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EXISTENCE AND REGULARITY FOR A CLASS OF NON-UNIFORMLY ELLIPTIC EQUATIONS IN TWO DIMENSIONS

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Abstract. We prove some existence and regularity results for solutions of equations in the form $-\text{div}(a(x,u)\nabla u) = f$, where $a(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded Carathéodory function satisfying the inequality $a(x,s) \geq (1+|s|)^{-\theta}$ with $0 \leq \theta \leq 1$ and Ω is a bounded open set of $\mathbb{R}^2$.

1. Introduction. Let Ω be a bounded open set of $\mathbb{R}^2$, and let us consider the following elliptic problem:

$$\begin{cases} -\text{div}(a(x,u)\nabla u) = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

where $a(x,s) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a bounded Carathéodory function, i.e., $a(x,\cdot)$ is continuous for almost every $x \in \Omega$ and $a(\cdot,s)$ is measurable for every $s \in \mathbb{R}$, which, for some $\theta \in [0,1]$, satisfies the following assumption :

$$\frac{1}{(1+|s|)^\theta} \leq a(x,s). \tag{1.2}$$

The datum f is assumed to belong to the space $L^1(\Omega)$.

Our aim is to prove some existence and regularity results for entropy solutions of (1.1) (for the definition of entropy solution see Section 2 and [5]).

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A problem similar to (1.1) has been considered by many authors in the case when Ω is a bounded open set of $\mathbb{R}^N$, $N \geq 3$. In [7] the existence of a solution of (1.1), when $0 \leq \theta < 1$, is proven when f belongs to $L^m(\Omega)$, $m \geq 1$. Depending on m suitable regularity results for such a solution are also given. For example one can prove that if $f \in L^1(\Omega)$ then there exists an entropy solution u of (1.1) which belongs to $W_0^{1,s}(\Omega)$ for every $s < \frac{N(1-\theta)}{N-1-\theta}$. In [4] a priori estimates for solutions of (1.1) are proven when $a(x,s)$ satisfies assumptions more general than (1.2). In [9] the properties of minimizers of a functional related to problem (1.1) are studied. Finally we recall that the case when the differential operator (1.1) behaves like the $p-$Laplacian has been considered for example in [10], [12].

In the bidimensional case if $f \in L^m(\Omega)$, $m > 1$, some a priori estimates for solutions of (1.1) are given, for example, in [4], and such estimates can be used to prove the existence of a solution. The case $f \in L^1(\Omega)$ appears to be a limit case, where the regularity of the solutions of (1.1) has to be described in a more accurate way. Such a phenomenon already appears in the uniformly elliptic case ($\theta = 0$). For example if one considers a problem of the following type

$$\begin{cases} -(a_{ij}(x)u_{x_j})_{x_i} = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $a_{ij}(x)$ satisfies the assumption

$$a_{ij}(x)\xi_i\xi_j \geq |\xi|^2 \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^2$$

one can prove (see [11], [2]) that, when $f \in L^1(\Omega)$, if u solves (1.3), then

$$\int_\Omega \exp(\beta|u|)dx < +\infty$$

for some positive β.

In this paper we prove some existence and regularity results for solutions of (1.1) under assumption (1.2) when $f \in L^1(\Omega)$. In Section 2 we prove the existence of an entropy solution of (1.1), when (1.2) is satisfied for some $0 \leq \theta < 1$. The method we use is similar to the one used in [7]; we firstly consider a sequence of approximate, nondegenerate problems and then, using suitable a priori estimates, we pass to the limit obtaining a solution of (1.1).

Such a solution u is such that $u \in W_0^{1,q}(\Omega)$, $\forall q < 2$, and $u \in L^p(\Omega)$, $\forall p < +\infty$.(see also [16] for a particular case).

In Section 3, using accurate estimates on the entropy solutions u of (1.1), we prove that if $||f||_{L^1} \leq 1$, then u is such that

$$\int_\Omega \exp[\beta|u|^{1-\theta}]dx < c|\Omega|, \qquad \forall \beta < \bar{\beta} = \frac{4\pi}{(1-\theta)}$$

where c is a constant depending on β and θ only. Such a result is sharp (see Remark 3.3) and it is proven making use of techniques which are similar to those used, for example, in [15], [1], [2].

The same techniques allow us to study in Section 4 intermediate regularity of entropy solutions of (1.1), when f is in a Lorentz space $L(1,p)$, $1 \leq p < +\infty$. The precise definition of such spaces can be found in Section 4, we only recall that the following inclusion properties hold:

$$L^p(\Omega) \subset L\log L(\Omega) = L(1,1) \subset L(1,p) \subset L^1(\Omega), \quad \forall 1 < p < +\infty. \tag{1.4}$$

We prove that if $f \in L(1,1)$, then any entropy solution is bounded. This result is sharp in the scale of Lorentz spaces in the sense that if $f \in L(1,p)$, $p > 1$, then unbounded solutions of problems satisfying the assumptions made on (1.1) may exist. On the other hand, if $f \in L(1,p)$, with $||f||_{L(1,p)} \leq 1$, then any entropy solution of (1.1) is such that

$$\int_\Omega \exp[\beta|u|^{(1-\theta)p'}]dx < c|\Omega|, \qquad \forall \beta < \bar{\beta}_p = \Big(\frac{4\pi}{(1-\theta)}\Big)^{p'}, \tag{1.5}$$

where c is a constant which depends on β, θ, p. It is interesting to observe that when $\theta = 0$ inequality (1.5) can be proven for any $\beta \leq (4\pi)^{p'}$ (see for example [2]), while in the case $0 < \theta < 1$ the restriction $\beta < \bar{\beta}_p$ in (1.5) is sharp (see Remark 4.4).

Finally, in Section 5, we prove the existence of an entropy solution of (1.1) when (1.2) is satisfied for $\theta = 1$. Such a solution u is such that u belongs to $W_0^{1,q}(\Omega)$ for every $1 \leq q < 2$ and u belongs to $L^p(\Omega)$ for every $p \geq 1$. Differently from the case $0 \leq \theta < 1$, we cannot obtain, see Remark 5.2, that a suitable exponential function of an entropy solution of (1.1) is summable.

2. Existence of a solution. In this section we prove the existence of an entropy solution of (1.1). In order to give the definition of entropy solution (see for example [5]) of problem (1.1) let us define the truncation function $T_k(s)$: for every $k > 0$ we put

$$T_k(s) : s \in \mathbb{R} \mapsto \max\{-k, \min\{k, s\}\}.$$

Let us recall the following result (see [5]) :

Proposition 2.1. *Let u be a measurable function such that $T_k(u) \in H_0^1(\Omega)$ for every $k > 0$. Then there exists a unique measurable function $v : \Omega \to \mathbb{R}^2$ such that*

$$v\chi_{\{|u|<k\}} = \nabla T_k(u) \quad \text{a.e in } \Omega, \quad \forall k > 0.$$

If u belongs to $W_0^{1,1}(\Omega)$, then $v = \nabla u$.

We can define the weak gradient of a measurable function u, such that $T_k(u) \in H_0^1(\Omega)$ for every $k > 0$, as the function v of Proposition 2.1.

Definition 2.2. Let $f \in L^1(\Omega)$. A function $u \in L^1(\Omega)$ is an entropy solution of (1.1) if, for every $k > 0$,

$$T_k(u) \in H_0^1(\Omega)$$

and

$$\int_\Omega a(x,u)\nabla u \nabla T_k(u-\varphi)dx \leq \int_\Omega f T_k(u-\varphi)dx \quad \forall \varphi \in H_0^1(\Omega) \cap L^\infty(\Omega). \tag{2.1}$$

Definition 2.3. The Marcinkiewicz space $M^p(\Omega)$, $p > 0$, is the set of all measurable functions $u : \Omega \to \mathbb{R}$ such that, for some positive constant c, the following inequality holds

$$|\{x \in \Omega : |u| > k\}| \leq \frac{c}{k^p} \quad \forall k > 0. \tag{2.2}$$

We put

$$||u||_{M^p(\Omega)} = \inf\{c > 0 : (2.2) \text{ holds}\}.$$

It is clear that $L^p(\Omega) \subset M^p(\Omega) \subset L^{p-\varepsilon}(\Omega)$ for every $p > 0$ and $0 < \varepsilon < p$.

We will prove that:

Theorem 2.4. *Under the assumption* (1.2) *and $f \in L^1(\Omega)$, then there exists u entropy solution of* (1.1) *such that u belongs to $L^p(\Omega)$ for every $p \geq 1$, $\nabla u \in M^q(\Omega)$ for every $1 \leq q < 2$.*

First of all we give some preliminary lemmas.

Lemma 2.5. *Let u be a measurable function in $M^p(\Omega)$ with $p > 0$. Let us assume that there exists $\rho > 0$ and $c > 0$ such that*

$$\int_\Omega |\nabla T_k(u)|^2 dx \leq ck^\rho \quad \forall k > k_0 > 0,$$

then $\nabla u \in M^q(\Omega)$ with $q = \frac{2p}{p+\rho}$.

The proof of the above Lemma can be found for example in [7].

Lemma 2.6. *Let f be a function in $L^1(\Omega)$, and let u be a measurable function such that $T_k(u) \in H_0^1(\Omega)$ for every $k > 0$. If*

$$\int_\Omega \frac{|\nabla T_k(u)|^2}{(1+|u|)^\theta} dx \leq c \int_\Omega |f||T_k(u)| dx \quad \forall k > 0, \tag{2.3}$$

where c is a positive constant, then u belongs to $M^p(\Omega)$ for every $p \geq 1$ and $|\nabla u|$ belongs to $M^q(\Omega)$ for every $1 \leq q < 2$.

Proof. Inequality (2.3) implies that

$$\int_\Omega \frac{|\nabla T_k(u)|^2}{(1+|u|)^\theta} dx \leq c_1 k, \tag{2.4}$$

where $c_1 = c||f||_{L^1(\Omega)}$. In what follows we will denote by $\{c_i\}_{i=2,3,4}$ any constant which depends only on the data. Observing that

$$\begin{aligned}\int_\Omega \frac{|\nabla T_k(u)|^2}{(1+|u|)^\theta} dx &= \int_{\{|u|\leq k\}} \frac{|\nabla T_k(u)|^2}{(1+|u|)^\theta} dx \\ &\geq \int_{\{|u|\leq k\}} \frac{|\nabla T_k(u)|^2}{(1+k)^\theta} dx = \int_\Omega \frac{|\nabla T_k(u)|^2}{(1+k)^\theta} dx,\end{aligned}$$

we have for $k \geq 1$

$$\int_\Omega |\nabla T_k(u)|^2 dx \leq c_1(1+k)^\theta k \leq c_2 k^{(1+\theta)}. \tag{2.5}$$

If we put $A_k = \{x \in \Omega : |u| > k\}$, by imbedding Theorem and (2.5), we get:

$$\Big(\int_{A_k} |T_k(u)|^\sigma\Big)^{\frac{2}{\sigma}} \leq c_3||T_k(u)||_{W_0^{1,2}(\Omega)} \leq c_4 k^{(1+\theta)} \quad \forall k \geq 1, \quad \forall \sigma \geq 1.$$

Thus,

$$|A_k| \leq c_4\Big(\frac{1}{k}\Big)^{\frac{(1-\theta)\sigma}{2}} \quad \forall k \geq 1, \quad \forall \sigma \geq 1. \tag{2.6}$$

Being σ arbitrary, by (2.6) we deduce that u belongs to $M^p(\Omega)$ for every $p \geq 1$. Moreover, by (2.5) and Lemma 2.5 it follows that $|\nabla u| \in M^q(\Omega)$ for every $1 \leq q < 2$. □

Lemma 2.7. *Let $a(x,s)$ be a Carathéodory function satisfying* (1.2), $\{v_n\} \subset H_0^1(\Omega)$ *and* $v \in H_0^1(\Omega)$ *be such that* $v_n \to v$ *weakly in* $H_0^1(\Omega)$. *Let* $\{u_n\}$ *be a sequence of measurable functions converging almost everywhere in* Ω *to a measurable function* u, *then*

$$\int_\Omega a(x,u)|\nabla v|^2 dx \leq \liminf_n \int_\Omega a(x,T_n(u_n))|\nabla v_n|^2 dx \leq c,$$

where c *is a constant.*

Proof. See Lemma 2.8 in [7]. □

The proof of the following Lemma can be found in [5].

Lemma 2.8. *Let* $\{u_n\}$ *be a sequence of measurable functions such that* $\{T_k(u_n)\}$ *is bounded in* $H_0^1(\Omega)$ *for every* $k > 0$. *Then there exists a measurable function* u *such that for every* $k > 0$ $T_k(u)$ *belongs to* $H_0^1(\Omega)$ *and there exists a subsequence, still denoted by* $\{u_n\}$, *such that* $u_n \to u$ *almost everywhere in* Ω *and* $T_k(u_n) \to T_k(u)$ *weakly in* $H_0^1(\Omega)$.

Proof of Theorem 2.4. Let $\{f_n\} \subseteq L^2(\Omega)$ be a sequence such that

$$\begin{cases} f_n \to f & \text{in } L^1(\Omega) \\ ||f_n||_{L^1(\Omega)} \leq ||f||_{L^1(\Omega)} \quad \forall n \in \mathbb{N}. \end{cases}$$

For every $n \in \mathbb{N}$, let us consider the problem

$$\begin{cases} -\text{div}(a(x,T_n(u_n))\nabla u_n) = f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.7}$$

It is known, (see for example [13]), that there exists $u_n \in H_0^1(\Omega)$ weak solution of (2.7), i.e.,

$$\int_\Omega a(x,T_n(u_n))\nabla u_n \nabla v dx = \int_\Omega f_n v\, dx \quad \forall v \in H_0^1(\Omega). \tag{2.8}$$

Let us choose $T_k(u_n)$ as test function in (2.8). We have

$$\int_\Omega a(x,T_n(u_n))|\nabla T_k(u_n)|^2 dx = \int_\Omega f_n T_k(u_n) dx. \tag{2.9}$$

Assumption (1.2) implies

$$\frac{1}{(1+k)^\theta}\int_\Omega |\nabla T_k(u_n)|^2 dx \le \int_\Omega a(x,T_n(u_n))\nabla u_n \nabla T_k(u_n)dx \\ = \int_\Omega f_n T_k(u_n)dx \le k||f||_{L^1(\Omega)}. \tag{2.10}$$

By Lemma 2.8, being by (2.10) $\{T_k(u_n)\}$ bounded in $H_0^1(\Omega)$, we get that there exists a measurable function u such that $T_k(u) \in H_0^1(\Omega)$ for every $k > 0$ and

$$\begin{cases} u_n \to u & \text{a.e. in } \Omega \\ T_k(u_n) \to T_k(u) & \text{weakly in } H_0^1(\Omega). \end{cases}$$

On the other hand by Lemma 2.7 and (2.9)

$$\int_\Omega a(x,u)|\nabla T_k(u)|^2 dx \le \liminf_n \int_\Omega f_n T_k(u_n)dx \le k||f||_{L^1(\Omega)}. \tag{2.11}$$

By (1.2), (2.11) and Lemma 2.6 we get that $u \in M^p(\Omega)$ for every $p \ge 1$ and $|\nabla u| \in M^q(\Omega)$ for every $1 \le q < 2$.

Let us now prove that u is an entropy solution of (1.1). Let $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$ and let us choose $T_k(u_n - \varphi)$ as test function in (2.8). We get

$$\int_\Omega a(x,T_n(u_n))\nabla u_n \nabla T_k(u_n-\varphi)dx = \int_\Omega f_n T_k(u_n - \varphi)dx. \tag{2.12}$$

From $\nabla u_n \nabla T_k(u_n - \varphi) = [\nabla(u_n - \varphi) + \nabla\varphi]\nabla T_k(u_n - \varphi)$, we have,

$$\int_\Omega a(x,T_n(u_n))\nabla u_n \nabla T_k(u_n-\varphi)dx \tag{2.13}$$
$$= \int_\Omega a(x,T_n(u_n))|\nabla T_k(u_n-\varphi)|^2 dx + \int_\Omega a(x,T_n(u_n))\nabla\varphi\nabla T_k(u_n-\varphi)dx.$$

By Lemma 2.7 applied with $v_n = T_k(u_n - \varphi)$ it results

$$\int_\Omega a(x,u)|\nabla T_k(u-\varphi)|^2 dx \le \liminf_n \int_\Omega a(x,T_n(u_n))|\nabla T_k(u_n-\varphi)|^2 dx. \tag{2.14}$$

We observe that $a(x,T_n(u_n)) \to a(x,u)$ almost everywhere in Ω and $a(x,s)$ is bounded. The fact that for every fixed $k > 0$ $T_k(u_n)$ is bounded in

$H_0^1(\Omega)$ implies that $\nabla T_k(u_n-\varphi)$ is bounded in $L^2(\Omega;\mathbb{R}^2)$ and then, up to a subsequence, $\nabla T_k(u_n-\varphi)\to\nabla T_k(u-\varphi)$ weakly in $L^2(\Omega;\mathbb{R}^2)$. Thus

$$\lim_n \int_\Omega a(x,T_n(u_n))\nabla\varphi\nabla T_k(u_n-\varphi)dx = \int_\Omega a(x,u)\nabla\varphi\nabla T_k(u-\varphi)dx. \quad (2.15)$$

Finally observing that

$$\lim_n \int_\Omega f_n T_k(u_n-\varphi)dx = \int_\Omega f T_k(u-\varphi)dx \quad (2.16)$$

and that

$$\begin{aligned}&\int_\Omega a(x,u)\nabla u\nabla T_k(u-\varphi)dx \quad (2.17)\\ &= \int_\Omega a(x,u)|\nabla T_k(u-\varphi)|^2dx + \int_\Omega a(x,u)\nabla\varphi\nabla T_k(u-\varphi)dx,\end{aligned}$$

by (2.13), (2.14), (2.15), (2.16) and (2.17) we have

$$\int_\Omega a(x,u)\nabla u\nabla T_k(u-\varphi)dx \le \int_\Omega f T_k(u-\varphi)dx.$$

3. An improvement on the regularity of solutions. In this section we prove that actually any entropy solution of (1.1) is more regular than simply being in $L^p(\Omega)$ for any p.

Theorem 3.1. *Under assumption* (1.2) *and* $f\in L^1(\Omega)$ *with* $||f||_{L^1(\Omega)}\le 1$, *if* u *is an entropy solution of* (1.1), *then for every* $\beta<\bar\beta=\frac{4\pi}{(1-\theta)}$ *there exists a constant* $c=c(\theta,\beta)$ *such that*

$$\int_\Omega \exp(\beta|u|^{(1-\theta)})dx \le c|\Omega|.$$

Proof. If u is an entropy solution of (1.1), choosing, for $h>0$, $T_h(u)$ as test function in (2.1) we have

$$\frac{1}{k}\int_{\{h<|u|\le h+k\}} a(x,u)|\nabla u|^2dx \le \int_{\{|u|>h\}} |f|dx. \quad (3.1)$$

Inequalities (3.1) and (1.2) imply that

$$\frac{1}{k(1+h+k)^{\theta}}\int_{\{h<|u|\le h+k\}}|\nabla u|^2dx \le \int_{\{|u|>h\}}|f|dx. \tag{3.2}$$

By Sobolev imbedding Theorem, we have

$$\omega||T_{h,k}(u)||_{L^2(\Omega)} \le ||T_{h,k}(u)||_{W_0^{1,1}(\Omega)} = \int_{\{h<|u|\le h+k\}}|\nabla u|dx, \tag{3.3}$$

where $\omega = 2\sqrt{\pi}$ (see for example [17]) and $T_{h,k} = T_k(u - T_h(u))$. If we put $\phi(h) = |\{x \in \Omega : |u| > h\}|$, we get

$$\phi(h+k) = \int_{\{|u|>h+k\}}dx \le \frac{1}{k^2}||T_{h,k}(u)||^2_{L^2(\Omega)}. \tag{3.4}$$

Then by (3.4) and (3.3) we get

$$\begin{aligned}\phi(h+k) &\le \frac{1}{\omega^2k^2}\Big(\int_{\{h<|u|\le h+k\}}|\nabla u|dx\Big)^2 \\ &\le \frac{1}{\omega^2}\Big(\frac{1}{k}\int_{\{h<|u|\le h+k\}}|\nabla u|^2dx\Big)\Big(\frac{|\{h<|u|\le h+k\}|}{k}\Big) \\ &= \frac{1}{\omega^2}\Big(\frac{1}{k}\int_{\{h<|u|\le h+k\}}|\nabla u|^2dx\Big)\Big(\frac{\phi(h)-\phi(h+k)}{k}\Big)\end{aligned} \tag{3.5}$$

and by (3.2)

$$\phi(h+k) \le \frac{1}{\omega^2}(1+h+k)^{\theta}\Big(\frac{\phi(h)-\phi(h+k)}{k}\Big)\big(\int_{\{|u|>h\}}|f|dx\big). \tag{3.6}$$

If $k \to 0^+$, by the assumption $||f||_{L^1(\Omega)} \le 1$ we get, for a.e. $h > 0$,

$$\phi(h) \le \frac{(1+h)^{\theta}}{\omega^2}(-\phi'(h))\Big(\int_{\{|u|>h\}}|f|dx\Big) \le \frac{(1+h)^{\theta}}{\omega^2}(-\phi'(h)). \tag{3.7}$$

An integration gives, for $h > 0$,

$$\phi(h) \le |\Omega|\exp\big\{-\frac{\omega^2}{(1-\theta)}\big((1+h)^{1-\theta}-1\big)\big\}. \tag{3.8}$$

For any $\beta > 0$ we have

$$\begin{aligned}\int_\Omega (\exp(\beta|u|^{(1-\theta)}) - 1)dx &= \int_\Omega dx \int_0^{|u|} \beta(1-\theta)t^{-\theta}\exp(\beta t^{(1-\theta)})dt \\ &= \int_\Omega dx \int_0^{+\infty} \beta(1-\theta)t^{-\theta}\exp(\beta t^{(1-\theta)})\chi_{\{|u|>t\}}dt \\ &= \beta(1-\theta)\int_0^{+\infty} t^{-\theta}\exp(\beta t^{(1-\theta)})\phi(t)dt \qquad (3.9)\end{aligned}$$

By (3.8) and (3.9) we obtain

$$\begin{aligned}&\int_\Omega (\exp(\beta|u|^{(1-\theta)}) - 1)dx \qquad (3.10)\\ &\leq \beta(1-\theta)|\Omega| \int_0^{+\infty} t^{-\theta}\exp\big[\beta t^{(1-\theta)} - \frac{\omega^2}{(1-\theta)}((1+t)^{(1-\theta)} - 1)\big]dt.\end{aligned}$$

We can observe that

$$\lim_{t\to+\infty} \frac{\beta t^{(1-\theta)} - \frac{\omega^2}{(1-\theta)}((1+t)^{(1-\theta)} - 1)}{t^{1-\theta}} = \beta - \frac{\omega^2}{(1-\theta)}.$$

If $\beta < \frac{\omega^2}{(1-\theta)}$ then the integrand on the right hand side of (3.10) behaves, as $t \to +\infty$, as $t^{-\theta}\exp[-\gamma t^{1-\theta}]$ with $\gamma > 0$, so the integral on the right hand side of (3.10) is finite and it is a constant which depends only on β and θ. □

As a matter of fact the argument in the proof of Theorem 3.1 can be used to prove that $\exp[\beta|u|^{(1-\theta)}]$ belongs to $L^1(\Omega)$ for every $\beta > 0$ (see also [11] for the case $\theta = 0$).

Proposition 3.2. *Under assumption* (1.2) *and* $f \in L^1(\Omega)$, *if* u *is an entropy solution of* (1.1), *then* $\exp[\beta|u|^{(1-\theta)}] \in L^1(\Omega)$ *for every* $\beta > 0$.

Proof. Let $\beta > 0$, for every $\varepsilon > 0$ we may split f as $f = f_1 + f_2$ with $||f_1||_{L^1(\Omega)} < \varepsilon$ and $||f_2||_{L^\infty(\Omega)} = M < +\infty$. Inequality (3.7) can be proven also in the present case and it gives:

$$\phi(h) \leq \frac{(1+h)^\theta}{\omega^2}(-\phi'(h))[\varepsilon + M\phi(h)].$$

An integration gives, for $h > 0$,

$$\phi(h) \leq |\Omega| \exp\Big[\frac{M}{\varepsilon}|\Omega| - \frac{\omega^2}{\varepsilon(1-\theta)}((1+h)^{1-\theta} - 1)\Big].$$

As in the proof of Theorem 3.1 we get

$$\int_\Omega (\exp(\beta|u|^{(1-\theta)}) - 1)dx$$
$$\leq \beta(1-\theta)|\Omega| \int_0^{+\infty} t^{-\theta} \exp\Big[\beta t^{(1-\theta)} + \frac{M}{\varepsilon}|\Omega| - \frac{\omega^2}{\varepsilon(1-\theta)}((1+t)^{(1-\theta)} - 1)\Big]dt.$$

The conclusion follows from the observation that, as in the proof of Theorem 3.1, the above integral is finite if $\varepsilon < \frac{\omega^2}{\beta(1-\theta)}$. □

Remark 3.3. In view of Proposition 3.2 the statement of Theorem 3.1 is sharp in the following sense. When

$$||f||_{L^1(\Omega)} \leq 1$$

the integral $\int_\Omega \exp[\beta|u|^{(1-\theta)}]dx$ is finite for any $\beta > 0$, but, in general, if $\beta \geq \frac{4\pi}{1-\theta}$, it is not possible to bound it with a constant which does not depend on f.

Indeed we can show that there exists a sequence $\{f_k\}$ with $||f_k||_{L^1(\Omega)} = 1$ and a problem satisfying assumption (1.2) such that for the corresponding entropy solution $\{u_k\}$ it results

$$\lim_k \int_\Omega \exp\Big[\frac{4\pi}{1-\theta}|u_k|^{(1-\theta)}\Big]dx = +\infty.$$

Namely consider the problem

$$\begin{cases} -\text{div}\big(\frac{\nabla u_k}{(1+|u_k|)^\theta}\big) = f_k & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases} \tag{3.11}$$

where $B \subset \mathbb{R}^2$ is the ball centered at the origin with $|B| = 1$ and, for $k \in \mathbb{N}$, let

$$f_k(x) = f_k^{\#}(x) = \begin{cases} 0 & \text{if } e^{-k} \leq \pi|x|^2 \leq 1 \\ e^k & \text{if } 0 \leq \pi|x|^2 < e^{-k}. \end{cases} \tag{3.12}$$

Observe that $||f_k||_{L^1(\Omega)} = 1$. The function u_k given by

$$u_k(x) = [(1-\theta)v_k(\pi|x|^2)+1]^{\frac{1}{1-\theta}} - 1 \tag{3.13}$$

when v_k is the function

$$v_k(s) = \begin{cases} \frac{1}{4\pi}\log(1/s) & \text{if } e^{-k} \le s \le 1 \\ \frac{1}{4\pi}(k+1-se^k) & \text{if } 0 \le s < e^{-k} \end{cases} \tag{3.14}$$

is an entropy solution of (3.11). We have

$$\begin{aligned} &\int_B \exp\big[\frac{4\pi}{1-\theta}|u_k(x)|^{1-\theta}\big]dx \\ &= \int_0^1 \exp\big[\frac{4\pi}{1-\theta}\big\{((1-\theta)v_k(s)+1)^{\frac{1}{(1-\theta)}} - 1\big\}^{(1-\theta)}\big]ds \\ &\ge \int_{e^-k}^1 \exp\big[\frac{4\pi}{1-\theta}\big\{((1-\theta)v_k(s)+1)^{\frac{1}{(1-\theta)}} - 1\big\}^{(1-\theta)}\big]ds. \end{aligned}$$

Observing that $(\frac{x}{a}+1)^a - 1 \ge (\frac{x}{a})^a$ for every $a \ge 1$ and $x \ge 0$ we get

$$\begin{aligned} &\int_B \exp\big[\frac{4\pi}{1-\theta}|u_k(x)|^{1-\theta}\big]dx \ge \int_{e^-k}^1 \exp\big[\frac{4\pi}{1-\theta}(1-\theta)v_k(s)\big]ds \\ &= \int_{e^-k}^1 \exp\big[\frac{4\pi}{1-\theta}\big(\frac{(1-\theta)}{4\pi}\log(1/s)\big)\big]ds = k \end{aligned}$$

and the claim follows. □

4. An intermediate case. Now we prove estimates for entropy solutions of (1.1) when the datum f belongs to spaces which are included in $L^1(\Omega)$ but are not included in any L^q with $q > 1$. We will consider the Lorentz spaces $L(1,p)$ with $1 \le p < +\infty$.

Let $\varphi : \Omega \to \mathbb{R}$ be a measurable function, we denote by

$$\varphi^*(s) = \sup\{t > 0 : |\{|\varphi| > t\}| > s\}$$

the decreasing rearrangement of φ and by

$$\bar{\varphi}(s) = \frac{1}{s}\int_0^s \varphi^*(t)dt$$

the average function of φ^*. A function $\varphi \in L^1(\Omega)$ belongs to the space $L(1,p)$, $1 \le p < +\infty$, if the quantity:

$$||f||_{L(1,p)} = \Big(\int_0^{|\Omega|} \Big(\int_0^s \varphi^*(t)dt\Big)^p \frac{ds}{s}\Big)^{\frac{1}{p}}$$

is finite. It results (see for example [6]) that if $1 < p < q < +\infty$ then

$$L(1,1) \subset L(1,p) \subset L(1,q) \subset L^1(\Omega).$$

Usually $L(1,1)$ is denoted by $L \log L$.

Let us first consider the case $f \in L \log L$.

Theorem 4.1. *Under the assumption* (1.2) *and* $f \in L \log L(\Omega)$*, then there exists* $u \in H_0^1(\Omega) \cap L^\infty(\Omega)$ *weak solution of* (1.1) *in the sense that*

$$\int_\Omega a(x,u)\nabla u \nabla v dx = \int_\Omega f v dx \quad \forall v \in H_0^1(\Omega). \tag{4.1}$$

Proof. For every $n \in \mathbb{N}$ let us consider the problem :

$$\begin{cases} -\mathrm{div}(a(x,T_n(u_n))\nabla u_n) = f & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.2}$$

Since $L \log L(\Omega) \subseteq H^{-1}(\Omega)$ (see for example [3]) it results that (see [13]) for every $n \in \mathbb{N}$ there exists $u_n \in H_0^1(\Omega)$ weak solution of (4.2); i.e.,

$$\int_\Omega a(x,T_n(u_n))\nabla u_n \nabla v dx = \int_\Omega f v dx \quad \forall v \in H_0^1(\Omega). \tag{4.3}$$

Moreover, by Corollary 2.2 and Remark 2.1 in [4], there exists a constant $c > 0$, independent on n, such that $||u_n||_{L^\infty(\Omega)} \le c$ for every $n \in \mathbb{N}$. If $\bar{n} > c$ the solution $u_{\bar{n}}$ is clearly a weak solution of (1.1) in the sense of (4.1) and the theorem is proved. $\square$

We now consider the case $L(1,p)$ with $1 < p < +\infty$.

Let us consider the function

$$B(s) = \int_0^s \frac{1}{(1+t)^\theta} dt = \frac{1}{(1-\theta)}[(1+s)^{(1-\theta)} - 1], \tag{4.4}$$

let v be the solution of the problem

$$\begin{cases} -\Delta v = f^{\#} & \text{in } \Omega^{\#} \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

where $f^{\#}(x) = f^*(\pi|x|^2)$ and $\Omega^{\#} = \{x \in \mathbb{R}^2 : \pi|x|^2 < |\Omega|\}$. We observe explicitly that the solution of (4.5) is given by

$$v(x) = v^{\#}(x) = \frac{1}{4\pi}\int_{\pi|x|^2}^{|\Omega|} \bar{f}(t)dt. \tag{4.6}$$

If u is an entropy solution of (1.1), by (3.1) and (1.2) we get

$$\frac{1}{h}\int_{\{t<|u|\le t+h\}} \frac{1}{(1+|u|)^{\theta}}|\nabla u|^2 dx \le \int_{\{|u|>t\}} |f|dx \tag{4.7}$$

and so, by Theorem 2.1 and Remark 2.2 in [4], we have the following estimate

$$B(u^*(s)) \le v^*(s) \quad \forall s \in (0, |\Omega|) \tag{4.8}$$

so by (4.4) and (4.6)

$$[(1+u^*(s))^{(1-\theta)} - 1] \le \frac{(1-\theta)}{4\pi}\int_s^{|\Omega|} \bar{f}(t)dt \quad \forall s \in (0, |\Omega|). \tag{4.9}$$

Theorem 4.2. *Under the assumption* (1.2), $1 < p < +\infty$ *and* $f \in L(1,p)$ *with* $||f||_{L(1,p)} \le 1$, *if* u *is an entropy solution of* (1.1), *then for every* $\beta < [\frac{4\pi}{(1-\theta)}]^{p'}$ *there exists a constant* $c = c(\theta, p, \beta)$ *such that*

$$\int_\Omega \exp[\beta|u|^{(1-\theta)p'}]dx \le c|\Omega|.$$

Proof. Let us observe that for every $s \in (0, |\Omega|)$, by Hölder inequality, we get

$$\begin{aligned} \int_s^{|\Omega|} \bar{f}(t)dt &\le \Big(\int_s^{|\Omega|} \bar{f}^p(t)t^{\frac{p}{p'}}dt\Big)^{\frac{1}{p}}\Big(\int_s^{|\Omega|} \frac{1}{t}dt\Big)^{\frac{1}{p'}} \\ &= \Big(\int_s^{|\Omega|} (\bar{f}^p(t)t^p)\frac{dt}{t}\Big)^{\frac{1}{p}}(\log|\Omega|/s)^{\frac{1}{p'}} \le (\log|\Omega|/s)^{\frac{1}{p'}}. \end{aligned} \tag{4.10}$$

By (4.9) we get

$$[(1+u^*(s))^{(1-\theta)}-1] \leq \frac{(1-\theta)}{4\pi}(\log|\Omega|/s)^{\frac{1}{p'}} \tag{4.11}$$

so

$$(1+u^*(s))^{(1-\theta)p'} \leq \big[1+\frac{(1-\theta)}{4\pi}(\log|\Omega|/s)^{\frac{1}{p'}}\big]^{p'}. \tag{4.12}$$

We recall that for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that

$$(1+y)^\alpha \leq c(\varepsilon)+(1+\varepsilon)y^\alpha \quad \forall y \geq 0,\ \forall \alpha \geq 1.$$

Let $\beta < \big[\frac{4\pi}{1-\theta}\big]^{p'}$ and let us fix $\varepsilon > 0$ such that

$$(1+\varepsilon)\beta < \big[\frac{4\pi}{(1-\theta)}\big]^{p'}. \tag{4.13}$$

Then

$$\big[1+\frac{(1-\theta)}{4\pi}(\log|\Omega|/s)^{\frac{1}{p'}}\big]^{p'} \leq c(\varepsilon)+(1+\varepsilon)\big[\frac{(1-\theta)}{4\pi}\big]^{p'}(\log|\Omega|/s) \tag{4.14}$$

Inequalities (4.12) and (4.14) imply

$$\beta(1+u^*(s))^{(1-\theta)p'} \leq \beta c(\varepsilon)+\beta(1+\varepsilon)\big[\frac{(1-\theta)}{4\pi}\big]^{p'}(\log|\Omega|/s). \tag{4.15}$$

Finally by (4.13) and (4.15) we have

$$\begin{aligned}\int_\Omega \exp(\beta|u|^{(1-\theta)p'})dx &= \int_0^{|\Omega|}\exp(\beta u^*(s)^{(1-\theta)p'})ds\\ &\leq \int_0^{|\Omega|}\exp(\beta[u^*(s)+1]^{(1-\theta)p'})ds\\ &\leq \int_0^{|\Omega|}\exp(\beta c(\varepsilon))\exp\big[\beta(1+\varepsilon)\big[\frac{(1-\theta)}{4\pi}\big]^{p'}(\log|\Omega|/s)\big]ds\\ &= e^{\beta c(\varepsilon)}\int_0^{|\Omega|}\big[\frac{|\Omega|}{s}\big]^{\beta(1+\varepsilon)\big[\frac{(1-\theta)}{4\pi}\big]^{p'}}ds = c(\theta,p,\beta)|\Omega|.\end{aligned}$$

Proposition 4.3. *Under assumption* (1.2) *and* $f \in L(1,p)$ *with* $1 < p < +\infty$*, if* u *is an entropy solution of* (1.1)*, then* $\exp[\beta|u|^{(1-\theta)p'}] \in L^1(\Omega)$ *for every* $\beta > 0$.

Proof. For every $\varepsilon > 0$ we can split f as $f = f_1 + f_2$ with $||f_1||_{L(1,p)} < \varepsilon$ and $f_2 \in L^\infty(\Omega)$. Then using inequality (4.10) and arguments similar to those in the proof of Proposition 3.2, we get the theorem. $\square$

Remark 4.4. Theorem 4.2 states that when $||f||_{L(1,p)} \leq 1$ with $1 < p < +\infty$, for an entropy solution of (1.1) the integral

$$\int_\Omega \exp\left[\beta|u|^{(1-\theta)p'}\right]dx \tag{4.16}$$

can be bounded by a constant, which depends on p and β only, every time $\beta < \bar\beta_p = \left(\frac{4\pi}{1-\theta}\right)^{p'}$. If $\beta \geq \bar\beta_p$ Proposition 4.3 shows that the integral (4.16) is still finite for every f but it is no possible to bound (4.16) independently of f. Indeed one can find a sequence $\{f_k\}_{k\in\mathbb{N}}$ such that $||f_k||_{L(1,p)} \leq 1$ and a problem satisfying the assumptions made on (2.1), such that for the sequence of corresponding entropy solutions $\{u_k\}_{k\in\mathbb{N}}$ it holds

$$\lim_k \int_\Omega \exp\left[\bar\beta|u_k|^{(1-\theta)p'}\right] = +\infty, \qquad \bar\beta_p = \left[\frac{4\pi}{(1-\theta)}\right]^{p'}. \tag{4.17}$$

Namely consider the problem (3.11) with f_k given by

$$f_k(x) = f_k^\#(x) = \begin{cases} 0 & \text{if } e^{-k} \leq \pi|x|^2 \leq 1 \\ e^k(1/p+k)^{-1/p} & \text{if } 0 \leq \pi|x|^2 < e^{-k} \end{cases}$$

observe that $||f_k||_{L(1,p)} \leq 1$. The function u_k given by

$$u_k(x) = [(1-\theta)v_k(\pi|x|^2)+1]^{\frac{1}{1-\theta}} - 1$$

is an entropy solution of (3.11) when v_k is given by

$$v_k(s) = \begin{cases} \left(4\pi\left(\frac{1}{p}+k\right)^{\frac{1}{p}}\right)^{-1}\log\left(\frac{1}{s}\right) & \text{if } e^{-k} \leq s \leq 1 \\ (k+1-e^k s)/(4\pi(1/p+k)^{1/p}) & \text{if } 0 \leq s < e^{-k}. \end{cases} \tag{4.18}$$

A calculation similar to the one given in Remark 3.3 gives (4.17).

5. The case $\theta = 1$. In this section we prove the existence of an entropy solution of (1.1) when assumption (1.2) is satisfied with $\theta = 1$; that is,

$$\frac{1}{(1+|s|)} \leq a(x,s). \tag{5.1}$$

Theorem 5.1. *Under assumption* (5.1) *and* $f \in L^1(\Omega)$ *there exists an entropy solution* u *of* (1.1) *such that* u *belongs to* $L^p(\Omega)$ *for every* $p \geq 1$ *and* $\nabla u \in L^q$ *for every* $1 \leq q < 2$.

Proof. Let $\varepsilon > 0$; we may split f as $f = f_\varepsilon + \tilde{f}$ with $||f_\varepsilon||_{L^1(\Omega)} < \varepsilon$ and $||\tilde{f}||_{L^\infty(\Omega)} = M < +\infty$. For every $n \in \mathbb{N}$ let us consider the problems:

$$\begin{cases} -\mathrm{div}(a(x, T_n(u_n))\nabla u_n) = \tilde{f} + f_n & \text{in } \Omega \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2}$$

where $f_n \in L^2(\Omega)$, $||f_n||_{L^1} \leq ||f_\varepsilon||_{L^1}$ and $f_n \to f_\varepsilon$ in $L^1(\Omega)$. It is known, see for example [13], that for every $n \in \mathbb{N}$ there exists $u_n \in H_0^1(\Omega)$ solution of (5.2). This means that:

$$\int_\Omega a(x, T_n(u_n))\nabla u_n \nabla \varphi = \int_\Omega (\tilde{f} + f_n)\varphi, \quad \forall \varphi \in H_0^1(\Omega). \tag{5.3}$$

Choosing $T_k(u_n)$ as test function in (5.3) we get

$$\int_\Omega a(x, T_n(u_n))|\nabla T_k(u_n)|^2 = \int_\Omega (\tilde{f} + f_n)T_k(u_n),$$

and assumption (5.1) implies

$$\frac{1}{(1+k)} \int_\Omega |\nabla T_k(u_n)|^2 dx \leq k||f||_{L^1(\Omega)} + 2k\varepsilon. \tag{5.4}$$

By Lemma 2.8, being by (5.4) $\{T_k(u_n)\}$ bounded in $H_0^1(\Omega)$, we get that there exists a measurable function u such that $T_k(u) \in H_0^1(\Omega)$ for every $k > 0$ and

$$\begin{cases} u_n \to u & \text{a.e. in } \Omega \\ T_k(u_n) \to T_k(u) & \text{weakly in } H_0^1(\Omega). \end{cases}$$

Arguing as in the proof of Theorem 2.4 u is an entropy solution of (1.1). As proven in [4] inequality (4.10) holds also when we assume (5.1). In such a case $B(s) = \log(1 + s)$ and we have

$$\log(1 + u_n^*(s)) \leq \frac{1}{4\pi} \int_s^{|\Omega|} \Big(\int_0^t (\tilde{f} + f_n)^*(r)dr\Big)\frac{dt}{t}, \quad \forall s \in (0, |\Omega|). \tag{5.5}$$

Using the fact that

$$\frac{1}{t}\int_0^t (\tilde{f}+f_n)^*(r)dr \le \frac{1}{t}\int_0^t \tilde{f}^*(r)dr + \frac{1}{t}\int_0^t {f_n}^*(r)dr$$

we get

$$\log(1+u_n^*(s)) \le \frac{1}{4\pi}\big[\varepsilon\log(\frac{|\Omega|}{s}) + M|\Omega|\big]. \tag{5.6}$$

Moreover, $u_n^*(s) \to u^*(s)$ a.e., then

$$\log(1+u^*(s)) \le \frac{1}{4\pi}\big[\varepsilon\log(\frac{|\Omega|}{s}) + M|\Omega|\big].$$

If ε is chosen to be small enough we deduce that u belongs to $L^r(\Omega)$ for some $r \ge 2$. Being u an entropy solution of (1.1) we get

$$\frac{1}{1+k}\int_\Omega |\nabla T_k(u)|^2 \le k||f||_{L^1(\Omega)} \tag{5.7}$$

and by Lemma 2.5 we deduce that $|\nabla u|$ belongs to $M^{\frac{2r}{2+r}}(\Omega)$.

Being u an entropy solution of (1.1), as in [4] we have that estimate (5.5) holds also for u, i.e.,

$$\log(1+u^*(s)) \le \frac{1}{4\pi}\int_s^{|\Omega|}\Big(\int_0^t f^*(r)dr\Big)\frac{dt}{t}, \quad \forall s \in]0,|\Omega|[. \tag{5.8}$$

Observing that f can be split in the sum of a bounded function and an L^1-function with arbitrarily small norm, inequality (5.8) implies that u belongs to $L^p(\Omega)$ for every $p \ge 1$ and, again by (5.7) and Lemma 2.5, that $|\nabla u|$ belongs to $L^q(\Omega)$ for every $1 \le q < 2$. $\square$

Remark 5.2. A natural question is to ask if, as in Section 3, a suitable exponential function of an entropy solution u of (1.1) is summable, under the assumptions of Theorem 5.1. In general this is not true. Namely let us consider the problem

$$\begin{cases} -\mathrm{div}\big(\frac{\nabla u}{1+|u|}\big) = f & \text{in } B \\ u = 0 & \text{on } \partial B, \end{cases} \tag{5.9}$$

where

$$f(x) = \frac{1}{\pi|x|^2\log\frac{1}{\pi|x|^2}\big(\log\big(\log\frac{1}{\pi|x|^2}\big)\big)^2}$$

and $B \subset \mathbb{R}^2$ is the ball centered at the origin with $|B|$ small enough in such a way that $f = f^\#$. Function u given by

$$u(x) = \exp(v(\pi|x|^2)) - 1,$$

where

$$v(s) = \frac{1}{4\pi}\int_s^{|B|} \frac{1}{t\log(\log(\frac{1}{t}))}dt,$$

is an entropy solution of (5.9). With a change of variable we get that

$$\begin{aligned}\int_B \exp(\alpha u^\gamma(x))dx &= \int_0^{|B|} \exp(\alpha(e^{v(s)} - 1)^\gamma)ds \\ &= \int_{\log(\frac{1}{|B|})}^{+\infty} \exp\big(\alpha\big(e^{v(e^{-t})} - 1\big)^\gamma - t\big)dt.\end{aligned}$$

Observing that for every $\alpha, \gamma > 0$

$$\lim_{t\to+\infty} \alpha(e^{v(e^{-t})} - 1)^\gamma - t = \lim_{t\to+\infty} \alpha\big(\exp\big(\frac{1}{4\pi}\int_{\log(\frac{1}{|B|})}^{t} \frac{1}{\log r}dr\big) - 1\big)^\gamma - t = +\infty$$

the claim follows.

REFERENCES

[1] J.A. Aguilar and I. Peral, *An a priori estimate for the N-Laplacian,* C.R. Acad. Sci. Paris, 319 (1994), 161–166.

[2] A. Alberico and V. Ferone, *Regularity properties of solutions of elliptic equations in $\mathbb{R}^2$ in limit cases,* Rend. Mat. Acc. Lincei, s.9, 6 (1995), 237-250.

[3] A. Alvino, *Un caso limite della disuguaglianza di Sobolev in Spazi di Lorentz,* Rend. Acc. Sci. Fis. Mat. Napoli, 44 (1977), 105–112.

[4] A. Alvino, V. Ferone, and G. Trombetti, *A priori estimates for a class of non uniformly elliptic equations,* Atti Sem. Mat. Fis. Univ. Modena, 46 (1998), 381–391.

[5] P. Benilan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre, and J. L. Vazquez, *An L^1 theory of existence and uniqueness of nonlinear elliptic equations,* Ann. Scuola Norm. Sup. Pisa Cl. Sci., 22 (1995), 240–273.

[6] C. Bennett, *Intermediate spaces and the class $Llog^+L$,* Ark. Mat., 11 (1973), 215–228.

[7] L. Boccardo, A. Dall' Aglio, and L. Orsina, *Existence and regularity results for some elliptic equations with degenerate coercivity,* Atti Sem. Mat. Fis. Univ. Modena, 46 (1998), 51–81.

[8] L. Boccardo and T. Gallouët, *Non-linear elliptic and parabolic equations involving measures data,* J. Funct. Anal., 87 (1989), 149–169.
[9] L. Boccardo and L. Orsina, *Existence and regularity of minima for integral functionals non coercive in the energy space,* Preprint.
[10] L. Boccardo, I. Peral, and J.L. Vazquez, *The N-Laplacian elliptic equation: variational versus entropy solutions,* J. Math. Anal. Appl., 201 (1996), 671–688.
[11] H. Brezis and F. Merle, *Uniform estimates and blow–up behavior for solution of $-\Delta u = V(x)e^u$ in two dimensions,* Comm. in P.D.E., 16 (1991), 1223–1253.
[12] T. Del Vecchio, *Nonlinear elliptic equations with measure data,* Potential Analysis, 4 (1995), 185–203.
[13] J.L. Lions, "Quelques Méthodes de Resolution de Problèmes aux Limites non Linéaires," Dunod, Gauthier-Villars, Paris, (1969).
[14] G. Stampacchia, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficicients discontinus,* Ann. Inst. Fourier (Grenoble), 15 (1965), 189–258.
[15] G. Talenti, *Elliptic equations and rearrangments,* Ann. Scuola Norm. Sup. Pisa, 3 (1976), 697–718.
[16] J.L. Vazquez, *Monotone perturbations of the Laplacian in $L^1(\mathbb{R}^n)$,* Israel J. Math., 43 (1982), 255–272.
[17] W.P. Ziemer, " Weakly Differentiable Functions," Springer, New York, (1989).