Differential and Integral Equations Volume 15, Number 5, May 2002, Pages 567–586

THE ASYMPTOTIC BEHAVIOUR OF PERTURBED EVOLUTION FAMILIES

Valentina Casarino
Politecnico di Torino, Dipartimento di Matematica
Corso Duca degli Abruzzi 24, I-10129 Torino, Italy

Lahcen Maniar
University Cadi Ayyad, Faculty of Sciences Semlalia
Department of Mathematics, 40000 Marrakesh, Morocco

Susanna Piazzera
Mathematisches Institut, Universität Tübingen
Auf der Morgenstelle 10, D-72076 Tübingen, Germany

(Submitted by: J.A. Goldstein)

Abstract. Given an evolution family $\mathcal{U} := (U(t,s))_{t\geq s}$ on a Banach space X, we present some conditions under which asymptotic properties of $\mathcal{U}$ are stable under small perturbations by a family

$$\mathcal{B} := (B(t), D(B(t))_{t\in\mathbb{J}},$$

$\mathbb{J} = \mathbb{R}$ or $\mathbb{R}_+$, of linear closed operators on X. Our results concern asymptotic properties like periodicity, (asymptotic) almost periodicity (even in the sense of Eberlein), uniform ergodicity and total uniform ergodicity. We present, moreover, an application of the abstract results to non-autonomous partial differential equations with delay.

1. Introduction

Evolution families appear as solutions of non-autonomous Cauchy problems

$$\begin{cases} u'(t) = A(t)u(t), & t \geq s, \\ u(s) = x, \end{cases} \tag{NCP}$$

where $(A(t), D(A(t)))_{t\in\mathbb{R}}$ are closed linear operators on a Banach space X.

Given an evolution family $\mathcal{U} := (U(t,s))_{t\geq s}$ and a family of (possibly unbounded) operators $(B(t), D(B(t)))_{t\in\mathbb{R}}$, it is well-known that, under suitable

Accepted for publication: April 2001.
AMS Subject Classifications: 47D06, 47A55, 34D05, 34G10.
The authors would like to thank Frank Räbiger and Abdelaziz Rhandi for many helpful discussions.

assumptions, there exists a unique evolution family $\mathcal{U}_B := (U_B(t,s))_{t\geq s}$, related to $\mathcal{U}$ by the variation of constants formula (see [17, 18] for the bounded case and [22, 23] for the unbounded case). This means that if the evolution family $\mathcal{U}$ is given by the solutions of a non-autonomous Cauchy problem (NCP), then the perturbed evolution family $\mathcal{U}_B$ gives us the mild solutions of the perturbed Cauchy problem

$$\begin{cases} u'(t) = A(t)u(t) + B(t)u(t), & t \geq s, \\ u(s) = x. \end{cases} \tag{PCP}$$

There are a lot of papers dealing with the asymptotic behaviour of evolution families (see for example [8, 13, 15, 16, 27, 29, 30]). For the asymptotic behaviour of non-autonomous inhomogeneous Cauchy problems we refer to [6].

In this paper we investigate how asymptotic properties of an evolution family $\mathcal{U} = (U(t,s))_{t\geq s}$ of bounded linear operators on a Banach space X may persist (or not) under small perturbations. The properties we have in mind, are e.g. uniform boundedness, periodicity, almost periodicity (even in the sense of Eberlein), uniform ergodicity and total uniform ergodicity.

Differently from the autonomous case, that is when $U(t,s) := T(t-s)$ for a $\mathcal{C}_0$-semigroup $(T(t))_{t\geq 0}$ on X and $B(t) \equiv B$, where such an investigation of "stable" and "unstable" properties started with the pioneering work of R. S. Phillips [21] in 1953 and lead to an almost complete classification (see [7] for a survey on this topic), not much is known in the case of perturbed evolution families, except for properties like exponential dichotomy and hyperbolicity, which have been recently studied in an extensive way in [8, 15, 16, 30]. As expected, the problem is more difficult in the non-autonomous case. We proceed as follows.

In Section 2 we fix some notations and define the asymptotic properties. We consider both asymptotic properties on the real line and on the half-line.

In Section 3, we consider a $\mathcal{C}_0$-semigroup $\mathcal{T} := (T(t))_{t\geq 0}$, perturbed by a family $\mathcal{B}$ of closed linear operators $(B(t), D(B(t)))_{t\geq 0}$ on X, fulfilling a suitable Miyadera condition, and we prove that the perturbed evolution family $\mathcal{U}_B = (U_B(t,s))_{t\geq s}$, related to $\mathcal{T}$ by a variation of constants formula ([23, Thm 3.4]), inherits, without any additional assumption, the asymptotic behaviour of $\mathcal{T}$. The method used is inspired by the recent work [7], where analogous stability results are proved in the autonomous case, and it relies on recent results of C. J. Batty and R. Chill [5], showing that the convolution product between a strongly continuous function L from $\mathbb{R}_+$ to the Banach space $\mathcal{L}(X)$ of all linear bounded operators on X and a map $g \in L^1(\mathbb{R}, X)$ preserves some asymptotic properties of L. It is worth noticing that results in

Section 3 are independent on the particular asymptotic property considered, the method works indeed for any homogeneous subspace of the Banach space $\mathrm{BUC}(\mathbb{R}_+, X)$ of all bounded uniformly continuous functions from $\mathbb{R}_+$ to X.

In Section 4 we apply our previous results to the following non-autonomous partial differential equation with delay

$$\begin{cases} u'(t) = Au(t) + L(t)u_t, \quad t \geq 0, \\ u(0) = x, \quad u_0 = f\,, \end{cases} \tag{DE}$$

where $(A, D(A))$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X, $x \in X$, $f \in L^p([-1,0], X)$ for some $1 \leq p < \infty$ and $\big(L(t)\big)_{t\geq 0}$ is a family of bounded linear operators from $L^p([-1,0], X)$ to X. We first prove that for every initial value in a suitable Banach space $\mathcal{X}$ there is a unique mild solution of (DE). Then we study, by using results of Section 3, the asymptotic behaviour of the evolution family associated to the evolution equation on $\mathcal{X}$. For the autonomous version of the equation (DE) the reader is referred to [4, 7]. We also mention [26, 28] for the non-linear case and [10] for the fully non-autonomous (i.e., $A = A(t)$) and inhomogeneous case.

Finally, in Section 5 we extend results obtained in Section 3, by considering a generic evolution family $\mathcal{U}$ perturbed by a family $\mathcal{B} := (B(t))_{t\in\mathbb{R}}$ of bounded linear operators on X. More specifically, we assume that the map $U(t+\cdot, s+\cdot)x$ belongs, for every $x \in X$, to some closed subspace $\mathcal{F}$ of the Banach space $\mathrm{BUC}(\mathbb{R}, X)$ (*e.g.*, $U(t+\cdot, s+\cdot)x$ is assumed to be, for all $x \in X$, almost periodic or Eberlein weakly almost periodic with relatively compact range). We find in Section 5 conditions on $(B(t))_{t\in\mathbb{R}}$ and $\mathcal{U}$ such that the (perturbed) map $U_B(t+\cdot, s+\cdot)x$ also belongs to $\mathcal{F}$ for every $x \in X$. We follow, in this case, the *evolution semigroup* approach, that is we associate to $\mathcal{U}$ a strongly continuous semigroup $\mathcal{T}$ acting on a suitable Banach space. We recall that evolution semigroups on spaces of almost periodic functions were introduced in [3] and they have been intensively used in [6, 11, 12, 13, 14] to study the asymptotic behaviour of solutions of non-autonomous Cauchy problems.

2. Preliminaries

In this section we introduce some definitions and notations which will be used in the following.

Let X be a complex Banach space and denote by $\mathcal{L}(X)$ the Banach algebra of all bounded linear operators on X. If $\mathcal{L}(X)$ is endowed with the strong topology, it will be denoted by $\mathcal{L}_s(X)$.

A family of bounded linear operators $\mathcal{U} := (U(t,s))_{t\geq s}$ on X is called *evolution family* if

(1) $U(t,s) = U(t,r)U(r,s)$ and $U(s,s) = Id$ for all $t \geq r \geq s$,
(2) the mapping $\{t \geq s\} \ni (t,s) \mapsto U(t,s)$ is strongly continuous.

The *exponential growth bound* of an evolution family $\mathcal{U}$ is defined by

$$\omega_0(U) := \inf\{w \in \mathbb{R} : \text{there exists } M_w \geq 1 \text{ with} \\ ||U(t,s)|| \leq M_w e^{w(t-s)} \text{ for } t \geq s \}. \tag{2.1}$$

An evolution family $\mathcal{U}$ is called *exponentially bounded* if $\omega_0(U) < +\infty$.

In the following $\mathbb{J}$ will denote either $\mathbb{R}$ or $\mathbb{R}_+$. Let $\mathrm{BC}(\mathbb{J}, X)$ be the Banach space of all bounded continuous functions from $\mathbb{J}$ to X, endowed with the uniform norm. The closed subspace of uniformly continuous functions will be denoted by $\mathrm{BUC}(\mathbb{J}, X)$.

If $f : \mathbb{J} \to X$, the set of all translates is $H(f) := \{f(\cdot + \omega) : \omega \in \mathbb{J}\}$. A function $f \in \mathrm{BC}(\mathbb{R}, X)$ is called *almost periodic* (abbreviated as *a.p.*) if $H(f)$ is relatively compact in $\mathrm{BC}(\mathbb{R}, X)$.

A function $f \in \mathrm{BC}(\mathbb{R}_+, X)$ is said to be *asymptotically almost periodic* (abbreviated as *a.a.p.*) if $H(f)$ is relatively compact in $\mathrm{BC}(\mathbb{R}_+, X)$.

If $H(f)$ is weakly relatively compact in $\mathrm{BC}(\mathbb{J}, X)$, the bounded continuous map $f : \mathbb{J} \to X$ is called *weakly (asymptotically) almost periodic in the sense of Eberlein*.

We shall consider different classes of asymptotic properties in Section 3 and Section 5.

Results in Section 3 hold for any homogeneous subspace of $\mathrm{BUC}(\mathbb{R}_+, X)$. We recall that a closed subspace $\mathcal{E}$ of $\mathrm{BUC}(\mathbb{R}_+, X)$ is said to be *translation invariant* if

$$\mathcal{E} = \{f \in \mathrm{BUC}(\mathbb{R}_+, X) : \ f(\cdot + t) \in \mathcal{E}\ \} \text{ for all } \ t \geq 0\,,$$

and *operator invariant* if $M \circ f \in \mathcal{E}$ for every $f \in \mathcal{E}$ and $M \in \mathcal{L}(X)$, where $M \circ f$ is defined by $(M \circ f)(t) = M(f(t))$, $t \geq 0$. A closed subspace $\mathcal{E}$ of $\mathrm{BUC}(\mathbb{R}_+, X)$ is said to be *homogeneous* if it is translation invariant and operator invariant.

From [5] the following classes of X-valued functions are homogeneous subspaces of $\mathrm{BUC}(\mathbb{R}_+, X)$:

- the space $C_0(\mathbb{R}_+, X)$ of all continuous functions vanishing at infinity;
- the class of all asymptotically almost periodic functions from $\mathbb{R}_+$ to X;
- the class of all weakly asymptotically almost periodic functions in the sense of Eberlein;

- the class of uniformly ergodic functions from $\mathbb{R}_+$ to X;
- the class of totally (uniformly) ergodic functions from $\mathbb{R}_+$ to X.

For the sake of completeness, we recall the definitions.

A function $f \in \mathrm{BUC}(\mathbb{R}_+, X)$ is said to be *uniformly ergodic* if the limit

$$\lim_{\alpha \to 0^+} \alpha \int_0^\infty e^{-\alpha s} f(\cdot + s)\, ds$$

exists and defines an element of $\mathrm{BUC}(\mathbb{R}_+, X)$.

A function $f \in \mathrm{BUC}(\mathbb{R}_+, X)$ is said to be *totally (uniformly) ergodic* if the function $e^{i\theta \cdot} f(\cdot)$ is uniformly ergodic for all $\theta \in \mathbb{R}$. Since f is uniformly bounded, this is also equivalent to the existence of the Cesáro limit

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t e^{i\theta s} f(\cdot + s)\, ds$$

in $\mathrm{BUC}(\mathbb{R}_+, X)$ (see [2]).

We introduce now the classes denoted by BUC_r, P_q, AP, $AAP_r{}^+$, W_r and W_r^+, for which results in Section 5 hold. Given a function space $\mathcal{F}(\mathbb{J}, X) \subset \mathrm{BC}(\mathbb{J}, X)$, we set

$$\mathcal{F}_r(\mathbb{J}, X) := \{f \in \mathcal{F}(\mathbb{J}, X) : \text{f has relatively compact range}\}.$$

If X is a finite dimensional space these two spaces coincide.

In particular, the following subspaces of $\mathrm{BUC}(\mathbb{R}, X)$ will be considered:

- the class $\mathrm{BUC}_r(\mathbb{R}, X)$ of all functions in $\mathrm{BUC}(\mathbb{R}, X)$ with relatively compact range (denoted by *b.u.c.r.*);
- the set $P_q(\mathbb{R}, X)$ of all periodic maps with period q from $\mathbb{R}$ to X (denoted by $q - p.$);
- the class $AP(\mathbb{R}, X)$ of all almost periodic functions from $\mathbb{R}$ to X;
- the class $AAP^+(\mathbb{R}, X)$ of all functions $f \in \mathrm{BUC}(\mathbb{R}, X)$ such that the restriction $f_{|\mathbb{R}_+}$ is a.a.p.;
- the set $AAP_r{}^+(\mathbb{R}, X)$ of all functions (denoted by *a.a.p.r.*) $f \in AAP^+(\mathbb{R}, X)$ with relatively compact range;
- the space $W(\mathbb{R}, X)$ of all Eberlein weakly almost periodic maps;
- the space $W_r(\mathbb{R}, X)$ of all Eberlein weakly almost periodic maps with relatively compact range (denoted by $E - w.a.p.r.$);
- the class $W^+(\mathbb{R}, X)$ of all functions $f \in \mathrm{BUC}(\mathbb{R}, X)$ such that the restriction $f_{|\mathbb{R}_+}$ is weakly asymptotically almost periodic in the sense of Eberlein;
- the class $W_r^+(\mathbb{R}, X)$ of functions in $W^+(\mathbb{R}, X)$ with relatively compact range (denoted by $E - w.a.a.p.r$).

All these subspaces are Banach spaces endowed with the supremum norm. Observe that, if $f \in BC(\mathbb{J}, X)$, then f is uniformly continuous on the compacta of $\mathbb{J}$, but not necessarily on the whole set $\mathbb{J}$. Thus the fact that f belongs to $\mathrm{BUC}(\mathbb{J}, X)$ (and, in particular, to $\mathrm{BUC}_r(\mathbb{J}, X)$) defines a particular type of asymptotic behaviour.

3. The convolution product approach

Let $\mathcal{T} := (T(t))_{t\geq 0}$ be a strongly continuous semigroup of bounded linear operators on X with generator $(A, D(A))$. The symbol X_1 will denote the Sobolev space of order one associated to $\mathcal{T}$, that is the Banach space $\big(D(A), ||\cdot||_1\big)$, where $||x||_1 := ||x|| + ||Ax||$ for every $x \in D(A)$.

We will use, in a slightly modified form, a perturbation result of Räbiger, Rhandi, Schnaubelt and Voigt ([23, Thm 3.4]). For a nice presentation of this result, we refer to [8, p.157] as well.

Take a family $\mathcal{B}$ of linear closed operators $(B(t), D(B(t)))_{t\geq 0}$, such that for all $s \geq 0$ and $x \in D(A)$ the following assumptions are satisfied:

(1) $T(\delta)x$ belongs to $D(B(s+\delta))$; for a.e. $\delta \geq 0$;
(2) the map $B(\cdot + s)T(\cdot)x$ is measurable on $[0, +\infty)$;
(3) $\int_0^{+\infty} ||B(s+\sigma)T(\sigma)x||d\sigma \leq q||x||$ for some constant $0 \leq q < 1$.

We recall that assumption (3) was firstly introduced in the autonomous case in [19] and [31], and in the non-autonomous case in [22].

It follows from [23, Thm 3.4 c)] (by choosing $U(t,s) = T(t-s)$, $t \geq s$) that, under these assumptions, there exists a unique exponentially bounded evolution family $\mathcal{U}_B = (U_B(t,s))_{t\geq s}$ such that, for every $x \in X$ and $s \in \mathbb{R}_+$, $U_B(t,s)x \in D(B(t))$ for a.e. $t \geq s$, $B(\cdot)U_B(\cdot, s)x$ is locally integrable and the following variation of constants formula holds

$$U_B(t,s)x = T(t-s)x + \int_s^t T(t-\sigma)B(\sigma)U_B(\sigma,s)xd\sigma\,, \quad t \geq s. \tag{3.1}$$

Moreover, it is proved in [23] that (3) holds for every $x \in X$.

We assume in this section that every orbit $t \mapsto T(t)x$, $x \in X$, of the semigroup $\mathcal{T}$ belongs to a homogeneous subspace $\mathcal{E}$ of the space $\mathrm{BUC}(\mathbb{R}_+, X)$ and we prove that under the hypotheses (1), (2) and (3) the perturbed evolution family $\mathcal{U}_B$ inherits the same asymptotic behaviour of $\mathcal{T}$.

From a technical point of view the following lemma ([5, Lemma 3.4]) will be crucial in the proof of Theorem 3.2, which is the main result in this section.

Lemma 3.1. *Let $L : \mathbb{R}_+ \to \mathcal{L}(X)$ be a bounded strongly continuous function. Let $\mathcal{E}$ be a homogeneous subspace of $BUC(\mathbb{R}_+, X)$, such that the functions $\mathbb{R}_+ \ni t \mapsto L(t)x \in X$ belong to $\mathcal{E}$ for every $x \in X$.*

If $g \in L^1(\mathbb{R}, X)$, then $(L * g)_{|\mathbb{R}_+} \in \mathcal{E}$.

Here, the convolution between L and g is defined as

$$(L * g)(t) = \int_0^{+\infty} L(s)g(t-s)ds = \int_{-\infty}^{t} L(t-s)g(s)ds\,,\ t \geq 0\,.$$

We can now prove the main result of this section.

Theorem 3.2. *Let $\mathcal{T} := (T(t))_{t\geq 0}$ be a uniformly bounded, strongly continuous semigroup generated by $(A, D(A))$ and let $(B(t), D(B(t)))_{t\geq 0}$ be a family of linear closed operators satisfying conditions (1), (2) and (3).*

If $\mathcal{E}$ is a homogeneous subspace of $BUC(\mathbb{R}_+, X)$ and if the maps, from $\mathbb{R}_+$ to X, $t \mapsto T(t)x$ belong to $\mathcal{E}$ for all $x \in X$, then the functions $t \mapsto U_B(t,0)x$ belong to $\mathcal{E}$ for all $x \in X$.

Proof. For $x \in X$, we have

$$U_B(t,0)x = T(t)x + \int_0^t T(t-\sigma)B(\sigma)U_B(\sigma,0)x\,d\sigma,\ t \geq 0.$$

Let $f_x(t) := \int_0^t T(t-\sigma)B(\sigma)U_B(\sigma,0)x d\sigma$. Then, since the function $t \mapsto T(t)x$ belongs to $\mathcal{E}$, it suffices to show that the map f_x, from $\mathbb{R}_+$ to X, belongs to $\mathcal{E}$ as well. Observe that $f_x(t) = (\mathcal{T} * g_x)(t)$, $t \geq 0$, where g_x is the function defined by

$$g_x(t) := \begin{cases} B(t)U_B(t,0)x, & t \geq 0 \\ 0, & t < 0. \end{cases}$$

Hence, by Lemma 3.1, it is sufficient to show that g_x belongs to $L^1(\mathbb{R}, X)$. From [23, Thm. 3.4c)] we have that g_x is locally integrable and hence measurable. Now from variation of constants formula, from the closedness of the operators $B(t)$ and assumption (3), we have

$$\begin{aligned} &\int_0^t ||B(s)U_B(s,0)x||ds \\ &\leq \int_0^t ||B(s)T(s)x||ds + \int_0^t \int_0^s ||B(s)T(s-\tau)B(\tau)U_B(\tau,0)x||d\tau ds \\ &\leq q\|x\| + \int_0^t \int_0^{t-\tau} ||B(s+\tau)T(s)B(\tau)U_B(\tau,0)x||ds d\tau \end{aligned}$$

$$\leq q||x|| + q\int_0^t ||B(s)U_B(s,0)x||ds.$$

Now since $0 \leq q < 1$ it holds

$$\int_0^t ||B(s)U_B(s,0)x||ds \leq \frac{q}{1-q}\|x\|.$$

Therefore, $g_x \in L^1(\mathbb{R}, X)$. As a consequence of Lemma 3.1, the function $t \longmapsto U_B(t,0)x$ belongs to $\mathcal{E}$. □

Observe that, if every orbit of $\mathcal{T}$ belongs to some closed, translation invariant subspace of $\mathrm{BUC}(\mathbb{R}_+, X)$, then $\mathcal{T}$ is, as a consequence of the uniform boundedness principle, uniformly bounded. Thus, if $(U_B(t,s))_{t\geq s\geq 0}$ inherits the same asymptotic behaviour as $\mathcal{T}, \mathcal{U}_B$ is uniformly bounded as well. The fact that, if $\mathcal{T}$ is uniformly bounded, then $\mathcal{U}_B$ is also uniformly bounded under assumptions (1), (2) and (3), could also be proved more directly and independently on Theorem 3.2 by means of the representation of $\mathcal{U}_B$ through the Dyson-Phillips series. In fact, by adapting semigroup theory techniques, it is not difficult to see that the "perturbed" evolution family $\mathcal{U}_B$ can be expressed by means of the so-called Dyson-Phillips series

$$U_B(t,s) = \sum_{n=0}^{\infty} U_n(t,s)\ , t \geq s \geq 0, \tag{3.2}$$

where $U_0(t,s)x := T(t-s)x$ and $U_n(t,s)x := \int_s^t U_{n-1}(t,\sigma)B(\sigma)T(\sigma-s)xd\sigma$ for all $n \geq 1$ and $x \in D(A)$. Exactly as in semigroups case, this series converges uniformly on the compacta $\{(t,s) : 0 \leq s \leq t \leq T\}$, for each $T > 0$. Following the proof of Theorem 2.1 in [7], one can prove that under the assumptions (1), (2) and (3) the series converges uniformly on the whole set $\{(t,s) : t \geq s \geq 0\}$ and that the evolution family $\mathcal{U}_B$ is bounded as well.

4. APPLICATION TO NON-AUTONOMOUS PARTIAL DIFFERENTIAL EQUATIONS WITH DELAY

Let $(A, D(A))$ be the infinitesimal generator of a strongly continuous, uniformly bounded semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Consider the following non-autonomous differential equation with delay

$$\begin{cases} u'(t) = Au(t) + L(t)u_t, & t \geq s \geq 0, \\ u(s) = x, \quad u_s = f, \end{cases} \tag{DEs}$$

where

- $x \in X$,

- $f \in L^p([-1,0],X)$, for some $1 \le p < \infty$,
- $\big(L(t)\big)_{t\ge 0}$ is a family of bounded linear operators from $L^p([-1,0],X)$ to X such that $L(\cdot) \in L^\infty_{loc}(\mathbb{R}_+, \mathcal{L}_s(L^p([-1,0],X),X))$,
- $u : [s-1,\infty) \to X$ and $u_t : [-1,0] \to X$, $t \ge s$, is defined by $u_t(\sigma) := u(t+\sigma)$ for all $\sigma \in [-1,0]$.

Equations of type (DEs) have recently been studied in [25] using extrapolation methods. However, in order to apply the results of the previous section, we need a different setting.

In order to study the well-posedness of (DEs), we proceed in analogy to [4] and consider the Banach space $\mathcal{X} := X \times L^p([-1,0],X)$, and the family of operators

$$\mathcal{A}(t) := \begin{pmatrix} A & L(t) \\ 0 & \frac{d}{d\tau} \end{pmatrix}, \ t \ge 0,$$

with domains

$$D(\mathcal{A}(t)) := \left\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(A) \times W^{1,p}([-1,0],X) \ : \ f(0) = x \right\}, \quad t \ge 0\,.$$

For each $t \ge 0$, the operator $\mathcal{A}(t)$ can be written as

$$\mathcal{A}(t) = \mathcal{A} + \mathcal{B}(t) = \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\tau} \end{pmatrix} + \begin{pmatrix} 0 & L(t) \\ 0 & 0 \end{pmatrix}.$$

The operator $\mathcal{A}$ generates the strongly continuous semigroup $(\mathbf{T}(t))_{t\ge 0}$ on $\mathcal{X}$, given by

$$\mathbf{T}(t) := \begin{pmatrix} T(t) & 0 \\ T_t & T_0(t) \end{pmatrix}, \tag{4.1}$$

where $(T_0(t))_{t\ge 0}$ is the nilpotent left translation semigroup on $L^p([-1,0],X)$ and $T_t : X \to L^p([-1,0],X)$ is defined as

$$(T_t\, x)(\sigma) := \begin{cases} T(t+\sigma)x, & \sigma > -t, \\ 0 & \text{otherwise}. \end{cases}$$

Observe that $\big(\mathcal{B}(t)\big)_{t\ge 0}$ defines a family of bounded linear operators on $\mathcal{X}$, since

$$||\mathcal{B}(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)||_{\mathcal{X}} = ||\left(\begin{smallmatrix} L(t)f \\ 0 \end{smallmatrix}\right)||_{\mathcal{X}} = ||L(t)f||_X \le ||L(t)|| \cdot ||f||_{L^p([-1,0],X)}.$$

Since $L(\cdot) \in L^\infty_{loc}(\mathbb{R}_+, \mathcal{L}_s(L^p([-1,0],X),X))$ then the assumptions (1) and (2) of Section 3 are satisfied and from [24, Thm. 2.1] there is a unique "perturbed" evolution family $\big(U_B(t,s)\big)_{t\ge s\ge 0}$ on $\mathcal{X}$, related to $(\mathbf{T}(t))_{t\ge 0}$ by the variation of constants formula (3.1).

We will say that a function $u : [s-1, \infty) \longrightarrow X$ is a *mild solution* of the delay equation (DEs) if the map $r \mapsto L(r)u_r$ is locally integrable on $[s, \infty)$ and

$$u(t) = \begin{cases} T(t-s)x + \int_s^t T(t-r)L(r)u_r \, dr & \text{for every } t \geq s, \\ x & \text{for } t = s, \\ f(t-s) & \text{for a.e. } t \in [s-1, s). \end{cases} \tag{4.2}$$

In particular, we note that u is continuous for $t \geq s$.

In the following proposition we relate the solutions of the delay differential equation with the perturbed evolution family $\big(U_B(t,s)\big)_{t \geq s \geq 0}$. Let $\pi_1 : \mathcal{X} \longrightarrow X$ be the projection into the first variable, $\pi_1 \binom{x}{f} := x$.

Proposition 4.1. *Let* $\binom{x}{f} \in \mathcal{X}$ *and let* $u : [s-1, \infty) \to X$ *be the function defined by*

$$u(t) := \begin{cases} \pi_1(U_B(t,s)\binom{x}{f}), & t \geq s, \\ x & \textit{for } t = s, \\ f(t-s) & \textit{for a.e. } t \in [s-1, s). \end{cases} \tag{4.3}$$

Then u is a mild solution of Equation (DEs).

Conversely, if $u : [s-1, \infty) \to X$ is a mild solution of Equation (DEs) then for each $\binom{x}{f} \in \mathcal{X}$, we have

$$U_B(t,s)\binom{x}{f} = \binom{u(t)}{u_t}, \; t \geq s \geq 0.$$

In particular, we have existence and uniqueness of the mild solution of Equation (DEs).

Proof. Let $\binom{x}{f} \in \mathcal{X}$, we define the function $\binom{u}{v} : [s, \infty) \to \mathcal{X}$ by

$$\binom{u(t)}{v(t)} := U_B(t,s)\binom{x}{f}, \quad t \geq s.$$

Then, from Section 3, the map $t \longmapsto \mathcal{B}(t)\binom{u(t)}{v(t)}$ is locally integrable and the variation of constants formula (3.1) is satisfied, that means

$$\begin{aligned} \binom{u(t)}{v(t)} &= \mathbf{T}(t-s)\binom{x}{f} + \int_s^t \mathbf{T}(t-r)\mathcal{B}(r)\binom{u(r)}{v(r)} \, dr \\ &= \begin{pmatrix} T(t-s)x \\ T_{t-s}x + T_0(t-s)f \end{pmatrix} + \int_s^t \mathbf{T}(t-r) \begin{pmatrix} 0 & L(r) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u(r) \\ v(r) \end{pmatrix} dr \\ &= \begin{pmatrix} T(t-s)x \\ T_{t-s}x + T_0(t-s)f \end{pmatrix} + \int_s^t \begin{pmatrix} T(t-r) & 0 \\ T_{t-r} & T_0(t-r) \end{pmatrix} \begin{pmatrix} L(r)v(r) \\ 0 \end{pmatrix} dr \end{aligned}$$

$$= \begin{pmatrix} T(t-s)x \\ T_{t-s}x + T_0(t-s)f \end{pmatrix} + \int_s^t \begin{pmatrix} T(t-r)L(r)v(r) \\ T_{t-r}L(r)v(r) \end{pmatrix} dr.$$

If we show that $u_t = v(t)$ for all $t \geq s$ then the function u defined in (4.2) will satisfy (4.3). To this purpose, we need the following Lemma that was essentially proved in [20, Thm. 4.2] and that we quote from [29, Lemma 2.11].

Lemma 4.2. *Let X be a Banach space and let I be an interval in $\mathbb{R}$. Assume that $f : [c,d] \to L^p(I,X)$, $1 \leq p < \infty$, is Bochner integrable and that $f(t) = g(t,\cdot)$ for a.e. $t \in [c,d]$, where $g : [c,d] \times I \to X$ is measurable. Then $g(\cdot,\sigma)$ is Bochner integrable and*

$$\Big(\int_c^d f(t)\,dt\Big)(\sigma) = \int_c^d g(t,\sigma)\,dt$$

for a.e. $s \in I$.

Now we have that

$$\begin{aligned} v(t)(\sigma) &= (T_{t-s}x + T_0(t-s)f)(\sigma) + \Big(\int_s^t T_{t-r}L(r)v(r)\,dr\Big)(\sigma) \\ &= (T_{t-s}x + T_0(t-s)f)(\sigma) + \int_s^t (T_{t-r}L(r)v(r))(\sigma)\,dr, \end{aligned}$$

where

$$(T_{t-r}L(r)v(r))(\sigma) = \begin{cases} T(t-r+\sigma)L(r)v(r) & \text{for } t-r+\sigma > 0, \\ 0 & \text{otherwise.} \end{cases}$$

So, for $t+\sigma \leq s$ we have that $\int_s^t (T_{t-r}L(r)v(r))(\sigma)\,dr = 0$, thus $u_t(\sigma) = u(t+\sigma) = f(t+\sigma) = v(t)(\sigma)$ for a.e. $\sigma \in [-1,0]$.

While for $t+\sigma > s$, we obtain

$$v(t)(\sigma) = T(t+\sigma-s)x + \int_s^{t+\sigma} T(t-r+\sigma)L(r)v(r)\,dr = u(t+\sigma)$$

for a.e. $\sigma \in [-1,0]$. The converse follows by the definition of a mild solution given by (4.2) and by the same computations done above. □

It will now be shown that the asymptotic behaviour of the semigroup $\mathcal{T}$ forces the asymptotic behaviour of $\mathcal{U}_B$. In particular, integral conditions of Miyadera type shall be explicited for the stability of properties like asymptotic almost periodicity (even in the sense of Eberlein), uniform ergodicity and total uniform ergodicity, by means of the method illustrated in Section 3. In particular, the following result can be stated.

Theorem 4.3. *Assume that there exists a constant $0 \leq q < 1$ such that*

$$\int_0^\infty \|L(t+s)(T_s\, x + T_0(s)\, f)\|\, ds \leq q\, \left\|\binom{x}{f}\right\|$$

for all $t \geq 0$ and all $\binom{x}{f} \in D(\mathcal{A})$. Assume that for every $x \in X$ the map $\mathbb{R}_+ \ni t \mapsto T(t)x$ is

(1) *continuous and vanishing at infinity, or*
(2) *asymptotically almost periodic, or*
(3) *uniformly ergodic, or*
(4) *totally uniformly ergodic.*

Then $\mathbb{R}_+ \ni t \longmapsto U_B(t,0)\binom{x}{f}$ belongs, for every $\binom{x}{f} \in \mathcal{X}$, to the same class of the map $\mathbb{R}_+ \ni t \longmapsto T(t)x$.

Proof. From the definition of $\mathbf{T}(t)$, we have

$$\int_0^\infty ||\mathcal{B}(t+s)\mathbf{T}(s)\binom{x}{f}||\, ds = \int_0^\infty ||L(t+s)(T_s\, x + T_0(s) f)||\, ds$$

for all $t \geq 0$. Then $(\mathbf{T}(t))_{t\geq 0}$ and $\big(\mathcal{B}(t)\big)_{t\geq 0}$ satisfy assumptions of Theorem 3.2. Furthermore, it follows [7, Lemma 4.1] that $t \mapsto \mathbf{T}(t)\binom{x}{f}$ belongs to the same class as $t \mapsto T(t)x$ for all $\binom{x}{f} \in \mathcal{X}$. We conclude, by Theorem 3.2 that the map $t \longmapsto U_B(t,0)\binom{x}{f}$ belongs to the same class of $t \mapsto \mathbf{T}(t)\binom{x}{f}$ for all $\binom{x}{f} \in \mathcal{X}$. □

At the light of Proposition 4.1, Theorem 4.3 yields the asymptotic behaviour of the solution of the delay equation (DE0).

Corollary 4.4. *Under the assumptions of Theorem* 4.3, *the mild solution of Equation (DE0)* $(s = 0)$ *belongs to the same class as the map $\mathbb{R}_+ \ni t \mapsto T(t)x$, for every initial value $\binom{x}{f} \in X \times L^p([-1,0], X)$.*

For the asymptotic behaviour of mild solutions of inhomogeneous non-autonomous partial differential delay equations, we refer to [10].

Example 4.5. As an example of a non-autonomous delay equation satisfying the conditions of Theorem 4.3, we take the operators $(L(t))_{t\geq 0}$ given by

$$L(t)f := \int_{-1}^0 k(t,\sigma) f(\sigma)\, d\sigma,$$

where $k \in L^\infty(\mathbb{R}_+ \times [-1,0], \mathcal{L}_s(X))$. In this case, if the operator $(A, D(A))$ generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ such that the maps $t \mapsto T(t)x$ are continuous and vanishing at infinity, or asymptotically almost periodic, or uniformly ergodic, or totally uniformly ergodic, and if there exists a constant $0 \le q < 1$ such that

$$\int_0^\infty \| \int_{-1}^0 k(t+s,\sigma)(T_s x + T_0(s) f)\, d\sigma \| \, ds \le q \, \left\| \binom{x}{f} \right\|$$

for all $t \ge 0$ and all $\binom{x}{f} \in D(\mathcal{A})$, then the mild solution of the delay equation

$$\begin{cases} u'(t) = Au(t) + \int_{-1}^0 k(t,\sigma) u(t+\sigma)\, d\sigma, & t \ge 0, \\ u(0) = x, \quad u_0 = f, \end{cases}$$

has the same asymptotic behaviour as the semigroup $(T(t))_{t\ge 0}$ for all initial values $\binom{x}{f} \in \mathcal{X}$.

5. The evolution semigroup approach

In this section we consider the permanence under bounded perturbations of the asymptotic properties corresponding to the spaces: BUC_r, P_q, AP, $AAP_r{}^+$, W_r and W_r^+.

As remarked in [12, Section 2], if we consider the non-autonomous problem $u'(t) = A(t)u(t)$, $t \in \mathbb{R}$, the periodicity of $A(t)$ (that is the condition that $A(t+q) = A(t)$ for some $q > 0$ and for all $t \in \mathbb{R}$) leads to the periodicity of the evolution family $\mathcal{U}$ related to $A(t)$, in the sense that $U(t+q, s+q) = U(t,s)$ for every $t \ge s$. This observation motivates the following definition.

If $\mathcal{U} := (U(t,s))_{t\ge s \in \mathbb{R}}$ is an evolution family on X, we say that $\mathcal{U}$ satisfies one of the asymptotic properties listed at the beginning of this section if the map, from $\mathbb{R}$ to X, $U(t+\cdot, s+\cdot)x$ has the corresponding property for every $t \ge s \in \mathbb{R}$ and $x \in X$. Define now

$$\big(T(t)f\big)(\cdot) := U(\cdot, \cdot - t) f(\cdot - t) \tag{5.1}$$

for $t \ge 0$ and $f \in \mathrm{BC}(\mathbb{R}, X)$. It is well-known (see [9, Section VI.9], [12, Section 2.2]) that $\mathcal{T} := (T(t))_{t\ge 0}$ is a (not necessarily strongly continuous) semigroup of linear operators on $\mathrm{BC}(\mathbb{R}, X)$. Given a function space $\mathcal{F}(\mathbb{R}, X) \subseteq \mathrm{BC}(\mathbb{R}, X)$, $\mathcal{T}$ is called the *evolution semigroup* induced by $\mathcal{U}$ on $\mathcal{F}(\mathbb{R}, X)$ if $\mathcal{T}$ is strongly continuous on $\mathcal{F}(\mathbb{R}, X)$ and $T(t)\mathcal{F}(\mathbb{R}, X) \subseteq \mathcal{F}(\mathbb{R}, X)$ for all $t \ge 0$.

Denote by $(G, D(G))$ the infinitesimal generator of the evolution semigroup $\mathcal{T}$ on $\mathcal{F}(\mathbb{R}, X)$.

The set $\mathcal{F}(\mathbb{R}, \mathcal{L}_s(X))$ will denote the set of all maps $F : \mathbb{R} \to \mathcal{L}(X)$ such that the functions $\mathbb{R} \ni t \mapsto F(t)x$ satisfy the asymptotic property $\mathcal{F}$ for every $x \in X$.

The following lemma holds.

Lemma 5.1. *Let $F \in \mathcal{F}(\mathbb{R}, \mathcal{L}_s(X))$, where $\mathcal{F}$ stands for BUC_r, P_q, AP, $AAP_r{}^+$, W_r and W_r^+. Then $F(\cdot)f(\cdot) \in \mathcal{F}(\mathbb{R}, X)$ for every $f \in \mathcal{F}(\mathbb{R}, X)$.*

Proof. The cases $\mathcal{F} = P_q$, AP, $AAP_r{}^+$, W_r and W_r^+ are proved in [13, Corollary 2.5] and [12, Lemma 1.1.11]. We prove the case $\mathcal{F} = \mathrm{BUC}_r$.

Observe, first of all, that $||F(t)|| \le M$ for all $t \in \mathbb{R}$, for some $M > 0$, as a consequence of the Banach-Steinhaus theorem. Thus $t \mapsto F(t)f(t)$ is bounded. Moreover, this map is uniformly continuous, since, if $x \in X$ and $t, h \in \mathbb{R}$, then

$$\begin{aligned}
&\lim_{h\to 0} \sup_{t\in\mathbb{R}} ||F(t+h)f(t+h) - F(t)f(t)|| \\
&\le \lim_{h\to 0} \sup_{t\in\mathbb{R}} ||F(t+h)\big(f(t+h) - f(t)\big)|| + \lim_{h\to 0} \sup_{t\in\mathbb{R}} ||\big(F(t+h) - F(t)\big)f(t)|| \\
&\le M \lim_{h\to 0} \sup_{t\in\mathbb{R}} ||f(t+h) - f(t)|| + \lim_{h\to 0} \sup_{t\in\mathbb{R}} ||\big(F(t+h) - F(t)\big)f(t)||,
\end{aligned}$$

and both limits in the last expression are zero, the first one as a consequence of the uniform continuity of f, the latter one in view of the relative compactness of the range of f and of the uniform continuity of $F(\cdot)x$ for all $x \in X$.

Finally, it is very easy to show that $F(\cdot)f(\cdot)$ has relatively compact range, and we give the proof only for the sake of completeness. Let $\{t_n\}$ be any sequence in $\mathbb{R}$. There exists a subsequence $\{t_{n_j}\}$, such that $\{f(t_{n_j})\}$ converges to some $y \in X$. Since $t \mapsto F(t)y$ has relatively compact range, there exists a subsequence $\{t_{n_{j_k}}\}$, such that $\{F(t_{n_{j_k}})y\}$ converges to some $z \in X$, when $k \to +\infty$. Thus $\{F(t_{n_{j_k}})f(t_{n_{j_k}})\}$ converges to z, for $k \to +\infty$, proving the thesis. □

It is worth noticing that the assumption on the relative compactness of the range of the maps is essential in the proof of the previous lemma, as shown in [13, Section 2.3].

In the following, a central role will be played by this result.

Proposition 5.2. *Let $\mathcal{U} := (U(t,s))_{t\ge s}$ be an exponentially bounded evolution family on a Banach space X. The following assertions are equivalent.*

(1) *The evolution family is b.u.c.r., a.p., a.a.p.r., E-w.a.p.r., E-w.a.a.p.r., respectively, and*

$$\lim_{t\to 0} U(s, s-t)x = x \text{ for all } x \in X$$

uniformly for $s \in \mathbb{R}$;

(2) *$\mathcal{T}$ is an evolution semigroup on $BUC_r(\mathbb{R}, X)$, $AP(\mathbb{R}, X)$, $AAP_r{}^+(\mathbb{R}, X)$, $W_r(\mathbb{R}, X)$ and $W_r^+(\mathbb{R}, X)$ respectively.*

Proof. Also in this case, one can find the proof for the classes AP, ${AAP_r}^+$, W_r and W_r^+ in [12, Prop. 2.2.2]. We consider the case $\mathcal{F} = \mathrm{BUC}_r$ and we prove firstly that (1) implies (2). The semigroup $\mathcal{T}$, defined by (5.1), acts on $\mathrm{BUC}_r(\mathbb{R}, X)$, whenever $U(t+\cdot, s+\cdot)x \in \mathrm{BUC}_r(\mathbb{R}, X)$ for all $t \geq s$, $x \in X$, in view of Lemma 5.1. Moreover, it holds

$$\begin{aligned}
&\lim_{t\to 0} \sup_{s\in\mathbb{R}} ||\big(T(t)f\big)(s) - f(s)|| = \lim_{t\to 0} \sup_{s\in\mathbb{R}} ||U(s, s-t)f(s-t) - f(s)|| \\
&\leq \lim_{t\to 0} \sup_{s\in\mathbb{R}} ||U(s, s-t)f(s-t) - U(s, s-t)f(s)|| \\
&\quad + \lim_{t\to 0} \sup_{s\in\mathbb{R}} ||U(s, s-t)f(s) - f(s)|| \\
&\leq M \lim_{t\to 0} \sup_{s\in\mathbb{R}} ||f(s-t) - f(s)|| + \lim_{t\to 0} \sup_{s\in\mathbb{R}} ||U(s, s-t)f(s) - f(s)||;
\end{aligned}$$

the first limit in the last term is zero, since f is uniformly continuous, the second one is zero as well, since f has relatively compact range and $\lim_{t\to 0} U(s, s-t)x = x$ for all $x \in X$, uniformly for $s \in \mathbb{R}$. Thus $\mathcal{T}$ is strongly continuous, entailing that $\mathcal{T}$ is an evolution semigroup on $\mathrm{BUC}_r(\mathbb{R}, X)$.

To prove that (2) implies (1) it suffices to observe, as in [12, Lemma 2.2.1], that $U(s, s-t)x = \big(T(t)\mathbf{1}\otimes x\big)(s)$, where $(\mathbf{1}\otimes x)(t) = x$ for all $t \in \mathbb{R}$. $\square$

We remark that the condition on the limit of $U(s, s-\cdot)x$ was introduced by J. Kreulich [13, Sections 2.30 and 2.31] and is necessary for the strong continuity of the semigroup $\mathcal{T}$, with the exception of the case $\mathcal{F} = P_q$, as the following proposition (see [12, Prop. 2.2.1]) shows.

Proposition 5.3. *Let $\mathcal{U} := (U(t,s))_{t\geq s}$ be an exponentially bounded evolution family on a Banach space X. The following assertions are equivalent.*

(1) *The evolution family is q-periodic.*
(2) *$\mathcal{T}$ is an evolution semigroup on $P_q(\mathbb{R}, X)$.*

In the following, we shall need a characterization of multiplication operators on the space $\mathrm{BUC}_r(\mathbb{R}, X)$. Recall that, if $\mathcal{F}$ stands for BUC_r, P_q, AP, ${AAP_r}^+$, W_r and W_r^+, an operator $T \in \mathcal{L}(\mathcal{F}(\mathbb{R}, X))$ is said to be a multiplication operator on $\mathcal{F}(\mathbb{R}, X)$ if there exists a function $F : \mathbb{R} \to \mathcal{L}(X)$ such that $\big(Tf\big)(s) = F(s)f(s)$ for every $f \in \mathcal{F}(\mathbb{R}, X)$ and $s \in \mathbb{R}$.

Lemma 5.4. *Let T belong to $\mathcal{L}(BUC_r(\mathbb{R}, X))$. The following assertions are equivalent.*

(1) *The operator T is a multiplication operator on $BUC_r(\mathbb{R}, X)$.*
(2) *$T(\varphi f) = \varphi T f$ for all $f \in BUC_r(\mathbb{R}, X)$ and $\varphi \in BUC(\mathbb{R}, \mathbb{C})$.*
(3) *If $f \in BUC_r(\mathbb{R}, X)$ is such that $f(s) = 0$ for some $s \in \mathbb{R}$, then $\big(Tf\big)(s) = 0$.*

Proof. The proof that (1)⇒ (2) and (3)⇒ (1) is the same as in [12, Theorem 3.1.4], if one observes that, if $f \in \mathrm{BUC}_r(\mathbb{R}, X)$ and $\varphi \in \mathrm{BUC}(\mathbb{R}, \mathbb{C})$, then $\varphi f \in \mathrm{BUC}_r(\mathbb{R}, X)$.

(2)⇒ (3). Let $f \in \mathrm{BUC}_r(\mathbb{R}, X)$ be such that $f(s) = 0$ for some $s \in \mathbb{R}$. Take a sequence of functions $\{\varphi_n\}$, such that $\varphi_n(s) = 1$ for all $n \in \mathbb{N}$ and $\mathrm{supp}\varphi_n \subseteq [s - \frac{1}{n}, s + \frac{1}{n}]$. Now the thesis follows as in [9, Prop. VI.9.13]. □

We now present a characterization of evolution semigroups on the space of almost periodic functions.

In the following, the symbol $(T_r(t))_{t\geq 0}$ will denote the translation group on $\mathcal{F}(\mathbb{R}, \mathbb{C})$.

Denote by $\tilde{\mathcal{F}}$ the set

$$\tilde{\mathcal{F}} := \{ f \text{ differentiable } : f\,, f' \in \mathcal{F}(\mathbb{R}, \mathbb{C})\,\} \subseteq \mathcal{F}(\mathbb{R}, \mathbb{C}),$$

if $\mathcal{F}$ stands for BUC_r, AP, ${AAP_r}^+$, W_r and W_r^+. In particular, when $\mathcal{F} = AP$, it results

$$\tilde{\mathcal{F}} = \{ f \in \mathrm{BUC}(\mathbb{R}, \mathbb{C}) : \, f \in AP\,, f' \in \mathrm{BUC}(\mathbb{R}, \mathbb{C})\},$$

at the light of [1, Prop. 1.VI].

Proposition 5.5. *Let $\mathcal{T} := (T(t))_{t\geq 0}$ be a strongly continuous semigroup on $\mathcal{F}(\mathbb{R}, X)$, where $\mathcal{F}$ stands for BUC_r,P_q, AP, ${AAP_r}^+$, W_r and W_r^+, with infinitesimal generator $(G, D(G))$. Then the following are equivalent*

1. *$\mathcal{T}$ is an evolution semigroup on $\mathcal{F}(\mathbb{R}, X)$.*
2. *$T(t)(\varphi f) = (T_r(t)\varphi)T(t)f$ for all $f \in \mathcal{F}(\mathbb{R}, X)$, $t \geq 0$ and $\varphi \in \mathcal{F}(\mathbb{R}, \mathbb{C})$.*
3. *If $f \in D(G)$ and $\varphi \in \tilde{\mathcal{F}}$, then $\varphi\, f \in D(G)$ and $G(\varphi\, f) = \varphi G f - \varphi' f$.*

Proof. We give the proof for the case $\mathcal{F} = AP$. In this case, the right-translation semigroup $\mathcal{T}_r$ is strongly continuous and its generator $(G_0, D(G_0))$ is given by $G_0\varphi := -\varphi'$, $\varphi \in D(G_0) := \tilde{AP}$.

(1)⇒ (3). If $f \in D(G)$ and $\varphi \in \tilde{AP}$, then

$$\frac{1}{t}\big(T(t)(\varphi\, f) - \varphi\, f\big) = \frac{1}{t}\big(T_r(t)\varphi - \varphi\big)T(t)f + \varphi\frac{1}{t}\big(T(t)f - f\big)$$

converges in AP to $-\varphi' f + \varphi G f$ as $t \to 0$.

(3)⇒ (2). It can be proved exactly as in [9, Theorem VI.9.14], by observing that $\tilde{AP}$ is dense in $AP(\mathbb{R}, \mathbb{C})$ (since the class of all trigonometric polynomials, which is contained in $\tilde{AP}$, is dense in $AP(\mathbb{R}, \mathbb{C})$).

(2)⇒(1). This can be proved as in [9, Theorem VI.9.14] as well, by using the characterization of multiplication operators on $AP(\mathbb{R}, X)$, given by Hutter in [12, Theorem 3.1.5].

The case $\mathcal{F} = \mathrm{BUC}_r$ can be proved by using Lemma 5.4 above, whereas the cases $\mathcal{F} = P_q$, $AAP_r{}^+$, W_r, W_r^+ can be treated in the same way, with the aid of Theorems 3.1.4, 3.1.5 and 3.1.7 in [12]. □

We can now prove the main theorem of this section, giving conditions under which the perturbed evolution family $\mathcal{U}_B$ inherits from $\mathcal{U}$ some asymptotic properties.

Theorem 5.6. *Let $\mathcal{U} := (U(t,s))_{t\geq s}$ be an exponentially bounded evolution family on a Banach space X. Let $B(t)$, $t \in \mathbb{R}$, be bounded linear operators on X. Let $\mathcal{F}$ stand for BUC_r, P_q, AP, $AAP_r{}^+$, W_r and W_r^+. Assume that:*

1. *$\lim_{t\to 0} U(s, s-t)x = x$ for all $x \in X$, uniformly for $s \in \mathbb{R}$ (except in the case $\mathcal{F} = P_q$, where this condition is superfluous).*
2. *$B(\cdot)$ belongs to $\mathcal{F}(\mathbb{R}, \mathcal{L}_s(X))$.*
3. *The map $U(t+\cdot, s+\cdot)x$, from $\mathbb{R}$ to X, belongs to $\mathcal{F}(\mathbb{R}, X)$ for all $x \in X$.*

Then the perturbed evolution family $\mathcal{U}_B := (U_B(t,s))_{t\geq s}$, satisfying

$$U_B(t,s)x = U(t,s)x + \int_s^t U_B(t,\tau)B(\tau)U(\tau,s)x\,d\tau \tag{5.2}$$

for all $t \geq s$ and $x \in X$, is such that the map $U_B(t+\cdot, s+\cdot)x$ belongs to $\mathcal{F}(\mathbb{R}, X)$ for every $x \in X$.

Proof. Let $\mathcal{T} := (T(t))_{t\geq 0}$ be the semigroup defined on $\mathcal{F}(\mathbb{R}, X)$ by (5.1). As a consequence of Proposition 5.2 and assumption (1) for the case $\mathcal{F} = \mathrm{BUC}_r$, AP, $AAP_r{}^+$, W_r and W_r^+, and in view of Proposition 5.3 when $\mathcal{F} = P_q$, $\mathcal{T}$ turns out to be an evolution semigroup on $\mathcal{F}(\mathbb{R}, X)$. Let $(G, D(G))$ be the infinitesimal generator of $\mathcal{T}$ on $\mathcal{F}(\mathbb{R}, X)$.

Define the multiplication operator $\mathcal{B} := B(\cdot)$ on $\mathcal{F}(\mathbb{R}, X)$ by

$$(\mathcal{B}f)(s) := B(s)f(s) \quad \text{for all } s \in \mathbb{R},\ f \in \mathcal{F}(\mathbb{R}, X).$$

It follows from Lemma 5.1 and assumption (2) that $B(\cdot)f(\cdot)$ belongs to $\mathcal{F}(\mathbb{R}, X)$ for every $f \in \mathcal{F}(\mathbb{R}, X)$. Observe that $\mathcal{B} \in \mathcal{L}(\mathcal{F}(\mathbb{R}, X))$, indeed, if $B(\cdot) \in \mathcal{F}(\mathbb{R}, \mathcal{L}_s(X))$, then it holds $\sup_{t\in\mathbb{R}} ||B(t)|| < \infty$ as a consequence of the uniform boundedness principle, and therefore

$$||\mathcal{B}f|| := \sup_{t\in\mathbb{R}} ||B(t)f(t)|| \leq \sup_{t\in\mathbb{R}} ||B(t)|| \cdot ||f||_\infty \text{ for all } f \in \mathcal{F}(\mathbb{R}, X).$$

Thus a well-known result in perturbation theory yields that the operator $(G + \mathcal{B}, D(G))$ generates a strongly continuous semigroup $\mathcal{T}_B := (T_B(t))_{t\geq 0}$ on the Banach space $\mathcal{F}(\mathbb{R}, X)$.

Since $T_B(\cdot)f \in \mathcal{F}(\mathbb{R}, X)$ for all $f \in \mathcal{F}(\mathbb{R}, X)$, the Banach-Steinhaus Theorem yields that $\mathcal{T}_B$ is uniformly bounded.

It will now be shown that $(T_B(t))_{t\geq 0}$ is an evolution semigroup, associated to some evolution family $(V(t,s))_{t\geq s\in\mathbb{R}}$. Take $f \in D(G_B)$ and $\varphi \in \tilde{\mathcal{F}}$, then $f \in D(G)$ and therefore $\varphi f \in D(G)$ and $G(\varphi f) = \varphi G f - \varphi' f$, as a consequence of Proposition 5.5. Thus φf belongs to $D(G_B)$ and

$$G_B(\varphi f) = G(\varphi f) + \mathcal{B}(\varphi f) = \varphi G f - \varphi' f + \varphi \mathcal{B} f = \varphi G_B f - \varphi' f.$$

Hence, Proposition 5.5 entails that $\mathcal{T}_B$ is an evolution semigroup on $\mathcal{F}(\mathbb{R}, X)$. Moreover, the evolution family $(V(t,s))_{t\geq s\in\mathbb{R}}$ associated to $\mathcal{T}_B$ is in $\mathcal{F}(\mathbb{R}, X)$, as a consequence of Proposition 5.2 and of Proposition 5.3.

In view of [24, Thm 2.1] there is a unique exponentially bounded evolution family $(U_B(t,s))_{t\geq s\in\mathbb{R}}$ such that (5.2) holds. It remains to show that $(U_B(t,s))_{t\geq s\in\mathbb{R}} = (V(t,s))_{t\geq s\in\mathbb{R}}$. From the variation of constants formula for the perturbed semigroup $\mathcal{T}_B$ we obtain

$$V(t,s)x = U(t,s)x + \int_s^t V(t,\tau)B(\tau)U(\tau,s)x\,d\tau$$

for all $t \geq s$ and $x \in X$. Hence $(U_B(t,s))_{t\geq s\in\mathbb{R}} = (V(t,s))_{t\geq s\in\mathbb{R}}$. □

The results of this section may also be applied to the study of the asymptotic behaviour of the solutions of the partial differential equation with delay (DEs) for $s \in \mathbb{R}$.

In this case, the unperturbed evolution family is given by

$$\mathcal{U}(t,s) := \mathbf{T}(t-s) \quad \text{for } t \geq s\,, \tag{5.3}$$

where $\mathbf{T}$ is the semigroup on $\mathcal{X}$ defined by (4.1).

Observe that the map $r \mapsto \mathcal{U}(t+r, s+r)\binom{x}{f} = \mathbf{T}(t-s)\binom{x}{f}$ is constant for all fixed $t \geq s$ and $\binom{x}{f} \in \mathcal{X}$, and therefore it belongs to $\mathrm{BUC}_r(\mathbb{R}, \mathcal{X})$, $P_q(\mathbb{R}, \mathcal{X})$, $AP(\mathbb{R}, \mathcal{X})$, $AAP_r{}^+(\mathbb{R}, \mathcal{X})$, $W_r(\mathbb{R}, \mathcal{X})$ and $W_r^+(\mathbb{R}, \mathcal{X})$.

Observe moreover that, if $\mathcal{U}$ is defined by (5.3), then it satisfies condition (1) in Theorem 5.6, since $\mathbf{T}$ is strongly continuous. Finally, we remark that, if $L(\cdot)$ is such that $t \mapsto L(t)f$ belongs to $\mathcal{F}(\mathbb{R}, X)$ for every $f \in L^p([-1,0], X)$, then $\mathcal{B}(\cdot)$ belongs to $\mathcal{F}(\mathbb{R}, \mathcal{L}_s(\mathcal{X}))$, since

$$\mathcal{B}(t)\binom{x}{f} = \binom{L(t)\varphi}{0}, \text{ for all } t \in \mathbb{R}$$

Thus the following can be stated.

Proposition 5.7. *Let* $\mathbf{T}$ *be the strongly continuous semigroup defined by* (4.1). *Let* $\big(L(t)\big)_{t\in\mathbb{R}}$ *be a strongly continuous family of bounded linear operators from* $L^p([-1,0], X)$ *to* X*, such that* $L(\cdot)f$ *belongs, for all* $f \in L^p([-1,0],$

X), to $\mathcal{F}(\mathbb{R}, X)$, where $\mathcal{F}$ stands for BUC_r, P_q, AP, $AAP_r{}^+$, W_r or W_r^+. Then, for all $t \geq s \in \mathbb{R}$, the map $U_B(t+\cdot, s+\cdot)\binom{x}{f}$ belongs, for every $\binom{x}{f} \in \mathcal{X}$, to $\mathcal{F}(\mathbb{R}, \mathcal{X})$.

As a consequence of Proposition 4.1, we obtain the following result.

Corollary 5.8. *Let $\binom{x}{f} \in X \times L^p([-1,0], X)$ and $s \in \mathbb{R}$. Let $u(\cdot, s) : [s-1, \infty) \to X$ be the (unique) mild solution of Equation (DEs). Assume that $L(\cdot)f$ belongs, for all $f \in L^p([-1,0], X)$, to $\mathcal{F}(\mathbb{R}, X)$, where $\mathcal{F}$ stands for BUC_r, P_q, AP, $AAP_r{}^+$, W_r or W_r^+.*

Then, for all $t \geq s \in \mathbb{R}$, the map $\mathbb{R} \ni r \mapsto u(t+r, s+r) \in X$ belongs to the same class $\mathcal{F}(\mathbb{R}, X)$

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