applies to system (80), thanks to the compatibility of the second member in $(80)_2$ (i.e., $\int_\Omega -\frac{1}{\beta}\Pi((\tilde{w}.\nabla)p)dx = 0$). Therefore, we get the estimates on $(\tilde{u}, \tilde{p}, \tilde{\tau})$:

$$\|\tilde{u}\|_2 + \|\tilde{p}\|_1 \leq c^1\big(\|\tilde{F}\| + \|\nabla\big((L[w]^{-1} - L[w_1]^{-1})p\big)\| + \|(\tilde{w}.\nabla)p\|\big), \quad (81)$$

$$\begin{aligned}\|\tilde{\tau}\|_1 \leq \epsilon c^2\big(\|\tilde{F}\| + \|\nabla(L[w]^{-1} - L[w_1]^{-1})p\| + \|(\tilde{w}.\nabla)p\|\big)\\ + \mathrm{We}\|(\tilde{w}.\nabla)\tau\| + \|\mathbf{g}(\nabla\tilde{w}, \tau)\|.\end{aligned} \quad (82)$$

After a rather lengthy calculation (see appendix B) we obtain the inequality

$$\|\tilde{u}\|_2 + \|\tilde{p}\|_1 + \|\tilde{\tau}\|_1 \leq (1 + \epsilon)\Phi_2\|\tilde{w}\|_2 + 2\|w_1\|_3\|\tilde{\tilde{\tau}}\|_1 + c\|w\|_2\|\tilde{\mu}\|_1. \quad (83)$$

The functions Φ_2, $\|w_1\|_3$ and $\|w\|_2$ depend on γ_0 and γ_1: we choose γ_0 and γ_1 small enough so that N is a contraction in $B^{k+1}_{\gamma_0}$. As $B^{k+1}_{\gamma_0}$ is a closed convex set of $\mathbb{H}^2 \times H^1 \times \mathbb{H}^1$, we can apply the Banach fixed point theorem, which implies the existence of (u, p, τ) such that $N(u, p, \tau) = (u, p, \tau)$.

Remark 8. The expression of Φ_2 contains the term $\epsilon(\tilde{w})$ (see appendix B) which can be bounded without difficulty by $c\gamma_0$, where c is a positive constant. Indeed $\lim_{\|\tilde{w}\|_2 \to 0} \epsilon(\tilde{w}) = 0$.

Remark 9. We can infer from (83) the continuity of the mapping N for the topology of $\mathbb{H}^2 \times H^1 \times \mathbb{H}^1$. Then we deduce from the compactness of $B^{k+1}_{\gamma_0}$ in $\mathbb{H}^2 \times H^1 \times \mathbb{H}^1$, for $k \geq 1$, the compactness of the mapping N. Then we can apply Schauder's fixed point theorem to establish the existence of a fixed point N. This gives less restriction on the size of γ_0 and γ_1, but it does not give the uniqueness of the solution.

Interpretation of problem (17). We have established the existence of a solution of problem (17). From the results of appendix A, $(17)_1$ is equivalent to

$$\begin{aligned}-(\Delta u + \nabla \mathrm{div}\, u) + L[u]^{-1}\nabla p = L[u]^{-1}f + L[u]^{-1}(I + \mathrm{We}(u.\nabla))^{-1}F(u, \tau)\\ - L[u]^{-1}(\mathrm{Re}(u.\nabla)u).\end{aligned} \quad (84)$$

From the regularity of f, u, and τ, we deduce that $(I + \mathrm{We}(u.\nabla))^{-1}F(u, \tau)$ and $(u.\nabla)u$ are in $\mathbb{H}^k$. Moreover, $L[u] : \mathbb{H}^k \to \mathbb{H}^k$ is bijective ($k = -1, 0, 1, ...$). We multiply (84) by $L[u]$, we take into account equalities (9)–(11) and the fact $u|_{\partial\Omega} = 0$, so that we precisely obtain (5)–(6). Therefore, we have established Theorem 1.

Remark 10. The size of β_1 depends on ϵ, We, and γ_0 (see the proof of Theorem 4 of existence of solutions for the linear problem (19)–(20)).

5. Study of the Maxwell model.

5.1. Transformation of the problem.

We study the following problem

$$\begin{cases} \mathrm{Re}\,(u.\nabla)u + \nabla p = f + \operatorname{div}\tau, \\ \frac{1}{\beta}\Pi((u.\nabla)p) + \operatorname{div} u = 0, \\ \mathrm{We}\,((u.\nabla)\tau + \mathbf{g}(\nabla u, \tau)) + \tau = 2\mathbf{D}[u]. \end{cases} \tag{85}$$

Problem (85) is posed in a regular bounded domain $\Omega \in \mathbb{R}^N$ $(N = 2, 3)$, and associated to the homogeneous boundary condition

$$u|_{\partial\Omega} = 0. \tag{86}$$

To transform problem (85)–(86), we start, like in the Jeffreys case $(0 < \epsilon < 1)$, by calculating

$$\operatorname{div}\tau = (I + \mathrm{We}(u.\nabla))^{-1}[(\Delta u + \nabla \operatorname{div} u) + F(u, \tau)]. \tag{87}$$

We replace $\operatorname{div}\tau$ in $(85)_1$ by its expression (87), so that

$$\mathrm{Re}\,(u.\nabla)u + \nabla p = f + (I + \mathrm{We}(u.\nabla))^{-1}[(\Delta u + \nabla \operatorname{div} u) + F(u, \tau)]. \tag{88}$$

We introduce a parameter $\lambda \in \mathbb{R}^*_+$, and we apply the operator $(\lambda I + \mathrm{We}(u.\nabla))$ to equation (88):

$$\begin{aligned} &\mathrm{Re}(\lambda I + \mathrm{We}(u.\nabla))(u.\nabla)u + (\lambda I + \mathrm{We}(u.\nabla))\nabla p \\ &\quad = (\lambda I + \mathrm{We}(u.\nabla))f + (\lambda I + \mathrm{We}(u.\nabla))(I + \mathrm{We}(u.\nabla))^{-1} \\ &\qquad [(\Delta u + \nabla \operatorname{div} u) + F(u, \tau)]. \end{aligned} \tag{89}$$

In this section the expression of the operator $L[u]$ is given by

$$L[u] = (\lambda I + \mathrm{We}(u.\nabla))(I + \mathrm{We}(u.\nabla))^{-1} = I + (\lambda - 1)(I + \mathrm{We}(u.\nabla))^{-1}.$$

(The dependence on λ is understood). If $\lambda \geq 1$ then $L[u]$ has the same properties as in the Jeffreys case. We multiply equation (89) by $L[u]^{-1}$, and we obtain

$$\begin{aligned} &-(\Delta u + \nabla \operatorname{div} u) + L[u]^{-1}\nabla[(\lambda I + \mathrm{We}(u.\nabla))p] \\ &\quad = F(u, \tau) + L[u]^{-1}((\lambda I + \mathrm{We}(u.\nabla))f \\ &\quad + \mathrm{We}\nabla u^T \nabla p - \mathrm{Re}(\lambda I + \mathrm{We}(u.\nabla))(u.\nabla)u), \end{aligned} \tag{90}$$

so that

$$
\begin{aligned}
&-(\Delta u+\nabla \operatorname{div} u)+\nabla[L[u]^{-1}(\lambda I+\mathrm{We}(u.\nabla))p]\\
&\quad=L[u]^{-1}(\lambda I+\mathrm{We}(u.\nabla))(f-\mathrm{Re}((u.\nabla)u)\\
&\quad\quad+L[u]^{-1}\{\mathrm{We}\ \mathrm{Com}(\nabla u,\nabla p)+\mathrm{We}\nabla u^T.\nabla p\}+F(u,\tau),
\end{aligned}
\tag{91}
$$

where $L[u]^{-1}$ Com$(\nabla u, \nabla p)$ is the commutator operator of $L[u]^{-1}$ and ∇.

By writing

$$
L[u]=\lambda\big(\frac{1}{\lambda}+\big(1-\frac{1}{\lambda}\big)(I+\mathrm{We}(u.\nabla))^{-1}\big),
$$

we deduce from the calculations of Appendix A that

$$
\begin{aligned}
\mathrm{Com}(\nabla u,\nabla p)=&(\lambda-1)(I+\mathrm{We}(u.\nabla))^{-1}(\nabla u)^T\nabla\big((I+\mathrm{We}(u.\nabla))^{-1}\\
&L[u]^{-1}(\lambda I+\mathrm{We}(u.\nabla))p\big).
\end{aligned}
\tag{92}
$$

We are going to study the system deduced from (85) by replacing $(85)_1$ by (91):

$$
\begin{cases}
-(\Delta u+\nabla \operatorname{div} u)+\nabla[L[u]^{-1}(\lambda I+\mathrm{We}(u.\nabla))p]\\
=L[u]^{-1}\{(\lambda I+\mathrm{We}(u.\nabla))(f-\mathrm{Re}((u.\nabla)u)\\
\quad+\mathrm{We}\ \mathrm{Com}(\nabla u,\nabla p)+\mathrm{We}\nabla u^T.\nabla p\}+F(u,\tau)\\
\frac{1}{\beta}\Pi((u.\nabla)p)+\operatorname{div} u=0\\
\mathrm{We}\{(u.\nabla)\tau+\mathbf{g}(\nabla u,\tau)\}+\tau=2D[u].
\end{cases}
\tag{93}
$$

5.2. Resolution of problem (92). The method used in Jeffreys' case can be applied for system (93) and (86), except that in the resolution of the linearized problem, equation (28) is replaced by the equation

$$
L[w]^{-1}(\lambda I+\mathrm{We}(w.\nabla))p+\frac{2}{\beta}(w.\nabla)p=p_1.
\tag{94}
$$

We apply the operator $L[w]$ to equation (94), and we obtain

$$
\lambda p+\big(\mathrm{We}+\frac{2}{\beta}\big)(w.\nabla)p+\frac{2(\lambda-1)}{\beta}(I+\mathrm{We}(w.\nabla))^{-1}(w.\nabla)p=L[w]p_1.
\tag{95}
$$

We can now apply Lemma 4 to obtain the existence and uniqueness of the solution to equation (95). The procedure used in the Jeffreys case shows Theorem 2.

Remark 11. The introduction of a fixed parameter $\lambda \neq 1$ and the multiplication by the operator $(\lambda I + \mathrm{We}(u.\nabla))$ is not necessary to give an equivalent form of problem (85). As a matter of fact we might calculate $\mathrm{div}\, \tau$ by using $(85)_1$, and replace it in

$$(I + \mathrm{We}(w.\nabla))\mathrm{div}\, \tau = (\Delta u + \nabla \mathrm{div}\, u) + F(u, \tau).$$

We then obtain the equation

$$-(\Delta u + \nabla \mathrm{div}\, u) = -(I + \mathrm{We}(w.\nabla))(\mathrm{Re}(u.\nabla)u - f + \nabla p) + F(u, \tau).$$

By commuting the operators ∇ and $(I+\mathrm{We}(u.\nabla))$, we transform the previous equation into

$$\begin{aligned} &-(\Delta u + \nabla \mathrm{div}\, u) + \nabla[(I + \mathrm{We}(w.\nabla))p] \\ &\quad = -(I + \mathrm{We}(w.\nabla))(\mathrm{Re}(u.\nabla)u - f) + \mathrm{We}\nabla u^T \nabla p + F(u, \tau). \end{aligned} \tag{96}$$

We observe that equation (96) is identical to equation $(93)_1$ with $\lambda = 1$.

In this case the transport equation (95) is simpler. In fact this is a classical transport equation, and the condition $c_5^{'}(\mathrm{We} + \frac{2}{\beta})\|w\|_{k+2} < 1$ suffices to give the same results as the ones given by Lemma 4.

Remark 12. The introduction of a real parameter $\lambda \neq 1$ and of the operator $(\lambda I + \mathrm{We}(u.\nabla))$ allows us to apply the previous method to more general situations. In fact this method can be used if we replace the constitutive law $(1)_3$ by the following constitutive law with several relaxation time parameters:

$$\begin{cases} \tau = 2\eta_s D[u] + \sum_{k=1}^{n} \tau_k, \\ \mathrm{We}_k\{(u.\nabla)\tau_k + \mathbf{g}(\nabla u, \tau_k)\} + \tau_k = 2\epsilon_k D[u],\ k = 1, ..., n. \end{cases}$$

6. The incompressible problem as limit of the compressible one. We are going to prove Theorem 3 in the case $0 < \epsilon < 1$. Actually the case $\epsilon = 1$ can be dealt with exactly in the same way.

Remark 13. In the case $\epsilon = 0$, i.e., the case of a Newtonian fluid, the convergence of the compressible problem to the incompressible one is given in Temam [11].

From Theorem 1, we know that a unique solution $(u_\beta, p_\beta, \tau_\beta) \in B_{\gamma_0}^{k+2}$, $k = 1, 2, ...$ to the following problem exists,

$$\begin{cases} \mathrm{Re}\,(u_\beta.\nabla)u_\beta - (1-\epsilon)(\Delta u_\beta + \nabla \mathrm{div}\, u_\beta) + \nabla p_\beta = f + \mathrm{div}\, \tau_\beta, \\ \frac{1}{\beta}\Pi((u_\beta.\nabla)p_\beta) + \mathrm{div}\, u_\beta = 0, \\ \mathrm{We}\,((u_\beta.\nabla)\tau_\beta + \mathbf{g}(\nabla u_\beta, \tau_\beta)) + \tau_\beta = 2\epsilon \mathbf{D}[u_\beta], \end{cases} \tag{97}$$

$$u_\beta|_{\partial\Omega} = 0. \tag{98}$$

We recall

$$B_{\gamma_0}^{k+2} = \big\{(w,\mu,\tau) \in \mathbb{H}^{k+2} \cap \mathbb{H}_0^1 \times H^{k+1} \times \mathbb{H}^{k+1}; \\ \|w\|_{k+2} + \|\mu\|_{k+1} + \|\tau\|_{k+1} \leq \gamma_0\big\}.$$

We now verify that γ_0 and γ_1 given by (78)–(79) are independent of β, for β sufficiently large. We recall the following expressions $\gamma_0 \leq \frac{1}{c_\gamma}$ where $c_\gamma = (c^1 + \epsilon c^2)(2c_3\mathrm{We} + \mathrm{Re} + \epsilon c_4)$, and $\gamma_1 \leq \frac{(1-c_\gamma)\gamma_0}{(c^1+\epsilon c^2)}$. Then, according to Remark 6, γ_0 and γ_1 only depend on β through c^1 and c^2. These constants come from estimates (38) and (39) in the solving of the linearized problem. If we examine the proof of these estimates we notice that we have made the following hypotheses

$$\|w\|_{k+2} \ < \big(\frac{1}{c_5^{'}} - \frac{2\epsilon}{\beta}\big), \qquad \beta \ > 2c_5^{'}\epsilon.$$

(see (51)) where $c_5^{'}$ is a constant which only depends on Ω (Sobolev's injection). The constant c_6 intervening in estimate (57) verifies

$$c_6 = \frac{c_5^{'}(1 + \|w\|_{k+2} + \frac{2\epsilon}{\beta})}{1 - c_5^{'}\big(\|w\|_{k+2} + \frac{2\epsilon}{\beta}\big)}.$$

Therefore we see that all the constants depend only on the lower bound of β, which will be denoted by β_1. This is the same situation for γ_1 because the same constants intervene. As γ_0 and γ_1 only depend on β_1, the classical results of compactness in Sobolev spaces imply that $(u_\beta, p_\beta, \tau_\beta)$ is weakly convergent in $\mathbb{H}^{k+2} \times H^{k+1} \times \mathbb{H}^{k+1}$ and strongly in $\mathbb{H}^{k+1} \times H^k \times \mathbb{H}^k$, $k = 1, ...$ toward (u, p, τ). These convergences are sufficient to pass to the limit in system (97). The limit verifies the system

$$\begin{cases} \mathrm{Re}\,(u.\nabla)u - (1-\epsilon)\Delta u + \nabla p = f + \mathrm{div}\,\tau, \\ \mathrm{div}\,u = 0, \\ \mathrm{We}\,((u.\nabla)\tau + \mathbf{g}(\nabla u, \tau)) + \tau = 2\epsilon\mathbf{D}[u]. \end{cases}$$

Moreover, u verifies the boundary condition (98). Indeed the convergence in $\mathbb{H}^2(\Omega)$ implies the convergence in $C(\overline{\Omega})$, which implies $u = 0$ on $\partial\Omega$. We can prove that the whole sequence $(u_\beta, p_\beta, \tau_\beta)$ converge. In fact this results from the uniqueness of the solution to the incompressible problem, so that this solution is the unique limit point of the sequence $(u_\beta, p_\beta, \tau_\beta)$. This ends the proof of Theorem 3.

Appendix A. In this appendix, we give the expressions of the commutator operators of $L[w]^{-1}$ and ∇ on the first hand, $L[w]^{-1}$ and $\nabla^2 = \nabla \circ \nabla$ on the other hand. We have the following result.

Proposition 5. *If w is regular enough, say $\mathbb{H}^{k+2}$, $k = 1, 2, ...$, then we have the relations*

$$\begin{aligned}
\nabla(L[w]^{-1}p) &= L[w]^{-1}\nabla p \\
&\quad + \epsilon \mathrm{We} L[w]^{-1}(I + \mathrm{We}(w.\nabla))^{-1}(\nabla w)^T \nabla[(I + \mathrm{We}(w.\nabla))^{-1} L[w]^{-1} p] \\
&= L[w]^{-1}\nabla p + (D_w L[w]^{-1}.\nabla w)p; \qquad \text{(A1)}\\
\nabla^2(L[w]^{-1}p) &= L[w]^{-1}\nabla^2 p + 2(D_w L[w]^{-1}.\nabla w))\nabla p + (D_w L[w]^{-1}.\nabla^2 w)p \\
&\quad - \epsilon(1-\epsilon)\mathrm{We}^2 L[w]^{-1}(I + \mathrm{We}(w.\nabla))^{-1}(\nabla w)^T \nabla \left[L[w]^{-1}(I + \mathrm{We}(w.\nabla))^{-1}\right. \\
&\quad \left.(\nabla w)^T \nabla[(I + \mathrm{We}(w.\nabla))^{-1} L[w]^{-1} p]\right] \qquad \text{(A2)}
\end{aligned}$$

Proof. We make a Taylor expansion to the second order of the operator $L[w]^{-1}$, so that we recognize $D_w L[w]^{-1}$ and $D_w^2 L[w]^{-1}$. To this end we calculate

$$\begin{aligned}
L[w]^{-1} &= (I + (1-\epsilon)\mathrm{We}(w.\nabla))^{-1}(I + \mathrm{We}(w.\nabla)) \\
L[w+H]^{-1} &= [(1-\epsilon)I + \epsilon(I + \mathrm{We}(w+H).\nabla)^{-1}]^{-1} \\
&= [(I + \mathrm{We}(w+H).\nabla)^{-1}((1-\epsilon)(I + \mathrm{We}(w+H).\nabla) + \epsilon I))]^{-1} \\
&= [(I + \mathrm{We}(w+H).\nabla)^{-1}((I + (1-\epsilon)\mathrm{We}(w+H).\nabla)]^{-1} \\
&= [(I + (1-\epsilon)\mathrm{We}(w.\nabla))(I + (1-\epsilon)\mathrm{We}(I + (1-\epsilon)\mathrm{We}(w.\nabla))^{-1}(H.\nabla)]^{-1} \\
&\quad \circ (I + \mathrm{We}(w+H).\nabla) \\
&= \left(I + (1-\epsilon)\mathrm{We}(I + (1-\epsilon)\mathrm{We}(w.\nabla))^{-1}(H.\nabla)\right)^{-1} \\
&\quad \circ (I + (1-\epsilon)\mathrm{We}(w.\nabla))^{-1}(I + \mathrm{We}(w+H).\nabla)).
\end{aligned}$$

We set

$$J_\epsilon[w] = (I + (1-\epsilon)\mathrm{We}(w.\nabla))^{-1}, \ \forall \epsilon \geq 0.$$

In particular, $L[w]^{-1} = J_\epsilon J_0[w]^{-1}$. We calculate

$$\begin{aligned}
&L[w+H]^{-1} - L[w]^{-1} \\
&= \{(I + (1-\epsilon)\mathrm{We}J_\epsilon[w](w.\nabla))^{-1} - I\}J_\epsilon[w](I + \mathrm{We}(w.\nabla)) \\
&\quad + \mathrm{We}(I + (1-\epsilon)\mathrm{We}J_\epsilon[w](H.\nabla))^{-1}J_\epsilon[w](H.\nabla)) \\
&= -(1-\epsilon)\mathrm{We}J_\epsilon[w](H.\nabla)L[w]^{-1} + \mathrm{We}L[w]^{-1}(I + \mathrm{We}w.\nabla)^{-1}(H.\nabla) + \\
&\quad + (1-\epsilon)^2\mathrm{We}^2 J_\epsilon[w](H.\nabla)L[w]^{-1}(I + \mathrm{We}(w.\nabla))^{-1}(H.\nabla)L[w]^{-1} - \\
&\quad - (1-\epsilon)\mathrm{We}^2 J_\epsilon[w](H.\nabla)L[w]^{-1}(I + \mathrm{We}(w.\nabla))^{-1}(H.\nabla) + O(\left|H\right|^3).
\end{aligned}$$

We now factorize $L[w]^{-1}$ on the right and the left sides of the previous equality, and we obtain

$$\begin{aligned}
&L[w+H]^{-1} - L[w]^{-1} \\
&\quad = L[w]^{-1}\{\epsilon\mathrm{We}(I + \mathrm{We}(w.\nabla))^{-1}(H.\nabla)(I + \mathrm{We}(w.\nabla))^{-1}\}L[w]^{-1} \\
&\qquad - (1-\epsilon)\mathrm{We}L[w]^{-1}\{\epsilon\mathrm{We}(I + \mathrm{We}(w.\nabla))^{-1}(H.\nabla)L[w]^{-1}\circ \\
&\qquad (I + \mathrm{We}(w.\nabla))^{-1}(H.\nabla)(I + \mathrm{We}(w.\nabla))^{-1}\}L[w]^{-1} + O(\left|H\right|^3).
\end{aligned}$$

We then easily deduce expressions (A1) and (A2).

Appendix B. The purpose of this appendix is to show estimate (83) which is needed to prove that the mapping N defined in section 4 is a contraction.

(i) Estimate of $\nabla\left[\left(L[w]^{-1} - L[w_1]^{-1}\right)p\right]$. We calculate

$$\begin{aligned}
&\nabla[\left(L[w]^{-1} - L[w_1]^{-1}\right)p] = \left(L[w]^{-1} - L[w_1]^{-1}\right)\nabla p \\
&\quad + D_w L[w_1]^{-1}(D_x w - D_x w_1)p + \left(D_w L[w]^{-1} - D_w L[w_1]^{-1}\right)D_x w\, p.
\end{aligned}$$

Using the Taylor expansion (see appendix A) of the operator $L[w]^{-1}$, we obtain

$$\|\left(L[w]^{-1} - L[w_1]^{-1}\right)\nabla p\| \le c\epsilon\mathrm{We}\|\tilde{w}\|_2\|\nabla p\|_1 + \left|\epsilon(\tilde{w})\right|\|\tilde{w}\|_2\|\nabla p\|,$$

$$\begin{aligned}
&\|(D_w L[w]^{-1} - D_w L[w_1]^{-1})D_x w\, p\| \\
&\quad \le \|D_w^2 L[w]^{-1}\tilde{w}D_x w\, p\| + \left|\epsilon(\tilde{w})\right|\|\tilde{w}\|_1\|w\|_2\|p\| \\
&\quad \le \epsilon(1-\epsilon)\mathrm{We}^2\left|\left|L[w]^{-1}(I + \mathrm{We}(w.\nabla))^{-1}\right.\right. \\
&\qquad \left.\left.(\tilde{w}.\nabla)L[w]^{-1}(I + \mathrm{We}(w.\nabla))^{-1}D_x w\ \nabla(I + \mathrm{We}(w.\nabla))^{-1}L[w]^{-1}p\right|\right| \\
&\quad \le c(1 + \left|\epsilon(\tilde{w})\right|)\|\tilde{w}\|_2\|w\|_2\|p\|_2.
\end{aligned}$$

Finally, we have

$$\|\nabla \left[\big(L[w]^{-1} - L[w_1]^{-1}\big)p\right]\| \leq \epsilon c \mathrm{We}\big(1 + \big|\epsilon(\tilde{w})\big|\big)\big(2 + \|w\|_2\big)\|\tilde{w}\|_2\|p\|_2. \quad \text{(B1)}$$

(ii) Estimate of $\|\tilde{F}\|$, where $\tilde{F} = F_1(w, \mu, \check{\tau}) - F_1(w_1, \mu_1, \check{\tau}_1)$, and the expression of F_1 is given by (18).

$$\begin{aligned}\tilde{F} &= \big(L[w]^{-1} - L[w_1]^{-1}\big)f + L[w]^{-1}(I + \mathrm{We}(w.\nabla))^{-1}F(w, \check{\tau})\\ &- L[w_1]^{-1}(I + \mathrm{We}(w_1.\nabla))^{-1})F(w_1, \check{\tau}_1) - \mathrm{Re}[(w.\nabla)w - (w_1.\nabla)w_1]\\ &+ \epsilon L[w]^{-1}(I + \mathrm{We}(w.\nabla))^{-1}(\nabla w)^T\nabla[(I + \mathrm{We}(w.\nabla))^{-1}\mu]\\ &- \epsilon L[w_1]^{-1}(I + \mathrm{We}(w_1.\nabla))^{-1}(\nabla w_1)^T\nabla[(I + \mathrm{We}(w_1.\nabla))^{-1}\mu_1].\end{aligned}$$

We are going to estimate the different terms. The first one is estimated as follows,

$$\|\big(L[w]^{-1} - L[w_1]^{-1}\big)f\| \leq \epsilon c \mathrm{We}\|f\|\|\tilde{w}\|_2(1 + \big|\epsilon(\tilde{w})\big|). \quad \text{(B2)}$$

For the terms containing $F(w, \check{\tau})$ and $F(w_1, \check{\tau}_1)$ we write

$$\begin{aligned}&L[w]^{-1}(I + \mathrm{We}(w.\nabla))^{-1}F(w, \check{\tau}) - L[w_1]^{-1}(I + \mathrm{We}(w_1.\nabla))^{-1}F(w_1, \check{\tau}_1)\\ &= \big(L[w]^{-1} - L[w_1]^{-1}\big)\{(I + \mathrm{We}(w.\nabla))^{-1}F(w, \check{\tau})\}\\ &\quad + L[w_1]^{-1}\{(I + \mathrm{We}(w.\nabla))^{-1}F(w, \check{\tau}) - (I + \mathrm{We}(w_1.\nabla))^{-1}F(w_1, \check{\tau}_1)\}\\ &= L[w_1]^{-1}[(I + \mathrm{We}(w.\nabla))^{-1} - (I + \mathrm{We}(w_1.\nabla))^{-1}]F(w, \check{v}\tau)\\ &\quad + L[w_1]^{-1}(I + \mathrm{We}(w_1.\nabla))^{-1}(F(w, \check{\tau}) - F(w_1, \check{\tau}_1))\\ &\quad + \big(L[w]^{-1} - L[w_1]^{-1}\big)(I + \mathrm{We}(w.\nabla))^{-1}F(w, \check{\tau}).\end{aligned}$$

The corresponding estimates of the terms on the right are:

$$\begin{aligned}&\|\big(L[w]^{-1} - L[w_1]^{-1}\big)(I + \mathrm{We}(w.\nabla))^{-1}F(w, \check{\tau})\|\\ &\quad \leq \epsilon c \mathrm{We}(1 + \big|\epsilon(\tilde{w})\big|)\|\tilde{w}\|_2\|w\|_3\|\check{\tau}\|_2\end{aligned} \quad \text{(B3)}$$

(see the expression of $F(w, \check{\tau})$ in (9), (10) and (11)). Besides

$$\begin{aligned}&F(w, \check{\tau}) - F(w_1, \check{\tau}_1)\\ &\quad = \mathrm{We}\{l_1(\partial w, \partial\tau) + l_2(\partial^2 w, \tau) - l_1(\partial w_1, \partial\tau_1) - l_2(\partial^2 w_1, \tau_1)\}.\end{aligned}$$

We easily show that, according to the expressions of l_1, l_2, given in (10) and (11), we obtain

$$\|l_1(\partial w, \partial\tau) - l_1(\partial w_1, \partial\tau_1)\| \leq 2\|\tilde{w}\|_2\|\check{\tau}\|_2 + 2\|w_1\|_3\|\tilde{\check{\tau}}\|_1,$$
$$\|l_2(\partial^2 w, \tau) - l_2(\partial^2 w_1, \tau_1)\| \leq 2\|\tilde{w}\|_2\|\check{\tau}\|_2 + 2\|w_1\|_3\|\tilde{\check{\tau}}\|_1,$$

so that we obtain the following estimate

$$\begin{aligned}&\|L[w_1]^{-1}(I + \mathrm{We}(w_1.\nabla))^{-1}(F(w, \check{\tau}) - F(w_1, \check{\tau}_1))\| \\ &\quad \leq 2c\big(2\|\tilde{w}\|_2\|\check{\tau}\|_2 + 2\|w_1\|_3\|\tilde{\check{\tau}}\|_1\big).\end{aligned} \tag{B4}$$

We now define the functions g and g_1 by

$$\mathrm{We}(w.\nabla)g + g = F(w, \check{\tau});\ \ \mathrm{We}(w_1.\nabla)g_1 + g_1 = F(w, \check{\tau}).$$

We make the difference of these two relations:

$$\mathrm{We}(w_1.\nabla)(g - g_1) + (g - g_1) = -\mathrm{We}(\tilde{w}.\nabla)g.$$

Hence, $\|g - g_1\| \leq c\mathrm{We}\|\tilde{w}\|_2\|g\|_1$, which implies

$$\begin{aligned}&\|L[w_1]^{-1}\big((I + \mathrm{We}(w.\nabla))^{-1} - (I + \mathrm{We}(w_1.\nabla))^{-1}\big)F(w, \check{\tau})\| \\ &\quad \leq c\mathrm{We}\|\check{\tau}\|_2\|w\|_3\|\tilde{w}\|_2.\end{aligned}$$

Therefore, we obtain the estimate

$$\begin{aligned}&\|L[w]^{-1}(I + \mathrm{We}(w.\nabla))^{-1}F(w, \check{\tau}) - L[w_1]^{-1}(I + \mathrm{We}(w_1.\nabla))^{-1}F(w_1, \check{\tau}_1)\| \\ &\quad \leq (2 + c\mathrm{We}(1 + \epsilon)\|w\|_3)\|\check{\tau}\|_2\|\tilde{w}\|_2 + \|w_1\|_3\|\tilde{\check{\tau}}\|_1.\end{aligned} \tag{B5}$$

To end estimating $\tilde{F}$, we still need to study two terms. First,

$$\begin{aligned}\mathrm{Re}\|(w.\nabla)w - (w_1.\nabla)w_1\| &\leq \mathrm{Re}\|(\tilde{w}.\nabla)w + (w_1.\nabla)\tilde{w}\| \\ &\leq \mathrm{Re}(\|w\|_2 + \|w_1\|_2)\|\tilde{w}\|_1,\end{aligned} \tag{B6}$$

and, second,

$$\begin{aligned}&\epsilon||L[w]^{-1}(I + \mathrm{We}(w.\nabla))^{-1}(\nabla w)^T\nabla[(I + \mathrm{We}(w.\nabla))^{-1}\mu] \\ &- L[w_1]^{-1}(I + \mathrm{We}(w_1.\nabla))^{-1}(\nabla w_1)^T\nabla[(I + \mathrm{We}(w_1.\nabla))^{-1}f\mu_1]|| \\ &\leq (c(1 + \epsilon)\mathrm{We}\|w\|_2\|\mu\|_1 + \|\mu_1\|_2 + \|w_1\|_3\|\mu_1\|_2)\|\tilde{w}\|_2 + c\|w\|_2\|\tilde{\mu}\|_1.\end{aligned} \tag{B7}$$

Gathering inequalities (B2)–(B7), we obtain the estimate of $\tilde{F}$,

$$\begin{aligned}\|\tilde{F}\| \leq \{\epsilon c\mathrm{We}(1+|\epsilon(\tilde{w})|)\|f\|_1 + (c\mathrm{We}(1+\epsilon)\|w\|_3 + 2)\|\check{\tau}\|_2 \\ + \mathrm{Re}(\|w\|_2 + \|w_1\|_2) + c\mathrm{We}(1+\epsilon)\|w\|_2\|\mu\|_1 + \|\mu_1\|_2 \\ + \|w_1\|_3\|\mu_1\|\}\|\tilde{w}\|_2 + 2\|w_1\|_3\|\tilde{\check{\tau}}\|_1 + c\|w\|_2\|\tilde{\mu}\|_1.\end{aligned} \tag{B8}$$

After taking into account $(B1)$ and $(B8)$ inequalities (81) and (82) become

$$\begin{aligned}\|\tilde{u}\|_2 + \|\tilde{p}\|_1 \leq \Phi_1(\|f\|_1, \|w\|_3, \|w_1\|_3, \|\mu\|_2, \|\mu_1\|_2, \|\check{\tau}\|_2)\|\tilde{w}\|_2 \\ + 2\|w_1\|_3\|\tilde{\check{\tau}}\| + c\|w\|_2\|\tilde{\mu}\|_1, \\ \|\tilde{\tau}\|_1 \leq \epsilon\Phi_1(\|f\|_1, \|w\|_3, \|w_1\|_3, \|\mu\|_2, \|\mu_1\|_2, \|\check{\tau}\|_2)\|\tilde{w}\|_2 \\ + \mathrm{We}\|\tau\|_1\|\tilde{w}\|_2 + c_{\mathbf{g}}\|\tau\|_1\|\tilde{w}\|_2,\end{aligned}$$

so that

$$\|\tilde{u}\|_2 + \|\tilde{p}\|_1 + \|\tilde{\tau}\|_1 \leq (1+\epsilon)\Phi_2\|\tilde{w}\|_2 + 2\|w_1\|_3\|\tilde{\check{\tau}}\|_1 + c\|w\|_2\|\tilde{\mu}\|_1, \tag{B9}$$

where Φ_2, $\|w_1\|_3$ and $\|w\|_2$ depend on γ_0 and on γ_1.

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