Differential and Integral Equations Volume 13 (4–6) April–June 2000, pp. 649–667

EXISTENCE AND UNIQUENESS OF SOLUTIONS TO THE KURAMOTO–SAKAGUCHI NONLINEAR PARABOLIC INTEGRODIFFERENTIAL EQUATION*

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(Submitted by: Emmanuele DiBenedetto)

Abstract. Global in time existence and uniqueness of classical solutions to a certain nonlinear parabolic partial differential equation, containing an integral term, are proved. Smoothness regularity and time-independent estimates for all partial derivatives are also obtained. Such an equation is of a non-standard type, and governs the time evolution of certain populations of infinitely many nonlinearly coupled random oscillators, described by a model first proposed by Kuramoto and Sakaguchi.

Introduction. In the recent years, several mathematical models based on *partial differential equations* containing *integral terms* have appeared in the literature. For some general results, see [12, 17], e.g., apart from the perhaps most famous case of the Boltzmann equation [7]. The nature and form of the integral term is usually clear from the context of each specific application. Indeed, integral in time terms may model memory properties of the material where a certain dynamical process takes place, while a space-integral may represent an average, global phenomenon distributed rather than localized in some portion or on the whole of the space-domain.

Without any claim of giving here an exhaustive account of the entire literature, though confined to some subclass of partial integro-differential equations, in this paper we analyze a model, earlier derived formally by Kuramoto [14] and Sakaguchi [18] to describe the dynamical behavior of a population of infinitely many nonlinearly coupled random oscillators.

It seems that numerous phenomena in Biology, Medicine, and Physics, can be modeled by a system of Langevin equations, sometimes in the so-called mean-field

Accepted for publication November 1998.

*This work was supported, in part, by the Russian Foundation for Basic Research under grant no. 96-01-01618 and by the GNFM of the Italian C.N.R.

AMS Subject Classifications: 35K10, 35K99.

coupling model, where each oscillator feels the global effects of the others essentially in the same way (see [19, 18], e.g.). Indeed, it was first proposed by Winfree [20] to model large (but finite) populations of nonlinearly coupled oscillators in such a way, and then, Kuramoto [14] and Sakaguchi [18] defined a model applicable to the limiting-case of infinitely many nonlinearly coupled random oscillators (see [19], e.g.).

In [5, 19], stability and bifurcations of the so-called Kuramoto-Sakaguchi model for nonlinearly coupled oscillators with randomly distributed frequencies, and subject to independent external white noises, has been analyzed, in the "thermodynamic limit" of infinitely many oscillators. Following [4, 19], in such a limit, a nonlinear Fokker-Planck-type equation for the one-oscillator probability density, $\rho(\theta, t, \omega)$, was obtained. This model equation has the form

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial \theta^2} - \frac{\partial}{\partial \theta} (v\rho), \tag{0.1}$$

where the drift velocity is given by

$$v(\theta, t, \omega) := \omega + Kr \sin(\psi - \theta), \tag{0.2}$$

and the so-called order-parameter amplitude, $r(t)$, and phase, $\psi(t)$, are given in terms of ρ and the frequency distribution density, $g(\omega)$, (see [5, 19]) by

$$r\mathrm{e}^{i\psi} = \int_0^{2\pi} \int_{-\infty}^{+\infty} \mathrm{e}^{i\varphi} \rho(\varphi, t, \omega)\, g(\omega)\, d\omega\, d\varphi. \tag{0.3}$$

It is also said that the nonlinear equation in (0.1) describes nonlinear Markov (also called Vlasov-McKean) stochastic processes [9]. The probability density is assumed to be 2π-periodic in the angle, θ, and normalized,

$$\int_0^{2\pi} \rho(\theta, t, \omega)\, d\theta = 1, \tag{0.4}$$

for all times and frequencies. For more details about the derivation of this model, and for the physical meaning of the parameters involved, we refer the reader to the original papers [5, 19] and to the references therein.

The authors of [5] pay attention to stationary and time-periodic states, and investigate questions of linear and nonlinear stability of these states. However, we cannot refer to the literature ([3, 12, 13, 15, 22, 21], e.g.) to show solvability and boundedness of solutions, uniformly in time. These questions are discussed in the present paper.

The rest of the paper is organized as follows. Section 1 is devoted to a restatement of the problem (0.1)–(0.3) in terms more convenient for a qualitative analysis, and some of the basic properties of solutions and auxiliary results are established there. In Section 2, we give a local existence theorem of classical solutions. Sections 3 and 4 are devoted to obtaining time-independent estimates for the solutions; global in time existence then follows. These sections represent the high points of the paper.

1. The basic problem and some properties of solutions. First of all, we reformulate the basic problem (0.1)–(0.3) in a more convenient form. Separating the real from the imaginary part in (0.3), the drift term in (0.2) can be rewritten as

$$v(\theta,t,\omega)=\omega+K\int_0^{2\pi}\int_{-\infty}^{+\infty}\sin(\varphi-\theta)\,\rho(\varphi,t,\omega)\,g(\omega)\,d\varphi\,d\omega,$$

and thus, from (0.1)–(0.3), we obtain

$$\frac{\partial\rho}{\partial t}=D\frac{\partial^2\rho}{\partial\theta^2}-\omega\frac{\partial\rho}{\partial\theta}-K\frac{\partial}{\partial\theta}\Big[\rho(\theta,t,\omega)\int_0^{2\pi}\int_{-\infty}^{+\infty}\sin(\varphi-\theta)\rho(\varphi,t,\omega)g(\omega)d\omega d\varphi\Big]. \quad (1.1)$$

This must be accompanied by the initial and the boundary conditions

$$\rho(\theta,0,\omega)=\rho_0(\theta,\omega), \quad (1.2)$$

$$\rho\big|_{\theta=0}=\rho\big|_{\theta=2\pi}, \qquad \frac{\partial\rho}{\partial\theta}\Big|_{\theta=0}=\frac{\partial\rho}{\partial\theta}\Big|_{\theta=2\pi}. \quad (1.3)$$

Here and in the sequel we shall assume the following properties for the data.

Conditions I. The initial value of the probability density, $\rho_0(\theta,\omega)$, is supposed to be:

(a_1) 2π-periodic in θ;

(a_2) smooth, say $\rho_0\in C^\infty$ in both variables, for simplicity (this implies, in particular, that ρ will be uniformly bounded with respect to the frequency parameter, ω);

(a_3) positive,

$$\rho_0(\theta,\omega)>0; \quad (1.4)$$

(a_4) normalized,

$$\int_0^{2\pi}\rho_0(\theta,\omega)\,d\theta=1. \quad (1.5)$$

The frequency distribution density $g(\omega)$, is assumed to be:

(b_1) nonnegative, $g(\omega)\ge 0$;

(b_2) integrable, $g\in L^1(\mathbf{R})$; and

(b_3) compactly supported, $\operatorname{supp} g\subset[-N,N]$.

It is also understood that the "free parameter" ω in (1.1) must picked up from $\operatorname{supp} g$.

We do *not* discuss, in this paper, the singular case when the coefficient ω in equation (1.1) is allowed to grow unboundedly. Moreover, the assumption $\omega \in \operatorname{supp} g$ suffices to describe a number of phenomena in Physics and Biology [5, 19].

In [8, 10, 16], an apparently similar equation was studied. There are, however, several important differences, since in (0.1)–(0.2) an additional integral with respect to ω, explicit dependence on ω, and periodicity with respect to θ, appear. Moreover, in [10] an additional nonlinear term played a special role in proving existence in the sup-norm. Here we must resort to different techniques.

In this section, we prove first a simple but important property of solutions to problem (1.1)–(1.3), that we will use in the sequel.

Lemma 1.1. *If there is a smooth solution $\rho(\theta, t, \omega)$ to problem* (1.1)–(1.3), *under condition* (1.4), *then it remains positive for all times $t \geq 0$,*

$$\rho(\theta, t, \omega) > 0. \tag{1.6}$$

Sketch of the proof. This lemma is a simple corollary of the "strong maximum principle" (see, e.g., [15]). Indeed, we can treat equation (1.1) as a linear equation, say

$$\frac{\partial \rho}{\partial t} = D\, \frac{\partial^2 \rho}{\partial \theta^2} - [\omega + B(\theta, t)]\, \frac{\partial \rho}{\partial \theta} + A(\theta, t)\, \rho,$$

with an obvious choice of the coefficients, A, B. Setting

$$\rho(\theta, t, \omega) := u(\theta, t, \omega)\, e^{\lambda t},$$

we obtain for the new unknown function $u(\theta, t, \omega)$, the equation

$$\frac{\partial u}{\partial t} = D\, \frac{\partial^2 u}{\partial \theta^2} - [\omega + B(\theta, t)]\, \frac{\partial u}{\partial \theta} + [A(\theta, t) - \lambda]\, u.$$

As shown in [15], choosing $\lambda > |A(\theta, t)|$, the solution $u(\theta, t, \omega)$ to such an equation does not attain a negative minimum if $t > 0$. Thus, from

$$u\Big|_{t=0} = \rho\Big|_{t=0} > 0$$

(cf. (1.4)), (1.6) follows.

Lemma 1.2. *The solution $\rho(\theta, t, \omega)$ to the problem* (1.1)–(1.3) *remains normalized for all times $t \geq 0$ and all ω's,*

$$\|\rho\|_{L^1} := \int_0^{2\pi} \rho(\theta, t, \omega)\, d\theta = 1, \tag{1.7}$$

provided that the initial distribution $\rho_0(\theta,\omega)$ *is normalized as in* (1.5).

Lemma 1.2 can be immediately proved integrating (1.1) in θ, and taking into account the "divergent form" of equation, i.e that its right-hand side is the θ-derivative of

$$D\frac{\partial\rho}{\partial\theta} - \omega\,\rho - K\,\rho \int_0^{2\pi}\int_{-\infty}^{+\infty} \sin(\varphi-\theta)\,\rho\, g(\omega)\, d\varphi\, d\omega,$$

the periodic boundary conditions (1.3), and the 2π-periodicity in θ of the kernels in the integral term in (1.1).

Below, we shall use often the following simple corollary of Lemmas 1.1 and 1.2.

Corollary 1. *For any smooth function* $f(\varphi,\theta,t,\omega)$*, such that*

$$|f(\varphi,\theta,t,\omega)| \le 1,$$

the inequality

$$\Big|\int_0^{2\pi}\int_{-\infty}^{+\infty} f(\varphi,\theta,t,\omega)\,\rho(\varphi,t,\omega)\,g(\omega)\,d\omega\,d\varphi\Big| \le A \tag{1.8}$$

holds, where the constant A *is given by*

$$A := \int_{-\infty}^{+\infty} g(\omega)\,d\omega. \tag{1.9}$$

In particular, Corollary 1 implies the following:

Remark 1.1. For any integrable nonnegative function $F(\theta,t,\omega)$, we have

$$\begin{aligned} &\Big|\int_0^{2\pi} F(\theta,t,\omega)\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty}\cos(\varphi-\theta)\rho\, g(\omega')\,d\varphi\,d\omega'\Big)\,d\theta\Big| \\ &\le \int_0^{2\pi} F\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty}|\cos(\varphi-\theta)|\rho g(\omega')\,d\varphi\,d\omega'\Big)\,d\theta \le A\int_0^{2\pi} F\,d\theta. \end{aligned} \tag{1.10}$$

2. Existence and uniqueness of solutions. The basic idea for proving existence of solutions to problem (1.1)–(1.5) is standard. Theorem 2.1 below establishes local solvability. Unfortunately, we cannot refer to well-known results on solvability because of the points given in Section 1, by which the present problem differs from those already studied in the literature. Therefore, we shall give a sketch of the proof, paying attention to the peculiarities of our problem.

Theorem 2.1. *In the assumptions above* (*Conditions I in Section* 1), *there exists* $T > 0$*, such that there is a classical solution,* $\rho(\theta, t, \omega)$ *to the problem* (1.1)–(1.3) *for* $t \in [0, T]$*. Such a solution is bounded uniformly in* ω*, along with all its space derivatives,* $\dfrac{\partial^m \rho}{\partial \theta^m}$.

Sketch of the proof. We construct a sequence of successive approximations and prove its convergence, as well as appropriate estimates, by a compactness argument. Consider the sequence $\{\rho_n(\theta, t, \omega)\}$ of solutions to the linear problems

$$\frac{\partial \rho_n}{\partial t} = D\,\frac{\partial^2 \rho_n}{\partial \theta^2} - \Big[\omega + B_n(\theta, t)\Big]\frac{\partial \rho_n}{\partial \theta} - \frac{\partial B_n}{\partial \theta}(\theta, t)\rho_n, \quad n = 1, 2, \dots , \tag{2.1}$$

with initial and boundary conditions as in (1.2)–(1.3), the coefficients B_n being given by

$$B_n(\theta, t) := K \int_0^{2\pi} \int_{-\infty}^{+\infty} \sin(\varphi - \theta)\,\rho_{n-1}(\varphi, t, \omega) g(\omega)\, d\omega\, d\varphi, \quad n = 1, 2, \dots . \tag{2.2}$$

As a first approximation, we choose the initial distribution $\rho_0(\theta, t, \omega) \equiv \rho_0(\theta, \omega)$. Note that the *linear* equation in (2.1) is *homogeneous* and in the *divergence form*:

$$\frac{\partial \rho_n}{\partial t} = \frac{\partial}{\partial \theta}\Big(D\frac{\partial \rho_n}{\partial \theta} - [\omega + B_n(\theta, t)]\,\rho_n\Big).$$

Thus, Lemmas 1.1 and 1.2 can be applied, and hence the successive approximations, $\rho_n(\theta, t, \omega)$ above, are all **(a)** positive,

$$\rho_n(\theta, t, \omega) > 0, \tag{2.3}$$

and **(b)** normalized

$$\|\rho_n\|_{L^1} := \int_0^{2\pi} \rho_n(\theta, t, \omega)\, d\theta = 1. \tag{2.4}$$

Clearly, for any given ω there exists a smooth solution $\rho_1(\theta, t, \omega)$ to problem (2.1), (1.2), (1.3) (see [15], e.g.). As follows from the maximum principle for linear equations [15], for any given $t > 0$ such a term, ρ_1, is bounded for $\omega \in [-N, N]$, uniformly in $N > 0$, and $\sup_\theta |\rho_1(\theta, t, \omega)|$ depends only on the quantity $B_{1\theta}(\theta, t)$. The $\sup_\theta |\rho_1(\theta, t, \omega)|$, however, may grow exponentially in time.

Note that, due to the definition (2.2) and the properties (2.3), (2.4) (see also Corollary 1), for any two integers $n \geq 1$ and $m \geq 0$, we have

$$\max\Big\{ \Big|\frac{\partial^m B_n(\theta, t)}{\partial \theta^m}\Big| \Big\} \leq KA. \tag{2.5}$$

Therefore, by the arguments above, for every positive integer, n,

- there exists a solution, $\rho_n(\theta, t, \omega)$, to (2.1), (1.2), (1.3);
- such a solution is smooth ($\rho_n \in C^\infty$ for $t > 0$);
- ρ_n is uniformly bounded in n and N, for any ω (cf. (b_3) in Conditions 1), for any given $t > 0$;
- the estimate
$$\sup_{\theta\in[0,2\pi]} |\rho_n(\theta, t, \omega)| \le e^{KA\,t} \tag{2.6}$$
holds.

Taking the derivatives with respect to θ in (2.1), and using the arguments above, in view of (2.5), it is easy to see that all these derivatives are also bounded,

$$\sup_{\theta\in[0,2\pi]} \left| \frac{\partial^m \rho_n}{\partial \theta^m} \right| \le C_m(KA, t), \qquad m = 1, 2, \dots, \tag{2.7}$$

uniformly in n and N, for $\omega \in [-N, N]$, and *may* grow in time. The only difference with estimating ρ_n is that the corresponding equations (unlike (2.1)) are now inhomogeneous, in fact, for instance, $\sigma_n := \frac{\partial \rho_n}{\partial \theta}$ satisfies

$$\frac{\partial \sigma_n}{\partial t} = D\,\frac{\partial^2 \sigma_n}{\partial \theta^2} - \Big[\omega + B_n(\theta, t)\Big]\,\frac{\partial \sigma_n}{\partial \theta} - 2\frac{\partial B_n(\theta, t)}{\partial \theta}\,\sigma_n - \frac{\partial^2 B_n(\theta, t)}{\partial \theta^2}\,\rho_n.$$

Clearly, an estimate on $\rho_{nt}(\theta, t, \omega)$ can be also derived from (2.1), given (2.7).

As is known, the fundamental solution to equation

$$u_t = u_{\theta\theta} - \omega\, u_\theta$$

is given by

$$G(\theta, t, \omega) := \frac{1}{2\sqrt{\pi\, t}}\, \mathrm{e}^{-\frac{(\theta-\omega t)^2}{4t}}, \tag{2.8}$$

see [6], e.g. Extending all functions $\rho_n(\theta, t, \omega)$ periodically on $\mathbf{R}$ as functions of θ, we obtain from (2.1), (2.2), the representation

$$\begin{aligned} \rho_n(\theta, t, \omega) = &\int_{-\infty}^{+\infty} G(\theta - \xi, t, \omega)\, \rho_0(\xi, \omega)\, d\xi - K \int_0^t \int_{-\infty}^{+\infty} G(\theta - \xi, t - \tau, \omega) \\ &\times \frac{\partial}{\partial \xi}\Big[\, \rho_n(\xi, \tau, \omega) \int_0^{2\pi} \int_{-\infty}^{+\infty} \sin(\varphi - \xi) \rho_{n-1}(\varphi, \tau, \omega) g(\omega) d\varphi d\omega \Big] d\xi\, d\tau. \end{aligned} \tag{2.9}$$

Therefore, we have for the differences $\rho_n(\theta, t, \omega) - \rho_{n-1}(\theta, t, \omega)$, integrating by parts,

$$\begin{aligned} \rho_n - \rho_{n-1} = -K \int_0^t \int_{-\infty}^{+\infty} &\frac{\partial G(\theta - \xi, t - \tau, \omega)}{\partial \xi} \Big[\rho_n \int_0^{2\pi} \int_{-\infty}^{+\infty} \sin(\varphi - \xi) \rho_{n-1} g\, d\varphi d\omega \\ &- \rho_{n-1} \int_0^{2\pi} \int_{-\infty}^{+\infty} \sin(\varphi - \xi) \rho_{n-2} g\, d\varphi d\omega \Big]\, d\xi\, d\tau \end{aligned}$$

or also

$$\rho_n - \rho_{n-1} = -K \int_0^t \int_{-\infty}^{+\infty} \frac{\partial G}{\partial \xi} \Big[(\rho_n - \rho_{n-1}) \int_0^{2\pi} \int_{-\infty}^{+\infty} \sin(\varphi - \xi) \rho_{n-1} g \, d\varphi \, d\omega + \rho_{n-1} \int_0^{2\pi} \int_{-\infty}^{+\infty} \sin(\varphi - \xi)(\rho_{n-1} - \rho_{n-2}) g \, d\varphi \, d\omega \Big] d\xi \, d\tau. \tag{2.10}$$

Therefore by (1.8), (2.8), the estimates

$$\begin{aligned} &\Big| \int_0^t \int_{-\infty}^{+\infty} \frac{\partial G}{\partial \xi} (\rho_n - \rho_{n-1}) \Big(\int_0^{2\pi} \int_{-\infty}^{+\infty} \sin(\varphi - \xi) \rho_{n-1} g \, d\varphi \, d\omega \Big) d\xi \, d\tau \Big| \\ \le & A \int_0^t \int_{-\infty}^{+\infty} \Big| \frac{\partial G}{\partial \xi} \Big| \, |\rho_n - \rho_{n-1}| \, d\xi \, d\tau \le M_1 \int_0^t \frac{\max_{\theta,\omega} |\rho_n - \rho_{n-1}|}{\sqrt{t-\tau}} d\tau, \\ &\Big| \int_0^t \int_{-\infty}^{+\infty} \frac{\partial G}{\partial \xi} \rho_{n-1} \Big(\int_0^{2\pi} \int_{-\infty}^{+\infty} \sin(\varphi - \xi)(\rho_{n-1} - \rho_{n-2}) g \, d\varphi \, d\omega \Big) d\xi \, d\tau \Big| \\ \le & M_2 \int_0^t \frac{\max_{\theta,\,\omega} |\rho_{n-1} - \rho_{n-2}|}{\sqrt{t-\tau}} \, g \, d\tau \end{aligned} \tag{2.11}$$

follow, with M_i depending on $\max_{\theta,\,\omega} \rho_n(\theta, t, \omega)$ and on the known quantity

$$\max_{\theta} \sqrt{t-\tau} \int_{-\infty}^{+\infty} \Big| \frac{\partial G(\theta - \xi, t-\tau, \omega)}{\partial \xi} \Big| \, d\xi \le M_3,$$

cf. (2.8). Using the estimates (2.11) it is easy to derive from (2.10) for

$$s_n(t) := \max_{\theta,\,\omega} |\rho_n - \rho_{n-1}|$$

that

$$s_n(t) \le M_4 \int_0^t \frac{1}{\sqrt{t-\tau}} \, [s_n(\tau) + s_{n-1}(\tau)] \, d\tau. \tag{2.12}$$

By (a version of) the Grownwall's lemma, (2.12) yields

$$s_n(t) \le M_5(t) \int_0^t \frac{s_{n-1}(\tau)}{\sqrt{t-\tau}} \, d\tau \le 2 M_5(t) \sqrt{t} \max_{0 \le \tau \le t} s_{n-1}(\tau), \tag{2.13}$$

where $M_5(t) := M_4 \, (1 + \exp(2\sqrt{t}))$. Repeating this process, we can prove that

$$\max_{\theta,\,\omega,\,\tau} |\rho_n - \rho_{n-1}| \le [2M_5(t)]^{n-1} \left(\sqrt{t}\right)^{n-1} \max_{\theta,\,\omega,\,\tau} |\rho_1 - \rho_0|. \tag{2.14}$$

Therefore, the sequence of successive approximations $\{\rho_n(\theta,t,\omega)\}$ converges for $t\in[0,T]$ with T sufficiently small, such that $2M_5(T)\sqrt{T}<1$. As before, we can show that all derivatives of ρ_n are bounded uniformly in n, and hence an existence theorem follows by the usual compactness arguments, which completes the proof.

Remark 2.1. Theorem 2.1 remains true if we replace the term $-\omega\rho_\theta$ in the governing equation (1.1) with the more general term

$$-\,\omega\;\frac{\partial}{\partial\theta}\Big(\alpha(\theta)\,\rho\Big),$$

where $\alpha(\theta)$ is a 2π–periodic, smooth function, say C^∞. Similarly, we can replace the kernel $\sin(\varphi-\theta)$ in the integral in (1.1) with any 2π–periodic, smooth function.

Most of the results of the present paper can be applied to certain generalizations of the Kuramoto–Sakaguchi equation, that is that proposed in [1, 2],

$$\begin{aligned}\frac{\partial\rho}{\partial t}&=D\frac{\partial^2\rho}{\partial\theta^2}-\omega\frac{\partial\rho}{\partial\theta}-h\,\frac{\partial}{\partial\theta}(\sin\theta\,\rho)\\&\quad-K\frac{\partial}{\partial\theta}\big[\int_0^{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sin(\varphi-\theta)\rho(\varphi,t,\omega,h)g(\omega)f(h)d\varphi\,d\omega\,dh\big],\end{aligned}$$

where both densities, $g(\omega)$ and $f(h)$, are positive, integrable, and compactly supported.

Remark 2.2. By direct computations, clearly, the solution $\rho(\theta,t,\omega)$ to (1.1)–(1.3) is continuous with respect to the parameter ω. Moreover, upon the appropriate smoothness of initial function, ρ_0, with respect to ω, there exist all derivatives $\frac{\partial^m\rho}{\partial\omega^m}$, which are smooth with respect to θ and t.

In fact, let us treat $\rho(\theta,t,\omega)$ as the solution to the linear problem

$$\frac{\partial\rho}{\partial t}=D\frac{\partial^2\rho}{\partial\theta^2}-\omega\frac{\partial\rho}{\partial\theta}-\frac{\partial}{\partial\theta}\Big[\mathcal{K}(\theta,t)\;\rho\Big],$$

where

$$\mathcal{K}(\theta,t):=K\int_0^{2\pi}\int_{-\infty}^{+\infty}\sin(\varphi-\theta)\rho\,g(\omega)\,d\omega\,d\varphi.$$

Then it is easy to see that

$$\frac{\partial r}{\partial t}=D\frac{\partial^2 r}{\partial\theta^2}-\omega\frac{\partial r}{\partial\theta}-\frac{\partial}{\partial\theta}\Big[\mathcal{K}(\theta,t)\;r\Big]-\frac{\partial\rho(\theta,t,\omega+\Delta\omega)}{\partial\theta},\tag{2.15}$$

where r is the differential quotient

$$r(\theta,t,\omega,\Delta\omega):=\frac{\rho(\theta,t,\omega+\Delta\omega)-\rho(\theta,t,\omega)}{\Delta\omega}.$$

Consequently, the representation

$$
\begin{aligned}
r(\theta, t, \omega, \Delta\omega) = & \int_{-\infty}^{+\infty} G(\theta - \xi, t, \omega)\, r(\theta, 0, \omega, \Delta\omega)\, d\xi \\
& - \int_0^t \int_{-\infty}^{+\infty} G(\theta - \xi, t - \tau, \omega)\ \rho_\theta(\xi, \tau, \omega + \Delta\omega)\, d\xi\, d\tau
\end{aligned} \tag{2.16}
$$

holds, with the Green's function G being independent of $\Delta\omega$. Passing to the limit as $\Delta\omega \to 0$, we obtain from (2.16) that the derivative $\frac{\partial \rho}{\partial \omega}$ does exist and is smooth in θ and t. By a similar procedure we can conclude that $\rho \in C^\infty$ as a function of ω.

The next basic result can now be established.

Theorem 2.2. *Under conditions of Theorem* 2.1 *the solution to problem* (1.1)–(1.3) *is unique.*

Again, the proof is standard, but because of the nonclassical form of equation (1.1), we cannot refer to the existing literature. We shall give a sketch of the proof, omitting all details.

Sketch of the proof. Suppose that there exist two solutions, $\rho_1(\theta, t, \omega)$ and $\rho_2(\theta, t, \omega)$. The "weighted difference" $u := (\rho_1 - \rho_2)\, e^{\lambda t}$ then satisfies

$$
\begin{aligned}
\frac{\partial u}{\partial t} = & D \frac{\partial^2 u}{\partial \theta^2} - \Big[\omega + B^*(\theta, t)\Big] \frac{\partial u}{\partial \theta} + [H_1(\theta, t) - \lambda]\, u \\
& + \int_0^{2\pi} H_2(\theta, \varphi, t, \omega) \int_{-\infty}^{+\infty} u(\varphi, t, \omega)\, g(\omega)\, d\omega\, d\varphi,
\end{aligned} \tag{2.17}
$$

with an appropriate (smooth) $B^*(\theta, t)$, and H_i, $i = 1, 2$, given by

$$
\begin{aligned}
H_1(\theta, t) := & K \int_0^{2\pi} \int_{-\infty}^{+\infty} \cos(\varphi - \theta)\, \rho_2(\varphi, t, \omega)\, g(\omega)\, d\varphi\, d\omega, \\
H_2(\theta, \varphi, t, \omega) := & K\Big(\rho_1(\theta, t, \omega)\ \cos(\varphi - \theta) - \frac{\partial \rho_2(\theta, t, \omega)}{\partial \theta} \sin(\varphi - \theta)\Big).
\end{aligned} \tag{2.18}
$$

Due to Corollary 1, it is easy to see that the inequalities

$$
|H_1(\theta, t)| \le KA,
$$

$$
\begin{aligned}
\Big| \int_0^{2\pi} H_2(\theta, \varphi, t, \omega) \int_{-\infty}^{+\infty} u(\varphi, t, \omega) g(\omega)\, d\omega\, d\varphi \Big| & \le 2\pi\, A\, \sup |H_2| \sup_{\theta,\, \omega} |u(\theta, t, \omega)| \\
& \le 2\pi\, KA\Big(\sup |\rho_1| + \sup \Big|\frac{\partial \rho_2}{\partial \theta}\Big|\Big) \sup_{\theta, \omega} |u(\theta, t, \omega)|,
\end{aligned} \tag{2.19}
$$

hold. Choose

$$\lambda \geq \big\{KA\big[1+2\pi\big(\sup|\rho_1|+\sup|\frac{\partial\rho_2}{\partial\theta}|\big)\big]+1\big\}, \tag{2.20}$$

and consider the quantity

$$m(\omega) := \max_{\theta,t}|u(\theta,t,\omega)|$$

on the interval of existence of solutions, $t\in[0,T]$. Choose the point $\omega^*\in[-N,N]$, such that

$$m(\omega^*) = \max_{\omega} m(\omega).$$

Consider then the point (θ^*,t^*), such that

$$|u(\theta^*,t^*,\omega^*)| = \max_{\theta,t}|u(\theta,t,\omega^*)|.$$

For simplicity, assume that $u(\theta^*,t^*,\omega^*)>0$. Obviously, the inequalities

$$u_t \geq 0, \qquad u_\theta = 0, \qquad u_{\theta\theta}\leq 0 \tag{2.21}$$

hold at the point (θ^*,t^*,ω^*). At the same time (see (2.19) and (2.20)), we have

$$(H_1-\lambda)u(\theta^*,t^*,\omega^*)+\int_0^{2\pi}H_2\int_{-\infty}^{+\infty}u(\varphi,t^*,\omega)g(\omega)\,d\omega\,d\varphi \leq -u(\theta^*,t^*,\omega^*). \tag{2.22}$$

Both (2.21) and (2.22) now contradict (2.17), and this completes the proof since $\max|u|=0$, and hence $u\equiv 0$.

3. Basic estimates of solutions. In this section we prove that the solution ρ to problem (1.1)–(1.3) is bounded, *uniformly in t*. The key technique we shall use is based on obtaining "energy-like estimates". Multiply both sides of equation (1.1) by ρ and integrate with respect to θ. Note that, because of smoothness and periodicity of $\rho(\theta,t,\omega)$, in all calculations below, all terms out of the integrals (arising from integration by parts) vanish. Thus, we obtain after simple calculations

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\int_0^{2\pi}\rho^2\,d\theta\\
&=-D\int_0^{2\pi}\rho_\theta^2 d\theta+\frac{1}{2}K\int_0^{2\pi}\frac{\partial\rho^2}{\partial\theta}\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty}\sin(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)d\theta\\
&=-D\int_0^{2\pi}\rho_\theta^2 d\theta+\frac{K}{2}\int_0^{2\pi}\rho^2\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty}\cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)d\theta,
\end{aligned}$$

and finally the identity

$$\begin{aligned}&\frac{\partial}{\partial t}\int_0^{2\pi}\rho^2\,d\theta+2D\int_0^{2\pi}\rho_\theta^2\,d\theta\\=&K\int_0^{2\pi}\rho^2\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty}\cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta.\end{aligned}\tag{3.1}$$

The next integral identity can be obtained multiplying both sides of equation (1.1) by $\rho_{\theta\theta}$ and integrating with respect to θ. We first need the following auxiliary calculations:

$$\int_0^{2\pi}\rho_t\rho_{\theta\theta}\,d\theta=-\int_0^{2\pi}\rho_\theta\rho_{\theta t}\,d\theta=-\frac{1}{2}\frac{\partial}{\partial t}\int_0^{2\pi}\rho_\theta^2\,d\theta;\qquad\int_0^{2\pi}\rho_\theta\rho_{\theta\theta}\,d\theta=0;$$

$$\begin{aligned}&\int_0^{2\pi}\rho_{\theta\theta}\,\frac{\partial}{\partial\theta}\big[\rho\int_0^{2\pi}\int_{-\infty}^{+\infty}\sin(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\big]\,d\theta\\
&=\frac{1}{2}\int_0^{2\pi}\frac{\partial}{\partial\theta}\big(\rho_\theta^2\big)\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty}\sin(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta\\
&-\int_0^{2\pi}\rho_{\theta\theta}\rho\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty}\cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta\\
&=\frac{1}{2}\int_0^{2\pi}\rho_\theta^2\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty}\cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta\\
&+\int_0^{2\pi}\rho_\theta^2\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty}\cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta\\
&+\int_0^{2\pi}\rho_\theta\rho\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty}\sin(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta\\
&=\frac{3}{2}\int_0^{2\pi}\rho_\theta^2\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty}\cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta\\
&+\frac{1}{2}\int_0^{2\pi}\rho^2\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty}\cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta.\end{aligned}$$

Therefore we obtain the identity

$$\begin{aligned}&\frac{\partial}{\partial t}\int_0^{2\pi}\rho_\theta^2\,d\theta+2D\int_0^{2\pi}\rho_{\theta\theta}^2\,d\theta\\=&K\int_0^{2\pi}(3\rho_\theta^2+\rho^2)\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty}\cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta.\end{aligned}\tag{3.2}$$

Multiplying both sides of equation (3.1) by the constant $3KA/(2D)+1$, and summing side by side to equation (3.2), we conclude that the "energy–like" functional

$$R(t,\omega):=\int_0^{2\pi}\big[\big(\frac{3KA}{2D}+1\big)\rho^2+\rho_\theta^2\big]\,d\theta\tag{3.3}$$

satisfies

$$\begin{aligned}&\frac{\partial R(t,\omega)}{\partial t}+(2D+3KA)\int_0^{2\pi}\rho_\theta^2\,d\theta+2D\int_0^{2\pi}\rho_{\theta\theta}^2\,d\theta\\&=K\int_0^{2\pi}\big[3\rho_\theta^2+\big(\frac{3KA}{2D}+2\big)\rho^2\big]\int_0^{2\pi}\int_{-\infty}^{+\infty}\cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\,d\theta.\end{aligned}\tag{3.4}$$

Now we need to estimate the right-hand side of (3.4). Setting first $F:=\rho_\theta^2(\theta,t,\omega)$, and then $F:=\rho^2(\theta,t,\omega)$ in (1.10), we get

$$\Big|\int_0^{2\pi}\rho_\theta^2\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty}\cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta\Big|\le A\int_0^{2\pi}\rho_\theta^2\,d\theta,\tag{3.5}$$

and

$$\Big|\int_0^{2\pi}\rho^2\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty}\cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta\Big|\le A\int_0^{2\pi}\rho^2\,d\theta,\tag{3.6}$$

respectively. Using (3.5) and (3.6) in (3.4) and dropping the integral of $\rho_{\theta\theta}^2$, the inequality

$$\frac{\partial R(t,\omega)}{\partial t}+2D\int_0^{2\pi}\rho_\theta^2\,d\theta\le K_1\int_0^{2\pi}\rho^2\,d\theta\tag{3.7}$$

can be established with $K_1:=KA\left(\frac{3KA}{2D}+2\right)$. The right-hand side of this relation could be estimated with the help of results of Section 1. Due to the "normalization property" in Lemma 1.2 (see (1.5)), for any $t\ge 0$ there exists a point $\theta_0\in[0,2\pi]$ such that $\rho(\theta_0,t,\omega)\le 1/(2\pi)$. Thus,

$$\rho(\theta,t,\omega)=\rho\big|_{\theta=\theta_0}+\int_{\theta_0}^{\theta}\frac{\partial\rho}{\partial\theta}\,d\theta\le\frac{1}{2\pi}+\int_0^{2\pi}|\rho_\theta|\,d\theta.\tag{3.8}$$

Taking into account Lemma 1.1 and, again, Lemma 1.2, and the inequality

$$ab\le\frac{\varepsilon}{2}a^2+\frac{1}{2\varepsilon}b^2$$

with $\varepsilon=K_1/2D$, and the Cauchy-Schwarz inequality

$$\int_0^{2\pi}|\rho_\theta|\,d\theta\le\sqrt{2\pi}\Big(\int_0^{2\pi}\rho_\theta^2\,d\theta\Big)^{1/2},\tag{3.9}$$

it follows from (3.8) that

$$\begin{aligned}K_1\int_0^{2\pi}\rho^2\,d\theta&\le K_1\int_0^{2\pi}\rho\Big(\frac{1}{2\pi}+\int_0^{2\pi}|\rho_\theta|\,d\theta\Big)d\theta\\&\le\frac{K_1}{2\pi}+\frac{\pi}{D}\,\frac{K_1^2}{2}+D\int_0^{2\pi}\rho_\theta^2\,d\theta.\end{aligned}\tag{3.10}$$

Thus, from (3.7) and (3.10) we conclude that

$$\frac{\partial R(t,\omega)}{\partial t} + D\int_0^{2\pi} \rho_\theta^2\, d\theta \le K_2 \tag{3.11}$$

with $K_2 := \frac{K_1}{2\pi} + \frac{\pi K_1^2}{2D}$. A similar estimate could be derived from (3.7) by using the Nash inequality [11]; such inequality is, however, very general, while we took advantage of the peculiarities of the present problem. Consider the two possibilities, that either the function $R(t,\omega)$ satisfies the inequality

$$\frac{\partial R(t,\omega)}{\partial t} \ge 0, \tag{3.12}$$

or

$$\frac{\partial R(t,\omega)}{\partial t} < 0. \tag{3.13}$$

In the first case, (3.11) yields

$$\int_0^{2\pi} \rho_\theta^2\, d\theta \le \frac{K_2}{D}, \tag{3.14}$$

and, consequently, from (3.8) and (3.9) we conclude that for the time t in (3.12), the inequality

$$\rho(\theta,t,\omega) \le C_1 \tag{3.15}$$

holds, with $C_1 := \frac{1}{2\pi} + \sqrt{2\pi\,\frac{K_2}{D}}$. From (3.15) and (3.14), it is easy to see that quantity $R(t,\omega)$ defined in (3.3) is bounded,

$$R(t,\omega) \le K_3 := 2\pi\left(\frac{3KA}{2D} + 1\right) C_1^2 + \frac{K_2}{D}, \tag{3.16}$$

for every t where $\frac{\partial R(t,\omega)}{\partial t} \ge 0$ (cf. (3.12)). If (3.13) holds, instead, for some open interval $t \in (t_0, t_1)$, obviously $R(t) \le R(t_0)$ and for $t = t_0$ estimate (3.16) holds. Therefore we have the following lemma.

Lemma 3.1. *The quantity $R(t,\omega)$ in* (3.3) *remains uniformly bounded as in* (3.16) *as long as the solution $\rho(\theta,t,\omega)$ to problem* (1.1)–(1.2) *remains smooth.*

We stress that (3.11) would give a time-dependent estimate for R, after integrating in time. Lemma 3.1 yields, instead, a time-independent estimate.

As the function $R(t,\omega)$ remains bounded for all times t in the interval of existence of the solution $\rho(\theta,t,\omega)$, the estimate

$$\int_0^{2\pi} \rho_\theta^2\, d\theta \le K_3 \tag{3.17}$$

follows from (3.3). Finally, the main result of this section can be obtained as a theorem, by (3.8), (3.9), and (3.17):

Theorem 3.1. *The classical solution* $\rho(\theta,t,\omega)$ *to problem* (1.1)-(1.2) *remains bounded, uniformly in* t, ω, *and* N, *for* $\omega \in [-N, N]$,

$$\rho(\theta,t,\omega) \le C_2 := \frac{1}{2\pi} + \sqrt{2\pi\, K_3}, \tag{3.18}$$

as long as it exists.

Remark 3.1. In proving Theorem 3.1 we have also established that

$$\int_0^{2\pi} \big[\rho^2 + \big(\frac{\partial\rho}{\partial\theta}\big)^2\,\big]\, d\theta \le K_4, \tag{3.19}$$

where the constant K_4 does depend neither on t nor on N, $K_4 := 2\pi\, C_2^2 + K_3$.

4. Estimating the derivatives of solutions. In this Section we obtain estimates for the derivatives of the unique solution, $\rho(\theta,t,\omega)$, to problem (1.1)–(1.3), independent of t, ω, and N, for $\omega \in [-N, N]$. The basic idea being similar to that in Section 3, we give only a sketchy account.

Multiplying both sides of equation (1.1) by the fourth derivative $\frac{\partial^4\rho}{\partial\theta^4}$, and integrating in θ, we get

$$\begin{aligned}\int_0^{2\pi} \frac{\partial\rho}{\partial t}\,\frac{\partial^4\rho}{\partial\theta^4}\, d\theta &= D\int_0^{2\pi} \frac{\partial^2\rho}{\partial\theta^2}\,\frac{\partial^4\rho}{\partial\theta^4}\, d\theta - \omega\int_0^{2\pi} \frac{\partial\rho}{\partial\theta}\,\frac{\partial^4\rho}{\partial\theta^4}\, d\theta \\ &- K\int_0^{2\pi} \frac{\partial^4\rho}{\partial\theta^4}\,\frac{\partial}{\partial\theta}\Big(\rho\int_0^{2\pi}\int_{-\infty}^{+\infty} \sin(\varphi-\theta)\,\rho\, g(\omega)\, d\varphi\, d\omega\Big)\, d\theta.\end{aligned} \tag{4.1}$$

We now transform all the summands of (4.1) into a more convenient form. In the following, integration by parts will be used repeatedly; all terms out of the integrals vanish due to the periodic boundary conditions (1.3) and the smoothness of solution.

$$\int_0^{2\pi} \frac{\partial\rho}{\partial t}\,\frac{\partial^4\rho}{\partial\theta^4}\, d\theta = \int_0^{2\pi} \frac{\partial^3\rho}{\partial^2\theta\partial t}\,\frac{\partial^2\rho}{\partial\theta^2}\, d\theta = \frac{1}{2}\frac{\partial}{\partial t}\int_0^{2\pi} \Big(\frac{\partial^2\rho}{\partial\theta^2}\Big)^2\, d\theta, \tag{4.2}$$

$$\int_0^{2\pi} \frac{\partial^2\rho}{\partial\theta^2}\,\frac{\partial^4\rho}{\partial\theta^4}\, d\theta = -\int_0^{2\pi} \Big(\frac{\partial^3\rho}{\partial\theta^3}\Big)^2\, d\theta; \tag{4.3}$$

$$\int_0^{2\pi} \frac{\partial\rho}{\partial\theta}\,\frac{\partial^4\rho}{\partial\theta^4}\, d\theta = -\int_0^{2\pi} \frac{\partial^2\rho}{\partial\theta^2}\,\frac{\partial^3\rho}{\partial\theta^3}\, d\theta = -\frac{1}{2}\int_0^{2\pi} \frac{\partial}{\partial\theta}\Big(\frac{\partial^2\rho}{\partial\theta^2}\Big)^2\, d\theta = 0; \tag{4.4}$$

$$\begin{aligned}&\int_0^{2\pi} \frac{\partial^4\rho}{\partial\theta^4}\,\frac{\partial}{\partial\theta}\Big(\rho\int_0^{2\pi}\int_{-\infty}^{+\infty} \sin(\varphi-\theta)\,\rho\, g(\omega)\, d\varphi\, d\omega\Big)\, d\theta \\ &= \int_0^{2\pi} \frac{\partial^4\rho}{\partial\theta^4}\,\frac{\partial\rho}{\partial\theta}\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \sin(\varphi-\theta)\,\rho\, g(\omega)\, d\varphi\, d\omega\Big)\, d\theta \\ &- \int_0^{2\pi} \frac{\partial^4\rho}{\partial\theta^4}\,\rho\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \cos(\varphi-\theta)\,\rho\, g(\omega)\, d\varphi\, d\omega\Big)\, d\theta.\end{aligned} \tag{4.5}$$

The following calculations are a little tedious, but merely technical.

$$
\begin{aligned}
&\int_0^{2\pi} \frac{\partial^4\rho}{\partial\theta^4}\,\frac{\partial\rho}{\partial\theta}\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \sin(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta \\
&= -\int_0^{2\pi} \frac{\partial^3\rho}{\partial\theta^3}\,\frac{\partial^2\rho}{\partial\theta^2}\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \sin(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta \\
&+\int_0^{2\pi} \frac{\partial^3\rho}{\partial\theta^3}\,\frac{\partial\rho}{\partial\theta}\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta \\
&= -\frac{1}{2}\int_0^{2\pi} \Big(\frac{\partial^2\rho}{\partial\theta^2}\Big)^2\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta \\
&-\int_0^{2\pi} \Big(\frac{\partial^2\rho}{\partial\theta^2}\Big)^2\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta \\
&-\int_0^{2\pi} \frac{\partial^2\rho}{\partial\theta^2}\,\rho_\theta\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \sin(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta \\
&= -\frac{3}{2}\int_0^{2\pi} \Big(\frac{\partial^2\rho}{\partial\theta^2}\Big)^2\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta \\
&-\frac{1}{2}\int_0^{2\pi} \rho_\theta^2\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta;
\end{aligned}
\tag{4.6}
$$

$$
\begin{aligned}
&\int_0^{2\pi} \frac{\partial^4\rho}{\partial\theta^4}\,\rho\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta \\
&= -\int_0^{2\pi} \frac{\partial^3\rho}{\partial\theta^3}\,\rho_\theta\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta \\
&-\int_0^{2\pi} \frac{\partial^3\rho}{\partial\theta^3}\,\rho\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \sin(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta \\
&= \int_0^{2\pi} \Big(\frac{\partial^2\rho}{\partial\theta^2}\Big)^2\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta \\
&+2\int_0^{2\pi} \frac{\partial^2\rho}{\partial\theta^2}\,\rho_\theta\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \sin(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta \\
&-\int_0^{2\pi} \frac{\partial^2\rho}{\partial\theta^2}\,\rho\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta \\
&= \int_0^{2\pi} \Big[\Big(\frac{\partial^2\rho}{\partial\theta^2}\Big)^2 + 2\rho_\theta^2\Big]\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \cos(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta \\
&+\int_0^{2\pi} \rho\,\rho_\theta\Big(\int_0^{2\pi}\int_{-\infty}^{+\infty} \sin(\varphi-\theta)\,\rho\,g(\omega)\,d\varphi\,d\omega\Big)\,d\theta
\end{aligned}
$$

$$= \int_0^{2\pi} \Big[\Big(\frac{\partial^2 \rho}{\partial \theta^2} \Big)^2 + 2\rho_\theta^2 + \frac{1}{2}\rho^2 \Big] \Big(\int_0^{2\pi} \int_{-\infty}^{+\infty} \cos(\varphi - \theta)\, \rho\, g(\omega)\, d\varphi\, d\omega \Big)\, d\theta. \tag{4.7}$$

Collecting (4.1) and (4.2)–(4.7), we conclude that

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t} \int_0^{2\pi} \big(\frac{\partial^2 \rho}{\partial \theta^2}\big)^2 d\theta = -D \int_0^{2\pi} \big(\frac{\partial^3 \rho}{\partial \theta^3}\big)^2 d\theta \\ &+ \frac{K}{2} \int_0^{2\pi} \big[5\big(\frac{\partial^2 \rho}{\partial \theta^2}\big)^2 + 5\big(\frac{\partial \rho}{\partial \theta}\big)^2 + \rho^2\big]\big(\int_0^{2\pi} \int_{-\infty}^{+\infty} \cos(\varphi - \theta)\, \rho\, g(\omega)\, d\varphi\, d\omega \big) d\theta. \end{aligned} \tag{4.8}$$

Multiplying both sides of equation (3.2) by $\frac{5KA}{2D} + 1$, both sides of equation (4.8) by 2, and summing up, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t} \int_0^{2\pi} \big[\big(\frac{5KA}{2D} + 1\big)\big(\frac{\partial \rho}{\partial \theta}\big)^2 + \big(\frac{\partial^2 \rho}{\partial \theta^2}\big)^2\big] d\theta + 2D\big(\frac{5KA}{2D} + 1\big) \int_0^{2\pi} \big(\frac{\partial^2 \rho}{\partial \theta^2}\big)^2 d\theta \\ &+2D \int_0^{2\pi} \big(\frac{\partial^3 \rho}{\partial \theta^3}\big)^2 d\theta = K \int_0^{2\pi} \big[5\rho_{\theta\theta}^2 + \big(5 + 3\big(\frac{5KA}{2D} + 1\big)\big)\rho_\theta^2 + \big(\frac{5KA}{2D} + 2\big)\rho^2\big] \\ &\times\big(\int_0^{2\pi} \int_{-\infty}^{+\infty} \cos(\varphi - \theta)\, \rho\, g(\omega)\, d\varphi\, d\omega \big)\, d\theta. \end{aligned} \tag{4.9}$$

Thus, by Corollary 1, choosing successively $F := \rho_{\theta\theta}^2$, ρ_θ^2, ρ^2, and recalling (3.19), we conclude that the "energy-like" functional

$$R_1(t, \omega) := \int_0^{2\pi} \Big(K_5 \Big(\frac{\partial \rho}{\partial \theta}\Big)^2 + \Big(\frac{\partial^2 \rho}{\partial \theta^2}\Big)^2 \Big)\, d\theta, \tag{4.10}$$

where $K_5 = \frac{5KA}{2D} + 1$ satisfies

$$\frac{\partial R_1(t, \omega)}{\partial t} + 2D \int_0^{2\pi} \Big(\frac{\partial^2 \rho}{\partial \theta^2}\Big)^2 d\theta + 2D \int_0^{2\pi} \Big(\frac{\partial^3 \rho}{\partial \theta^3}\Big)^2 d\theta \le K_6, \tag{4.11}$$

for some positive constant K_6.

We can use now an argument similar to that in Section 3. First of all, due to the periodicity of $\rho(\theta, t, \omega)$, for every t and ω there exists a point $\theta = \theta_0$ such that $\rho_\theta(\theta_0, t, \omega) = 0$. Therefore (see(3.8)),

$$\Big|\frac{\partial \rho}{\partial \theta}\Big| \le \int_{\theta_0}^{\theta} \Big|\frac{\partial^2 \rho}{\partial \theta^2}\Big|\, d\theta \le \sqrt{2\pi} \Big(\int_0^{2\pi} \Big(\frac{\partial^2 \rho}{\partial \theta^2}\Big)^2 d\theta \Big)^{1/2}. \tag{4.12}$$

It is easy to establish that $R_1(t, \omega)$ is bounded uniformly in t and ω. Indeed, for every t such that

$$\frac{\partial R_1(t, \omega)}{\partial t} \ge 0,$$

inequality (4.11) above implies that

$$\int_0^{2\pi} \left(\frac{\partial^2 \rho}{\partial \theta^2}\right)^2 d\theta \leq K_7.$$

Consequently, (3.19) along with (4.10) show that $R_1(t, \omega)$ is uniformly bounded in t and ω. As R_1 is bounded, the following inequality

$$\int_0^{2\pi} \left(\frac{\partial^2 \rho}{\partial \theta^2}\right)^2 d\theta \leq K_8 \tag{4.13}$$

holds for all $t \geq 0$ and ω. Finally, by (4.12) and (4.13) we conclude that

$$\sup_{\theta \in [0, 2\pi]} \left|\frac{\partial \rho}{\partial \theta}\right| \leq C_3. \tag{4.14}$$

It follows from equation (1.1), that the time-derivative $\rho_t(\theta, t, \omega)$ is bounded uniformly in t. Summarizing, we conclude with the following *global existence* theorem.

Theorem 4.1. *The classical solution, $\rho(\theta, t, \omega)$ to problem* (1.1)–(1.3) *exists for all times $t > 0$, and is uniformly bounded in t*

$$||\rho(\theta, t, \omega)||_{C^m} \leq M, \quad m \geq 0. \tag{4.15}$$

Clearly, we proved (4.15) only for $m = 0$, 1, but we can achieve the general result for all $m > 1$ by methods similar to those adopted above.

In closing, we observe that mass is conserved (by (1.7)), and, moreover, by (4.15), ρ itself is bounded (along with all its derivatives) for all times, so no "spikes" arbitrarily large could develop anywhere.

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