Differential and Integral Equations Volume 15, Number 2, February 2002, Pages 147–165

MULTIPLE SIGN CHANGING SOLUTIONS IN A CLASS OF QUASILINEAR EQUATIONS*

J. V. Goncalves
Universidade de Brasília, Departamento de Matemática
70910-900 Brasília(DF)-Brasil

A. L. Melo
Universidade Federal da Paraíba, Departamento de Matemática e Estatística,
Campina Grande(PB)-Brasil

(Submitted by: Klaus Schmitt)

Abstract. This paper deals with finding multiple sign-changing solutions of the following class of quasilinear problems:

$$\begin{cases} -(r^{\alpha}|u'(r)|^{\beta}u'(r))' = \lambda\, r^{\gamma} f(u(r)),\ 0 < r < R \\ u(R) = u'(0) = 0, \end{cases}$$

where α, β and γ are given real numbers, $\lambda > 0$ is a parameter, $f : \mathbf{R} \to \mathbf{R}$ is some continuous function and $0 < R < \infty$. A result on existence of infinitely many sign-changing solutions is obtained by considering a family of associated initial value problems which are solved through a shooting argument and a counting of zeroes.

1. Introduction

Our purpose in this work is to find multiple sign-changing solutions of the problem

$$\begin{cases} -(r^{\alpha}|u'(r)|^{\beta}u'(r))' = \lambda\, r^{\gamma} f(u(r)),\ 0 < r < R \\ u(R) = u'(0) = 0 \end{cases} \tag{$(*)_\lambda$}$$

where α, β and γ are given real numbers, $\lambda > 0$ is a parameter, $f : \mathbf{R} \to \mathbf{R}$ is continuous, $0 < R < \infty$ and $u'(r) = \frac{du}{dr}$. The conditions below will be assumed in our main results:

$$\beta > -1,\ \gamma \geq \max\{\alpha, \frac{-\alpha}{\beta+1}\}. \tag{$h1$}$$

*Research partially supported by CNPq/PRONEX/Brasil.
Accepted for publication: October 2000.
AMS Subject Classifications: 34A15, 35J25.

As was pointed out by Clement, Figueiredo and Mitidieri [4] the quasilinear operator $Lu \equiv (r^\alpha |u'(r)|^\beta u'(r))'$ includes as a special case the following operators when considered acting on radial functions:

(1.1) Laplacian: $\alpha = N-1,\ \beta = 0$,
(1.2) p-Laplacian $(1 < p < N)$: $\alpha = N-1,\ \beta = p-2$,
(1.3) k-Hessian $(1 \le k \le N):\ \alpha = N-k,\ \beta = k-1$

In the present paper, we look at $(*)_\lambda$, as the radial form of the problem

$$(P) \quad \begin{cases} -div(a(x)|\nabla u(x)|^\beta \nabla u(x)) = \lambda\ h(x,u),\ x \in B_R \\ u(x) = 0,\ x \in \partial B_R, \end{cases}$$

where $a(x) = a(r),\ h(x,u) = b(x)f(u)$ with $b(x) = b(r)$, where $r = |x|$ and a, b are suitably given functions and $B_R \subset \mathbf{R^N}$ is the ball of radius R centered at the origin. The following conditions will be imposed on f:

$(h2)$ $tf(t) > 0$ for $t \neq 0$,
$(h3)$ there exists $d_\infty > 0$ such that f is nondecreasing in $(-\infty, d_\infty]$,
$(h4)$ $\underline{\lim}_{\nu\to 0^\pm} \int_0^\nu |f(t)|^{\frac{-1}{\beta+1}} sgn(f(t))dt < \infty$.

Solutions in this paper are intended as differentiable functions u such that

$$r^\alpha |u'(r)|^\beta u'(r) \quad \text{is differentiable}$$

and u satisfies the equations in the classical sense. (So we look for classical solutions.)

We establish next our main result:

Theorem 1.1. *Let $f \in C(\mathbf{R}) \cap \mathbf{C^1}(\mathbf{R}\backslash\{\mathbf{0}\})$ and assume $(h1)$–$(h4)$. Then there exists a positive number Λ, and for each $\lambda \in (0, \Lambda]$, there exist an infinite sequence of real numbers $\{d_\ell\}_{\ell=0}^\infty$ and a sequence $\{u_\ell\}_{\ell=0}^\infty$ of solutions of $(*)_\lambda$ satisfying*

(1.4) $d_0 > d_1 > \cdots > d_\ell > \cdots > 0$,
(1.5) $u_\ell(0) = d_\ell$,
(1.6) *u_ℓ has precisely ℓ zeroes in $(0, R)$.*

Remark. Making $a = b \equiv 1$ and $\beta = p-2$ with $1 < p < \infty$ and $\lambda = 1$, (P) becomes

$$-(r^{N-1}|u'(r)|^{p-2}u'(r))' = r^{N-1}f(u(r)),\ 0 < r < R,\ u(R) = u'(0) = 0.$$

It was shown by Iaia [2] that, if $f(t) = |t|^{\delta-1}t,\ t \in \mathbf{R}$ and $1 < \delta+1 < p < N$, then this problem has infinitely many nodal solutions. Theorem 1.1 extends the main result of [2] in the sense that we were able to treat both a more general class of quasilinear operators L and a broader class of terms f. In this respect we point out that any continuous function f satisfying $(h2)$–$(h3)$ and the condition

$$\lim_{t\to 0}\frac{f(t)}{|t|^{\delta-1}t}>0 \quad \text{for some} \quad \delta\in(0,\beta+1)$$

satisfies $(h4)$ as well. On the other hand, letting $\alpha=\gamma\equiv\widehat{c}>1$, and taking $\zeta,\eta\in C^1(\mathbf{R})$ such that $\zeta>0$, $t\zeta'(t)\geq 0$, η is nondecreasing, $\eta(0)=0$, we get the following examples of functions satisfying $(h2)$–$(h4)$:

(1.7) $f(t)=\zeta(t)|t|^{\delta-1}t+\eta(t)$, $\beta=0$, $0<\delta<1$,
(1.8) $f(t)=\zeta(t)\sin h(t)+\eta(t)$, $\beta>0$,
(1.9) $f(t)=\arctan g(t)$, $\beta>0$.

This class of problems has been investigated by many authors, and we would like to refer the reader to Saxton and Wei [10], Castro and Kurepa [3], Cheng [1], Strauss [8], Ni and Serrin [6], Castro, Cóssio and Neuberger [5], and their references.

Our proof of Theorem 1.1, which will be presented in Section 5, is based on the shooting method, and we have adapted many of the ideas in [2]. We consider, at first, the following family of initial value problems:

$$(*)_{\lambda,d} \qquad \begin{cases} -\left(r^{\alpha}|u'(r)|^{\beta}u'(r)\right)'=\lambda r^{\gamma}f(u(r)),\ r>0\\ u(0)=d,\ u'(0)=0,\end{cases}$$

where $d\in(0,d_\infty]$. By a solution of $(*)_{\lambda,d}$ we mean a classical solution. We shall state in the sequel of this section three crucial results. The first one, Theorem 1.2 below, which will be proved in Section 4, is the basis for the proof of our main result, Theorem 1.1. In its turn Theorem 1.2 will be proved using Proposition 1.4 and Theorem 1.3, whose proofs will be given in Sections 2 and 3 respectively.

Theorem 1.2. *Let $f\in C(\mathbf{R})\cap\mathbf{C^1}(\mathbf{R}\backslash\{\mathbf{0}\})$ and assume $(h1)$–$(h4)$. Then there is $\Lambda>0$ such that for each $d\in(0,d_\infty]$, and for each $\lambda\in(0,\Lambda]$, problem $(*)_{\lambda,d}$ has a unique solution $u(\cdot,d,\lambda)\equiv u(\cdot,d)\equiv u(\cdot)\in C^1([0,\infty))$ satisfying the following:*

(1.10) *If $d_0\in(0,d_\infty]$, then $u(r,d)\to u(r,d_0)$ as $d\to d_0$, uniformly in $[0,T]$, for all $T>0$.*
(1.11) *If $d_0\in(0,d_\infty]$, then $u'(r,d)\to u'(r,d_0)$ as $d\to d_0$, uniformly on compact subsets of $(0,\infty)$.*
(1.12) *If $d\in(0,d_\infty]$, then there is an increasing sequence of zeroes of $u(\cdot,d)$, namely, $z_\ell\equiv z_\ell(d)$, $\ell\geq 1$, with $z_1(d_\infty)\geq R$, and such that $u(r,d)>0$*

for $0 < r < z_1(d)$, $u'(r,d) < 0$ *for* $0 < r \le z_1(d)$, $u(r,d) \neq 0$ *for* $z_\ell < r < z_{\ell+1}$ *and* $u'(z_\ell, d) \neq 0$. *In addition,*

$(1.12)_i$ $z_\ell(d) \to 0$ *as* $d \to 0$,

$(1.12)_{ii}$ $z_\ell(d) \to z_\ell(d_0)$ *as* $d \to d_0$.

(1.13) *If* $d_0 \in (0, d_\infty)$ *and* $u(r, d_0)$ *has* ℓ *zeroes in* $(0,R)$, *then* $u(r,d)$ *has at most* $\ell + 1$ *zeroes in* $(0,R)$ *whenever* $d < d_0$, d *close enough to* d_0.

The result below gives existence and uniqueness of the solution of problem $(*)_{\lambda,d}$.

Theorem 1.3. *Let* $f \in C(\mathbf{R}) \cap \mathbf{C^1}(\mathbf{R}\backslash\{\mathbf{0}\})$ *and assume* $(h1)$–$(h3)$ *and* $\lambda > 0$. *Then problem* $(*)_{\lambda,d}$ *has a unique solution* $u(\cdot, d, \lambda) \equiv u(\cdot, d) \equiv u(\cdot) \in C^1([0,\infty))$.

To close this section we state a technical result which will be used in the proofs of Theorems 1.2 and 1.3.

Proposition 1.4. *Assume* $(h1)$ *and let let* $d \in (0, d_\infty)$, $\lambda > 0$ *and* $T > 0$. *If* u *is a solution of* $(*)_{\lambda,d}$ *in* $[0,T]$ *and* $F(t) \equiv \int_0^t f(s)ds$, *then*

(1.14) $|u'(r)|^{\beta+2} \le \lambda r^{\gamma-\alpha} \frac{\beta+2}{\beta+1} F(d)$, $0 \le r \le T$,

(1.15) $F(u(r)) \le F(d)$, $0 \le r \le T$.

2. Proof of Proposition 1.4

The solution u of $(*)_{\lambda,d}$ satisfies

$$-\Big(r^\alpha |u'(r)|^\beta u'(r)\Big)' = \lambda r^\gamma f(u(r)),\ 0 < r \le T \tag{2.1}$$

so that

$$r^\alpha |u'(r)|^\beta u'(r) = -\int_0^r \lambda t^\gamma f(u(t))dt,\ 0 < r \le T. \tag{2.2}$$

Hence $u \in C^1([0,\infty))$, and further

$$u'(r) = -r^{-\frac{\alpha}{\beta+1}} \Big| \int_0^r \lambda t^\gamma f(u(t))dt \Big|^{-\frac{\beta}{\beta+1}} \int_0^r \lambda t^\gamma f(u(t))dt. \tag{2.3}$$

We infer from (2.3) that $u \in C^2$ at the points $r > 0$ where $u'(r) \neq 0$, and from (2.1),

$$-\alpha r^{\alpha-1}|u'(r)|^\beta u'(r) - r^\alpha(\beta+1)|u'(r)|^\beta u''(r) = \lambda r^\gamma f(u(r)),$$

and further

$$-\alpha r^{\alpha-1}|u'(r)|^{\beta+2} = r^\alpha(\beta+1)|u'(r)|^\beta u'(r)u''(r) + \lambda r^\gamma f(u(r))u'(r) \tag{2.4}$$

at those points. Consider the "energy functional" $E : [0,\infty) \to \mathbf{R}$ defined by

$$E(0) = \lambda F(d) \text{ and } E(r) = \frac{\beta+1}{\beta+2} r^{\alpha-\gamma}|u'(r)|^{\beta+2} + \lambda F(u(r)),\ r > 0.$$

We have that

$$E'(r) = (\alpha-\gamma) r^{\alpha-\gamma-1}\frac{\beta+1}{\beta+2}|u'(r)|^{\beta+2} + (\beta+1) r^{\alpha-\gamma}\ |u'(r)|^{\beta} u'(r) u''(r) \\ + \lambda f(u(r)) u'(r),$$

and using (2.4), we obtain

$$\begin{aligned} E'(r) &= \tfrac{(\alpha-\gamma)\cdot(\beta+1)}{\beta+2} r^{\alpha-\gamma-1}|u'(r)|^{\beta+2} - r^{-\gamma}\alpha r^{\alpha-1}|u'(r)|^{\beta+2} \\ &= -\tfrac{\alpha+\gamma(\beta+1)}{\beta+2} r^{\alpha-\gamma-1}|u'(r)|^{\beta+2} \end{aligned} \tag{2.5}$$

at the points $r > 0$ where $u'(r) \neq 0$. On the other hand, from (2.2) we get

$$|u'(r)|^{\beta+2} \leq \left(\lambda \frac{f(d)}{\gamma+1}\right)^{\frac{\beta+2}{\beta+1}} r^{\frac{(\gamma+1-\alpha)(\beta+2)}{\beta+1}}. \tag{2.6}$$

Now using (2.6) we find that

$$\lim_{r\to 0} E(r) = \lambda F(d)$$

so that $E \in C([0,\infty))$. By (2.5) we infer that $E(r) \leq \lambda F(d)$, and consequently (1.14) and (1.15) hold true.

3. Proof of Theorem 1.3

Proposition 1.4 shall largely be used here. We need two steps. In the first one we find local solutions, and in the second one we find global solutions of $(*)_{\lambda,d}$. So, let $d \geq 0$, $\lambda > 0$ and $\epsilon > 0$, which will be taken small, and consider the following family of problems:

$$(*)_{\lambda,d,\epsilon} \quad \begin{cases} -\left(r^{\alpha}|u'(r)|^{\beta}u'(r)\right)' = \lambda r^{\gamma} f(u(r)),\ 0 < r < \epsilon \\ u(0) = d,\ u'(0) = 0. \end{cases}$$

Step 1. $(*)_{\lambda,d,\epsilon}$ has a unique solution $u(\cdot) \equiv u(\cdot,d,\lambda,\epsilon) \in C^2([0,\epsilon])$, ϵ small enough.

Indeed, when $d = 0$, it is easily shown with the aid of (1.14) that $u \equiv 0$ is the only solution. So let $d > 0$, and take $\rho \in (0,d)$ and $\epsilon > 0$. The set

$$K^{\epsilon}_{\rho}(d) \equiv \{u \in C([0,\epsilon])\ :\ u(0) = d,\ \|u-d\|_{\infty} \leq \rho\}$$

is closed in $C([0,\epsilon]) \equiv (C([0,\epsilon]), \|\cdot\|_\infty)$ and is, as a matter of fact, a complete metric space. Now, if ρ and ϵ are small and $u \in K_\rho^\epsilon(d)$, it follows that $u(r) > 0,\ 0 \le r \le \epsilon$, and the solutions of $(*)_{\lambda,d,\epsilon}$ are the fixed points of the operator

$$Tu(r) = d - \int_0^r t^{\frac{-\alpha}{\beta+1}} \left(\int_0^t \lambda \tau^\gamma f(u(\tau)) d\tau \right)^{\frac{1}{\beta+1}} dt,\ 0 \le r \le \epsilon.$$

We will show in the Appendix that there are $\epsilon, \rho > 0$ and $k \in (0,1)$ such that

$$T\left(K_\rho^\epsilon(d)\right) \subset K_\rho^\epsilon(d) \tag{3.1}$$

$$\|Tu - Tv\|_\infty \le k\|u - v\|_\infty,\ u, v \in K_\rho^\epsilon(d). \tag{3.2}$$

Using (3.1) and (3.2) we infer by the Banach fixed-point theorem that T has a unique fixed point which is thus the unique C^2-solution of $(*)_{\lambda,d,\epsilon}$.

Step 2. $(*)_{\lambda,d}$ has exactly one solution $u \in C^1([0,\infty))$.

Indeed, when $d = 0$ we infer with the help of Proposition 1.4 that the only solution is $u = 0$. So, let $d > 0$. At first we show that $(*)_{\lambda,d}$ has at most one solution $u \in C^1([0,\infty))$. To this end, assume u_1, u_2 are two C^1 solutions. Let $r > 0$ and define

$$S_0 \equiv \left\{ r \ge 0\ :\ u_1(t) = u_2(t) \text{ and } u'_1(t) = u'_2(t),\ 0 \le t \le r \right\}.$$

We claim that

$$S_0 \ne \emptyset \text{ and } S_0 \text{ is both open and closed in } [0,\infty). \tag{3.3}$$

Let also

$$S_\infty \equiv \{ r > 0\ :\ (*)_{\lambda,d} \text{ has a solution in } [0,r) \} \text{ and } T_\infty \equiv \sup S_\infty.$$

We claim that

$$T_\infty = \infty. \tag{3.4}$$

Assuming (3.3) and (3.4) we infer that $(*)_{\lambda,d}$ has a unique solution. This proves Theorem 1.3 up to the verification of (3.1)–(3.4). We leave the verifications of those items to the Appendix since they are quite technical and in some respects fairly standard.

4. Proof of Theorem 1.2

Here, we shall use Proposition 1.4 and Theorem 1.3. Let $u(\cdot, d, \lambda) \equiv u(\cdot, d) \equiv u(\cdot) \in C^1([0,\infty))$ be the unique (classical) solution of $(*)_{\lambda,d}$ as asserted in Theorem 1.3. We remark that u satisfies

$$r^\alpha |u'(r)|^\beta u'(r) = -\int_0^r \lambda t^\gamma f(u(t)) dt. \tag{4.1}$$

Verification of (1.10)**.** Take $d_n \to d_0$. We will show that $|u_n - u_0|_{L^\infty[0,T]} \to 0$, where $u_n(r) \equiv u(r, d_n)$ and $u_0(r) \equiv u(r, d_0)$. Indeed, we have by (1.14) that $|u_n{}'(r)| \le C$, $0 \le r \le T$, and by the mean value theorem, $|u_n(r)| \le C$ for some constant $C > 0$ and for all $n \ge 1$. Applying the Arzéla–Áscoli theorem, there is a subsequence, still denoted u_n, such that $u_n \to v$ uniformly in $[0, T]$ for some $v \in C[0, T]$. We will show next that $v \equiv u_0$ on $[0, T]$. Indeed, at first we have $t^\gamma f(u_n(t)) \to t^\gamma f(v(t))$ and $|t^\gamma f(u_n(t))| \le Ct^\gamma$ for all $0 \le t \le T$ and some $C > 0$. Using Lebesgue's Theorem, (4.1) and the above, we get the following:

$$\int_0^r \lambda t^\gamma f(u_n(t))dt \to \int_0^r \lambda t^\gamma f(v(t))dt,$$

$$r^\alpha |u_n{}'(r)|^\beta u_n{}'(r) \to -\int_0^r \lambda t^\gamma f(v(t))dt,\ 0 \le r \le T.$$

As a matter of fact, $u_n{}'(r) \to \omega(r)$, pointwise, for some function ω, and, on the other hand, from $\displaystyle\int_0^r u_n{}'(t)dt = u_n(r) - d_n$ we infer that $\omega(r) = v^{'}(r)$ and $v^{'}(0) = 0$. Hence

$$|v'(r)|^\beta v^{'}(r) = -r^{-\alpha} \int_0^r \lambda t^\gamma f(v(t))dt,$$

and since by the above also $v(0) = d_0$ it follows by uniqueness of solution given by Theorem 1.3 that $v = u_0$. This shows (1.10).

Verification of (1.11)**.** Letting $0 < a \le r \le b < \infty$ and taking $d_n \to d_0$ we have

$$r^\alpha \left| |u_n{}'(r)|^\beta u_n{}'(r) - |u_0{}'(r)|^\beta u_0{}'(r) \right| \le \int_a^b \lambda t^\gamma |f(u_n(t)) - f(u_0(t))| dt,$$

where the expression on the right tends to 0 by Lebesgue's theorem again. Hence,

$$\Big(|u_n{}'(r)|^\beta u_n{}'(r) - |u_0{}'(r)|^\beta u_0{}'(r)\Big) \cdot \big(u_n{}'(r) - u_0{}'(r)\big) \to 0 \text{ uniformly in } [a, b],$$

and applying the following inequality (see Simon [7] or Tolksdorff [9]),

$$(|x|^\beta x - |y|^\beta y)(x - y) \ge \begin{cases} c\ |x - y|^{\beta+2} \text{ if } \beta \ge 0 \\ c\ \frac{|x-y|^2}{(1+|x|+|y|)^{-\beta}} \text{ if } -1 < \beta \le 0, \end{cases}$$

where $c > 0$ is some constant, we conclude that $|u_n{}' - u_0{}'|_{L^\infty[a,b]} \to 0$ as $n \to \infty$.

Verification of (1.12)**.** We shall construct step by step a sequence $\{z_\ell\}$ with the asserted properties. In the first step we will show the following:

There is $z_1 \equiv z_1(d) > 0$ such that $u(z_1) = 0$, $u'(z_1) < 0$ and both

$$u(r) > 0, \text{ and } u^{'}(r) < 0, \; 0 < r < z_1. \tag{4.2}$$

Assume, to the contrary, that $u(r) > 0$ for all $r > 0$. By (4.1), $u'(r) < 0$, and

$$|u'(r)|^{\beta+1} \geq \lambda \frac{r^{\gamma-\alpha+1}}{\gamma+1} f(u(r)).$$

Integrating from 0 to r we obtain

$$-\int_0^r u'(\tau) f(u(\tau))^{\frac{-1}{\beta+1}} d\tau \geq \left(\frac{\lambda}{\gamma+1}\right)^{\frac{1}{\beta+1}} \int_0^r \tau^{\frac{\gamma-\alpha+1}{\beta+1}} d\tau,$$

which yields, by the change of variables $t = u(\tau)$,

$$\int_{u(r)}^d f(t)^{\frac{-1}{\beta+1}} dt \geq (\frac{\lambda}{\gamma+1})^{\frac{1}{\beta+1}} \left(\frac{\beta+1}{\gamma+\beta-\alpha+2}\right) r^{\left(\frac{\gamma+\beta-\alpha+2}{\beta+1}\right)}.$$

Now, taking $\underline{\lim}_{r\to\infty}$ on both sides above, and using $(h1)$ and $(h2)$, which give in particular that $\gamma+\beta-\alpha+2 > 0$, we arrive at a contradiction to $(h4)$. We label $z_1(d) \equiv z_1(d, \lambda)$ the first zero of $u(r)$. Of course, $u(r) > 0$, for $0 < r < z_1(d)$. Now, we show that there is $\Lambda > 0$ such that $z_1(d_\infty) \geq R$ provided $0 < \lambda \leq \Lambda$. For this, let $\nu \in (0,1)$ and label $r_\infty(\nu)$ the point in $(0, z_1(d_\infty))$ such that $u(r_\infty(\nu), d_\infty) = \nu d_\infty$. Now, define $\Lambda > 0$ by

$$\Lambda = \frac{(\gamma+1) d_\infty^{\beta+1}}{f(d_\infty)} \left(\frac{\gamma+\beta-\alpha+2}{\beta+1}\right)^{\beta+1} \frac{1}{2R^{\gamma+\beta-\alpha+2}} \tag{4.3}$$

and choose $\nu \in (0,1)$ such that

$$(1-\nu)^{\beta+1} = \frac{f(d_\infty)\ \Lambda\ R^{\gamma+\beta-\alpha+2}}{(\gamma+1) d_\infty^{\beta+1}} \left(\frac{\beta+1}{\gamma+\beta-\alpha+2}\right)^{\beta+1}. \tag{4.4}$$

From (4.1) we have

$$(-u'(r))^{\beta+1} \leq r^{-\alpha} \int_0^r t^\gamma \lambda f(d_\infty) dt, \; 0 \leq r \leq r_\infty(\nu)$$

so that

$$-u'(r) \leq \left(\frac{\lambda f(d_\infty)}{\gamma+1}\right)^{\frac{1}{\beta+1}} r^{\frac{\gamma-\alpha+1}{\beta+1}}. \tag{4.5}$$

Integrating from 0 to $r_\infty(\nu)$ in (4.5) we get

$$(1-\nu)^{\beta+1} \leq \frac{\lambda f(d_\infty)}{(\gamma+1) d_\infty^{\beta+1}} \left(\frac{\beta+1}{\gamma+\beta-\alpha+2}\right)^{\beta+1} r_\infty(\nu)^{\gamma+\beta-\alpha+2},$$

and finally, combining this with (4.4), we obtain $R \leq r_\infty(\nu) \leq z_1(d_\infty)$ provided $0 < \lambda \leq \Lambda$. So (4.2) is proved. In the second step we shall prove the following:

There is $z_2 \equiv z_2(d) > z_1$ such that $u(z_2) = 0$, $u^{'}(z_2) > 0$ and

$$u(r) < 0, \ z_1 < r < z_2. \tag{4.6}$$

Indeed, since $u^{'}(z_1) < 0$, $u^{'}(r) < 0$ for all $r > z_1$, r close to z_1. Now we claim that

$$u^{'}(m_1) = 0 \text{ for some } m_1 > z_1.$$

Indeed, otherwise, $u^{'}(r) < 0$, for $r \geq z_1$, and by (1.15), $F(u(r)) \leq F(d)$; that is,

$$\int_0^{u(r)} f(t)dt \leq F(d), \ r \geq 0.$$

We contend that $u(r) \geq -C$ for some $C > 0$ and for all $r \geq z_1$ because, if not, $u(r_n) \to -\infty$ for some $r_n \nearrow \infty$, and so

$$\int_{-\infty}^0 f(s)ds = \lim_n \int_{u(r_n)}^0 f(s)ds \geq -F(d).$$

But this is impossible since, by ($h2$)–($h3$), $\displaystyle\int_{-\infty}^0 f(s)ds = -\infty$. Thus, we have

$$u(r) \geq -C, \ u'(r) < 0, \text{ for } r \geq z_1,$$

and consequently $u(r) \to L$ as $r \to \infty,$ for some $L < 0$. Now, by (1.14),

$$\frac{|u'(r)|^{\beta+1}}{r^{\gamma-\alpha+1}} \to 0 \text{ as } r \to \infty.$$

This, together with (4.1) and ($h3$), produces a contradiction, namely,

$$\frac{1}{r^{\gamma+1}} \int_0^r t^\gamma f(u(t))dt \to 0 \text{ and } \frac{1}{r^{\gamma+1}} \int_0^r t^\gamma f(u(t))dt \to \frac{f(L)}{\gamma+1} < 0 \text{ as } r \to \infty.$$

Therefore, $u'(m_1) = 0$ for some $m_1 > z_1$, so that

$$u(r) < 0 \text{ for } z_1 < r < m_1 \text{ and } u'(r) < 0 \text{ for } z_1 \leq r < m_1.$$

Now, taking $r > m_1$, r close to m_1, and using (4.1), we get $\int_{m_1}^r t^\gamma f(u(t))dt < 0$, and so

$$u(r) < 0, \ u'(r) > 0 \text{ for all } r > m_1, \ r \text{ close to } m_1.$$

Assume for the sake of contradiction that $u(r) < 0$ for all $r > m_1$, so that $u'(r) > 0$. Since

$$-r^{\alpha}|u'(r)|^{\beta}u'(r) = \lambda \int_{m_1}^{r} t^{\gamma} f(u(t))dt \le \frac{\lambda f(u(r))}{\gamma+1}(r^{\gamma+1} - {m_1}^{\gamma+1}),$$

we get by taking $r > \bar{r} \equiv 2^{\frac{1}{\gamma+1}} m_1$ above that $r^{\gamma+1} - m_1^{\gamma+1} > \frac{r^{\gamma+1}}{2}$, and so

$$-r^{\alpha}|u'(r)|^{\beta}u'(r) \le \frac{\lambda f(u(r))}{2(\gamma+1)} r^{\gamma+1},$$

which gives

$$u'(r) \ge (-\frac{\lambda f(u(r))}{2(\gamma+1)})^{\frac{1}{\beta+1}} r^{\frac{\gamma-\alpha+1}{\beta+1}}, \ r > \bar{r}. \tag{4.7}$$

Integrating in (4.7) from $\bar{r}$ to r, we have

$$\int_{\bar{r}}^{r} u'(t)(-f(u(t)))^{-\frac{1}{\beta+1}}\, dt \ge (\frac{\lambda}{2(\gamma+1)})^{\frac{1}{\beta+1}} \int_{\bar{r}}^{r} t^{\frac{\gamma-\alpha+1}{\beta+1}} dt,$$

and, making the change of variables $y = u(t)$,

$$\int_{u(\bar{r})}^{u(r)} (-f(y))^{-\frac{1}{\beta+1}}\, dy \ge (\frac{\lambda}{2(\gamma+1)})^{\frac{1}{\beta+1}} (r^{\frac{\gamma+\beta-\alpha+2}{\beta+1}} - \bar{r}^{\frac{\gamma+\beta-\alpha+2}{\beta+1}}). \tag{4.8}$$

From (4.8) we get, since by $(h1)$ we have both $\gamma+\beta-\alpha+2 > 0$ and $\beta+1 > 0$, that

$$\underline{\lim}_{r\to\infty} \int_{u(\bar{r})}^{u(r)} (-f(y))^{-\frac{1}{\beta+1}}\, dy = \infty,$$

contradicting $(h4)$ and consequently showing (4.6).

In the third step we show the following:

There is $z_3 = z_3(d) > z_2$ such that $u(z_3) = 0$, $u'(z_3) < 0$ and

$$u(r) > 0 \text{ for all } r \in (z_2, z_3). \tag{4.9}$$

Indeed, since $u'(z_2) > 0$ (see claim (4.6) proved above), we infer that

$$u'(r) > 0 \text{ for all } r > z_2, \ r \text{ close to } z_2.$$

At first we claim that there is $m_2 > z_2$ such that $u'(m_2) = 0$. Indeed, otherwise, $u'(r) > 0,$ for all $r > z_2$, which gives that $u(r) > 0,$ for all $r > z_2$. By (1.15),

$$\int_0^{u(r)} f(t)dt \le \int_0^d f(t)dt$$

so that $u(r) \le d$ for all $r \ge 0$. Hence there is $L \in (0, d]$ such that

$$u(r) \to L \text{ and } u(r) \le L, \ r \ge z_2.$$

As earlier, we have both

$$\frac{|u'(r)|^{\beta+1}}{r^{\gamma-\alpha+1}} = \frac{\lambda}{r^{\gamma+1}} \int_0^r t^\gamma f(u(t))dt \to \frac{\lambda f(L)}{\gamma+1}$$

and

$$\frac{|u'(r)|^{\beta+1}}{r^{\gamma-\alpha+1}} = \frac{\lambda}{r^{\gamma+1}} \int_0^r t^\gamma f(u(t))dt \to 0$$

as $r \to \infty$, which is impossible. As a consequence, there is $m_2 > z_2$ such that $u'(m_2) = 0$, and $u'(r) > 0$, $z_2 \le r < m_2$, proving our claim. Assume again, for the sake of contradiction, that $u(r) > 0$ for all $r > m_2$ so that $u'(r) < 0$ also for all $r > m_2$. We have

$$-r^\alpha |u'(r)|^\beta u'(r) = \lambda \int_{m_2}^r t^\gamma f(u(t))dt \ge \frac{\lambda f(u(r))}{\gamma+1}(r^{\gamma+1} - m_2^{\gamma+1}).$$

Setting $\bar{r} = 2^{\frac{1}{\gamma+1}} m_2$ and taking $r > \bar{r}$,

$$|u'(r)|^{\beta+1} = r^{-\alpha} \int_{m_2}^r \lambda t^\gamma f(u(t))dt \ge \frac{\lambda f(u(r))}{2(\gamma+1)} r^{\gamma-\alpha+1}.$$

As before, integrating and using the change of variables $y = u(t)$ we get

$$\int_{u(r)}^{u(\bar{r})} (f(y))^{-\frac{1}{\beta+1}} dy \ge (\frac{\lambda}{2(\gamma+1)})^{\frac{1}{\beta+1}} \frac{\beta+1}{\gamma+\beta-\alpha+2} (r^{\frac{\gamma+\beta-\alpha+2}{\beta+1}} - \bar{r}^{\frac{\gamma+\beta-\alpha+2}{\beta+1}}).$$

Taking $\underline{\lim}_{r\to\infty}$ on both sides above we arrive at a contradiction to $(h4)$, finishing the proof of (4.9).

To end the proof we argue as in (4.6) and (4.9) to get zeroes z_4 and z_5 and, inductively, a sequence $\{z_\ell\}$ with the properties asserted in (1.12).

Verification of $(1.12)_i$**.** We will show first that $z_1(d) \to 0$ when $d \to 0$. Using both (4.2) and $(h3)$ in (4.1) there is some constant $c_1(\lambda) > 0$ such that

$$-u'(r)(f(u(r)))^{-\frac{1}{\beta+1}} \ge c_1(\lambda) r^{\frac{\gamma+1-\alpha}{\beta+1}}.$$

Now, at first, integrating from 0 to r and secondly, making the change of variables $y = u(t)$, we find some constant $c_2(\lambda) > 0$ such that

$$\int_{u(r)}^d (f(y))^{-\frac{1}{\beta+1}} dy \ge c_2(\lambda) r^{\frac{\gamma+\beta+2-\alpha}{\beta+1}}.$$

Making $r = z_1(d)$, we find

$$c_2(\lambda) z_1(d)^{\frac{\gamma+\beta+2-\alpha}{\beta+1}} \le \int_0^d (f(y))^{-\frac{1}{\beta+1}} dy,$$

and using $(h3)$ and $(h4)$ we infer that $z_1(d) \to 0$ as $d \to 0$.

Now, letting $\ell \geq 1$, we assume that $u(r) > 0$ in $(z_\ell(d), z_{\ell+1}(d))$, so that by the notation of (4.6) and (4.9) we have $u'(r) > 0$ in $(z_\ell(d), m_\ell(d))$ and $u'(r) < 0$ in $(m_\ell(d), z_{\ell+1}(d))$ (the case $u(r) < 0$ in $(z_\ell(d), z_{\ell+1}(d))$ is handled similarly). Now, using $(h3)$ in (4.1) and integrating from $m_\ell(d)$ to r we obtain successively

$$-r^\alpha |u'(r)|^\beta u'(r) \geq \lambda f(u(r)) \frac{(r^{\gamma+1} - m_\ell(d)^{\gamma+1})}{\gamma+1},$$

$$-u'(r)\Big(f(u(r))\Big)^{-\frac{1}{\beta+1}} \geq \Big(\lambda \frac{r^{\gamma+1} - m_\ell(d)^{\gamma+1}}{(\gamma+1)r^\alpha}\Big)^{\frac{1}{\beta+1}}.$$

Now, $r^{\gamma-\alpha} \geq m_\ell(d)^{\gamma-\alpha}$ since $\gamma \geq \alpha$, and consequently

$$r^{\gamma+1-\alpha} - r^{-\alpha} m_\ell(d)^{\gamma+1} \geq m_\ell(d)^{\gamma-\alpha}(r - m_\ell(d)),$$

which gives

$$-u'(r)(f(u(r)))^{-\frac{1}{\beta+1}} \geq \Big(\frac{\lambda m_\ell(d)^{\gamma-\alpha}}{\gamma+1}\Big)^{\frac{1}{\beta+1}} \Big(r - m_\ell(d)\Big)^{\frac{1}{\beta+1}}.$$

Integrating from $m_\ell(d)$ to $z_{\ell+1}(d)$ and making the change of variables $y = u(t)$, we get some constant $c_3(\lambda) > 0$ such that

$$c_3(\lambda) m_\ell(d)^{\frac{\gamma-\alpha}{\beta+1}} (z_{\ell+1}(d) - m_\ell(d))^{\frac{\beta+2}{\beta+1}} \leq \int_0^d (f(y))^{-\frac{1}{\beta+1}} dy. \tag{4.10}$$

Assume now $z_\ell(d) < r < m_\ell(d)$. Then by a similar argument, this time integrating from $z_\ell(d)$ to $m_\ell(d)$, we get some constant $c_4(\lambda) > 0$ such that

$$c_4(\lambda) m_\ell(d)^{\frac{\gamma-\alpha}{\beta+1}} (m_\ell(d) - z_\ell(d))^{\frac{\beta+2}{\beta+1}} \leq \int_0^d \Big(f(y)\Big)^{-\frac{1}{\beta+1}} dy. \tag{4.11}$$

Now since by $(h4)$, $\int_0^d (f(y))^{-\frac{1}{\beta+1}} dy \to 0$ when $d \to 0$, we infer by passing to the limit in both (4.10) and (4.11) as $d \to 0$, and using the inequality $z_\ell(d) < m_\ell(d) < z_{\ell+1}(d)$, that $\lim_{d\to 0} z_\ell(d) = \lim_{d \to 0} z_{\ell+1}(d)$ for every $\ell \geq 1$. Since, as proved earlier, $z_1(d) \to 0$ as $d \to 0$ it follows that $z_\ell(d) \to 0$ as $d \to 0$.

Verification of $(1.12)_{ii}$**.** We argue by induction over ℓ. Let us first show that $z_1(d) \to z_1(d_0)$ as $d \to d_0$. To do this, let $d_n \to d_0$, $u_n(\cdot) \equiv u(\cdot, d_n)$ and $u_0(\cdot) \equiv u(\cdot, d_0)$ so that we have by (1.10) that $u_n(r) \to u_0(r)$ uniformly in compact subsets of $(0, \infty)$. For each $\epsilon > 0$ small we find

$$u_0(r) > 0,\ 0 \leq r \leq z_1(d_0) - \epsilon \text{ and } u_0(z_1(d_0) + \epsilon) < 0$$

so that

$$u_n(r) > 0,\ 0 \le r \le z_1(d_0) - \epsilon \text{ and } u_n(z_1(d_0) + \epsilon) < 0$$

for sufficiently large n. As a consequence, $z_1(d_0) - \epsilon < z_1(d_n) < z_1(d_0) + \epsilon$, showing that $z_1(d_n) \to z_1(d_0)$. Assume by induction that $z_\ell(d_n) \to z_\ell(d_0)$ for some $\ell > 1$. We will show that $z_{\ell+1}(d_n) \to z_{\ell+1}(d_0)$. For that matter, we assume $u_0(t) < 0$ for $z_\ell(d_0) < t < z_{\ell+1}(d_0)$. (The other case is handled similarly.) Taking again $\epsilon > 0$ small we find that $u_n(t) < 0$ for $z_\ell(d_0) + \epsilon \le t \le z_{\ell+1}(d_0) - \epsilon$ and $u_n(z_{\ell+1}(d_0) + \epsilon) > 0$, showing that $z_{\ell+1}(d_0) - \epsilon < z_{\ell+1}(d_n) < z_{\ell+1}(d_0) + \epsilon$. Consequently, $z_{\ell+1}(d) \to z_{\ell+1}(d_0)$ as $d \to d_0$.

Verification of (1.13)**.** Let $d \in (0, d_0)$. It suffices to show that $z_{\ell+2}(d) > R$ whenever d is close enough to d_0. We assume that $u(r, d_0) < 0$ for $r \in (z_\ell(d_0), z_{\ell+1}(d_0))$ (the other case is handled similarly).

Notice that, by the construction of the zeroes in (1.12), $z_{\ell+1}(d_0) \ge R$ and $z_{\ell+2}(d_0) > R$, since u_0 has ℓ zeroes in $(0, R)$ and the sequence $z_\ell(d)$ is increasing in ℓ, for each $d \in (0, d_\infty]$. But, by $(1.12)_{ii}$, $z_{\ell+2}(d) \to z_{\ell+2}(d_0)$ as $d \to d_0$, so that $z_{\ell+2}(d) > R$ whenever d is close enough to d_0. This completes the proof of Theorem 1.3

5. Proof of Theorem 1.1

At first let $\lambda \in (0, \Lambda]$, where $\Lambda > 0$ was introduced in Theorem 1.2. The proof will require some steps, and we will skip the λ-dependence on u.

Step 1. Letting $A_0 \equiv \{d \in (0, d_\infty] \ :\ z_1(d) \ge R\}$ and $d_0 \equiv \inf A_0$, and remarking that by (1.12), $z_1(d_\infty) \ge R$ so that $A_0 \ne \emptyset$, we claim that

$$0 < d_0 \text{ and } z_1(d_0) = R. \tag{5.1}$$

Verification of (5.1)**.** Indeed, if on the contrary, $d_0 = 0$, there is a sequence $d^j \in A_0$ such that $d^j \searrow 0$, and by $(1.12)_i$, $z_1(d^j) \to 0$, which is a contradiction. On the other hand, assuming to the contrary, that $z_1(d_0) > R$, there is a sequence $d^j \nearrow d_0$, and by $(1.12)_{ii}$, $z_1(d^j) \to z_1(d_0)$. Hence $z_1(d^{j_0}) > R$ so that $d^{j_0} \in A_0$, contradicting $d^j < d_0$. This proves (5.1).

Step 2. Setting, for any integer $\ell \ge 1$, $A_\ell \equiv \{d \in (0, d_\infty] : z_\ell(d) < R,\ z_{\ell+1}(d) \ge R\}$ and $d_\ell \equiv \inf A_\ell$ we claim that

$$(c)_\ell \qquad \begin{cases} 0 < d_\ell \cdots < d_1 < d_0 \le d_\infty, \\ A_\ell \ne \emptyset,\ z_\ell(d_\ell) < R,\ z_{\ell+1}(d_\ell) = R. \end{cases}$$

Assuming $(c)_\ell$ holds for all $\ell \ge 1$, it follows by also using Step 1 that the functions given by Theorem 1.2, namely $u_\ell(\cdot) \equiv u(\cdot, d_\ell) \in C^1([0, \infty)),\ \ell \ge 0$, satisfy

$$r^\alpha |u_\ell{}'(r)|^\beta u_\ell{}'(r) \text{ is differentiable,}$$

$$-(r^\alpha|u_\ell{}'(r)|^\beta u_\ell{}'(r))' = \lambda r^\gamma f(u_\ell(r)),\ 0<r<R,$$
$$u_\ell{}'(0) = u'(0,d_\ell) = 0 \text{ and } u_\ell(R) = u(z_{\ell+1},d_\ell) = 0,$$

that is, u_ℓ is a classical solution of $(*)_\lambda$ and u_ℓ has precisely ℓ zeroes in $(0,R)$, and so $\{u_\ell\}_{\ell=0}^\infty$ is an infinite sequence of solutions of $(*)_\lambda$ as asserted in Theorem 1.1.

Verification of the claim in Step 2. We show, at first, that

$$A_1 \neq \emptyset,\ 0<d_1<d_0\le d_\infty,\ z_1(d_1)<R \text{ and } z_2(d_1)=R.$$

Indeed, since $z_1(d_0)=R$ we have by (1.13) that, if $d<d_0$ and d is sufficiently close to d_0, then $u(r,d)$ has at most one zero in $(0,R)$. Assume, for the sake of contradiction, it has no zero in $(0,R)$. Then $z_1(d)\ge R$ with $d<d_0$, which is impossible. Hence $u(r,d)$ has precisely one zero in $(0,R)$, and so $d\in A_1$, showing that $A_1\neq\emptyset$. Now, recalling that $d\in A_1$ if and only if both $z_1(d)<R$ and $z_2(d)\ge R$, it follows that $d_1<d_0$. Next, $d_1>0$, because otherwise one would find a sequence $d^j\in A_1$ with $d^j\searrow d_1\equiv 0$ so that by (1.12) $z_2(d^j)\to z_2(d_1)=0$, contradicting $z_2(d^j)\ge R$. Thus $0<d_1<d_0\le d_\infty$.

Next, we will show that $z_1(d_1)<R$ and $z_2(d_1)=R$, which gives, in particular, that $d_1\in A_1$. To show that take $d^j\in A_1$, $d^j\searrow d_1$. Then $z_1(d^j)\to z_1(d_1)\le R$, and $z_2(d^j)\to z_2(d_1)\ge R$. Now, assume for the sake of contradiction that $z_1(d_1)=R$; that is, $u(r,d_1)$ has no zeroes in $(0,R)$. By (1.13) if $d<d_1$ and d is near to d_1, $u(r,d)$ has at most one zero in $(0,R)$. In fact, if there is no zero in $(0,R)$ then $d\in A_1$, so that $d\ge d_0>d_1$ which is impossible. Hence $z_1(d_1)<R$. On the other hand, assume for the sake of contradiction that $z_2(d_1)>R$. Taking $d^j\nearrow d_1$ we have $z_1(d^j)\to z_1(d_1)<R$ and $z_2(d^j)\to z_2(d_1)>R$, so that $z_1(d^{j_0})<R$ and $z_2(d^{j_0})>R$ for some j_0, which is impossible. Thus $z_2(d_1)=R$.

Assume by induction $(c)_\ell$ holds for $\ell>1$. We will show that $(c)_{\ell+1}$ holds true; that is,

$$(c)_{\ell+1}\qquad \begin{cases} 0<d_{\ell+1}<d_\ell<\cdots<\cdots<d_1<d_0\le d_\infty \\ A_{\ell+1}\neq\emptyset,\ z_{\ell+1}(d_{\ell+1})<R,\ z_{\ell+2}(d_{\ell+1})=R. \end{cases}$$

Let us show first that $A_{\ell+1}\neq\emptyset$. Indeed, we have that $d_\ell\in A_\ell$, and so $u(r,d_\ell)$ has precisely ℓ zeroes in $(0,R)$. By 1.13, $u(r,d)$ has at most $\ell+1$ zeroes in $(0,R)$ provided $d<d_\ell$ and d is sufficiently close to d_ℓ. Assuming that $u(r,d)$ has only $m\le\ell$ zeroes in $(0,R)$ we infer that $d\in A_m$, and so $d\ge d_m\ge d_\ell$, which is impossible. Hence $u(r,d)$ has precisely $\ell+1$ zeroes in $(0,R)$ so that $d\in A_{\ell+1}$, showing that $A_{\ell+1}\neq\emptyset$. Now $d_{\ell+1}\equiv\inf A_{\ell+1}>0$, because, if not, take $d^j\in A_{\ell+1}$ such that $d^j\searrow d_{\ell+1}=0$ so that $z_{\ell+1}(d^j)\to 0$, contradicting $z_{\ell+1}(d^j)\ge R$. Thus $0<d_{\ell+1}<d_\ell<\cdots<d_1<d_0\le d_\infty$.

In the next step, we show that $z_{\ell+1}(d_{\ell+1}) < R$, and $z_{\ell+2}(d_{\ell+1}) = R$. Indeed, taking $d^j \nearrow d_{\ell+1}$ we have both $z_{\ell+1}(d_{\ell+1}) \leq R$ and $z_{\ell+2}(d_{\ell+1}) \geq R$. Indeed, if, on the contrary, $z_{\ell+1}(d_{\ell+1}) = R$, it follows that $u(r, d_{\ell+1})$ has precisely ℓ zeroes in $(0, R)$, and by (1.13) $u(r, d)$ has at most $\ell + 1$ zeroes in $(0, R)$ provided $d < d_{\ell+1}$ and d is close to $d_{\ell+1}$. If it has only $m \leq \ell$ zeroes in $(0, R)$ it follows that $d \in A_m$ showing that $d \geq d_m \geq d_\ell > d_{\ell+1}$, a contradiction. Hence $z_{\ell+1}(d_{\ell+1}) < R$. Again, if $z_{\ell+2}(d_{\ell+1}) > R$ take $d^j \nearrow d_{\ell+1}$ so that $z_{\ell+1}(d^j) \to z_{\ell+1}(d_{\ell+1}) < R$, and $z_{\ell+2}(d^j) \to z_{\ell+2}(d_{\ell+1}) > R$. Hence, there is some $d^{j_0} \in A_{\ell+1}$, which is impossible, ending the proof of Theorem 1.1 because, finally, $z_{\ell+1}(d_{\ell+1}) < R$ and $z_{\ell+2}(d_{\ell+1}) = R$.

6. Appendix

In this section we prove assertions (3.1)–(3.4) used in the proof of Theorem 1.3.

Verification of (3.1)**.** Let $\rho \in (0, \frac{d}{2}]$ so that $\frac{d}{2} \leq u(r) \leq 2d$. If $u \in K^\epsilon_\rho(d)$ we have

$$|T(u(r)) - T(u(0))| = \int_0^r s^{\frac{-\alpha}{\beta+1}} \Big(\int_0^s \lambda t^\gamma f(u(t)) dt \Big)^{\frac{1}{\beta+1}} ds, \ 0 < r \leq \epsilon.$$

Let

$$g(s) \equiv s^{\frac{-\alpha}{\beta+1}} \Big(\int_0^s \lambda t^\gamma f(u(t)) dt \Big)^{\frac{1}{\beta+1}}, \ 0 < s \leq \epsilon \text{ and } c_1 = \max_{s \in [\frac{d}{2}, 2d]} f(s)$$

so that $g(s) \leq (\lambda \frac{c_1}{\gamma+1})^{\frac{1}{\beta+1}} s^{\frac{\gamma+1-\alpha}{\beta+1}}$, and further

$$|T(u(r)) - T(u(0))| \leq \frac{\beta+1}{\gamma+\beta-\alpha+2} \big(\lambda \frac{c_1}{\gamma+1}\big)^{\frac{1}{\beta+1}} r^{\frac{\gamma+\beta-\alpha+2}{\beta+1}}, \ 0 < r \leq \epsilon.$$

Hence $Tu \in C([0, \epsilon])$. Moreover, taking $\epsilon > 0$ sufficiently small we infer (3.1).

Verification of (3.2)**.** Take $u, v \in K^\epsilon_\rho(d)$. There is $h \in (0,1)$ such that

$$\begin{aligned} & T(v(r)) - T(u(r)) \\ & = \int_0^r s^{\frac{-\alpha}{\beta+1}} \Big(\Big(\lambda \int_0^s t^\gamma f(u(t)) dt\Big)^{\frac{1}{\beta+1}} - \Big(\lambda \int_0^s t^\gamma f(v(t)) dt\Big)^{\frac{1}{\beta+1}} \Big) ds \\ & = \int_0^r \frac{s^{\frac{-\alpha}{\beta+1}}}{\beta+1} \Big(\int_0^s \lambda t^\gamma f(hu(t) + (1-h)v(t)) dt \Big)^{\frac{-\beta}{\beta+1}} \\ & \times \frac{d}{d\tau} \int_0^s \lambda t^\gamma f(\tau u(t) + (1-\tau)v(t)) dt|_{\tau=h} \, ds \end{aligned}$$

provided $\rho > 0$ is as in (3.1). Observing that

$$\frac{d}{d\tau}\int_0^s \lambda t^\gamma f(\tau u(t)+(1-\tau)v(t))dt$$
$$= \int_0^s \lambda t^\gamma f'(\tau u(t)+(1-\tau)v(t))(u(t)-v(t))dt,$$

and making

$$\widehat{\theta}(r)=\int_0^r \frac{s^{\frac{-\alpha}{\beta+1}}}{\beta+1}\Big[\Big(\int_0^s \lambda t^\gamma f(hu+(1-h)v)dt\Big)^{\frac{-\beta}{\beta+1}}\int_0^s \lambda t^\gamma |f'(hu+(1-h)v)|dt\Big]ds$$

we get

$$|T(u(r))-T(v(r))| \le \widehat{\theta}(r)\|u-v\|_\infty. \tag{6.1}$$

Let $c_2 \equiv \min_{s\in[\frac{d}{2},2d]} f(s)$ and $c_3 \equiv \max_{s\in[\frac{d}{2},2d]} |f'(s)|$. We find, by estimating,

$$\widehat{\theta}(r) \le \frac{\lambda^{\frac{1}{\beta+1}} c_1^{\frac{-\beta}{\beta+1}} c_3}{\gamma+2-\alpha+\beta}(\gamma+1)^{\frac{-1}{\beta+1}}\varepsilon^{\frac{\gamma+2-\alpha+\beta}{\beta+1}},\ 0<r\le\epsilon \text{ when } -1<\beta\le 0$$

and

$$\widehat{\theta}(r) \le \frac{\lambda^{\frac{1}{\beta+1}} c_2^{\frac{-\beta}{\beta+1}} c_3}{\gamma+2-\alpha+\beta}(\gamma+1)^{\frac{-1}{\beta+1}}\varepsilon^{\frac{\gamma+2-\alpha+\beta}{\beta+1}},\ 0<r\le\epsilon,\ \text{when } \beta>0.$$

Taking $\epsilon > 0$ small enough it follows that $\sup_{0\le r\le\epsilon}\widehat{\theta}(r) < 1$, and we get (3.2) from (6.1).

Verification of (3.3). $S_0 \neq \emptyset$ because $[0,\epsilon) \subset S_0$, and S_0 is closed in $[0,\infty)$ because $u_1, u_2 \in C^1([0,\infty))$. It remains to show that S_0 is open in $[0,\infty)$. To do this let $\widehat{r} \in S_0$, $\widehat{r} > 0$. We consider two cases:

Case 1. $u_1'(\widehat{r}) = u_2'(\widehat{r}) = 0$. Let $u_1(\widehat{r}) = u_2(\widehat{r}) \equiv \widehat{d}$. If $\widehat{d} = 0$ it follows using (1.14) that $u_1 = u_2 = 0$ so that $S_0 = [0,\infty)$, and consequently S_0 is open. Now, assume $\widehat{d} > 0$. Letting $\rho, \epsilon > 0$ and

$$\widehat{K}_\rho^\epsilon(\widehat{d}) \equiv \Big\{u \in C([\widehat{r},\widehat{r}+\epsilon]) : u(0)=\widehat{d},\ \|u-\widehat{d}\|_\infty \le \rho\Big\}$$

and defining $\widehat{T}$ by

$$\widehat{T}(u(r)) = \hat{d} - \int_{\widehat{r}}^r s^{\frac{-\alpha}{\beta+1}}\Big(|\lambda\int_{\widehat{r}}^s \lambda t^\gamma f(u(t))dt|^{\frac{-\beta}{\beta+1}}\int_{\widehat{r}}^s \lambda t^\gamma f(u(t))dt\Big)ds$$

it follows (see the proofs of (3.1) and (3.2)) that

$$\widehat{T}\left(\widehat{K}_\rho^\epsilon(\widehat{d})\right) \subset \widehat{K}_\rho^\epsilon(\widehat{d}) \tag{6.2}$$

$$\|\widehat{T}u - \widehat{T}v\|_\infty \le k\|u-v\|_\infty,\ u,v \in \widehat{K}^\epsilon_\rho(\widehat{d}) \tag{6.3}$$

for $\epsilon > 0$ sufficiently small and for some $k \in (0,1)$. Hence $\widehat{T}$ has a unique fixed point in $\widehat{K}^\epsilon_\rho(\widehat{d})$ so that $u_1 = u_2$, in $(\widehat{r}, \widehat{r}+\delta)$ for some $\delta > 0$ showing that $(\widehat{r}, \widehat{r}+\delta) \subset S_0$, and consequently S_0 is open in $[0,\infty)$.

Case 2. $u_1'(\widehat{r}) = u_2'(\widehat{r}) \neq 0$. We shall, again, strongly use Proposition 1.4 and its proof. At first we remark that both $u_1'(r)$ and $u_2'(r)$ do not vanish in a neighborhood of $\widehat{r}$. So, by (2.3) and the definition of $E(r)$, for $i = 1,2$,

$$(r^{\alpha-\gamma}\frac{\beta+1}{\beta+2}|u'_i(r)|^{\beta+2} + \lambda F(u_i(r)))' = -(\frac{\alpha+\gamma(\beta+1)}{\beta+2})r^{\alpha-\gamma-1}|u'_i(r)|^{\beta+2}.$$

Integrating from $\widehat{r}$ to r, we obtain

$$\begin{aligned}&\frac{\beta+1}{\beta+2}s^{\alpha-\gamma}|u'_i(s)|^{\beta+2}|_{\widehat{r}}^r + \lambda F(u_i(s))|_{\widehat{r}}^r\\ &= -\frac{\alpha+\gamma(\beta+1)}{\beta+2}\int_{\widehat{r}}^r s^{\alpha-\gamma-1}|u'_i(s)|^{\beta+2}ds\end{aligned} \tag{6.4}$$

and making $i = 1,2$ in (6.4) and subtracting the corresponding equations,

$$\begin{aligned}&\tfrac{\beta+1}{\beta+2}r^{\alpha-\gamma}\big[|u'_1(r)|^{\beta+2} - |u'_2(r)|^{\beta+2}\big] + \lambda\big[F(u_1(r)) - F(u_2(r))\big]\\ &= -\tfrac{\alpha+\gamma(\beta+1)}{\beta+2}\int_{\widehat{r}}^r s^{\alpha-\gamma-1}\big[|u'_1(s)|^{\beta+2} - |u'_2(s)|^{\beta+2}\big]ds.\end{aligned} \tag{6.5}$$

Now, it can be shown that the functions

$$A(r) \equiv \begin{cases} \frac{|u'_1(r)|^{\beta+2} - |u'_2(r)|^{\beta+2}}{u'_1(r) - u'_2(r)} \text{ if } u'_1(r) \neq u'_2(r),\\ (\beta+2)|u'_1(r)|^\beta u'_1(r) \text{ if } u'_1(r) = u'_2(r)\end{cases}$$

$$B(r) \equiv \begin{cases} \lambda\frac{F(u_1(r)) - F(u_2(r))}{u_1(r) - u_2(r)} \text{ if } u_1(r) \neq u_2(r),\\ \lambda f(u_1(r)) \text{ if } u_1(r) = u_2(r)\end{cases}$$

are continuous, A is continuously differentiable in a neighborhood of $\widehat{r}$ and in addition, $A'(\widehat{r}) = (\beta+1)(\beta+2)|u'_1(\widehat{r})|^\beta u''_1(\widehat{r})$. Moreover, making $v(r) \equiv u_1(r) - u_2(r)$ in (6.5),

$$\tfrac{(\beta+1)r^{\alpha-\gamma}}{(\beta+2)}A(r)v'(r) + B(r)v(r) = -\left(\tfrac{\alpha+\gamma(\beta+1)}{\beta+2}\right)\int_{\widehat{r}}^r s^{\alpha-\gamma-1}A(s)v'(s)ds, \tag{6.6}$$

and since $A(\widehat{r}) \neq 0$ it follows that $\frac{1}{A}$ is continuous and bounded in a neighborhood of $\widehat{r}$. Multiplying (6.6) by $\frac{1}{A(r)}\frac{(\beta+2)}{(\beta+1)}r^{\gamma-\alpha}$, integrating the resulting equation and further computing the result by parts we obtain

$$\begin{aligned} v'(r) &+ \frac{(\beta+2)}{(\beta+1)} r^{\gamma-\alpha} \frac{B(r)}{A(r)} v(r) + \frac{(\alpha+(\beta+1)\gamma)}{(\beta+1)A(r)} r^{\gamma-\alpha} \\ &= -\Big[s^{\alpha-\gamma-1} v(s) A(s) \Big|_{\widehat{r}}^{r} - \int_{\widehat{r}}^{r} \Big(s^{\alpha-\gamma-1} A(s) \Big)' v(s) ds \Big] \end{aligned}$$

so that

$$\begin{aligned} v'(r) &+ \Big[\frac{(\beta+2)B(r)}{(\beta+1)A(r)} r^{\gamma-\alpha} + \frac{(\alpha+(\beta+1)\gamma)}{(\beta+1)r} \Big] v(r) \\ &= \frac{(\alpha+(\beta+1)\gamma) r^{\gamma-\alpha}}{(\beta+1)A(r)} \int_{\widehat{r}}^{r} \big(s^{\alpha-\gamma-1} A(s) \big)' v(s) ds. \end{aligned}$$

We write $v'(r)$ as

$$v'(r) = C(r)v(r) + D(r) \int_{\widehat{r}}^{r} F(s) v(s) ds$$

where C, D and F are continuous and bounded in a neighborhood of $\widehat{r}$. Integrating from $\widehat{r}$ to r and estimating we get

$$\begin{aligned} |v(r)| &\leq C_1 \int_{\widehat{r}}^{r} |v(s)| ds + \int_{\widehat{r}}^{r} |D(s)| \int_{\widehat{r}}^{s} |F(t)||v(t)| dt\, ds \\ &\leq C_1 \int_{\widehat{r}}^{r} |v(s)| ds + C_2 \int_{\widehat{r}}^{r} |v(s)| ds \leq C \int_{\widehat{r}}^{r} |v(s)| ds. \end{aligned}$$

By the Gronwall inequality, $v = 0$ in a neighborhood of $\widehat{r}$. Therefore, there is some $\delta > 0$ such that $u_1 = u_2$ in $(\widehat{r} + \delta, \widehat{r})$, and consequently S_0 is open. So (3.3) is verified.

Verification of (3.4)**.** Assume for the sake of contradiction that $T_\infty < \infty$. We consider two cases:

Case 1**.** $u'(T_\infty) = 0$. If $u(T_\infty) = 0$, let $\tilde{u}(t) = u(t)$, if $0 \leq t \leq T_\infty$, and $\tilde{u}(t) = 0$, if $t > T_\infty$. Then $\tilde{u} \in C^1([0,\infty))$ is a solution of $(*)_{\lambda,d}$, a contradiction. If, on the other hand, $u(T_\infty) \equiv d^\infty > 0$, letting

$$T(u(r)) = d^\infty - \int_{T_\infty}^{r} s^{\frac{-\alpha}{\beta+1}} \Big(\Big| \int_{T_\infty}^{s} \lambda t^\gamma f(u(t)) dt \Big|^{\frac{-\beta}{\beta+1}} \int_{T_\infty}^{s} \lambda t^\gamma f(u(t)) dt \Big) ds$$

and

$$\widehat{K}_\rho^\epsilon(d^\infty) \equiv \{ u \in C[T_\infty, T_\infty + \epsilon] : u(T_\infty) = d^\infty,\ \|u - d^\infty\|_\infty \leq \rho \}$$

we infer that $T : \widehat{K}_\rho^\epsilon(d^\infty) \to \widehat{K}_\rho^\epsilon(d^\infty)$ is a contraction, and by the Banach fixed-point principle T has a unique fixed point, contradicting the maximality of T_∞

Case 2. $u'(T_\infty) \neq 0$. At first we remark that u extends uniquely to $[0, T_\infty]$. Let $u(T_\infty) \equiv d^\infty$ and $u'(T_\infty) \equiv d_1^\infty$. There is some $\delta > 0$ such that $u'(t) \neq 0,\ T_\infty - \delta \leq t \leq T_\infty$. From (2.3) we have

$$|u'(r)| = r^{-\frac{\alpha}{\beta+1}} | \int_0^r \lambda s^\gamma f(u(s)) ds |^{\frac{1}{\beta+1}}$$

so that

$$\int_0^r s^\gamma f(u(s)) ds \neq 0,\ r \in (T_\infty - \delta, T_\infty].$$

Since also

$$u'(r) = \pm r^{-\frac{\alpha}{\beta+1}} | \int_0^r \lambda s^\gamma f(u(s)) ds |^{\frac{1}{\beta+1}}$$

it follows that $u \in C^2(T_\infty - \delta, T_\infty]$, and by (2.1) we get that

$$u''(r) = -\frac{\alpha}{(\beta+1)r} u'(r) - \frac{1}{(\beta+1)} |u'(r)|^{-\beta} \lambda f(u(r)) r^{\gamma-\alpha},$$

and by Peano's theorem there is some $\delta > 0$ such that the above equation has a solution in $(T_\infty - \delta, T_\infty + \delta)$ satisfying $u(T_\infty) \equiv d^\infty$ and $u'(T_\infty) \equiv d_1^\infty$, contradicting the maximality of T_∞. This proves (3.4).

References

[1] Y. Cheng, *On the existence of radial solutions of a nonlinear elliptic equation on the unit ball*, Nonlinear Anal., 24 (1995), 287–307.

[2] J. Iaia, *Radial solutions to a p-Laplacian Dirichlet problem*, Applicable Anal., 58 (1995), 335–350.

[3] A. Castro and A. Kurepa, *Infinitely many radially symmetric solutions to a superlinear Dirichlet problem in a ball*, Proc. Amer. Math. Soc., 101 (1987), 57–64.

[4] Ph. Clément, D.G. de Figueiredo, and E. Mitidieri, *Quasilinear elliptic equations with critical exponents*, Top. Meth. Nonl. Anal., 7 (1996), 133–170.

[5] A. Castro, J. Cossio, and J.M. Neuberger, *A sign-changing solution for a superlinear Dirichlet problem*, Rocky Mountain J. Math., 27 (1997), 1041–1053.

[6] W.M. Ni and J. Serrin, *Nonexistence theorems for singular solutions of quasilinear partial differential equations*, Comm. Pure Appl. Math., 39 (1986) 379–399.

[7] J. Simon, *Regularité de la solution d'une equation non lineaire dans* $\mathbf{R}^\mathbf{N}$, Springer Lecture Notes in Math., # 665 (Ph. Benilan, editor) (1978), 203–227.

[8] W. Strauss, *Existence of solitary waves in higher dimensions*, Comm. Math. Phys., 55 (1977), 149–162.

[9] P. Tolksdorff, *On quasilinear boundary value problems in domains with corners*, Nonlinear Anal., 5 (1981), 721–735.

[10] R. Saxton and D. Wei, *Radial solutions to a nonlinear p-harmonic Dirichlet problem*, Applicable Anal., 51 (1993), 59–80.