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# BLOW-UP AND STABILITY OF A NONLOCAL DIFFUSION-CONVECTION PROBLEM ARISING IN OHMIC HEATING OF FOODS

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**Abstract.** We study the blow-up and stability of solutions of the equation $u_t + u_x = u_{xx} + \lambda f(u)/(\int_0^1 f(u)\,dx)^2$ with certain initial and boundary conditions. When $f$ is a decreasing function, we show that if $\int_0^\infty f(s)\,ds < \infty$, then there exists a $\lambda^* > 0$ such that for $\lambda > \lambda^*$, or for any $0 < \lambda \le \lambda^*$ but with initial data sufficiently large, the solutions blow up in finite time. If $\int_0^\infty f(s)\,ds = \infty$, then the solutions are global in time. The stability of solutions in both cases is discussed. We also study the case of $f$ being increasing.

## 1. Introduction

Our purpose is to study the nonlocal initial-boundary value problem

$$u_t + u_x = u_{xx} + \frac{\lambda f(u)}{\left(\int_0^1 f(u)\,dx\right)^2}, \quad 0 < x < 1, \quad t > 0, \tag{1.1a}$$

$$\mathcal{B}(u(x,t)) = 0, \quad x = 0, 1, \quad t > 0, \tag{1.1b}$$

$$u(x,0) = u_0(x), \quad 0 < x < 1, \tag{1.1c}$$

where $\lambda$ is a positive parameter, $u = u(x,t)$ is the dimensionless temperature and $\mathcal{B}$ represents a suitable linear boundary operator. The initial data $u_0(x)$ are taken to be so that $u_0(x)$ and $u_0'(x)$ are bounded and $u_0(x) \ge 0$ in $[0,1]$ ( the last requirement is a consequence of the fact that for any $u_0(x)$ the solution becomes nonnegative sometime).

The function $f$ is positive and monotonic; i.e.,

$$f(s) > 0\,, \quad f'(s) < 0 \ \text{ or } \ f'(s) > 0\,, \quad s \ge 0. \tag{1.2}$$

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Problem (1.1) models the temperature which arises when an electric current flows through a moving material. This kind of problem appears, for example, in the sterilization of food. The food passes through a conduit, a part of which lies between two electrodes. A high electric current passes through the electrodes into the food, resulting in Ohmic heating.

The problem was first considered in [14], where the stability of different models was studied. Related material can be found in [4, 7, 9, 15, 16, 18].

In some problems heat convection dominates heat conduction (see [14]) depending upon the velocity of the food. In this work we consider velocities such that heat convection and conduction coexist.

We now outline the derivation of the model. The problem can be described by the following system of p.d.e.'s:

$$\bigtriangledown \cdot (\sigma \bigtriangledown \phi) = 0, \qquad pc\Big[\frac{\partial T}{\partial t} + \underset{\sim}{v} \cdot \bigtriangledown T\Big] = \bigtriangledown \cdot (k \bigtriangledown T) + \sigma |\bigtriangledown \phi|^2.$$

More details can be found in [14]; see also [1, 2, 10, 12].

The first equation describes the conservation of charge, while the second one governs the variation of the temperature $T$ of the food. The food has density $p$, specific heat $c$, velocity $\underset{\sim}{v}$ and thermal conductivity $k$. The potential difference is $\phi$ and $\sigma$ is the electrical conductivity ($1/\sigma$ is the electrical resistivity).

Here, we assume that the curved surface is thermally and electrically insulated and that $k$, $p$, and $c$ are positive constants. Hence, following calculations similar to those in [10, 12], we obtain the one-dimensional equation

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial z} = \frac{k}{pc}\frac{\partial^2 T}{\partial z^2} + \frac{\sigma \phi_z^2}{pc} = \frac{k}{pc}\frac{\partial^2 T}{\partial z^2} + \frac{V^2}{pc}\frac{1}{\sigma}\left(\int_I \frac{1}{\sigma}\, dz\right)^{-2}, \quad z \in I.$$

It is convenient to take $v = \frac{k}{pc}$; then by rescaling the time and changing the temperature and space variables suitably, we obtain problem (1.1). The parameter $\lambda$ is equal to $V^2/k$, $V$ is the potential difference and $f(u) = 1/\sigma(u)$.

On the other hand, if we consider a complete electrical circuit with constant resistance and a fixed EMF, part of which is the conduit with the food, then following a similar calculation, we obtain instead of (1.1a) the equation

$$u_t + u_x = u_{xx} + \frac{\lambda f(u)}{\left(a + b\int_0^1 f(u)\, dx\right)^2}, \qquad a, b > 0; \tag{1.3}$$

see [10].

## 2. Existence and Monotonicity

The local existence of a solution of (1.1) can be obtained by using a Picard-type iteration scheme; see the appendix in [12]. The uniqueness is a consequence of $f$ being Lipschitz. More precisely, for $\lambda > 0$, $f(s) \geq c > 0$ and Lipschitz in $(a, b)$, there exists a unique solution $a < u < b$ where $a < \min\{0, \inf u_0\}$ and $b > \max\{0, \sup u_0\}$. This solution continues to exist as long as it remains less than or equal to $b$. This argument implies that $u$ ceases to exist only by blow-up; that is, there exists a sequence $(x_n, t_n) \to (x^*, t^*)$ with $t^* < \infty$ such that $u(x_n, t_n) \to \infty$ as $n \to \infty$.

If, in addition, $f$ is decreasing then we may use comparison methods. Indeed we define $\overline{u}$ as an upper solution to problem (1.1) if $\overline{u}$ satisfies

$$\overline{u}_t + \overline{u}_x \geq \overline{u}_{xx} + \frac{\lambda f(\overline{u})}{\left(\int_0^1 f(\overline{u})\,dx\right)^2}, \quad 0 < x < 1, \quad t > 0,$$
$$\mathcal{B}(\overline{u}(x,t)) \geq \mathcal{B}(u(x,t)), \quad x = 0, 1, \quad t > 0,$$
$$\overline{u}(x, 0) \geq u(x, 0), \quad 0 < x < 1,$$

and a lower solution $\underline{u}$ to (1.1) if the above inequalities are reversed. We now define $v = \overline{u} - \underline{u}$, where $v = v(x,t)$. Then there exists a time $T > 0$ such that $v \geq 0$ for $0 \leq t < T$ and

$$v_t + v_x \geq v_{xx} + \frac{\lambda f'(s)}{\left(\int_0^1 f(\underline{u})\,dx\right)^2}\, v\,, \quad 0 < x < 1, \quad 0 < t < T,$$

$$\mathcal{B}(v(x,t)) \geq 0\,, \ \ x = 0, 1\,, \ \ 0 < t < T\,, \ \ v(x, 0) \geq 0, \ \ 0 < x < 1,$$

for $s \in (\underline{u}, \overline{u})$. The latter implies, by the maximum principle, that $v \geq 0$ for $0 \leq t \leq T$; see [10, 12].

## 3. Decreasing Functions

We now consider problem (1.1) with different types of boundary conditions. The function $f$ satisfies (1.2) with $f'(s) < 0$. It is convenient to assume Dirichlet and Neumann boundary conditions at $x = 0, 1$ respectively. Then problem (1.1) becomes

$$u_t + u_x = u_{xx} + \frac{\lambda f(u)}{\left(\int_0^1 f(u)\,dx\right)^2}, \quad 0 < x < 1, \quad t > 0\,, \tag{3.1a}$$

$$u(0, t) = u_x(1, t) = 0, \quad t > 0, \tag{3.1b}$$

$$u(x, 0) = u_0(x), \quad 0 < x < 1. \tag{3.1c}$$

The corresponding steady problem is

$$w'' - w' + \mu f(w) = 0, \quad 0 < x < 1, \tag{3.2a}$$

$$w(0) = w'(1) = 0, \tag{3.2b}$$

where $\mu = \lambda / \big( \int_0^1 f(w)\, dx \big)^2$ is a positive local parameter ($\lambda$ is the nonlocal one). Problem (3.2) has a unique solution for every $\mu > 0$, since $f$ satisfies (1.2) with $f'(s) < 0$ (by using monotone-type arguments), [3, 5, 13], and Lemma 3.2 holds; see below. Here, due to the convection term, the steady problem is not symmetric, in contrast with the pure diffusion problem, [11]. In the sequel we shall investigate the spectrum of $\lambda$. Therefore, integrating (3.2a) we obtain

$$\lambda(M) = \frac{(w'(0) + M)^2}{\mu}, \tag{3.3}$$

where $M = \sup ||w|| = w(1)$. Also multiplying (3.2a) by $w'$ and integrating we get

$$\frac{(w'(0))^2}{\mu} = 2 \left[ \int_0^M f(s)\, ds - \frac{1}{\mu} \int_0^1 (w'(x))^2\, dx \right] \leq 2 \int_0^M f(s)\, ds. \tag{3.4}$$

We now have:

**Lemma 3.1.** *If* $\int_0^\infty f(s)\, ds < \infty$, *then* $\frac{(w'(0))^2}{\mu} \to 2 \int_0^\infty f(s)\, ds$ *as* $\mu \to \infty$.

The proof is similar to the proof of Lemma 5.1 in [17]. Also, by using some ideas from [5], we have the following:

**Lemma 3.2.** *If $w$ is the solution to* (3.2) *and* $\int_0^\infty f(s)\, ds < \infty$, *then* $w(x;\mu) \to \infty$ *as* $\mu \to \infty$ *for every $x$ in* $(0, 1]$.

**Proof.** We first prove that $\phi(\mu) = \int_0^1 f(w(x;\mu))\, dx \to 0$ as $\mu \to \infty$. Therefore, we construct a lower solution of (3.2) of the form $z = \beta\phi_1$ where $\phi_1$ is the principal eigenfunction of

$$-\phi'' + \phi' = \lambda\phi, \quad 0 < x < 1, \tag{3.5a}$$

$$\phi(0) = \phi'(1) = 0 \tag{3.5b}$$

and $\beta > 0$.

It is known that $\lambda_1 > 0$ and $\phi_1 > 0$; also we normalize $\phi_1$ by taking $||\phi_1||_\infty = 1$. Now on choosing $\beta$ to satisfy $\frac{\lambda_1 \beta}{f(\beta)} \leq \mu$, $\beta\phi_1$ becomes a lower solution of (3.2). Thus, it is sufficient to choose $\frac{\beta}{f(\beta)} = \frac{\mu}{\lambda_1}$.

This choice of $\beta$ is unique for each $\mu > 0$. Indeed, $F(\beta) = \frac{\lambda_1 \beta}{f(\beta)}$ is one-to-one since $F'(\beta) > 0$ and maps $\mathbb{R}_+$ onto $\mathbb{R}_+$ since $F((0,\infty)) = (0,\infty)$. Finally

$F$ is a diffeomorphism; hence, to each $\mu$ corresponds a unique $\beta(\mu)\phi_1$ which is a lower solution to (3.2). Thus we get

$$\phi(\mu) = \int_0^1 f(w(x;\mu))\,dx \leq \int_0^1 f(\beta(\mu)\phi_1(x))\,dx \ \to 0 \ \text{ as } \ \mu \to \infty.$$

The last limit implies that $w(x;\mu) \to \infty$ as $\mu \to \infty$ for $0 < x \leq 1$; otherwise, we could find an $x_0$ so that $w(x;\mu) < \infty$ in $(0, x_0)$, but this would imply that $\lim_{\mu\to\infty} \phi(\mu) > 0$, contradicting that $\phi(\mu) \to 0$ as $\mu \to \infty$. □

From the above we obtain that $M = w(1) \to \infty$ as $\mu \to \infty$. Moreover, on using the maximum principle in problem (3.2) we obtain the response local diagram; see Figure 1(d).

Now multiplying (3.2a) by $w' - w$ and integrating we obtain

$$\frac{(w'(0))^2}{\mu} = 2\left[\int_0^M f(s)\,ds - \int_0^1 f(w)w\,dx\right] + \frac{M^2}{\mu}. \tag{3.6}$$

In addition we have

$$\int_0^1 f(w)w\,dx = f(w(\xi;\mu))w(\xi;\mu), \quad \xi \in (0,1);$$

hence,

$$\int_0^1 f(w)w\,dx \to 0, \quad \text{as } \ \mu \to \infty\,,$$

on using Lemma 3.2 and the fact that $f$ is decreasing.

From (3.6) we obtain, by taking the limit as $\mu \to \infty$,

$$\frac{M^2}{\mu} \to 0. \tag{3.7}$$

It is convenient to normalize the integral

$$\int_0^\infty f(s)\,ds = 1. \tag{3.8}$$

Now we have the following:

**Proposition 3.3.** *If* (3.8) *holds, then* $\lambda(M) \to 2$ *as* $M \to \infty$ (*or equivalently as* $\mu \to \infty$).

The proof is a consequence of (3.3), (3.7) and Lemma 3.1.

We now discuss the case where

$$\int_0^\infty f(s)\,ds = \infty. \tag{3.9}$$

**Proposition 3.4.** *Let $f$ satisfy* (3.9) *and $w$ be the solution of* (3.2). *Then*

$$\frac{(w'(0))^2}{\mu} \to \infty \ \text{as}\ \mu \to \infty \quad \text{and} \quad \lambda(M) \to \infty \ \text{as}\ M \to \infty.$$

**Proof.** Let $z$ satisfy

$$z''(x) + \mu g(z(x)) = 0, \quad 0 < x < 1-\delta, \tag{3.10}$$

$$z(x) = M, \quad z'(x) = 0, \quad 1-\delta \le x \le 1, \quad z(0) = 0. \tag{3.11}$$

It is easily proved that $z$ is a lower solution of (3.2) provided that

(a) $0 < g(s) < f(s)$, $g'(s) < 0$ and $\int_0^\infty g(s)\,ds = \infty$ (for instance $g$ can be taken as $g(s) = \gamma f(s)$, $0 < \gamma < 1$),

(b) $\mu \ge \mu_0 = \sup_{z\in(0,M)}\{[2\int_z^M g(s)\,ds]/[f(z)-g(z)]^2\}$;

thus, we obtain

$$\frac{(w'(0))^2}{\mu} \ge \frac{(z'(0))^2}{\mu} = 2\int_0^M g(s)\,ds \to \infty$$

as $\mu \to \infty$. Hence,

$$\lambda(M) = \frac{(w'(0)+M)^2}{\mu} \ge \frac{(w'(0))^2}{\mu} \to \infty \ \text{ as }\ M \to \infty. \qquad \square$$

From the above analysis we can obtain the main possible response diagrams; see Figure 1. It is possible, in Figure 1(b), to have more than one turning point. This can occur even in the cases of Figures 1(a), (c); see also [13, 17].

**Stability.** We now study the stability of the steady solutions, for $0 < \lambda \le \lambda^* < \infty$ if $\int_0^\infty f(s)\,ds < \infty$ or for any $\lambda > 0$ if $\int_0^\infty f(s)\,ds = \infty$, by using comparison methods. Therefore we construct an upper solution $U(x,t) = w(x;\overline{\mu}(t))$ decreasing in time and a lower solution $z(x,t) = w(x;\underline{\mu}(t))$ increasing in time, of the $u$-problem. Both $\overline{\mu}$ and $\underline{\mu}$ satisfy the following initial value problem:

$$\dot{\nu} = h(\nu) \equiv \inf_{x\in(0,1)}\Big\{\frac{f(w)}{w_\nu}\Big\}\frac{\lambda - \lambda(\nu)}{\big(\int_0^1 f(w)\,dx\big)^2}, \qquad \nu(0) = \nu_0\,, \tag{3.12}$$

where $w = w(x;\nu(t))$ is the solution of (3.2) with $\mu = \nu(t)$. The existence of the solution $w = w(x;\nu(t))$ is a consequence of the fact that to each $t > 0$ there corresponds a $\mu$ so that $\mu = \nu(t)$ is a continuous function and maps $\mathbb{R}_+$ onto $[\nu_0, \infty)$; $w = w(x;\mu)$ is the unique solution of problem (3.2) ($(\mu, M) = (\nu(t), M(t))$ is a point of the diagram in Figure 1(d)). Moreover, the continuity of the function $h(\nu)$ and the form of diagrams in Figure 1

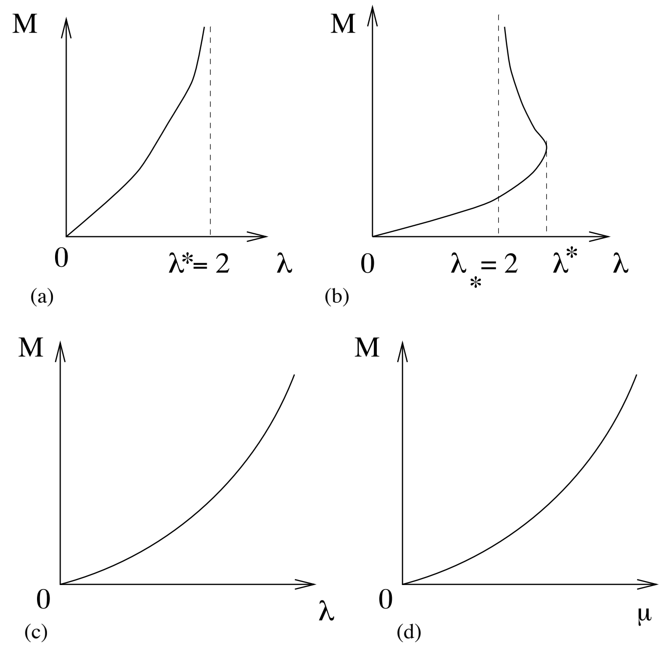


FIGURE 1. $\lambda$, $\mu$-diagrams represent the nonlocal, local diagrams respectively of problem (3.2)
(i) (a), (b) for the case $\int_0^\infty f(s)\,ds = 1$
(ii) (c) for the case $\int_0^\infty f(s)\,ds = \infty$.

imply the existence and uniqueness of the solution $\nu = \nu(t)$ of (3.12), see also [6]. Indeed, by using the function

$$G(\xi) = \int_{\nu_0}^{\xi} \frac{ds}{h(s)},$$

since $h(s)$ is continuous, it is easy to prove that $G$ is a $C^1$-diffeomorphism which maps $[\nu_0, \mu)$ onto $[0, \infty)$ $\left(G(\mu) = \int_{\nu_0}^{\mu} \frac{ds}{h(s)} = \infty\right.$, as long as $\displaystyle\frac{1}{\lambda'(\mu)} = \lim_{s \to \mu} \frac{s - \mu}{\lambda(s) - \lambda(\mu)} \neq 0\Big)$. Hence the integral equation $\int_{\nu_0}^{\nu(t)} \frac{ds}{h(s)} = t$, $t > 0$, as well as problem (3.12), has a unique solution $\nu_0 \le \nu(t) = G^{-1}(t) < \mu$, $t \ge 0$, [6]. The function $\nu = \nu(t)$ is a global (in time) solution since $\nu$ is bounded $(\underline{\mu}(0) \le \nu(t) \le \overline{\mu}(0))$. Note also that $\nu$ is strictly monotonic (more precisely $\overline{\mu}(t)$ $(\underline{\mu}(t))$ is decreasing (increasing)) in each case (see below).
Moreover, we have that $w_\nu > 0$; this is a consequence of the maximum

principle ($f'(s) < 0$). Also $w_\nu$ is finite; indeed, for a fixed $\nu$, any sufficiently large constant is an upper solution of the $w_\nu$-problem (the $w_\nu$-problem is defined by differentiating (3.2) with respect to $\mu$ and substituting $\nu$ for $\mu$), $0 \le w_\nu \le constant$. Hence $\inf_{x\in(0,1)} \{f(w)/w_\nu\}$ is always positive since also $f(s)$ is bounded away from zero for $s \le \sup w < \infty$.

For the upper (lower) solution we choose $\nu_0$ so that $w(x;\nu(0)) \ge (\le)\, u_0(x)$; this is possible since $w_\nu > 0$ and Lemma 3.2 holds. More precisely, this can clearly be done if we require that $u_0$ and $u_0'$ are bounded (admissible initial data: $u_0$ and $u_0'$ are bounded, $u_0 \ge 0$, $u_0 \in C^1([0,1])$; otherwise, we can choose $\nu(\epsilon)$ so that $w(\cdot;\nu(\epsilon)) \ge u(\cdot,\epsilon)$ for small $\epsilon > 0$).

Now if the response diagram is of the type (unique solution) such that, to each $\lambda$ corresponds a unique $M(\mu)$ (see Figure 1(a), (c), or (b) for $\lambda < \lambda_*$), then we take $\nu_0 > (<)\mu$, while if the response diagram is of the type (two solutions) such that, to each $\lambda$ corresponds two $M'$s, $M_1 = M(\mu_1)$ and $M_2 = M(\mu_2)$ (see Figure 1(b) for $\lambda_* < \lambda < \lambda^*$), then we take $\mu(\lambda_*) = \mu_* < \nu_0 < \mu_1$ or $\mu_1 < \nu_0 < \mu_2$, or $\nu_0 > \mu_2$ (the latter can be extended to cases with more turning points, in which to each $\lambda$ corresponds more than two $M'$s). The above choices, in all cases, have as a consequence that $\lambda - \lambda(\nu) < (>)0$, $\nu = \overline{\mu}(t)$ $(\underline{\mu}(t))$ for the upper (lower) solution; these inequalities hold in a proper region of $\nu$ (we recall that $\nu = \overline{\mu}(t)$ or $\underline{\mu}(t)$); i.e., for the case of the unique solution, $<$ $(>)$, if $\nu > \mu\,(\nu < \mu)$, while for the case of two solutions, $<$ $(>)$, if $\mu_1 < \nu < \mu_2\,(\mu_* < \nu < \mu_1$ or $\nu > \mu_2)$. These imply that $U(x,t) = w(x;\overline{\mu}(t))$ $(z(x,t) = w(x;\underline{\mu}(t)))$ is an upper (lower) solution since it has the form of a stationary one. Thus $\lambda - \lambda(\overline{\mu}) < 0 < \lambda^* - \lambda(\overline{\mu})$; similarly, $\lambda - \lambda(\underline{\mu}) > 0 > \lambda_* - \lambda(\underline{\mu})$. The latter implies $U_t = w_\nu\,\dot{\nu} < 0$ $\quad (z_t = w_\nu \dot{\nu} > 0)$, since $w_\nu > 0$; $\dot{\overline{\mu}}(t) < 0$, $\dot{\underline{\mu}}(t) > 0$.

Returning to the case of a unique steady-state $w(x) \equiv w_1(x)$, the above construction implies that $z(x,t) \le u(x,t) \le U(x,t)$. Then $u(x,t)$ is a global in-time solution and $z(x,t) \to w(x)-$ $(z(x,t) < w(x))$, $\;U(x,t) \to w(x)+$ $(U(x,t) > w(x))$ as $t \to \infty$ uniformly in $x$ since $\overline{\mu}(t) \to \overline{l} = \mu+$, $\underline{\mu}(t) \to \underline{l} = \mu-$ as $t \to \infty$, where $\mu+$ $(\mu-)$ means that $\overline{\mu}(t) \to \mu$, $(\underline{\mu}(t) \to \mu)$ and $\overline{\mu}(t) > \mu$ $(\underline{\mu}(t) < \mu)$. The latter is true; i.e., $\overline{l} = \underline{l} = \mu$, since assuming that $\overline{\mu}(t) \to \widehat{\mu}+ = \widehat{\mu} > \mu$, as $t \to \infty$, problem (3.12) would imply $\int_{\overline{\mu}(0)}^{\overline{\mu}(t)} \frac{ds}{h(s)} = t$, and by taking the limit as $t \to \infty$ we should have $\int_{\overline{\mu}(0)}^{\widehat{\mu}} \frac{ds}{h(s)} = \infty$. But this can occur if and only if $h(\widehat{\mu}) = 0$ or equivalently $\lambda(\widehat{\mu}) = \lambda$, which contradicts the uniqueness of the solution of the nonlocal steady problem (otherwise we would have $\lambda(\widehat{\mu}) = \lambda(\mu) = \lambda$ with $\widehat{\mu} > \mu$, which is a contradiction; see

Figure 1). The same argument applies to all other cases with more steady-states, giving always an extra steady solution. Similarly it is shown that $\underline{\mu}(t) \to \mu-$, as $t \to \infty$. Finally from the analysis above we get that $w$ is a globally asymptotically stable solution. (If $\int_0^\infty f(s)\,ds = \infty$, then we can also prove that $u(x,t)$ is a global-in-time solution, by using the equation which is satisfied by $M(t) = \max_{x\in[0,1]} u(\cdot,t)$. Indeed, we have

$$\dot{M}(t) \le \frac{\lambda f(M)}{\left(\int_0^1 f(u)\,dx\right)^2} < \frac{\lambda}{f(M)} \quad \text{or} \quad \int_{M(0)}^{M(t)} f(s)\,ds < \lambda t,$$

which implies that $u(x,t)$ is global in time.)

For the case of two steady solutions $w_1 < w_2$ in (0,1], we choose suitable initial data, i.e., $w_1 \le u_0 \le w_2$. As before we construct a lower and an upper solution, $z$ and $U$ respectively, so that $z(x,t) \le u(x,t) \le U(x,t)$ and $z \to w_1-$, $U \to w_1+$ ($\overline{\mu}(t) \to \mu_1+$, $\underline{\mu}(t) \to \mu_1-$) as $t \to \infty$ uniformly in $x$. On the other hand, if $u_0(x) > w_2(x)$ we take $\underline{\mu}(0)$ such that $z(x,0) = w(x;\underline{\mu}(0)) \le u_0(x)$ and $z(x,t) = w(x;\underline{\mu}(t)) \ge w_2(x)$ for $t \ge 0$; then $z(x,t) \le u(x,t)$ and $z(x,t) \to \infty$ as $t \to T^* \le \infty$ uniformly in $x$, since $\underline{\mu}(t) \to \infty$ as $t \to \infty$ (assuming the opposite we have a contradiction; indeed, if $\underline{\mu}(t)$ were bounded then $\underline{\mu}(t) \to \mu_3- = \mu_3$ as $t \to \infty$; this would imply the existence of an $M_3 = M(\mu_3)$, of a $\lambda_3 = \lambda(\mu_3)$, and of a third steady solution $w_3 > w_2 > w_1$) and Lemma 3.2 and $w_{\underline{\mu}} > 0$ hold (we recall that $w_\nu > 0$). Thus $w_2$ is unstable and $\sup u(\cdot,t) \to \infty$ as $t \to t^*- \le T^* \le \infty$; that is, $u$ is unbounded. When there is a unique steady-state $w^* = w_1 = w_2$ at $\lambda = \lambda^*$, $w^*$ is unstable; in fact, it is stable from below and unstable from above.

The above procedure can be carried over to cases with more steady-states, $w_1 < w_2 < w_3 \cdots$, for $0 < \lambda \le \lambda^*$. Then we find that $w_1$ is asymptotically stable, $w_2$ unstable, $w_3$ asymptotically stable, and so on.

For $\lambda \ge \lambda^*$, when $\lambda^*$ does not belong in the spectrum of (3.2), we can always construct a lower solution $z(x,t)$ since $\lambda > \mu(\int_0^1 f(w)\,dx)^2$ for all $\mu$. Moreover, as above, $z(x,t) \to \infty$ as $t \to T^* \le \infty$ uniformly in $x$, which implies that $u$ is unbounded as $t \to t^*- \le T^* \le \infty$.

Finally we have that, for $\lambda \in (\lambda_*, \lambda^*]$ and $u_0(x) > w_2(x)$ or for $\lambda > \lambda^*$, $u$ is unbounded.

**Blow-up of unbounded solutions.**

**1st case: $\boldsymbol{\lambda > \lambda^*}$ ($\boldsymbol{\int_0^\infty f(s)\,ds < \infty}$).** We shall prove that the unbounded solutions for $\lambda > \lambda^*$ blow up globally in finite time. Therefore following ideas similar to [11, 17] we construct again a lower solution $z = z(x,t)$

to the $u$-problem of the form

$$z_{xx} + \mu(t) f(z) = 0, \quad 0 < x < \delta(t), \quad t > 0, \tag{3.13a}$$

$$z(0,t) = 0, \quad t > 0, \tag{3.13b}$$

$$z(x,t) = M(t) = \sup_{x \in (0,\delta)} z(x,t), \quad z_x(x,t) = 0, \quad \delta(t) \le x \le 1, \quad t > 0. \tag{3.13c}$$

Multiplying (3.13a) by $z_x$ and integrating we get

$$z_x(x,t) = \sqrt{2\mu}\, [F(z) - F(M)]^{1/2}\,, \tag{3.14}$$

where $F(\sigma) = \int_\sigma^\infty f(s)\, ds$, so $F'(\sigma) = -f(\sigma) < 0$. The relation (3.14) implies

$$\int_0^M \frac{ds}{[F(s) - F(M)]^{1/2}} = \delta \sqrt{2\mu}. \tag{3.15}$$

Also

$$\int_0^1 f(z)\, dx = -\int_0^\delta \frac{z_{xx}}{\mu}\, dx + (1-\delta) f(M) =$$

$$= \left( \frac{2}{\mu} \int_0^M f(s)\, ds \right)^{1/2} + (1-\delta) f(M) \sim \sqrt{\frac{2}{\mu}} + f(M)$$

for $\delta \ll 1 \ll M$. By choosing

$$\delta(M) = \frac{a}{2} f(M) \int_0^M [F(s) - F(M)]^{-1/2}\, ds\,, \tag{3.16}$$

where $a$ is a suitable constant with $a > 1/[(\frac{\lambda}{2})^{1/2} - 1]$ (such an $a$ gives $\Lambda = \frac{1}{2}[\frac{\lambda}{(1+a)^2} - \frac{2}{a^2}] > 0$), we obtain

$$\sqrt{\frac{2}{\mu}} = a\, f(M), \tag{3.17}$$

and

$$\int_0^1 f(z)\, dx \sim (1+a) f(M) \quad \text{for } \delta \ll 1 \ll M\,. \tag{3.18}$$

Also by using (1.2), (3.15), and (3.17) we get

$$\delta = \frac{1}{\sqrt{2\mu}} \int_0^M [F(s) - F(M)]^{-1/2}\, ds \le a(M\, f(M))^{1/2}.$$

The last relation implies that $\delta \to 0$ as $M \to \infty$ (or equivalently as $\mu \to \infty$).

**Proposition 3.5.** *The solution $z$ of problem* (3.13*a*) *is a lower solution to problem* (3.1), *which blows up globally in finite time.*

**Proof.** For $\delta \le x \le 1$, we have

$$\begin{aligned}\mathcal{F}(z) \quad &:= z_t - z_{xx} + z_x - \tfrac{\lambda f(z)}{\left(\int_0^1 f(z)\,dx\right)^2} = \dot{M} - \tfrac{\lambda f(M)}{\left(\int_0^1 f(z)\,dx\right)^2} \sim \\ &\sim \dot{M} - \tfrac{\lambda}{(1+a)^2 f(M)} < \dot{M} - \tfrac{\Lambda}{f(M)},\end{aligned}$$

so $\mathcal{F}(z) \le 0$ taking $0 < \dot{M} \le \Lambda / f(M)$, for $M$ sufficiently large.

In the case where $0 < x < \delta$, we first integrate relation (3.14) with respect to $x$ and then differentiate with respect to $t$, so we obtain

$$z_t = \frac{x\dot{\mu}}{\sqrt{2\mu}}[F(z) - F(M)]^{\frac{1}{2}} + \tfrac{1}{2}\dot{M} f(M)[F(z) - F(M)]^{\frac{1}{2}} \int_0^z [F(s) - F(M)]^{-\frac{3}{2}}\,ds$$
$$:= A + B.$$

By using (3.15) and (3.17) $A$ becomes

$$\begin{aligned}A \quad &= \quad -\frac{f'(M)\dot{M}}{f(M)}[F(z) - F(M)]^{1/2} \int_0^z [F(s) - F(M)]^{-1/2}\,ds \\ &\le \quad -\frac{f'(M)\dot{M} f^{1/2}(z) M}{2 f^{3/2}(M)}, \quad \text{since } s(M^{1/2} - s) \le \frac{1}{4}M, \\ &\le \quad \frac{\Lambda f(z)}{2 f^2(M)} \quad \text{provided that } 0 \le \dot{M} \le -\Lambda / M f'(M).\end{aligned}$$

Also, for $B$ we have

$$\begin{aligned}B \quad &\le \quad \frac{1}{2}\dot{M} f(M) f^{1/2}(z)(M - z)^{1/2} f^{-3/2}(M) \int_0^z (M - s)^{-3/2} ds \\ &\le \quad \frac{\dot{M} f^{1/2}(z)}{f^{1/2}(M)} \le \frac{\Lambda f(z)}{2 f^2(M)}\end{aligned}$$

provided that $0 \le \dot{M} \le \frac{\Lambda}{2f(M)}$. Finally, in $(0, \delta]$, if

$$0 \le \dot{M}(t) \le \min\left\{\frac{\Lambda}{2f(M)}, \frac{-\Lambda}{M\, f'(M)}\right\},$$

we get (after some calculation) for $M \gg 1$,

$$\begin{aligned}\mathcal{F}(z) &\lesssim \frac{\Lambda f(z)}{f^2(M)} + \mu f(z) + \frac{2M^{1/2} f(z)}{a f^{3/2}(M)} - \frac{\lambda f(z)}{(1+a)^2 f^2(M)} \\ &\le \frac{f(z)}{f^2(M)}\left[\Lambda + \frac{2}{a^2} + \Lambda - \frac{\lambda}{(1+a)^2}\right] = \frac{f(z)}{f^2(M)}[2\Lambda - 2\Lambda] = 0,\end{aligned}$$

since $\frac{2(f(M)M)^{1/2}}{a} \le \Lambda$ for $M$ sufficiently large.

Now on choosing

$$\dot{M} = \min\left\{\frac{\Lambda}{2f(M)}, -\frac{\Lambda}{M\,f'(M)}\right\}, \tag{3.19}$$

$\mathcal{F}(z) \leq 0$ for $x \in (0,\delta)\cup(\delta,1)$, $z, z_x$ are continuous and $z(0,t) = z_x(1,t) = 0$, so $z$ is a lower solution for the $u$-problem if $M$ is large enough (after some time at which $u$ is sufficiently large).

We shall show now that $z$ blows up in finite time. Indeed, relation (3.19) implies

$$\Lambda\frac{dt}{dM} = \max\{2f(M), -M\,f'(M)\} \leq 2f(M) - M\,f'(M) \quad \text{or}$$

$$\Lambda t \leq 3\int_0^M f(s)\,ds - M\,f(M) < 3\int_0^\infty f(s)\,ds < \infty,$$

since $M\,f(M) \to 0$ as $M \to \infty$.

The latter implies that $z$ blows up globally at $T^* = \frac{3}{\Lambda}\int_0^\infty f(s)\,ds < \infty$. Hence $u$ must also blow up at some $t^* \leq T^* < \infty$. □

**2nd case:** $\boldsymbol{\lambda_* < \lambda \leq \lambda^*}$ $\left(\int_0^\infty \boldsymbol{f(s)\,ds < \infty}\right)$**.** The above construction can be used to prove that the unbounded solutions for $\lambda_* < \lambda \leq \lambda^*$ and $u_0(x) > w_2(x)$ also blow up in finite time.

The key to this proof is the form of $\Lambda = \frac{1}{2}[\frac{\lambda}{(1+a)^2} - \frac{2}{a^2}]$, which is positive if $\lambda > \lambda_* = 2$, on choosing $a$ suitably, i.e., $a > 1/[\left(\frac{\lambda}{2}\right)^{1/2} - 1]$.

This blow-up is global, meaning that $u(x,t) \to \infty$ as $t \to t^*$ for all $x$ in $(0,1]$ and $u_x(0,t) \to \infty$ as $t \to t^*$. Indeed, by noting that

$$M(t) = \sup_{[0,1]} u(\cdot,t) \quad \text{satisfies} \quad \dot{M} \leq \frac{\lambda f(0)}{\left(\int_0^1 f(u)\,dx\right)^2} = h(t),$$

we have $M(t) - M(0) \leq \int_0^t h(s)\,ds \to \infty$ as $t \to t^*-$, which implies $\int_0^1 f(u)\,dx \to 0$ as $t \to t^*-$. Thus for $\lambda > \lambda^*$ or for $\lambda_* < \lambda \leq \lambda^*$ and $u_0 > w_2$, $u$ blows up globally and $u_x(0,t) \to \infty$ as $t \to t^*-$.

**The Robin problem.** We now consider $u$ satisfying (3.1a) and (3.1c), and instead of (3.1b) we have

$$u_x(0,t) - a\,u(0,t) = u_x(1,t) = 0, \quad t > 0, \quad a > 0. \tag{3.20}$$

The corresponding steady problem is

$$w'' - w' + \mu f(w) = 0, \quad 0 < x < 1, \tag{3.21a}$$

$$w'(0) - a\,w(0) = w'(1) = 0. \tag{3.21b}$$

Integrating (3.21a) and using (3.21b) we get

$$\lambda(M) = \frac{[w(1) - (1-a)w(0)]^2}{\mu}. \tag{3.22}$$

Multiplying (3.21a) by $w'$ and integrating gives

$$\frac{(w'(0))^2}{\mu} = 2\left[\int_m^M f(s)\,ds - \frac{1}{\mu}\int_0^1 (w'(x))^2\,dx\right],$$

which implies that

$$\frac{a^2w^2(0)}{\mu} = \frac{(w'(0))^2}{\mu} \leq 2\int_m^M f(s)\,ds, \tag{3.23}$$

where $0 < m = \min w = w(0) < M = w(1)$ (by using the maximum principle in the equation $v'' - v' + \mu f'(w)v = 0$, $v = w'$).

Now we have the following:

**Lemma 3.6.** *If* $\int_0^\infty f(s)\,ds < \infty$, *then* $\frac{w^2(0)}{\mu} \to 0$ *as* $\mu \to \infty$.

**Proof.** We introduce the auxiliary problem

$$z'' + \mu g(z) = 0, \quad 0 < x < 1-\delta, \quad z'(0) = a\,z(0), \tag{3.24a}$$

$$z(x) = N = \max_{[0,1-\delta]} z, \quad z'(x) = 0, \quad 1-\delta \leq x \leq 1 \tag{3.24b}$$

where $g(s)$ is chosen as in Proposition 3.4.

By following the same steps as in [17] we show that $z$ is a lower solution to problem (3.22). Thus we have $(z'(x))^2 = 2\mu[G(z) - G(N)]$, and $z'(x) > 0$, for $0 < x < 1-\delta$, where $G(z) = \int_z^\infty g(s)\,ds < \infty$.

Now on taking $\mu \geq \mu_0 = \sup_{z\in(0,N)} \frac{2[G(z)-G(N)]}{[f(z)-g(z)]^2}$, $z$ is indeed a lower solution to problem (3.21), so $z(x) < w(x)$ for $x \in (0,1)$, and $N \leq M$. Moreover $N \to \infty$ as $\mu \to \infty$, on using the same ideas as in Lemma 3.2 for problem (3.21). Hence $M \to \infty$ as $\mu \to \infty$. Noting now that $z''(x) < 0$ in $(0, 1-\delta)$, we obtain $z(x) < z(0)(1+ax)$ in $(0, 1-\delta]$ and for $x = 1-\delta$, $N < z(0)[1 + a(1-\delta)] \leq m[1 + a(1-\delta)]$, which implies that $m \to \infty$ as $N \to \infty$ (or equivalently as $\mu \to \infty$). Thus (3.23) implies that $\frac{w^2(0)}{\mu} \to 0$ as $\mu \to \infty$. □

Multiplying equation (3.21a) by $w' - w$ and integrating we get

$$\frac{w^2(1)}{\mu} = \frac{(a-1)^2w^2(0)}{\mu} - 2\left[\int_m^M f(s)\,ds - \int_0^1 f(w)w\,dx\right]. \tag{3.25}$$

Using arguments similar to those of Lemma 3.2 we prove that

$$\int_0^1 f(w)w\,dx \to 0, \quad \text{as } \mu \to \infty,$$

and via (3.25) we obtain

$$\frac{w^2(1)}{\mu} \to 0, \quad \text{as} \quad \mu \to \infty, \tag{3.26}$$

since

$$\int_m^M f(s)\,ds \to 0, \quad \text{as} \ \ \mu \to \infty.$$

Finally we have:

**Proposition 3.7.** *If* $\int_0^\infty f(s)\,ds < \infty$*, then* $\lambda(M) \to 0$ *as* $M \to \infty$*.*

The proof is a consequence of Lemma 3.6 and relations (3.22) and (3.26).

The latter proposition implies that there exists a $0 < \lambda^* < \infty$ such that for $0 < \lambda < \lambda^*$ problem (3.21) has at least two solutions, whereas it has no solutions for $\lambda > \lambda^*$. Thus the main possible response diagrams are as in Figure 2; it is possible to have more turning points in each case.

The stability can be examined as in the case of boundary conditions (3.1b). Thus for $0 < \lambda < \lambda^*$ we again find that the minimal steady-state is asymptotically stable, the next greater one unstable, the next asymptotically stable and so on. At the critical value $\lambda^*$, $w^* = w(x;\lambda^*)$ is unstable; it is stable from below but unstable from above.

The unbounded solutions, which exist either for $\lambda > \lambda^*$ and any initial data, or for $0 < \lambda \le \lambda^*$ and $u_0 > w_{max}$, where $w_{max}$ is the greatest steady solution, blow up globally in finite time. This can be proved following the same steps as in [11, 17].

The above techniques may be applied to the study of problems with different kinds of boundary conditions.

## 4. Increasing Functions

Now we examine the case when $f$ satisfies (1.2) with $f'(s) > 0$. At first we consider problem (1.1) with Dirichlet boundary conditions,

$$u(0,t) = u(1,t) = 0\,, \tag{4.1}$$

and show that the solution $u(x,t)$ is global in time and uniformly bounded in $(x,t)$.

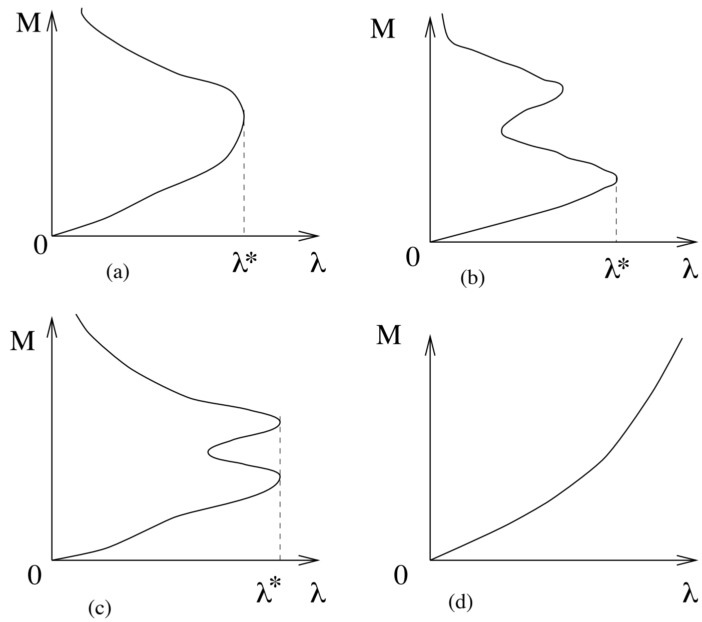


FIGURE 2. The response diagrams of problem (3.21).
(i) (a), (b), (c) for the case $\int_0^\infty f(s)\,ds < \infty$ ,
(ii) (d) for the case $\int_0^\infty f(s)\,ds = \infty$.

We prove this by using the same ideas as in [8]; see also [3] for an alternative method. Therefore we expand $u(x,t)$ in a complete system of eigenfunctions $y_n(x)$ of the operator $-\frac{\partial^2}{\partial x^2} + \frac{\partial}{\partial x}$, i.e.,

$$u(x,t) = \sum_{n=1}^{\infty} E_n(t) y_n(x), \tag{4.2}$$

where $y_n(x) = e^{x/2}\sin(n\pi x)$, $n = 1, 2, \ldots$. Substituting (4.2) in equation (1.1a) we obtain

$$E_n'(t) + \lambda_n E_n(t) = A_n(t), \tag{4.3}$$

where $\lambda_n = \frac{1}{4} + n^2\pi^2$ is the corresponding eigenvalue of $y_n(x)$ and

$$A_n(t) = \frac{2\lambda \int_0^1 f(u) e^{-x/2} \sin(n\pi x)\,dx}{\left(\int_0^1 f(u)\,dx\right)^2}\,.$$

Solving (4.3), and since $f$ is increasing, we obtain the following estimate:

$$|E_n(t)| \leq |E_n(0)|e^{-\lambda_n t} + \frac{2\lambda}{n^2\pi^2 f(0)}[1 - e^{-\lambda_n t}];$$

hence,

$$\limsup_{t\to\infty} |E_n(t)| \leq \frac{2\lambda}{n^2\pi^2 f(0)}\,.$$

From the series (4.2) we have

$$\limsup_{t\to\infty} u(x,t) \leq e^{1/2} \limsup_{t\to\infty} \sum_{n=1}^{\infty} |E_n(t)| \leq \frac{2\lambda e^{1/2}}{f(0)} \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} = \frac{\lambda e^{1/2}}{3f(0)}, \quad (4.4)$$

since the series of $|E_n(t)|$ converges uniformly; hence, the desired result follows. This result is different from what has been proven in [12], where blow-up is possible even if $f$ is increasing.

We now consider Neumann boundary conditions,

$$u_x(0,t) = u_x(1,t) = 0. \quad (4.5)$$

Then following the above analysis, we get

$$y_n(x) = e^{x/2}\big(\cos(n\pi x) - \frac{1}{2n\pi}\sin(n\pi x)\big), \quad n = 1, 2, \ldots$$

and finally the estimate

$$\lim_{t\to\infty} \sup |E_n(t)| \leq \frac{2\lambda e^{1/2}}{n^2\pi^2 f(0)}.$$

From the above estimate it is obvious that the coefficients $E_n(t)$ exist for every $0 < t < \infty$, and therefore $u(x,t)$ exists for all time. In this case $u$ is not necessarily bounded because the corresponding steady problem has no solution.

For Robin boundary conditions the solution $u(x,t)$ is global in time and uniformly bounded. The same analysis can be carried over to problems where $f$ is bounded away from zero ($f(s) \geq c > 0$) as well as in the case where $f$ is nonnegative and the equation of the problem is (1.3), to get that $u(x,t)$ is global in time and uniformly bounded.

## 5. DISCUSSION

In this work we study the equation $u_t + u_x = u_{xx} + \lambda f(u)/(\int_0^1 f(u)\,dx)^2$ with certain boundary and initial conditions, with $f$ positive and monotonic.

In the case where $f$ is decreasing, the main tools employed are comparison methods. Thus we get that if $\int_0^\infty f(s)\,ds < \infty$ then there exists a $\lambda^*$ such that for $\lambda > \lambda^*$ there is no steady solution and the time-dependent solutions blow up in finite time. This blow-up occurs uniformly with respect to the space variable. Blow-up can also occur when $0 < \lambda \leq \lambda^*$ for suitably large initial data and for those $\lambda$ for which there exist at least two solutions. Some of the results are similar to those for the standard reaction-diffusion problem $u_t = u_{xx} + \lambda e^u$, for which there also exists a $\lambda^*$. On the other hand, if $\int_0^\infty f(s)\,ds = \infty$, the time-dependent solutions are global in time and their stability is similar to the case $\lambda < \lambda^*$.

For the case of increasing $f$, where comparison methods cannot be used, we apply a different method (eigenfunctions expansion). Then we get that the solution is global in time and uniformly bounded.

Questions on the behaviour of the time-dependent solutions at $\lambda = \lambda^*$ (when there is a unique steady state for $0 < \lambda < \lambda^*$ and no solutions for $\lambda \geq \lambda^*$), as well as estimations of blow-up time, have not been completely answered yet.

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