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BLOW-UP PHENOMENON AND GLOBAL SOLUTION TO A COUPLED PARABOLIC-ELLIPTIC SYSTEM OF CHEMOTAXIS

A. Boy-Dalverny
Ecole Nationale d'Ingénieurs de Tarbes rue Azereix, 65000 Tarbes, France

M. Madaune-Tort
Laboratoire de Mathématiques Appliquées. ERS A 2055
Université de Pau et des Pays de L'Adour
Avenue de l'Université 64000 Pau France

(Submitted by: Haim Brezis)

Abstract. We study a model proposed by E.F. Keller and L.A. Segel in two dimensions. Under some assumptions on the initial datum we prove the global existence and uniqueness of the solution and for other initial values we show that there exists a radial symmetric solution exploding in finite time.

1. Introduction

The aim of this paper is the mathematical study of a simplified version of E.F. Keller and L.A. Segel's model [16], [17], [18] describing the movement of a population to the gradient of a chemical substance secreted by this very population.

We consider the following nonlinear coupled parabolic-elliptic system

$$(S_1) \quad \begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = -div(u \nabla \psi(a)) & \text{on } (0,+\infty)\times\Omega, \\ -\Delta a = \alpha\big(u - \frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx\big) & \text{on } (0,+\infty)\times\Omega, \\ d_1 \frac{\partial u}{\partial n} - u\frac{\partial \psi(a)}{\partial n} = 0,\ \frac{\partial a}{\partial n} = 0 & \text{on } (0,+\infty)\times\Gamma, \\ u(0,\cdot) = u_0 & \text{on } \Omega, \end{cases} \tag{1}$$

where Ω is a C^2 bounded domain in $\mathbb{R}^2$, Γ its boundary, u the population density and a a function of the chemotactic agent's concentration such that $\frac{1}{|\Omega|}\int_\Omega a\,dx = 0$. The function u_0 is given in $H^2(\Omega)$ such that $u_0 \geq 0$ and

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$\frac{\partial u_0}{\partial n} = 0$ on $\partial\Omega$. The function ψ is a Lipschitzian convex function on $\mathbb{R}$ with ψ' locally Lipschitzian on $\mathbb{R}$. Let $d_1 > 0$ and $\alpha > 0$.

Notation. For any $T > 0$, $Q_T = (0,T) \times \Omega$. For any function f, $f^+ = f_+ = \max(f, 0)$, when f is in $L^1(\Omega)$, $\overline{f} = \frac{1}{|\Omega|}\int_\Omega f dx$ and when f is in $L^\infty(\Omega)$, $\|f\|_{\infty,\Omega} = \|f\|_{L^\infty(\Omega)}$.

The local existence of a solution to (S_1) has been proved for $\Omega \subset \mathbb{R}^N$, $N \geq 1$ [5], [2], [24]. The existence of a unique global solution is already known in one dimension and has been shown by P. Biler [1] and A. Boy [6], as well as by T. Nagai and T. Senba [20] in the particular case of radial solutions with ψ linear.

After the Introduction, we will study the global existence and uniqueness of a solution in two space dimensions when $\overline{u}_0$ is small enough. Our purpose is to generalize W. Jager and S. Luckhaus' results [15] concerning the basic model $\psi(a) = a$.

In two space dimensions for the parabolic-elliptic system (S_1) we also find results of global-time radially symmetric solutions by P. Biler [1] and T. Nagai and T. Senba [20] for smooth functions ψ that are strictly sublinear sensitivity functions (that is $\psi'(a) > 0$, $\lim \psi'(a) = 0$ when $a \to \infty$ and the function $a \to a\psi'(a)$ is an increasing one) [1], or for ψ defined by $\psi(a) = \chi \log(a)$, with $\chi > 0$ [20]. Moreover, when ψ is linear, the existence of a global solution has been studied by T. Nagai, T. Senba, and K. Yoshida [21] and by H. Gajewski and K. Zacharias [12] even for the parabolic-parabolic system.

In three dimensions the only results obtained in the case of a homogeneous Neumann problem seems to be those of P. Biler [1]. He achieves a global-time existence result when $\psi(a) = \chi \log(a)$, with $\chi < 2/3$. In this situation we can observe that the attractive chemotactic effect is smaller than the one in the linear case as it decreases when a increases. The other global existence results in three space dimensions have been obtained with quite different systems. Thus under homogeneous Dirichlet boundary conditions in three dimensions, when $\overline{u}_0$ is small enough, the existence of a global solution was shown by J.I. Diaz and T. Nagai [10] when the function ψ is such that $\psi'(a)$ is a positive constant, and more generally for any convex function ψ by A. Boy and M. Madaune-Tort [7] (therefore the attractive chemotactic effect increases as a increases). Similarly under homogeneous Neumann boundary conditions in three dimensions if we add some growth terms in the first equation of the system (S_1) A. Boy and M. Madaune-Tort [7] highlight a global-time solution. Moreover, additional interesting results have been obtained for these parabolic-elliptic systems by A. Bonami, D. Hilhorst, E. Logak and M. Mimura [3], [4] when the growth term is given by

$$\frac{1}{\varepsilon}u(1-u)(u-a),\ a\in[0,1] \text{ a fixed constant},\ \varepsilon>0 \tag{2}$$

and ψ is a smooth function such that $\psi(a)>0$ and $\psi'(a)>0$ for $a>0$.

The goal of their study is to prove a unique local solution to a free-boundary associated problem describing the u behaviour when ε tends to 0^+.

In the last part we prove the existence of a radially symmetric solution exploding in finite time when $\overline{u}_0$ is large enough. We generalize W. Jager and S. Luckhaus' results. Indeed in [15] W. Jager and S. Luckhaus (as well as T. Nagai [19] or T. Nagai and T. Senba [20] or T. Senba and T. Suzuki [23] or J.J. Diaz, T. Nagai, and J.M. Rakotoson on unbounded domains [11]) showed a blow-up phenomenon when the function ψ is linear (note that, in [22], it is shown that blow-up can occur in the case where $\psi(a)=\chi a^p$, $p>1$). This paper aims to show the same result but in the more general case in which ψ is a convex function. As the proof can also be performed when Ω is an open set of $\mathbb{R}^3$ we will consider in this third part both the situations $N=2$ and $N=3$.

In the whole following study, like W. Jager and S. Luckhaus [15], we will work with the new problem

$$(S_2)\quad\begin{cases}\frac{\partial u^\diamond}{\partial t}-d_1\Delta u^\diamond=-div(u^\diamond\nabla\psi_\star(a^\diamond)) & \text{on }\Omega,\quad A\\ -\Delta a^\diamond=u^\diamond-1 & \text{on }\Omega,\quad B\\ d_1\frac{\partial u^\diamond}{\partial n}-u\frac{\partial\psi(a^\diamond)}{\partial n}=0 & \text{on }\Gamma,\\ \frac{\partial a^\diamond}{\partial n}=0 & \text{on }\Gamma,\\ u^\diamond(0,\cdot)=\frac{u_0}{\overline{u_0}}, & \text{on }\Omega.\end{cases} \tag{3}$$

W. Jager and S. Luckhaus have obtained (S_2) by putting $a^\diamond=\frac{a}{\alpha\overline{u_0}}$, $u^\diamond=\frac{u}{\overline{u_0}}$ and $\psi_\star(x)=\psi(\alpha\overline{u_0}x)$ with a and u the solutions to system (S_1).

The function $\psi_\star$ is Lipschitzian convex and $\psi'_\star$ locally Lipschitzian on $\mathbb{R}$. In the following we will drop the $^\diamond$ on u and a. One remarks that

$$\overline{u}=1. \tag{4}$$

2. Global existence in two space dimensions

According to [6] and [1] there exists a time T such that (S_1) (therefore (S_2)) admits a local solution $(u,a)\in L^\infty(0,T;H^2(\Omega))\times L^\infty(0,T;H^3(\Omega))$.

We note that

$$T^* = \sup\left\{0 < T \le +\infty : (S_1) \text{ admits a solution } (u,a) \in (L^\infty(Q_T))^2\right\}$$

and we will prove in this section that $T^* = +\infty$.

Theorem 1. *There exists a constant C_Ω, depending only on Ω, such that if*

$$\alpha \bar{u}_0 \sup(\psi')^+ < \frac{d_1}{12|\Omega| C_\Omega^2}, \tag{5}$$

then (S_1) admits a unique solution such that

$$\forall T \in (0,+\infty), \quad (u,a) \in L^\infty((0,T); H^2(\Omega)) \times L^\infty((0,T); H^3(\Omega)).$$

The hypothesis (5) shows that the more significant the chemical influence is, the weaker $\bar{u}_0$ has to be.

In order to prove this theorem, we first demonstrate the result below:

Proposition 1. *Let (u,a) be the local solution of (S_2) on $[0,T^*) \times \Omega$. If*

$$\lim_{k\to+\infty} \overline{\lim_{t\to T^*}} \sup(\psi'_\star)^+ \int_\Omega (u(t,\cdot)-k)^+ dx < \frac{d_1}{12 C_\Omega^2}, \tag{6}$$

then there exist a real $m_0 > 1$ and a constant K_{4,Ω,T^*}, such that

$$\forall m \in (1, m_0],\ \forall t \in (0,T^*), \quad \|u(t,\cdot)\|_{L^m(\Omega)} \le K_{4,\Omega,T^*}.$$

Proof. The solution (u,a) of (S_2) satisfies

$$(S_V)\begin{cases} \forall \varphi \in V \text{ for almost every } t \in (0,T^*), \\ \left\langle \frac{\partial u}{\partial t}, \varphi \right\rangle + d_1 \int_\Omega \nabla u \nabla \varphi \, dx = \int_\Omega u \nabla \psi_\star(a) \nabla \varphi \, dx, \qquad A \\ \int_\Omega \nabla \psi_\star(a) \nabla \varphi \, dx = \int_\Omega \psi'_\star(a)(u-1)\varphi \, dx - \int_\Omega (\nabla a)^2 \psi''_\star(a) \varphi \, dx. \ B \end{cases} \tag{7}$$

As for $T \in (0,T^*)$, $u \in L^\infty(0,T;H^2(\Omega))$ and $a \in L^\infty(0,T;H^3(\Omega))$, we can choose $\varphi = (u-k)_+^{m-1}$, where $k \ge 2$ and $m > 1$.

According to (7-A), we have

$$\frac{1}{m}\frac{d}{dt}\int_\Omega (u-k)_+^m \, dx + \frac{4(m-1)d_1}{m^2}\int_\Omega \left(\nabla (u-k)_+^{\frac{m}{2}}\right)^2 dx = J_1 + kJ_2$$

with

$$J_2 = \int_\Omega \psi'_\star(a) \nabla (u-k)_+^{m-1} \nabla a \, dx,$$

$$J_1 = \int_\Omega \psi'_\star(a)(u-k)_+ \nabla (u-k)_+^{m-1} \nabla a \, dx = \frac{m-1}{m}\int_\Omega \psi'_\star(a) \nabla (u-k)_+^m \nabla a \, dx.$$

Now according to the equation (7-B), as $\psi_\star$ is convex,

$$J_2 \leq \int_\Omega \psi'_\star(a)(u-1)(u-k)_+^{m-1}\,dx.$$

Then

$$J_2 \leq \int_\Omega \psi'_\star(a)(u-k)_+^m\,dx + (k-1)\int_\Omega \psi'_\star(a)(u-k)_+^{m-1}\,dx.$$

Similarly,

$$J_1 \leq \frac{(m-1)}{m}\int_\Omega \psi'_\star(a)(u-k)_+^{m+1}\,dx + \frac{(m-1)}{m}(k-1)\int_\Omega \psi'_\star(a)(u-k)_+^m\,dx.$$

We therefore obtain

$$\frac{1}{m}\frac{d}{dt}\int_\Omega (u-k)_+^m\,dx + \frac{4(m-1)d_1}{m^2}\int_\Omega \Big(\nabla(u-k)_+^{\frac{m}{2}}\Big)^2\,dx \leq K \tag{8}$$

with

$$K = \frac{(m-1)}{m}\int_\Omega \psi'_\star(a)(u-k)_+^{m+1}\,dx + k(k-1)\int_\Omega \psi'_\star(a)(u-k)_+^{m-1}\,dx$$
$$+ \Big(\frac{(2m-1)}{m}k - \frac{(m-1)}{m}\Big)\int_\Omega \psi'_\star(a)(u-k)_+^m\,dx.$$

Moreover, as $(u-k)_+^{\frac{m+1}{2}} \in W^{1,1}(\Omega)$, if we apply to this function the Poincaré-Wirtinger inequality valid in bounded domains of $\mathbb{R}^2$ ([8], p. 194), it follows that

$$\Big(\int_\Omega \Big((u-k)_+^{\frac{m+1}{2}} - \frac{1}{|\Omega|}\int_\Omega (u-k)_+^{\frac{m+1}{2}}dx\Big)^2 dx\Big)^{\frac{1}{2}} \leq C_\Omega \int_\Omega \Big|\nabla(u-k)_+^{\frac{m+1}{2}}\Big|dx.$$

This inequality implies

$$\Big(\int_\Omega (u-k)_+^{m+1}dx\Big)^{\frac{1}{2}} \leq \frac{|\Omega_k|^{\frac{1}{2}}}{|\Omega|^{\frac{1}{2}}}\Big|\int_\Omega (u-k)_+^{m+1}dx\Big|^{\frac{1}{2}} + C_\Omega\int_\Omega \Big|\nabla(u-k)_+^{\frac{m+1}{2}}\Big|dx$$

with $\Omega_k = \{x : (u-k)_+(x) \neq 0\}$. Thanks to the three properties $u \geq 0$, (4) and

$$\int_\Omega u(t,x)\,dx \geq \int_{\Omega_k} k\,dx + \int_{\Omega-\Omega_k} u(t,x)\,dx,$$

we see $k|\Omega_k| \leq |\Omega|$. Then $\forall k \geq 2$, $|\Omega_k| \leq \frac{1}{2}|\Omega|$ and

$$\begin{aligned}\int_\Omega (u-k)_+^{m+1} dx &\leq 12C_\Omega^2\Big(\frac{m+1}{2}\Big)^2\Big(\int_\Omega \Big|(u-k)_+^{\frac{m-1}{2}}\nabla(u-k)_+\Big|\,dx\Big)^2\\ &\leq 12C_\Omega^2\Big(\frac{m+1}{m}\Big)^2\Big(\int_\Omega \Big|(u-k)_+^{\frac{1}{2}}\nabla(u-k)_+^{\frac{m}{2}}\Big|\,dx\Big)^2\\ &\leq 12C_\Omega^2\Big(\frac{m+1}{m}\Big)^2\int_\Omega (u-k)_+\,dx\int_\Omega\Big(\nabla(u-k)_+^{\frac{m}{2}}\Big)^2 dx.\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{4(m-1)d_1}{m^2}\int_\Omega\Big(\nabla(u-k)_+^{\frac{m}{2}}\Big)^2 dx - \frac{(m-1)}{m}\int_\Omega \psi'_\star(a)(u-k)_+^{m+1}dx\\ \geq K_m\int_\Omega\Big(\nabla(u-k)_+^{\frac{m}{2}}\Big)^2 dx\end{aligned}$$

with

$$K_m = \frac{4(m-1)d_1}{m^2} - 12C_\Omega^2\Big(\frac{m+1}{m}\Big)^2\frac{(m-1)}{m}\sup(\psi'_\star)^+\int_\Omega (u-k)_+dx.$$

But according to the hypothesis (6) there exists $m_0 > 1$, $k_0 \geq 2$ and $t_0 \in (0,T^*)$ such that $\forall k \geq k_0$, $\forall t \in (t_0,T^*)$, $\forall m \in [1,m_0]$,

$$\sup(\psi'_\star)^+\int_\Omega (u-k)_+\,dx \leq \frac{md_1}{3\,(m+1)^2\,C_\Omega^2},$$

and then $K_m \geq 0$.

Let us consider any (k,t,m) such that $k \geq k_0$, $t \in (t_0,T^*)$ any $m \in [1,m_0]$. According to (8), we get

$$\begin{aligned}\frac{1}{m}\frac{d}{dt}\int_\Omega (u-k)_+^m\,dx \leq \Big(\frac{(2m-1)}{m}k - \frac{(m-1)}{m}\Big)\int_\Omega \psi'_\star(a)(u-k)_+^m\,dx\\ + k(k-1)\int_\Omega \psi'_\star(a)(u-k)_+^{m-1}\,dx.\end{aligned} \tag{9}$$

By using the Young inequality and (9), it follows that

$$\frac{d}{dt}\int_\Omega (u-k)_+^m dx \leq K_{1,\Omega} + K_{2,\Omega}\int_\Omega (u-k)_+^m dx$$

with $K_{1,\Omega} = k\,(k-1)\sup{(\psi'_\star)}^+|\Omega|$ and $K_{2,\Omega} = \sup{(\psi'_\star)}^+(k-1+k^2)m_0$.

After integrating on (t_0,t) and according to the Gronwall lemma, we have, $\forall t \in (t_0,T^*)$,

$$\int_\Omega (u(t,\cdot)-k)_+^m\,dx \leq \Big(\int_\Omega (u(t_0,\cdot)-k)_+^m\,dx + K_{1,\Omega}\Big)\exp(K_{2,\Omega}t).$$

Then there exists a constant K_{3,Ω,T^*} such that

$$\forall t \in (t_0, T^*) \text{ and } \forall m \in [1, m_0] : \int_\Omega (u(t,\cdot) - k)_+^m \, dx \le K_{3,\Omega,T^*} .$$

Finally, there exists a constant K_{4,Ω,T^*} such that

$$\forall t \in (t_0, T^*) \text{ and } \forall m \in [1, m_0] : \|u(t,.)\|_{L^m(\Omega)} \le K_{4,\Omega,T^*} .$$

Now let us prove that Theorem 1 is a consequence of this last result. Indeed as the relationship between the functions ψ and $\psi_\star$ is $\psi(\alpha\overline{u_0}x) = \psi_\star(x)$. Thus, $\alpha\overline{u_0}\psi'(\alpha\overline{u_0}x) = \psi'_\star(x)$, and if hypothesis (5) is satisfied, then hypothesis (6) is also satisfied. Therefore according to Proposition 1 there exists $m_0 \in \mathbb{R}$, $m_0 > 1$ such that

$$\forall m \in [1, m_0] \text{ and } t < T^* : \|u(t,\cdot)\|_{L^m(\Omega)} \le K_{4,\Omega,T^*} . \tag{10}$$

Through general theorems on parabolic-elliptic equations, equations (3-A) and (3-B) allow us to conclude that property (10) implies the existence of a constant $K_{5,\Omega}$ such that for all $t \in (t_0, T^*)$

$$\|u(t,\cdot)\|_{L^\infty(\Omega)} \le K_{5,\Omega} \text{ and } \|a(t,\cdot)\|_{L^\infty(\Omega)} \le K_{5,\Omega}. \tag{11}$$

Consequence. *We notice that if ψ is decreasing, the problem (S_1) admits a unique solution on $[0, +\infty) \times \Omega$.*

3. Blow-up of a radially symmetric solution

The biological study of chemotaxis shows that as soon as N (the dimension of the set Ω) is such that $N > 1$, an excessive concentration of population may appear in some places and the blow up phenomenon of the solution may occur in finite time [9]. In the preceding paragraph we have seen how to choose the initial datum u_0 to obtain a homogeneous balance of cells. Now we will present initial conditions implying the explosion of the solution to (S_2). This blow-up phenomenon of a local solution in finite time has already been studied in two dimension spaces if $\psi'(a)$ is a positive constant by W. Jager and S. Luckaus [15], T. Nagai [19] and T. Nagai and T. Senba [20], T. Senba and T. Suzuki [23] and for parabolic systems by M.A. Herrero and J.J.L. Velazquez [13], [14].

In this paragraph we want to prove the existence of a radially symmetric solution exploding in finite time when $\overline{u}_0$ is large enough in a more general case than those of W. Jager and S. Luckaus [15] in which ψ is a convex function.

Now, as the techniques used for that permit us to show the blow-up phenomenon in the three-dimensional case, in the following purpose we consider $R > 0$ such that $\Omega = \{x : \|x\| < R^{\frac{1}{N}}\}$ where $N = 2$ or $N = 3$ represents the dimension of the set Ω.

We assume that $u_0 \neq 0$ and ψ is an increasing function such that $\inf(\psi'_\star) > 0$. We then introduce $\varkappa = \inf(\psi'_\star)$.

As the operator is objective (does not vary) by rotation, it is easy to observe thanks to a uniqueness argument [5] that if u_0 is radially symmetric, then the solutions u and a of (S_2) are also radially symmetric. Accordingly we consider the following change of variables: $x = r\cos\theta\cos\alpha'$, $y = r\sin\theta\cos\alpha'$, $z = r\sin\alpha'$ with $\alpha' = (N-2)\alpha$, $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\theta \in [0, 2\pi]$ and $r \in [0, R^{\frac{1}{N}})$.

Now for all $(t,r) \in (0,T)\times(0,R^{\frac{1}{N}})$, as the solution (u,a) of (S_2) is radially symmetric, we may introduce, for each (θ, α'), the functions

$$U(t,r) = u(t,r,0,0) = u(t, r\cos\theta\cos\alpha', r\sin\theta\cos\alpha', r\sin\alpha')$$
$$A(t,r) = a(t,r,0,0) = a(t, r\cos\theta\cos\alpha', r\sin\theta\cos\alpha', r\sin\alpha').$$

The aim of this paragraph is to show the following theorem:

Theorem 2. *There exist a constant C and a function θ on $(0,R)$ such that if $u_0 > \theta$ on $(0,R)$ and $\alpha\overline{u}_0 \inf(\psi') > C$, there exists a time $\overline{t} < +\infty$, such that*

$$\lim_{\substack{t \to \overline{t} \\ <}} U(t,0) = +\infty. \tag{12}$$

For that purpose we write

$$\forall\,(t,\rho) \in (0,T)\times(0,R), \quad V(t,\rho) = \int_0^{\rho^{1/N}} (U(t,r)-1)r^{N-1}\,dr. \tag{13}$$

Indeed, given that

$$\frac{\partial V}{\partial \rho}(t,\rho) = \frac{1}{N}\left(U(t,\rho^{\frac{1}{N}}) - 1\right), \tag{14}$$

in order to obtain (12) it suffices to prove the existence of a time $\overline{t} < +\infty$ such that

$$\lim_{\substack{t \to \overline{t} \\ <}} \frac{\partial V(t,0)}{\partial \rho} = +\infty. \tag{15}$$

To achieve this last result we are going to construct a function $\widetilde{W}$ that bounds V from below and such that $\frac{\partial \widetilde{W}(t,0)}{\partial \rho}$ explodes in finite time.

We are thus going to begin by demonstrating the following intermediate result.

Lemma 1. *V defined by* (13) *satisfies* $V \in C([0,T]; H^2((0,R))) \cap L^\infty(0,T : H^3((0,R)))$,

$$\frac{\partial V}{\partial t} \in C\left([0,T]; H^1((0,R))\right) \cap L^2\left(0,T; H^2((0,R))\right)$$

and V is a solution to the equation

$$\frac{\partial}{\partial t}V(t,\rho) - d_1 N^2 \rho^{\frac{2(N-1)}{N}} \frac{\partial^2}{\partial \rho^2} V(t,\rho) - \psi_\star'(A)\frac{N}{2}\frac{\partial}{\partial \rho}V^2(t,\rho) - \psi_\star'(A)V(t,\rho) = 0 \qquad (16)$$

with $V(t,R) = V(t,0) = 0$ *and* $V(0,\rho) = \int_0^{\rho^{\frac{1}{N}}} (u_0 - 1) r^{N-1}\, dr$.

Proof. The regularity on V is a direct consequence of those satisfied by U. Then, we have to prove (16). Integrating the equation (3-A) on $S_{\rho^{\frac{1}{N}}} = \{x \in \Omega : \|x\| < \rho^{\frac{1}{N}}\}$, $0 < \rho < R$, we have

$$\int_{S_{\rho^{\frac{1}{N}}}} \frac{\partial u(t,\cdot)}{\partial t}\, dX - d_1 \int_{S_{\rho^{\frac{1}{N}}}} \Delta u(t,\cdot)\, dX + \int_{S_{\rho^{\frac{1}{N}}}} div(u(t,\cdot)\nabla\psi_\star(a(t,\cdot)))\, dX = 0 \qquad (17)$$

with $X = (x,y)$, if $N = 2$ and $X = (x,y,z)$, if $N = 3$.

Since u is radially symmetric, we obtain

$$\int_{S_{\rho^{\frac{1}{N}}}} \frac{\partial u(t,X)}{\partial t}\, dX = 2^{N-1}\pi \frac{\partial}{\partial t} V(t,\rho) \qquad (18)$$

and

$$\int_{S_{\rho^{\frac{1}{N}}}} \Delta u(t,X)\, dX = 2^{N-1}\pi \int_0^{\rho^{\frac{1}{N}}} \Big(\frac{\partial^2 U(t,r)}{\partial r^2} r^{N-1} + (N-1)\frac{\partial U(t,r)}{\partial r} r^{N-2}\Big) dr$$
$$= 2^{N-1}\pi \rho^{\frac{N-1}{N}} \frac{\partial U}{\partial r}\left(t, \rho^{\frac{1}{N}}\right). \qquad (19)$$

According to (14) it follows that

$$\int_{S_{\rho^{\frac{1}{N}}}} \Delta u(t,X)\, dX = 2^{N-1}\pi N^2 \rho^{\frac{2N-2}{N}} \frac{\partial^2 V}{\partial \rho^2}(t,\rho). \qquad (20)$$

We now consider the third term of (17) and thanks to Green's formula, it follows that

$$\int_{S_{\rho^{\frac{1}{N}}}} div\left[u\left(t,\cdot\right)\nabla\psi_\star\left(a\left(t,\cdot\right)\right)\right]dX = \psi_\star'(A)U(t,\rho^{\frac{1}{N}})\int_{S_{\rho^{\frac{1}{N}}}} \Delta a(t,X)\,dX.$$

Then the equality (3-B) implies

$$\begin{aligned}\int_{S_{\rho^{\frac{1}{N}}}} div\left[u\left(t,\cdot\right)\nabla\psi_\star\left(a\left(t,\cdot\right)\right)\right]dX &= -\psi_\star'(A)U(t,\rho^{\frac{1}{N}})\int_{S_{\rho^{\frac{1}{N}}}}(u-1)\,dX,\\ \int_{S_{\rho^{\frac{1}{N}}}} div\left[u\left(t,\cdot\right)\nabla\psi_\star\left(a\left(t,\cdot\right)\right)\right]dX &= -2^{N-1}\pi\psi_\star'(A)U(t,\rho^{\frac{1}{N}})V(t,\rho).\end{aligned}$$

Through equality (14) it ensues that

$$\int_{S_{\rho^{\frac{1}{N}}}} div[u(t,\cdot)\nabla\psi_\star(a(t,\cdot))]dX = -2^{N-1}\pi\psi_\star'(A)\Big(\frac{N}{2}\frac{\partial V^2}{\partial\rho}(t,\rho)+V(t,\rho)\Big). \tag{21}$$

Lastly according to (18), (20) and (21) the equality (17) gives us (16).

To finish with, we notice that

$$V(t,R) = \int_0^{R^{\frac{1}{N}}}(U(t,r)-1)r^{N-1}dr = \frac{1}{2^{N-1}\pi}\int_{S_{\rho^{\frac{1}{N}}}} u(t,X)\,dX - \int_0^{\rho^{\frac{1}{N}}} r^{N-1}dr$$

and as $\overline{u}=1$ and $|S_{\rho^{\frac{1}{N}}}| = \frac{2^{N-1}}{N}\pi\rho$, we find

$$V(t,R) = \frac{|S_{\rho^{\frac{1}{N}}}|}{2^{N-1}\pi} - \frac{\rho}{N} = 0.$$

The last assertions of Lemma 1 result from the definition of V.

Now we are going to construct the lower bound $\widetilde{W}$. Let ρ_1 and ρ_2 be two reals such that $0<\rho_1<\rho_2<R$ and

$$\widetilde{W}(t,\rho) = \begin{cases}\frac{\tilde{a}\rho}{\rho+\tau^3} & \text{for } \rho<\rho_1,\\ \gamma(R-\rho-\frac{(\rho_2-\rho)_+^2}{\rho_2})\frac{\tilde{a}\rho_1}{\rho_1+\tau^3} & \text{for } \rho_1\le\rho,\end{cases} \tag{22}$$

where $\tau=\rho_0-bt$, $\gamma=(R-\rho_1-\frac{(\rho_2-\rho_1)^2}{\rho_2})^{-1}$. Parameters $\widetilde{a}>0$, $b>0$, $\rho_0>0$, ρ_1 and ρ_2 will be determined so as to have $\widetilde{W}$ a lower bound of V.

We can notice that $\widetilde{W}$ is a positive function when $\rho_1 > \frac{\rho_2-\sqrt{4R\rho_2-3\rho_2^2}}{2}$.

Lemma 2. *There exist a constant $c > 0$ and parameters $\widetilde{a}$, b, ρ_0, ρ_1 and ρ_2 such that, if $\varkappa = \inf \psi'_\star(a) > c$, the function $\widetilde{W}$ defined by (22) fulfills the following conditions,* $\forall \varphi \in H_0^1((0,R))$, $\varphi \geq 0$, $\forall (t,\rho) \in (0, \frac{\rho_0}{b}) \times (0,R)$:

$$\int_0^R \frac{\partial}{\partial t}\widetilde{W}(t,\rho)\varphi\, d\rho + d_1 N^2 \int_0^R \rho^{\frac{2N-2}{N}} \frac{\partial \widetilde{W}}{\partial \rho}\frac{\partial \varphi}{\partial \rho}\, d\rho - \int_0^R \psi'_\star(A)\frac{N}{2}\frac{\partial \widetilde{W}^2}{\partial \rho}\varphi\, d\rho$$

$$- \int_0^R \psi'_\star(A)\widetilde{W}\varphi\, d\rho + 2d_1 N(N-1)\int_0^R \rho^{\frac{N-2}{N}}\frac{\partial \widetilde{W}}{\partial \rho}\varphi\, d\rho \leq 0, \tag{23}$$

$$\forall t \in (0, \frac{\rho_0}{b}), \qquad \widetilde{W}(t,R) = \widetilde{W}(t,0) = 0.$$

Proof. Let $\rho_0 > 0$ be fixed. According to the definition of $\widetilde{W}$, with a view to achieving (23) we begin by choosing parameters so that the following inequality,

$$\frac{\partial}{\partial t}\widetilde{W}(t,\rho) - d_1 N^2 \rho^{\frac{2N-2}{N}} \frac{\partial^2 \widetilde{W}}{\partial \rho^2}(t,\rho) - \psi'_\star(A)\frac{N}{2}\frac{\partial \widetilde{W}^2}{\partial \rho}(t,\rho) - \psi'_\star(A)\widetilde{W}(t,\rho) \leq 0, \tag{24}$$

is satisfied.

(i) Study of (24) when $\rho < \rho_1$. It follows that

$$\frac{\partial}{\partial t}\widetilde{W}(t,\rho) = \frac{3\tilde{a}\rho b(\rho_0 - bt)^2}{(\rho + (\rho_0 - bt)^3)^2} = \frac{3b\tau^2}{\rho + \tau^3}\widetilde{W}.$$

Similarly on $(0, \frac{\rho_0}{b}) \times (0, \rho_1)$,

$$\rho \frac{\partial^2}{\partial \rho^2}\widetilde{W}(t,\rho) = \frac{-2\tilde{a}\tau^3}{(\rho + \tau^3)^3}\rho = \frac{-2\tau^3}{(\rho + \tau^3)^2}\widetilde{W}$$

and

$$\frac{\partial}{\partial \rho}\widetilde{W}^2(t,\rho) = 2\widetilde{W}\frac{\tilde{a}\tau^3}{(\rho + \tau^3)^2}.$$

Thanks to the latter results, in order to prove (24), we have to show the existence of two constants b, $\widetilde{a}$ such that

$$\frac{3b\tau^2}{\rho + \tau^3} + \frac{(2d_1 N^2 \rho^{\frac{N-2}{N}} - \psi'_\star(A)N\tilde{a})\tau^3}{(\rho + \tau^3)^2} - \psi'_\star(A) \leq 0. \tag{25}$$

Now to prove (25) the cases $N = 2$ and $N = 3$ are different.

For $N = 2$, we find the case studied by W. Jager and S. Luckhaus [15].

We choose $\tilde{a}$ such that

$$\tilde{a}\varkappa - 4d_1 > 0. \tag{26}$$

We notice that the condition on $\widetilde{a}$ is independent of the other parameters. Then we study the function

$$f_1(X) = \frac{3b\tau^2}{X+\tau^3} + \frac{2\left(4d_1 - \varkappa\tilde{a}\right)\tau^3}{\left(X+\tau^3\right)^2} - \varkappa.$$

In so far as it admits a maximal value in $X_0 = \frac{4(\tilde{a}\varkappa - 4d_1)\tau}{3b} - \tau^3$, we choose b such that $f_1(X_0) < 0$. Then, as $\tau \le \rho_0$, it is sufficient to choose b such that

$$0 < b < \tfrac{2}{3}\sqrt{\tfrac{2(\tilde{a}\varkappa - 4d_1)\varkappa}{\rho_0}} \quad \text{where } \varkappa = \inf\left(\psi'_\star\right). \tag{27}$$

Thus, as $\widetilde{W}(t,\rho) = \frac{\tilde{a}\rho}{\rho + (\rho_0 - bt)^3}$ with $\widetilde{a}$ fixed satisfying (26) and b satisfying (27) we have (25) when $N = 2$.

Now, we study the case $N = 3$ and inequality (25) becomes

$$\frac{3b\tau^2}{\rho+\tau^3} + \frac{(18d_1\rho^{\frac{1}{3}} - 3\psi'_\star(A)\tilde{a})\tau^3}{(\rho+\tau^3)^2} - \psi'_\star(A) \le 0.$$

First of all we choose $\widetilde{a}$ and ρ_2 such that

$$\rho_2^{\frac{1}{3}} \le \min(\frac{\varkappa\tilde{a}}{6d_1}, R^{\frac{1}{3}}). \tag{28}$$

Then when $\rho < \rho_1$, $18d_1\rho^{\frac{1}{3}} - 3\psi'_\star(A)\tilde{a} < 0$. Similarly we study the function

$$f_2(X) = \frac{3b\tau^2}{X+\tau^3} - \frac{2\varkappa\tilde{a}\tau^3}{\left(X+\tau^3\right)^2} - \varkappa.$$

Since it admits a maximal value in $X_0 = \frac{4\tilde{a}\varkappa\tau}{3b} - \tau^3$, we choose b such that $f_2(X_0) < 0$. As $\tau \le \rho_0$, one just has to choose b such that

$$0 < b < \tfrac{2}{3}\varkappa\sqrt{\tfrac{2\tilde{a}}{\rho_0}} \text{ where } \varkappa = \inf\left(\psi'_\star\right). \tag{29}$$

With conditions (28) and (29) on $\widetilde{a}$, ρ_2 and b inequality (25) is satisfied when $\rho < \rho_1$ and $N = 3$.

(ii) Study of (24) when $\rho > \rho_1$. We then have

$$\frac{\partial}{\partial t}\widetilde{W}(t,\rho) = \frac{3b\tau^2}{\rho_1+\tau^3}\widetilde{W} \text{ so } \frac{\partial}{\partial t}\widetilde{W}(t,\rho) = \frac{3b\rho_0^2}{\rho_1}\widetilde{W}. \tag{30}$$

Similarly,

$$\frac{\partial}{\partial\rho}\widetilde{W}(t,\rho) = -\gamma\big(1 - \frac{2(\rho_2-\rho)_+}{\rho_2}\big)\frac{\widetilde{a}\rho_1}{\rho_1+\tau^3};$$

hence,

$$\frac{\partial^2}{\partial\rho^2}\widetilde{W}(t,\rho) = \begin{cases} \frac{-2\gamma}{\rho_2}\frac{\tilde{a}\rho_1}{\rho_1+\tau^3} & \text{for } \rho \le \rho_2, \\ 0 & \text{for } \rho_2 \le \rho, \end{cases}$$

and for $\rho_1 < \rho < R$

$$-\frac{\partial^2}{\partial\rho^2}\widetilde{W}(t,\rho) \le \frac{2\widetilde{W}}{\rho_2\,(R-\rho_2)}. \tag{31}$$

Eventually,

$$\frac{\partial}{\partial\rho}\widetilde{W}^2(t,\rho) = -2\widetilde{W}\big(R-\rho_1-\frac{(\rho_2-\rho_1)^2}{\rho_2}\big)^{-1}\frac{\tilde{a}\rho_1}{\rho_1+\tau^3}\big(1-\frac{2(\rho_2-\rho)_+}{\rho_2}\big).$$

But,

$$\frac{\rho_1}{\rho_1+\tau^3}\big(1-\frac{2(\rho_2-\rho)_+}{\rho_2}\big) \le 1.$$

Moreover, as $0<\rho_1<\rho_2<R$,

$$0 \le \big(R-\rho_1-\frac{(\rho_2-\rho_1)^2}{\rho_2}\big)^{-1} \le (R-\rho_2)^{-1}.$$

Then since $\widetilde{W} \ge 0$, we have

$$-\frac{\partial}{\partial\rho}\widetilde{W}^2(t,\rho) \le 2\widetilde{W}(R-\rho_2)^{-1}\tilde{a}. \tag{32}$$

Thanks to (30), (31) and (32) we can notice that inequality (24) is satisfied as soon as

$$\frac{3b\rho_0^2}{\rho_1} + \frac{2d_1N^2\rho^{\frac{2N-2}{N}}}{\rho_2\,(R-\rho_2)} \le \psi_\star'(A)\big(1-\frac{N\tilde{a}}{R-\rho_2}\big). \tag{33}$$

But as soon as $\varkappa > \frac{4N}{R}d_1$, we can either choose $\tilde{a}$ satisfying (26) or $\widetilde{a}$ and ρ_2, $0<\rho_2<R$, satisfying (28) such that $0<\tilde{a}<\frac{R-\rho_2}{N}$. Then, if $\varkappa > \frac{2N^2d_1R^{\frac{2N-2}{N}}}{\rho_2(R-\rho_2-N\tilde{a})}$, we can find $b>0$ satisfying (27), if $N=2$, or (29) if $N=3$ and such that b is small enough to obtain (33). Then we can choose ρ_1 such that $\rho_2>\rho_1>\frac{\rho_2-\sqrt{4R\rho_2-3\rho_2^2}}{2}$.

With these different conditions on parameters $\tilde{a}$, b, ρ_1, and ρ_2, if we assume

$$\varkappa > c \text{ where } c = \max\big(\frac{4N}{R}d_1, \frac{2N^2d_1R^{\frac{2N-2}{N}}}{\rho_2\,(R-\rho_2-N\tilde{a})}\big), \tag{34}$$

the function $\widetilde{W}$ satisfies for all $t \in (0,\frac{\rho_0}{b})$ inequality (24) on $(0,\rho_1)$ and on (ρ_1,R).

(iii) Study of (23). Finally, in order to prove result (23) when $\varphi \in H_0^1((0,R))$ with $\varphi \geq 0$, it remains to show that with the previous choice of parameters $\tilde{a}$, b, ρ_1, and ρ_2 and assumption (34) for all $t \in (0, \frac{\rho_0}{b})$

$$\mathcal{I} = \frac{\partial \widetilde{W}\left(t, (\rho_1)_+\right)}{\partial \rho} - \frac{\partial \widetilde{W}\left(t, (\rho_1)_-\right)}{\partial \rho} \geq 0. \tag{35}$$

In effect, according to (24),

$$\int_0^R \frac{\partial}{\partial t}\widetilde{W}(t,\rho)\varphi\, d\rho + d_1 N^2 \int_0^R \rho^{\frac{2N-2}{N}} \frac{\partial \widetilde{W}}{\partial \rho}\frac{\partial \varphi}{\partial \rho}\, d\rho - \int_0^R \psi_\star'(A)\frac{N}{2}\frac{\partial \widetilde{W}^2}{\partial \rho}\varphi\, d\rho$$
$$- \int_0^R \psi_\star'(A)\widetilde{W}\varphi\, d\rho + 2d_1 N(N-1)\int_0^R \rho^{\frac{N-2}{N}}\frac{\partial \widetilde{W}}{\partial \rho}\varphi\, d\rho \leq -d_1 N^2 \rho_1^{\frac{2N-2}{N}}\varphi(\rho_1)\mathcal{I}.$$

Thanks to definition (22) of $\widetilde{W}$, we have

$$\begin{aligned}\mathcal{I} &= \frac{\widetilde{a}}{(\rho_1+\tau^3)^2}\gamma\Big[\frac{1}{\rho_2}(\rho_2 - 2\rho_1)(\rho_1+\tau^3)\rho_1 - \gamma^{-1}\tau^3\Big] \\ &= \frac{\widetilde{a}}{(\rho_1+\tau^3)^2}\gamma\Big[\frac{1}{\rho_2}(\rho_2-2\rho_1)\rho_1^2 + (-\frac{\rho_1^2}{\rho_2} + \rho_2 - R)\tau^3\Big].\end{aligned}$$

Consequently, if we choose $\rho_1 \in \mathbb{R}$ such that $\frac{\rho_2 - \sqrt{4R\rho_2 - 3\rho_2^2}}{2} < \rho_1 < \frac{\rho_2}{2}$ and ρ_0 such that $0 < \rho_0^3 < \frac{\rho_1^2}{\rho_2}(\rho_2 - 2\rho_1)(\frac{\rho_1^2}{\rho_2} - \rho_2 + R)^{-1}$ then the inequality (35) is satisfied.

With these two additional conditions compatible with the other previously obtained on parameters $\tilde{a}$, b, ρ_1 and ρ_2, the constant

$$c = \max\Big(\frac{4N}{R}d_1, \frac{2N^2 d_1 R^{\frac{2N-2}{N}}}{\rho_2\,(R - \rho_2 - N\tilde{a})}\Big)$$

is such that if $\varkappa > c$ the corresponding function $\widetilde{W}$ satisfies (23). The function $\widetilde{W}$ also satisfies for any $t \in (0, \frac{\rho_0}{b})$, $\widetilde{W}(t,0) = 0$ and $\widetilde{W}(t,R) = 0$.

Remark. We note that if u_0 satisfies

$$\forall \rho \in (0, R^{\frac{1}{N}}), \qquad \frac{1}{N}(u_0\,(\rho) - 1) \geq \frac{\partial \widetilde{W}}{\partial \rho}\left(0, \rho^N\right) \tag{36}$$

as $\widetilde{W}(0,0) = V(0,0) = 0$, we have

$$\forall \rho \in (0,R), \qquad \widetilde{W}(0,\rho) \leq V(0,\rho). \tag{37}$$

Lemma 3. *We assume that* u_0 *satisfies* (36) *and that* $\psi_\star$ *satisfies* (34). *If* V *is the function defined by* (13), *then for all* T *such that* $T < \min\left(\frac{\rho_0}{b}, T^*\right)$, *the functions* $\widetilde{W}$ *and* V *are defined on* $[0,T] \times \Omega$ *and satisfy*

$$\forall t \in (0,T), \quad \forall \rho \in (0,R), \qquad \widetilde{W}(t,\rho) \le V(t,\rho).$$

Proof. Through the Lemma 1 and Lemma 2, $\widetilde{W}$ satisfies inequality (23) and V is a solution of (16). Moreover, V and $\widetilde{W}$ satisfy

$$\forall t \in (0,T), \quad V(t,0) = \widetilde{W}(t,0) = 0 \text{ and } V(t,R) = \widetilde{W}(t,R) = 0.$$

If $H = \widetilde{W} - V$, we obtain on $(0,T) \times \Omega : \forall \varphi \in H_0^1\left((0,R)\right), \varphi \ge 0$,

$$\int_0^R \frac{\partial}{\partial t} H(t,\rho)\varphi \, d\rho + d_1 N^2 \int_0^R \rho^{\frac{2(N-1)}{N}} \frac{\partial}{\partial \rho} H(t,\rho) \frac{\partial \varphi}{\partial \rho} \, d\rho$$
$$- \frac{N}{2} \int_0^R \psi_\star'(A) \frac{\partial}{\partial \rho}\left(\widetilde{W}^2(t,\rho) - V^2(t,\rho)\right) \varphi \, d\rho - \int_0^R \psi_\star'(A) H \varphi \, d\rho$$
$$+ 2 d_1 N(N-1) \int_0^R \rho^{\frac{N-2}{N}} \frac{\partial}{\partial \rho} H(t,\rho) \varphi \, d\rho \le 0.$$

For $t \in (0,T)$, we can choose $\varphi = \rho^{\frac{2-N}{N}} H^+$ where $H^+ = \max(H,0)$. It follows that

$$\int_0^R \frac{\partial}{\partial t} H(t,\rho) H^+(t,\rho) \rho^{\frac{2-N}{N}} d\rho + d_1 N^2 \int_0^R \rho \big(\frac{\partial}{\partial \rho} H^+(t,\rho)\big)^2 d\rho$$
$$+ d_1 N^2 \int_0^R \big(\frac{\partial}{\partial \rho} H^+(t,\rho)\big) H^+(t,\rho) d\rho - \int_0^R \psi_\star'(A) \rho^{\frac{2-N}{N}} (H^+(t,\rho))^2 d\rho$$
$$- N \int_0^R \big(\psi_\star'(A) \widetilde{W}(t,\rho) \frac{\partial}{\partial \rho} \widetilde{W}(t,\rho) - \psi_\star'(A) V(t,\rho) \frac{\partial}{\partial \rho} V(t,\rho)\big) \rho^{\frac{2-N}{N}} H^+(t,\rho) \, d\rho$$
$$\le 0. \tag{38}$$

We note successively by K_1, K_2, K_3, K_4 and K_5 the five terms of the left member of (38). We see that $K_5 = -N \ (K_6 + K_7)$, where

$$K_6 = \int_0^R \psi_\star'(A) \widetilde{W}(t,\rho) \rho^{\frac{2-N}{N}} H^+(t,\rho) \frac{\partial}{\partial \rho} H(t,\rho) \, d\rho,$$
$$K_7 = \int_0^R \psi_\star'(A) \rho^{\frac{2-N}{N}} \left(H^+(t,\rho)\right)^2 \frac{\partial}{\partial \rho} V(t,\rho) \, d\rho.$$

As for $\rho < \rho_1$, $\widetilde{W}(t,\rho) \leq \frac{\tilde{a}\rho}{\tau^3} = c_1(t)\rho$, for all $t \in (0,T)$, we get

$$|K_6| \leq c_1(T)\|\psi_\star'\|_\infty \int_0^R \left|\rho^{\frac{2}{N}} H^+(t,\rho)\frac{\partial}{\partial\rho}H^+(t,\rho)\right| d\rho$$
$$\leq c_1(T)\|\psi_\star'\|_\infty \Big(\frac{R^{\frac{2}{N}}}{2\alpha}\int_0^R \rho^{\frac{2-N}{N}}(H^+(t,\rho))^2 d\rho + \frac{\alpha}{2}\int_0^R \rho(\frac{\partial}{\partial\rho}H^+(t,\rho))^2 d\rho\Big)$$

with $\alpha > 0$ such that $d_1 N - \frac{\alpha\|\psi_\star'\|_\infty c_1(T)}{2} \geq 0$.

Now there exists a constant c_3 such that

$$|K_7| \leq \left\|\psi_\star'\right\|_\infty c_3 \int_0^R \rho^{\frac{2-N}{N}} \left(H^+(t,\rho)\right)^2 d\rho$$

because $\frac{\partial}{\partial\rho}V(t,\rho) = \frac{1}{N}(U(t,\rho^{\frac{1}{N}}) - 1)$ is bounded like the local solution U.

Obviously, we have

$$|K_4| \leq \left\|\psi_\star'\right\|_\infty \int_0^R \rho^{\frac{2-N}{N}} \left(H^+(t,\rho)\right)^2 d\rho.$$

Eventually, $K_3 = \frac{N^2 d_1}{2}((H^+)^2(t,R) - (H^+)^2(t,0)) = 0$, and the inequality (38) implies

$$\frac{d}{dt}\int_0^R \rho^{\frac{2-N}{N}} \left(H^+\right)^2(t,\rho)\, d\rho$$
$$\leq \left\|\psi_\star'\right\|_\infty (1 + Nc_3 + \frac{R^{\frac{2}{N}}}{2\alpha}c_1(T)N)\int_0^R \rho^{\frac{2-N}{N}} \left(H^+(t,\rho)\right)^2 d\rho.$$

According to the preceding remark, thanks to hypothesis (36) we have $H^+(0,\rho) = 0$. Then, after integrating from 0 to t, by the Gronwall lemma it follows that $\forall t \in (0,T)$, $\forall \rho \in [0,R)$, $\widetilde{W}(t,\cdot) \leq V(t,\cdot)$.

Proof of Theorem 2. Let us introduce the function θ defined on $(0, R^{\frac{1}{N}})$ by $\theta(\rho) = N\frac{\partial\widetilde{W}}{\partial\rho}(0,\rho^{\frac{1}{N}}) + 1$, where $\widetilde{W}$ is given by Lemma 2.

According to the definition of $\widetilde{W}$ for all $\rho \in [0,\rho_1)$, this function is well defined for all $t \in [0,t_0)$, with $t_0 = \frac{\rho_0}{b}$, and

$$\lim_{\substack{t \to t_0 \\ <}} \frac{\partial\widetilde{W}}{\partial\rho}(t,0) = +\infty.$$

Given that ψ_* is defined by the relationship $\psi_*(x) = \psi(\alpha\overline{u}_0 x)$, under the conditions $u_0 > \theta$, $\alpha\overline{u}_0\inf(\psi') = \inf(\psi_*') > c$ then u_0 satisfies (36) and $\psi_\star$ satisfies (34) for all $T < \frac{\rho_0}{b}$. Thus according to Lemma 3 for V defined on

$[0,T]\times\Omega$, we have $\forall\rho\in[0,\rho_1)$, $\widetilde{W}(t,\rho)\leq V(t,\rho)$. Now as $\widetilde{W}(t,0)=V(t,0)=0$, it follows that

$$\frac{\partial\widetilde{W}}{\partial\rho}(t,0)\leq\frac{\partial V}{\partial\rho}(t,0).$$

Then, there exists a time $\overline{t}$ such that $\overline{t}\leq t_0<+\infty$ and

$$\lim_{\substack{t\to\overline{t}\\ <}}\frac{\partial V}{\partial\rho}(t,0)=+\infty.$$

Thus, the function U explodes at this time $\overline{t}$, because $\frac{\partial V}{\partial\rho}(t,0)=\frac{1}{N}(U(t,0)-1)$, and Theorem 2 is proved.

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