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ASYMPTOTIC ANALYSIS OF A DIFFUSIVE MODEL FOR SHAPE MEMORY ALLOYS WITH CATTANEO-MAXWELL HEAT FLUX LAW*

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Abstract. This work is concerned with a diffusive model for some austenite-martensite phase transition processes ruled by the Cattaneo-Maxwell heat flux law, namely assuming for the heat flux a relaxed version of the classical Fourier law. A rigorous asymptotic analysis of the macroscopic model is performed and it is shown that such model is nothing but a singular perturbation of the standard Frémond model for shape memory alloys. Convergence results are proved along with error estimates.

1. Introduction

This paper deals with a diffusive model for shape memory alloys with thermal memory, for which, in a recent work (cf. [3]), we have discussed existence, uniqueness, and regularity of the solutions. This model, characterized by the Cattaneo-Maxwell heat flux law (cf. [4]) and diffusion for the phase proportions, can be formally obtained just by introducing some relaxation and diffusion terms in the standard Frémond model for shape memory alloys (cf. [9, 10]). Now, our aim is to find a rigorous justification to such modeling procedure by performing an asymptotic analysis of the derived system, as the relaxation and diffusive parameters tend to zero, in order to prove that it converges, in a suitable sense, to the standard Frémond one. Thus, for the sake of convenience, we first present the three-dimensional Frémond model with Fourier heat flux law and no diffusive effect for the phase proportions, then we describe the diffusive model with Cattaneo-Maxwell law, and finally, we investigate the relations between them.

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The phenomenon of shape memory can be ascribed to structural phase transitions between different configurations of the metallic lattice, known as austenite and martensite. Mechanical actions and dynamics of crystalline phases are strongly influenced by temperature. Indeed, shape memory alloys present a strong temperature dependence in their stress-strain relations, till the formation of hysteresis loops. The macroscopic model proposed by Frémond in [9, 10] aims to describe the thermomechanical evolution of a shape memory body in terms of the absolute temperature θ, the small displacements $\mathbf{u}$, and the local proportions of martensite and austenite. Moreover, Frémond's model refers to any dimension of space and allows the different variants to coexist at each point with suitable proportions.

More precisely, two martensitic (β_1, β_2) and one austenitic (β_3) variants are considered. In addition, the total free energy is given by a weighted sum of the specific free energies of each phase and of a mixture term, namely a thermally coupled indicator function forcing the phase variables to attain only meaningful values. As a consequence, it is possible to consider only two independent phase variables χ_1 and χ_2 linearly related to β_1, β_2, and β_3 (see [3, 7] for a detailed argumentation) and such that

$$(\chi_1, \chi_2) \in S := \{(\gamma_1, \gamma_2) \in \mathbf{R}^2 : |\gamma_2| \leq \gamma_1 \leq 1\}. \tag{1.1}$$

Constitutive equations follow from universal conservation laws for energy and momentum, as well as from the second principle of Thermodynamics, and they are supplied with suitable initial and boundary conditions. The resulting system turns out to be highly nonlinear, especially due to three terms in the energy balance coupling temperature, displacements and phase fractions. Here, we restrict our analysis to a linearized energy balance equation and also omit the inertial term in the momentum balance equation. The quasi-static linearized three-dimensional system has been studied in [7], to which we refer for a detailed description of the mathematical model and for specifying the physical meaning of the involved constants ν, λ, μ, ζ, l, θ^*, c_0, and k_0, that will appear soon and are required to be strictly positive. However, let us point out the papers [5, 6, 13] where other more complete versions of the Frémond model have been discussed from the point of view of existence of solutions.

Thus, let us consider a metallic alloy, which is supposed to be isotropic and homogeneous, located in a bounded domain Ω of $\mathbf{R}^3$ of class $C^{1,1}$, and let its boundary $\Gamma = \partial\Omega$ be split into two measurable sets Γ_0 and Γ_1, with strictly positive surface measure for Γ_0. Finally, let us fix some time interval $[0, T]$, with $T > 0$. Assuming an elastic behavior with respect to the strain tensor $\epsilon(\mathbf{u})$, we address the following equilibrium equations

$$\operatorname{div}(-\nu\Delta(\operatorname{div}\mathbf{u})\mathbf{I} + \lambda(\operatorname{div}\mathbf{u})\mathbf{I} + 2\mu\epsilon(\mathbf{u})\alpha(\theta)\chi_2\mathbf{I}) = 0, \tag{1.2}$$

where $\mathbf{I}$ is the identity matrix and α denotes a Lipschitz continuous and bounded function, which is related to the expansion coefficient, at least within a certain range of temperature, and vanishes over the critical point θ_c (the Curie temperature). Observe that no volume forces are applied. As to the phase dynamics, if one considers only linearly dissipative effects, the evolution behavior is described by

$$\zeta\partial_t\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} + \partial I_S(\chi_1,\chi_2) \ni \rho\begin{pmatrix}-l(\theta-\theta^*)/\theta^*\\ -\alpha(\theta)\operatorname{div}\mathbf{u}\end{pmatrix}, \tag{1.3}$$

where " ∂_t " denotes the time derivative, ρ represents the constant density (which is supposed to be the same for the three variants), and ∂I_S is the subdifferential of the indicator function of the convex set S. For the reader's convenience, let us recall that the indicator function $I_S(x) = 0$ if $x \in S$, $I_S(x) = +\infty$ otherwise, plays a role in the mixture term of the total free energy. Besides, it is well known (cf., for instance, [2]) that $\partial I_S(x) = \{y \in \mathbf{R}^2 : \sum_{i=1}^2 y_i(z_i - x_i) \le 0, \forall\, z \in S\}$. To complete the set of equations, we point out that the temperature field obeys the energy balance equation coupled with the heat flux law. As we have already noticed, in the energy balance equation only the linear terms are retained and so, in the spirit of [3], we neglect the mechanically induced heat sources in the energy balance that reads as follows

$$\partial_t e(x,t) + \operatorname{div}\mathbf{q}(x,t) = \rho f(x,t), \tag{1.4}$$

f denoting the external heat source and e the internal energy specified by

$$e(x,t) := \rho(c_0\theta(x,t) - l\chi_1(x,t)). \tag{1.5}$$

If the classical Fourier law is supposed for the heat flux, namely

$$\mathbf{q}(x,t) = -k_0\nabla\theta(x,t), \tag{1.6}$$

combining (1.4) and (1.5) with (1.6) yields the following equation

$$\rho\partial_t(c_0\theta - l\chi_1)(x,t) - k_0\Delta\theta(x,t) = \rho f(x,t), \tag{1.7}$$

where the parameter k_0 introduced in (1.6) is strictly positive. The system (1.2), (1.3), (1.7) is associated with suitable initial and boundary conditions.

Thus, assuming that an external traction $\mathbf{g}$ is applied on Γ_1 and that the part Γ_0 of the boundary is fixed, we have the boundary conditions

$$((-\nu\Delta(\operatorname{div}\mathbf{u}) + \lambda\operatorname{div}\mathbf{u} + \alpha(\theta)\chi_2)\mathbf{1} + 2\mu\epsilon(\mathbf{u}))\cdot\mathbf{n} = \mathbf{g} \quad \text{on } \Gamma_1\times(0,T) \tag{1.8}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_0\times(0,T), \tag{1.9}$$

where $\mathbf{n}$ denotes the outward normal unit vector on the boundary. Besides, no double forces are considered and, consequently, there holds

$$\partial_n(\operatorname{div}\mathbf{u}) = 0 \quad \text{on } \Gamma\times(0,T) \tag{1.10}$$

for the normal derivative of the divergence of $\mathbf{u}$. Concerning the heat flux on the boundary, it appears quite natural to prescribe it by $\mathbf{q}\cdot\mathbf{n} = -h$, and thus from (1.6) we infer the non homogeneous Neumann boundary condition

$$k_0\partial_n\theta = h, \quad \text{on } \Gamma\times(0,T). \tag{1.11}$$

Finally, the initial values to be given are

$$\theta(0) = \theta^0, \qquad (\chi_1(0),\chi_2(0)) = (\chi_1^0,\chi_2^0). \tag{1.12}$$

As we have already pointed out, the initial-boundary value problem (1.2), (1.3), (1.7), (1.8)-(1.12) has been studied in [7], where existence and uniqueness of the solution are proved for a weak formulation. Instead, in [3] we have followed a different approach, assuming the heat flux to be governed by the relaxed version of (1.6) known as Cattaneo-Maxwell law, namely

$$\varepsilon\partial_t\mathbf{q}(x,t) + \mathbf{q}(x,t) = -k_0\nabla\theta(x,t), \qquad (x,t)\in\Omega\times(0,T), \tag{1.13}$$

where ε is a strictly positive parameter accounting for relaxation. Note that this modified law has been generalized in the framework of the theory of materials with thermal memory (cf. [12]). Indeed, by setting

$$k_\varepsilon(t) := \frac{k_0}{\varepsilon}\exp(-t/\varepsilon), \tag{1.14}$$

and supplying (1.13) with the initial condition $\mathbf{q}(0) = \mathbf{q_0}$, relations (1.5), (1.4), (1.13) yield

$$\rho\partial_t(c_0\theta - l\chi_1)(x,t) - \Delta(k_\varepsilon*\theta)(x,t) = \rho f(x,t) - \exp(-t/\varepsilon)\operatorname{div}\mathbf{q_0}(x), \tag{1.15}$$

where "$*$" denotes the usual convolution product over $(0,t)$. Let us note that this approach is compatible with thermodynamics since k_ε is a positive type kernel, namely for any function $v\in L^2(0,T)$ and any t, $\int_0^t(k_\varepsilon*v)v\geq 0$ (cf. [11]). Besides, the hyperbolic structure of (1.15) can be easily put in evidence, taking into account that $k_0>0$ in (1.13). The introduction of thermal memory gives rise to some mathematical difficulties strictly connected with this assumption, and the existence of solutions for the resulting system seems not easy to be proved. This fact, along with other arguments on the dissipative behavior, induced us to mollify (1.3) for a smoother evolution of the phase proportions. Hence, we include some diffusion effects following, in this respect, the approach of [13], where some further physical motivation is presented. Thus, fixing a diffusive parameter $\eta > 0$, we replace (1.3) by

$$\zeta \partial_t \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} - \eta \begin{pmatrix} \Delta\chi_1 \\ \Delta\chi_2 \end{pmatrix} + \partial I_S(\chi_1, \chi_2) \ni \rho \begin{pmatrix} -l(\theta - \theta^*)/\theta^* \\ -\alpha(\theta) \operatorname{div} \mathbf{u} \end{pmatrix}, \tag{1.16}$$

and consequently we need to prescribe boundary condition also for (χ_1, χ_2). In particular, the no flux assumption for χ_1 and χ_2 turns out to be rather natural, i.e.,

$$\partial_n \chi_1 = \partial_n \chi_2 = 0, \quad \text{on } \Gamma \times (0, T). \tag{1.17}$$

In [3] we have proved existence and uniqueness of the solution for a weak formulation of an initial-boundary value problem similar to (1.2), (1.15), (1.16), (1.8)-(1.12), and (1.17). Indeed, contrary to [3] where a third type boundary condition was considered for θ, here we assume a non homogeneous Neumann boundary condition for $k_\varepsilon * \theta$. This is derived from (1.13) and $\mathbf{q} \cdot \mathbf{n} = -h$ (compare with (1.11)), and it reads

$$\partial_n(k_\varepsilon * \theta) = h + \exp(-t/\varepsilon)\mathbf{q_0} \cdot \mathbf{n}, \quad \text{on } \Gamma \times (0, T). \tag{1.18}$$

Now, our investigation aims to find some asymptotic relations between the solutions of the two models described above and, from now on, quoted as the parabolic and the diffusive hyperbolic ones. To this purpose, we perform an asymptotic analysis of the last problem as both the relaxation parameter ε and the diffusive parameter η tend to zero, and we discuss convergence, and rates of convergence, of the solution $(\theta_\varepsilon, \mathbf{u}_\varepsilon, \chi_{1\varepsilon}, \chi_{2\varepsilon})$ to the diffusive hyperbolic model in the case when the convergence of $\eta = \eta(\varepsilon)$ is related to that of ε in the sense that $\lim_{\varepsilon \searrow 0} \eta(\varepsilon) = 0$ and $\eta(\varepsilon) \geq c\,\varepsilon^{1/2}$, for some positive constant c. A motivation for this procedure is the fact that the kernel k_ε approximates (in a suitable sense) the measure $k_0\delta$, δ denoting the Dirac mass. Thus, the solution of (1.13) prescribed by $\mathbf{q}_\varepsilon(t) = -(k_\varepsilon * \nabla\theta_\varepsilon)(t) + \mathbf{q_0} \exp(-t/\varepsilon)$ should give the Fourier law in the limit. Moreover, in (1.15), the term $\Delta(k_\varepsilon * \theta)$ may be seen as an approximation of the diffusive term $k_0 \Delta\theta$ in (1.7). Analogously, it is a standard matter to check that the right hand sides of (1.15) and (1.18) converge to ρf and h as ε goes to zero.

We prove strong convergence of the solution $(\theta_\varepsilon, \mathbf{u}_\varepsilon, \chi_{1\varepsilon}, \chi_{2\varepsilon})$ to (1.2), (1.15), (1.16), (1.8)-(1.10), (1.12), (1.17), (1.18) to the solution $(\theta, \mathbf{u}, \chi_1, \chi_2)$

of (1.2), (1.3), (1.7), (1.8)-(1.12) in certain function spaces, along with error estimates, in the case when $\eta(\varepsilon)$ tends to zero of the same order as $\varepsilon^{1/2}$.

Here is the plan of the paper. In Section 2, we recall some known existence and uniqueness results concerning the two initial-boundary value problems, and we also present some regularity properties of the solutions (see Proposition 2.1 and Proposition 2.3). In Section 3, we state both the convergence results (Theorem 3.1) and the error estimates (Theorem 3.2). The proof of Theorem 3.1 is split into two parts: in Section 4 we perform some a priori estimates independent of the parameters ε and η, while in Section 5 we prove the needed convergences, arguing by compactness and monotonicity or verifying them directly. Finally, Section 6 is devoted to show Theorem 3.2, which requires stronger hypotheses on the regularity of the data and suitable assumptions on the boundedness properties of the approximating sequences.

2. Variational formulations

The problems we have presented in the Introduction will now be set in an abstract framework. Let us introduce the Hilbert triplet $V \hookrightarrow H \hookrightarrow V'$, with

$$V := H^1(\Omega) \quad \text{and} \quad H := L^2(\Omega), \tag{2.1}$$

where H is identified with its dual space H' as usual. Let $(\cdot,\cdot)$ stand for the standard scalar product in H, and $\langle\cdot,\cdot\rangle$ denote the duality pairing between V' and V. By abuse of notation, we indicate by $\|\cdot\|_H$ both the norm in H and in H^3. Besides, we consider the Hilbert space $\mathbf{W}$ defined by

$$\mathbf{W} := \{\mathbf{v} \in V^3 \,:\, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_0,\, \operatorname{div}\mathbf{v} \in V\} \tag{2.2}$$

and endowed with the norm

$$\|\mathbf{v}\|^2_{\mathbf{W}} = \int_0^t \int_\Omega |\nabla(\operatorname{div}\mathbf{v})|^2 + \sum_{i,j=1}^{3} \int_0^t \int_\Omega (\partial_{x_i} v_j)^2, \quad \mathbf{v} = (v_1, v_2, v_3). \tag{2.3}$$

In addition, let us introduce the bilinear form $a(\cdot,\cdot)$ on $\mathbf{W} \times \mathbf{W}$ prescribed by

$$\begin{aligned} a(\mathbf{w},\mathbf{v}) &= \nu \int_0^t \int_\Omega \nabla(\operatorname{div}\mathbf{w}) \cdot \nabla(\operatorname{div}\mathbf{v}) \\ &+ \lambda \int_0^t \int_\Omega \operatorname{div}\mathbf{w} \operatorname{div}\mathbf{v} + 2\mu \sum_{i,j=1}^{3} \int_0^t \int_\Omega \epsilon_{ij}(\mathbf{w})\epsilon_{ij}(\mathbf{v}) \end{aligned} \tag{2.4}$$

for $\mathbf{w}$,$\mathbf{v}$ in $\mathbf{W}$. Here, $\epsilon(\mathbf{v})$ denotes the linearized strain tensor related to $\mathbf{v}$, namely

$$\epsilon_{ij}(\mathbf{v}) = \frac{1}{2}\left(\partial_{x_i} v_j + \partial_{x_j} v_i\right), \qquad i,j = 1,2,3.$$

We observe that, as the constants λ and μ are positive, we can apply Korn's inequality and deduce that there exists a constant $C > 0$ such that

$$C\,\|\mathbf{w}\|_{\mathbf{W}}^2 \le a(\mathbf{w},\mathbf{w}) \qquad \forall \mathbf{w} \in \mathbf{W}. \tag{2.5}$$

In the sequel, for the sake of simplicity we assume

$$\rho = c_0 = l = \zeta = 1.$$

Then, in order to establish a precise formulation, we fix the bounded closed convex subset K of H^2 (cf. (1.1)) defined by

$$K := \{(\gamma_1,\gamma_2) \in H^2 : (\gamma_1,\gamma_2) \in S \text{ a.e. in } \Omega\}, \tag{2.6}$$

and we point out that there exists a positive constant C_K, depending only on S, such that

$$\left(|\gamma_1(x)|^2 + |\gamma_2(x)|^2\right)^2 \le C_K \quad \text{for a.a. } x \in \Omega, \tag{2.7}$$

for all pairs $(\gamma_1,\gamma_2) \in K$. To put the problem in the abstract setting of the above mentioned Hilbert spaces, we introduce the operators

$$A : V \to V', \tag{2.8}$$

$$\mathcal{H} : \mathbf{W} \to \mathbf{W}', \tag{2.9}$$

$$\mathcal{B} : H \to \mathbf{W}', \tag{2.10}$$

specified by

$$\langle Au, v\rangle = \int_\Omega \nabla u \cdot \nabla v, \quad u, v \in V, \tag{2.11}$$

$${}_{\mathbf{W}'}\langle \mathcal{H}\mathbf{w}, \mathbf{v}\rangle_{\mathbf{W}} = a(\mathbf{w},\mathbf{v}), \quad \mathbf{w},\mathbf{v} \in \mathbf{W}, \tag{2.12}$$

$${}_{\mathbf{W}'}\langle \mathcal{B}u, \mathbf{v}\rangle_{\mathbf{W}} = \int_\Omega u \operatorname{div} \mathbf{v}, \quad u \in H, \mathbf{v} \in \mathbf{W}. \tag{2.13}$$

Now, we begin to deal with the initial-boundary value problem related to the parabolic equation (1.7), namely we take into account the system constituted by equations (1.2), (1.3), (1.7) supplied with the Cauchy conditions (1.12) and the boundary conditions (1.8)-(1.11). In our analysis, such system corresponds to the limit situation. Thus, we consider the following assumptions on the data

$$f \in L^1(0,T; L^2(\Omega)), \tag{2.14}$$

$$h \in W^{1,1}(0,T;H^{-1/2}(\Gamma)) \cap L^1(0,T;H^{1/2}(\Gamma)), \tag{2.15}$$

$$\mathbf{g} \in L^\infty(0,T;L^2(\Gamma_1)^3), \tag{2.16}$$

$$\theta^0 \in H, \tag{2.17}$$

$$(\chi_1^0,\chi_2^0) \in K, \tag{2.18}$$

$$k_0 > 0, \tag{2.19}$$

$$\alpha \in W^{1,\infty}(\mathbf{R}) \tag{2.20}$$

and prescribe the functions

$$F \in L^1(0,T;H), \tag{2.21}$$

$$H \in W^{1,1}(0,T;V'), \tag{2.22}$$

$$\mathbf{G} \in L^\infty(0,T;\mathbf{W}') \tag{2.23}$$

by the definitions

$$\langle F(t),v\rangle = \int_\Omega f(t)v, \quad v \in V, \tag{2.24}$$

$$\langle H(t),v\rangle = {}_{H^{-1/2}(\Gamma)}\langle h(t), v_{|\Gamma}\rangle_{H^{1/2}(\Gamma)}, \quad v \in V, \tag{2.25}$$

$$_{\mathbf{W}'}\langle \mathbf{G}(t),\mathbf{v}\rangle_{\mathbf{W}} = \int_{\Gamma_1} \mathbf{g}(t)\cdot \mathbf{v}_{|\Gamma}, \quad \mathbf{v} \in \mathbf{W}, \tag{2.26}$$

for almost any t in $(0,T)$. Thus, the problem can be stated as follows.

Problem (P)**.** *Find* $\theta \in W^{1,1}(0,T;V') \cap C^0([0,T];H) \cap L^2(0,T;V)$*,* $\chi_i \in H^1(0,T;H)$ *for* $i=1,2$*, and* $\mathbf{u} \in L^\infty(0,T;\mathbf{W})$ *such that the conditions*

$$\theta(0) = \theta^0 \tag{2.27}$$

$$(\chi_1(0),\chi_2(0)) = (\chi_1^0,\chi_2^0) \tag{2.28}$$

are fulfilled, and $(\theta,\chi_1,\chi_2,\mathbf{u})$ *solves almost everywhere in* $(0,T)$ *the abstract equations*

$$\partial_t\theta + k_0 A\theta - \partial_t\chi_1 = F + H \tag{2.29}$$

$$\partial_t \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} + \begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} = \begin{pmatrix}-(\theta-\theta^*)/\theta^*\\ -\alpha(\theta)\operatorname{div}\mathbf{u}\end{pmatrix} \tag{2.30}$$

$$\mathcal{H}\mathbf{u} + \mathcal{B}(\alpha(\theta)\chi_2) = \mathbf{G}, \tag{2.31}$$

for some $\binom{\xi_1}{\xi_2} \in L^2(0,T;H^2)$ *with*

$$\begin{pmatrix}\xi_1\\ \xi_2\end{pmatrix} \in \partial I_K \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}, \quad a.e. \text{ in } (0,T). \tag{2.32}$$

A variation of the problem (P) has been treated in [7] in the case $F \equiv 0$, assuming (as boundary condition) that the heat flux on the boundary is proportionally related to the difference between the interior and (known) exterior temperatures. Nonetheless, it is not difficult to check that the existence and uniqueness result can be extended to our framework. Indeed, by easily adapting the fixed point argument used there, one can check the following statement.

Proposition 2.1. *Under the assumptions* (2.14)-(2.23)*, the Problem* (P) *admits a unique solution that in addition fulfils*

$$\|\operatorname{div}\mathbf{u}\|_{L^\infty(\Omega\times(0,T))} \le c_2, \tag{2.33}$$

where the constant c_2 *is positive and depends only on* Ω, C, $\|\mathbf{g}\|_{L^\infty(0,T;L^2(\Gamma_1)^3)}$, C_K, $\|\alpha\|_{W^{1,\infty}(\mathbf{R})}$, *and the constant given by the Sobolev inclusion* $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$. *Besides, if one requires*

$$f \in L^2(0,T;L^2(\Omega)) \tag{2.34}$$

$$\theta^0 \in V, \tag{2.35}$$

then there holds

$$\theta \in H^1(0,T;H) \cap C^0(0,T;V), \tag{2.36}$$

and if $(\chi_1^0,\chi_2^0) \in V^2$, *then*

$$\chi_i \in H^1(0,T;H) \cap L^\infty(0,T;V), \quad i = 1,2. \tag{2.37}$$

Moreover, taking in addition

$$\mathbf{g} \in H^1(0,T;L^2(\Gamma_1)^3), \tag{2.38}$$

then it follows that

$$\mathbf{u} \in H^1(0,T;\mathbf{W}) \tag{2.39}$$

$$\operatorname{div}\mathbf{u} \in H^1(0,T;H^2(\Omega)). \tag{2.40}$$

Remark 2.2. For the reader's convenience, we point out that, differentiating (2.31) with respect to time and referring to (2.20) and (2.38), one can deduce (2.39) and (2.40) by exploiting an argument similar to that presented in [7, Lemma 1], after observing that $(\alpha(\theta)\chi_2)' = (\alpha(\theta))'\chi_2 + \alpha(\theta)\chi_2'$ belongs to $L^2(0,T;H)$ (in the sequel, in order to simplify the notation, prime will denote the time derivative as well as ∂_t). Thus, an application of Lax-Milgram theorem ensures the existence and uniqueness of a function $\mathbf{u}' \in L^2(0,T;\mathbf{W})$ fulfilling the abstract equation

$$\mathcal{H}\mathbf{u}' + \mathcal{B}((\alpha(\theta)\chi_2)') = \mathbf{G}'.$$

Then, proceeding as in the proof of [7, Lemma 1], we deduce that $(-\Delta \operatorname{div} \mathbf{u}')$ belongs to $L^2(0,T;H)$ and that $\partial_n \operatorname{div} \mathbf{u}' = 0$. Consequently, we obtain (2.40) as a regularity result for elliptic boundary value problems.

The regularity result (2.37) will arise from the *approximating– passage to the limit procedure* we are going to exploit in the sequel of the paper (cf. Remark 6.1).

Now, we address ourselves to the approximating problem characterized by the nearly hyperbolic equation (1.15) and the diffusive variational inequality (1.16). We state it in terms of only one parameter, the coefficient ε, while we let the other parameter η vary in dependence of ε. Initial and boundary conditions (1.8)-(1.10), (1.12), (1.17), and (1.18) are assumed. Then, we consider some sequences f_ε, $\mathbf{g}_\varepsilon$, h_ε, θ_ε^0, $\chi_{1\varepsilon}^0$, $\chi_{2\varepsilon}^0$ which should approach, in a suitable sense, f, g, h, θ^0, χ_1^0, and χ_2^0 (see the later Section 3) and satisfy

$$f_\varepsilon \in W^{1,1}(0,T;L^2(\Omega)), \tag{2.41}$$

$$h_\varepsilon \in W^{2,1}(0,T;H^{-1/2}(\Gamma)) \cap W^{1,1}(0,T;H^{1/2}(\Gamma)), \tag{2.42}$$

$$\mathbf{g}_\varepsilon \in L^\infty(0,T;L^2(\Gamma_1)^3), \tag{2.43}$$

$$\theta_\varepsilon^0 \in V, \tag{2.44}$$

$$(\chi_{1\varepsilon}^0, \chi_{2\varepsilon}^0) \in V^2 \cap K, \quad i = 1,2. \tag{2.45}$$

Consequently, the functions

$$F_\varepsilon \in W^{1,1}(0,T;H), \tag{2.46}$$

$$H_\varepsilon \in W^{2,1}(0,T;V'), \tag{2.47}$$

$$\mathbf{G}_\varepsilon \in L^\infty(0,T;\mathbf{W}') \tag{2.48}$$

can be defined as in (2.24)-(2.26), by substituting $f, h, \mathbf{g}$ with $f_\varepsilon, h_\varepsilon, \mathbf{g}_\varepsilon$. In particular, let us observe that f_ε and h_ε correspond to the right side of (1.15) and (1.18), respectively, and are actually perturbations of f and h. Hence, we can state the abstract version of the diffusive hyperbolic problem (P_ε). Note that, here, the diffusive coefficient $\eta(\varepsilon)$, corresponding to η in (1.16), is supposed to be a function of ε and obviously it attains only positive values.

Problem (P_ε)**.** *Find* $\theta_\varepsilon \in H^1(0,T;H) \cap L^\infty(0,T;V)$*,* $\chi_{i\varepsilon} \in H^1(0,T;H) \cap C^0([0,T];V) \cap L^2(0,T;H^2(\Omega))$*, for* $i = 1,2$*, and* $\mathbf{u}_\varepsilon \in L^\infty(0,T;\mathbf{W})$*, such that the Cauchy conditions*

$$\theta_\varepsilon(0) = \theta_\varepsilon^0, \tag{2.49}$$

$$\chi_{i\varepsilon}(0) = \chi_{i\varepsilon}^0, \quad i = 1,2 \tag{2.50}$$

are satisfied and the abstract equations

$$\partial_t\theta_\varepsilon + A(k_\varepsilon * \theta_\varepsilon) - \partial_t\chi_{1\varepsilon} = F_\varepsilon + H_\varepsilon, \tag{2.51}$$

$$\partial_t \begin{pmatrix}\chi_{1\varepsilon}\\ \chi_{2\varepsilon}\end{pmatrix} + \eta(\varepsilon)\begin{pmatrix}A\chi_{1\varepsilon}\\ A\chi_{2\varepsilon}\end{pmatrix} + \begin{pmatrix}\xi_{1\varepsilon}\\ \xi_{2\varepsilon}\end{pmatrix} = \begin{pmatrix}-(\theta_\varepsilon - \theta^*)/\theta^*\\ -\alpha(\theta_\varepsilon)\operatorname{div}\mathbf{u}_\varepsilon\end{pmatrix}, \tag{2.52}$$

$$\mathcal{H}\mathbf{u}_\varepsilon + \mathcal{B}(\alpha(\theta_\varepsilon)\chi_{2\varepsilon}) = \mathbf{G}_\varepsilon \tag{2.53}$$

are fulfilled almost everywhere in $(0,T)$, *for some* $\begin{pmatrix}\xi_{1\varepsilon}\\ \xi_{2\varepsilon}\end{pmatrix} \in L^2(0,T;H^2)$ *with*

$$\begin{pmatrix}\xi_{1\varepsilon}\\ \xi_{2\varepsilon}\end{pmatrix} \in \partial I_K \begin{pmatrix}\chi_{1\varepsilon}\\ \chi_{2\varepsilon}\end{pmatrix}, \quad a.e.\ in\ (0,T). \tag{2.54}$$

Then, the following proposition holds.

Proposition 2.3. *Under assumptions* (2.19)-(2.20), (2.41)-(2.48), *and* (1.14), *the Problem* (P_ε) *admits one and only one solution which also fulfils*

$$\|\operatorname{div}\mathbf{u}_\varepsilon\|_{L^\infty(\Omega\times(0,T))} \le c_3, \tag{2.55}$$

where c_3 *denotes a positive constant depending only on* Ω, $\|\alpha\|_{W^{1,\infty}(\mathbf{R})}$, $\|\mathbf{g}_\varepsilon\|_{L^\infty(0,T;L^2(\Gamma_1)^3)}$, C, C_K, *and the constant given by the Sobolev inclusion* $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$.

Besides, if

$$(\chi^0_{1\varepsilon}, \chi^0_{2\varepsilon}) \in H^2(\Omega)^2, \quad \partial_n\chi^0_{i\varepsilon} = 0\ a.e.\ on\ \Gamma,\ i = 1,2, \tag{2.56}$$

then $\chi_{i\varepsilon}$ *fulfils*

$$\chi_{i\varepsilon} \in W^{1,\infty}(0,T;H) \cap H^1(0,T;V) \cap L^\infty(0,T;H^2(\Omega)). \tag{2.57}$$

Moreover, requiring in addition

$$\mathbf{g}_\varepsilon \in H^1(0,T;L^2(\Gamma_1)^3), \tag{2.58}$$

then $\mathbf{u}_\varepsilon$ *satisfies*

$$\mathbf{u}_\varepsilon \in H^1(0,T;\mathbf{W}), \tag{2.59}$$

$$\operatorname{div}\mathbf{u}_\varepsilon \in H^1(0,T;H^2(\Omega)). \tag{2.60}$$

Remark 2.4. The existence and uniqueness of the solution to Problem (P_ε) follows from a fixed point argument similar to that exploited in [3]. Nonetheless, here the Neumann boundary condition (1.18) (instead of the third type boundary condition assumed in [3]) allows us to refer directly to the results presented in [1, Theorem 8.1] and to avoid the argumentation presented in [3, Lemma 5.3]. Thus, in the first instance, recalling that $A(k_\varepsilon * \theta_\varepsilon) = k_\varepsilon(0)A(1*\theta_\varepsilon) + k'_\varepsilon * A(1*\theta_\varepsilon)$, we deduce that $1*\theta_\varepsilon \in C^1([0,T];H) \cap$

$C^0([0,T];V)$. The further property that $\theta_\varepsilon \in H^1(0,T;H) \cap L^\infty(0,T;V)$ is strictly connected with (2.46)-(2.47) and the regularity of $(\chi_{1\varepsilon}, \chi_{2\varepsilon})$ (which can be inferred as in [3, Lemma 8.1]). However, see also the later Remark 4.1. Finally, concerning (2.59) and (2.60) we refer to Remark 2.2.

Remark 2.5. We point out that assumption (2.15) can be weakened if one looks only for the existence of a solution $(\theta, \mathbf{u}, \chi_1, \chi_2)$ of the Problem (P), for which it is enough to require $h \in L^2(0,T;H^{-1/2}(\Gamma))$. Analogously, the problem (P_ε) admits a solution $(\theta_\varepsilon, \mathbf{u}_\varepsilon, \chi_{1\varepsilon}, \chi_{2\varepsilon})$ yet in the case (2.41) and (2.42) are substituted by the less stronger hypotheses $f_\varepsilon \in L^1(0,T;L^2(\Omega))$ and $h_\varepsilon \in W^{1,1}(0,T;H^{-1/2}(\Gamma))$ (cf. i.e., [1]). Nonetheless, the convergence, as $\varepsilon \searrow 0$, of the solution $(\theta_\varepsilon, \mathbf{u}_\varepsilon, \chi_{1\varepsilon}, \chi_{2\varepsilon})$ to the Problem (P_ε) to the solution $(\theta, \mathbf{u}, \chi_1, \chi_2)$ to the Problem (P) is strictly related to the regularities prescribed by (2.15), (2.41), and (2.42). Note that they are reasonable assumptions if one looks at the Problem (P_ε) as a singular perturbation of (P), in a sense that the data of (P_ε) turn out to approximate the data of (P) with sufficient regularity.

3. Main results

In this section, we state our convergence results along with error estimates. To this aim, the sequences $f_\varepsilon, h_\varepsilon, \mathbf{g}_\varepsilon, \chi^0_{1\varepsilon}, \chi^0_{2\varepsilon}, \theta^0_\varepsilon$ will be required to fulfil suitable boundedness and convergence properties. First, we assume that

$$\lim_{\varepsilon \searrow 0} \eta(\varepsilon) = 0, \tag{3.1}$$

$$d\,\varepsilon^{1/2} \le \eta(\varepsilon) \le \widetilde{\eta}, \qquad \forall \varepsilon \in (0,1), \tag{3.2}$$

for some constants $\widetilde{\eta}$ and d not depending on ε. By requirements (3.1) and (3.2) we are assuming that the phase diffusion is related to the relaxation parameter in the heat flux equation. Then, we suppose that there are a constant $c_4 > 0$ and a fixed $\widetilde{\varepsilon} \in (0,1)$ such that for any $\varepsilon \in (0, \widetilde{\varepsilon})$ there holds

$$\varepsilon \left\| f_\varepsilon \right\|_{W^{1,1}(0,T;L^2(\Omega))} + \left\| f_\varepsilon \right\|_{L^1(0,T;L^2(\Omega))} \le c_4 \tag{3.3}$$

$$\varepsilon \left\| h_\varepsilon \right\|_{W^{2,1}(0,T;H^{-1/2}(\Gamma)) \cap W^{1,1}(0,T;H^{1/2}(\Gamma))} + \left\| h_\varepsilon \right\|_{W^{1,1}(0,T;H^{-1/2}(\Gamma)) \cap L^1(0,T;H^{1/2}(\Gamma))} \le c_4 \tag{3.4}$$

$$\left\| \mathbf{g}_\varepsilon \right\|_{L^\infty(0,T;L^2(\Gamma_1)^3)} \le c_4 \tag{3.5}$$

$$\left\| \theta^0_\varepsilon \right\|_H + \varepsilon^{1/2} \left\| \theta^0_\varepsilon \right\|_V + \sum_{i=1}^{2} \left(\left\| \chi^0_{i\varepsilon} \right\|_H + \eta^{1/2}(\varepsilon) \left\| \chi^0_{i\varepsilon} \right\|_V \right) \le c_4. \tag{3.6}$$

Moreover, let the following strong convergences

$$f_\varepsilon \to f \quad \text{in } L^1(0,T;L^2(\Omega)), \tag{3.7}$$

$$h_\varepsilon \to h \quad \text{in } L^1(0,T;H^{-1/2}(\Gamma)), \tag{3.8}$$

$$\mathbf{g}_\varepsilon \to \mathbf{g} \quad \text{in } L^2(0,T;L^2(\Gamma_1)^3), \tag{3.9}$$

$$\theta^0_\varepsilon \to \theta^0 \quad \text{in } H, \tag{3.10}$$

$$\chi^0_{i\varepsilon} \to \chi^0_i \quad \text{in } H, \quad i=1,2 \tag{3.11}$$

be satisfied as ε tends to 0. It is worth observing that, if f, h, $\mathbf{g}$, θ^0, (χ^0_1, χ^0_2), fulfil (2.14)-(2.18), it is not difficult to construct some approximating sequences f_ε, h_ε, $\mathbf{g}_\varepsilon$, θ^0_ε, $(\chi^0_{1\varepsilon}, \chi^0_{2\varepsilon})$, which obey (3.3)-(3.11). Besides, by recalling (1.15) and (1.18) in the Introduction and comparing them with (1.7) and (1.11), it turns out that conditions (3.3)-(3.4) and (3.7)-(3.8) express some known boundedness and convergence properties of $\exp(-t/\varepsilon)$.

In agreement with the theory developed in [3], let us replace the unknown absolute temperature by another variable obtained by integrating in time. Namely, instead of θ and θ_ε, we consider

$$w = 1 * \theta \quad \text{and} \quad w_\varepsilon = 1 * \theta_\varepsilon. \tag{3.12}$$

Thus, we can rewrite problems (P) and (P_ε), and the related existence, uniqueness, and regularity results in terms of w and w_ε. Here, for the reader's convenience, we only specify the equivalent formulations for the constitutive equations of the two problems. Hence, equations (2.29)-(2.31) can be rewritten as

$$\partial_t^2 w + k_0 A \partial_t w - \partial_t \chi_1 = F + H, \tag{3.13}$$

$$\partial_t \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} + \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} -(\partial_t w - \theta^*)/\theta^* \\ -\alpha(\partial_t w)\operatorname{div}\mathbf{u} \end{pmatrix}, \tag{3.14}$$

$$\mathcal{H}\mathbf{u} + \mathcal{B}(\alpha(\partial_t w)\chi_2) = \mathbf{G} \tag{3.15}$$

and the Cauchy conditions corresponding to (2.27) turn out to be

$$w(0) = 0, \qquad \partial_t w(0) = \theta^0. \tag{3.16}$$

Analogously, (2.51)-(2.53) reduce to

$$\partial_t^2 w_\varepsilon + A(k_\varepsilon * \partial_t w_\varepsilon) - \partial_t \chi_{1\varepsilon} = F_\varepsilon + H_\varepsilon \tag{3.17}$$

$$\partial_t \begin{pmatrix} \chi_{1\varepsilon} \\ \chi_{2\varepsilon} \end{pmatrix} + \eta(\varepsilon) \begin{pmatrix} A\chi_{1\varepsilon} \\ A\chi_{2\varepsilon} \end{pmatrix} + \begin{pmatrix} \xi_{1\varepsilon} \\ \xi_{2\varepsilon} \end{pmatrix} = \begin{pmatrix} -(\partial_t w_\varepsilon - \theta^*)/\theta^* \\ -\alpha(\partial_t w_\varepsilon)\operatorname{div}\mathbf{u}_\varepsilon \end{pmatrix} \tag{3.18}$$

$$\mathcal{H}\mathbf{u}_\varepsilon + \mathcal{B}(\alpha(\partial_t w_\varepsilon)\chi_{2\varepsilon}) = \mathbf{G}_\varepsilon, \tag{3.19}$$

and (2.49) is replaced by

$$w_\varepsilon(0) = 0, \qquad \partial_t w_\varepsilon(0) = \theta_\varepsilon^0. \tag{3.20}$$

Now, we are in a position to state our main convergence result.

Theorem 3.1. *Assume that* (2.14)-(2.23), (2.41)-(2.48), (1.14), *and* (3.1)-(3.11) *hold. Let* $(w, \chi_1, \chi_2, \mathbf{u})$ *and* $(w_\varepsilon, \chi_{1\varepsilon}, \chi_{2\varepsilon}, \mathbf{u}_\varepsilon)$ *be the solutions to the problems* (P) *and* (P_ε)*, respectively, where* w *and* w_ε *are defined by* (3.12). *Then, there exist positive constants* c_w*,* c_χ*,* $c_{\mathbf{u}}$ *such that, for any* $\varepsilon \in (0, \widehat{\varepsilon})$*, the following stability estimates are satisfied*

$$\varepsilon \|w_\varepsilon\|_{H^2(0,T;H)\cap W^{1,\infty}(0,T;V)} + \varepsilon^{1/2} \|w_\varepsilon\|_{W^{1,\infty}(0,T;H)\cap H^1(0,T;V)} + \|w_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq c_w \tag{3.21}$$

$$\sum_{i=1}^{2} \Big(\eta(\varepsilon) \|\chi_{i\varepsilon}\|_{L^2(0,T;H^2(\Omega))} + \eta^{1/2}(\varepsilon) \|\chi_{i\varepsilon}\|_{L^\infty(0,T;V)} + \|\chi_{i\varepsilon}\|_{H^1(0,T;H)\cap L^\infty(Q)}\Big) \leq c_\chi \tag{3.22}$$

$$\|\mathbf{u}_\varepsilon\|_{L^\infty(0,T;\mathbf{W})} \leq c_{\mathbf{u}}, \tag{3.23}$$

where the constant c_w*,* c_χ*, and* $c_{\mathbf{u}}$ *depend only on* Ω*,* T*,* C*,* C_K*,* k_0*,* c_4*,* $\|\alpha\|_{W^{1,\infty}(\mathbf{R})}$*, and* c_3*. Moreover, as* ε *tends to 0, the following strong* $(\rightarrow)$*, weak* $(\rightharpoonup)$*, or weak star* $(\overset{*}{\rightharpoonup})$ *convergences hold*

$$w_\varepsilon \to w \quad in\ H^1(0,T;H) \cap L^\infty(0,T;V), \tag{3.24}$$

$$\chi_{i\varepsilon} \to \chi_i \quad in\ C^0([0,T];H), \quad \chi_{i\varepsilon} \overset{*}{\rightharpoonup} \chi_i \quad in\ H^1(0,T;H) \cap L^\infty(Q),\ i = 1,2, \tag{3.25}$$

$$\eta^{1/2}(\varepsilon)\chi_{i\varepsilon} \overset{*}{\rightharpoonup} 0 \quad in\ L^\infty(0,T;V) \tag{3.26}$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad in\ L^2(0,T;\mathbf{W}), \quad \mathbf{u}_\varepsilon \overset{*}{\rightharpoonup} \mathbf{u} \quad in\ L^\infty(0,T;\mathbf{W}). \tag{3.27}$$

Now, we are going to introduce the last statement of this section, which is concerned with suitable estimates for the difference of the solutions to (P_ε) and (P) with respect to ε. We restrict ourselves to consider more regular data. Indeed, in addition to (2.14)-(2.20), we require that

$$(\chi_1^0, \chi_2^0) \in V^2 \cap K. \tag{3.28}$$

We let the approximating sequences f_ε, h_ε, and θ_ε^0 be as in (2.41), (2.42), and (2.44), while for $\mathbf{g}_\varepsilon$ and $(\chi_{1\varepsilon}^0, \chi_{2\varepsilon}^0)$ we assume that

$$\mathbf{g}_\varepsilon \in H^1(0,T;L^2(\Gamma_1)^3), \tag{3.29}$$

$$(\chi_{1\varepsilon}^0, \chi_{2\varepsilon}^0) \in H^2(\Omega)^2 \cap K, \quad \partial_n \chi_{1\varepsilon}^0 = \partial_n \chi_{2\varepsilon}^0 = 0 \text{ a.e. on } \Gamma. \tag{3.30}$$

In the last part of this work, we will allow the diffusive parameter $\eta = \eta(\varepsilon)$ to satisfy (3.1) and (3.2), but in addition we ask that

$$d\,\varepsilon^{1/2} \le \eta(\varepsilon) \le d'\,\varepsilon^{1/2}, \tag{3.31}$$

where $d' \ge d$ is another constant independent of ε. Henceforth, for $\varepsilon \in (0, \widehat{\varepsilon})$, we let (3.3)-(3.6) and

$$\sum_{i=1}^{2} \left(\varepsilon \left\| \chi_{i\varepsilon}^0 \right\|_{H^2(\Omega)} + \left\| \chi_{i\varepsilon}^0 \right\|_V \right) \le c_4 \tag{3.32}$$

$$\varepsilon \left\| \mathbf{g}_\varepsilon \right\|_{H^1(0,T;L^2(\Gamma_1)^3)} \le c_4 \tag{3.33}$$

hold. Let us note that some sequence $(\chi_{1\varepsilon}^0, \chi_{2\varepsilon}^0)$ fulfilling the above condition can be easily arranged (see, e.g., [3, Lemma 8.1]) and that, in particular, it is possible to construct it in such a way that $\chi_{i\varepsilon}^0 \to \chi_i^0$ in V. Now, we need a prescribed rate of convergence for the sequences $\{f_\varepsilon - f\}$, $\{h_\varepsilon - h\}$, $\{\mathbf{g}_\varepsilon - \mathbf{g}\}$, and $\{\theta_\varepsilon^0 - \theta^0\}$, $\{\chi_{1\varepsilon}^0 - \chi_1^0\}$, $\{\chi_{2\varepsilon}^0 - \chi_2^0\}$, in suitable spaces. Thus, we assume that there exists a constant c_5, independent of ε, such that

$$\left\| \theta_\varepsilon^0 - \theta^0 \right\|_H^2 + \sum_{i=1}^{2} \left\| \chi_{i\varepsilon}^0 - \chi_i^0 \right\|_H \le c_5\,\varepsilon^{1/4}, \tag{3.34}$$

$$\left\| f_\varepsilon - f \right\|_{L^1(0,T;L^2(\Omega))} \le c_5\,\varepsilon^{1/4}, \tag{3.35}$$

$$\left\| h_\varepsilon - h \right\|_{L^1(0,T;H^{-1/2}(\Gamma))} \le c_5\,\varepsilon^{1/4}, \tag{3.36}$$

$$\left\| \mathbf{g}_\varepsilon - \mathbf{g} \right\|_{L^2(0,T;L^2(\Gamma_1)^3)} \le c_5\,\varepsilon^{1/4} \tag{3.37}$$

for all $\varepsilon \in (0, \widehat{\varepsilon})$. Then, the following theorem can be proved.

Theorem 3.2. *Let the hypotheses of Theorem 3.1 and in addition* (3.28)-(3.37) *hold. Then, there exist two constants $c_6, c_7 > 0$ depending on Ω, T, C, C_K, $\|\alpha\|_{W^{1,\infty}(\mathbf{R})}$, c_4, k_0, and c_5, such that, for any $\varepsilon \in (0, \widehat{\varepsilon})$, the following stability and error estimates are satisfied*

$$\begin{aligned} &\varepsilon^{3/4} \left\| w_\varepsilon \right\|_{H^2(0,T;H)} + \varepsilon^{1/2} \left\| w_\varepsilon \right\|_{W^{1,\infty}(0,T;V)} \\ &+ \varepsilon^{1/4} \left\| w_\varepsilon \right\|_{W^{1,\infty}(0,T;H)} + \left\| w_\varepsilon \right\|_{H^1(0,T;V) \cap L^\infty(0,T;H^2(\Omega))} + \varepsilon \left\| \mathbf{u}_\varepsilon \right\|_{H^1(0,T;\mathbf{W})} \\ &+ \sum_{i=1}^{2} \Big(\varepsilon^{3/4} \left\| \chi_{i\varepsilon} \right\|_{H^1(0,T;V)} + \varepsilon^{1/2} \left\| \chi_{i\varepsilon} \right\|_{W^{1,\infty}(0,T;H)} + \varepsilon^{1/4} \left\| \chi_{i\varepsilon} \right\|_{L^2(0,T;H^2(\Omega))} \\ &+ \left\| \chi_{i\varepsilon} \right\|_{L^\infty(0,T;V)} \Big) \le c_6 \end{aligned} \tag{3.38}$$

$$\|w_\varepsilon - w\|_{H^1(0,T;H)\cap C^0([0,T];V)} + \sum_{i=1}^{2} \|\chi_{i\varepsilon} - \chi_i\|_{C^0([0,T];H)}$$
$$+ \|\mathbf{u}_\varepsilon - \mathbf{u}\|_{L^2(0,T;\mathbf{W})} \le c_7\, \varepsilon^{1/4}, \tag{3.39}$$

where $(w, \chi_1, \chi_2, \mathbf{u})$ *and* $(w_\varepsilon, \chi_{1\varepsilon}, \chi_{2\varepsilon}, \mathbf{u}_\varepsilon)$ *denote the solutions to the problems* (P) *and* (P_ε)*, respectively.*

Remark 3.3. Let us point out that (3.3)-(3.6) and (3.32)-(3.33) combined with (3.34)-(3.37) are satisfied by functions θ^0, χ_1^0, χ_2^0, $\mathbf{g}$, f, h with more regularity than that prescribed by (2.14)-(2.18). Indeed, by assuming (3.3)-(3.6), (3.32)-(3.33) coupled with (3.34)-(3.37), for each function we implicitly define some real interpolation space between the space in which we require the boundedness of the approximating sequence (see (3.3)-(3.6) and (3.32)-(3.33)) and the space in which is prescribed the rate of convergence (see (3.34)-(3.37)). For a detailed presentation of the interpolation theory we refer to [14], where in particular the reader can find (see p. 35) the procedure to find explicitly these spaces. Here, as an example, let us specify the regularity implicitly required on f. Indeed, (3.3) and (3.35) imply that the function f belongs to a real interpolation space between $L^1(0,T;L^2(\Omega))$ and $W^{1,1}(0,T;L^2(\Omega))$ that can be specified, by using the notation of [14], as follows

$$f \in \big(W^{1,1}(0,T;L^2(\Omega)), L^1(0,T;L^2(\Omega))\big)_{\frac{4}{5},\infty} = B^{1/5}_{1,\infty}(0,T;L^2(\Omega)). \qquad 3.38$$

4. A priori estimates

In this section we prove the first part of Theorem 3.1, namely the uniform bounds (3.21)-(3.23), by performing the appropriate a priori estimates, independent of ε, on the solutions of Problem (P_ε). The procedure basically exploits the following identity (cf. (1.14) and [8])

$$\varepsilon k_\varepsilon + 1 * k_\varepsilon = k_0 \quad \text{in } (0,T). \tag{4.1}$$

Indeed, due to (4.1) we can integrate (3.17) with respect to time, add to (3.17) itself multiplied by ε, and, thanks to (2.49), (2.50), and (3.20), get

$$\varepsilon w_\varepsilon'' + w_\varepsilon' + k_0 A w_\varepsilon - \varepsilon \chi_{1\varepsilon}' - \chi_{1\varepsilon} = \theta_\varepsilon^0 - \chi_{1\varepsilon}^0 + \varepsilon F_\varepsilon + 1 * F_\varepsilon + \varepsilon H_\varepsilon + 1 * H_\varepsilon. \tag{4.2}$$

Throughout the following two sections, the symbol c will denote different constants not depending on ε. Moreover, in the sequel of the paper we always let $\varepsilon \in (0, \widetilde{\varepsilon})$, where the upper bound $\widetilde{\varepsilon} \in (0,1)$ has been introduced while prescribing the boundedness hypotheses of the previous section. We start by noting that (3.3) implies that

$$\varepsilon \|F_\varepsilon\|_{W^{1,1}(0,T;H)} + \|F_\varepsilon\|_{L^1(0,T;H)} \le c_4, \tag{4.3}$$

and analogously (3.4) entails

$$\varepsilon \|H_\varepsilon\|_{W^{2,1}(0,T;V')} + \|H_\varepsilon\|_{W^{1,1}(0,T;V')} \le c_4. \tag{4.4}$$

First a priori estimate. This is performed by testing (4.2) by w'_ε and then integrating over $(0,t)$, where t is arbitrary in $(0,T)$. Note that, by virtue of the regularity of w_ε (see Proposition 2.3 and Remark 2.4), this procedure turns out to be rigorous.

Hence, integrating by parts and recalling (3.20) yield

$$\frac{\varepsilon}{2}\left\|w'_\varepsilon(t)\right\|_H^2 - \frac{\varepsilon}{2}\left\|\theta_\varepsilon^0\right\|_H^2 + \left\|w'_\varepsilon\right\|^2_{L^2(0,t;H)} + \frac{k_0}{2}\left\|\nabla w_\varepsilon(t)\right\|_H^2 \le \sum_{j=1}^{7} I_j(t), \tag{4.5}$$

where $I_1(t)$, $I_2(t)$,..., $I_7(t)$ will be specified right below. The right hand of (4.5) can be easily estimated. Indeed, owing to Young's inequality

$$2ab \le \frac{1}{\sigma}a^2 + \sigma b^2, \quad \forall\, a,b \in \mathbf{R},\, \forall\, \sigma > 0,$$

there holds

$$|I_1(t)| = \left|\int_0^t \int_\Omega \varepsilon \chi'_{1\varepsilon} w'_\varepsilon\right| \le \frac{1}{4}\left\|\chi'_{1\varepsilon}\right\|^2_{L^2(0,t;H)} + c\varepsilon \left\|w'_\varepsilon\right\|^2_{L^2(0,t;H)}, \tag{4.6}$$

and besides,

$$|I_2(t)| = \left|\int_0^t \int_\Omega \chi_{1\varepsilon} w'_\varepsilon\right| \le \frac{1}{16}\left\|w'_\varepsilon\right\|^2_{L^2(0,t;H)} + 4\left\|\chi_{1\varepsilon}\right\|^2_{L^2(0,t;H)}. \tag{4.7}$$

By virtue of (3.6) we can estimate the term $I_3(t) = \int_0^t \int_\Omega (\theta_\varepsilon^0 - \chi_{1\varepsilon}^0) w'_\varepsilon$, namely

$$|I_3(t)| \le \frac{1}{16}\left\|w'_\varepsilon\right\|^2_{L^2(0,t;H)} + 4\left\|\theta_\varepsilon^0 - \chi_{1\varepsilon}^0\right\|^2_{L^2(0,T;H)} \le \frac{1}{16}\left\|w'_\varepsilon\right\|^2_{L^2(0,t;H)} + c. \tag{4.8}$$

Next, (4.3) allows us to infer that

$$|I_3(t)| = \left|\int_0^t \int_\Omega \varepsilon F_\varepsilon w'_\varepsilon\right| \le c\int_0^t \left\|F_\varepsilon(s)\right\|_H \left\|\varepsilon^{1/2} w'_\varepsilon(s)\right\|_H \, ds, \tag{4.9}$$

the term $\int_0^t \|F_\varepsilon(s)\|_H \, ds$ being bounded independently of ε. Similarly, by the Hölder inequality we get

$$|I_4(t)| = \left| \int_0^t \int_\Omega (1 * F_\varepsilon) w_\varepsilon' \right| \le \frac{1}{16} \left\| w_\varepsilon' \right\|^2_{L^2(0,t;H)} + 4T \left\| F_\varepsilon \right\|^2_{L^1(0,T;H)}$$
$$\le \frac{1}{16} \left\| w_\varepsilon' \right\|^2_{L^2(0,t;H)} + c. \tag{4.10}$$

Finally, owing to (4.4) and (3.20), after an integration by parts we can write

$$\begin{aligned} |I_5(t)| &= |\langle (\varepsilon H_\varepsilon + 1 * H_\varepsilon)(t), w_\varepsilon(t) \rangle| \\ &\le \left(\frac{k_0}{8} \|\nabla w_\varepsilon(t)\|^2_H + \frac{1}{16T} \|w_\varepsilon(t)\|^2_H \right) + c \|H_\varepsilon\|^2_{L^\infty(0,T;V')} \\ &\le \left(\frac{k_0}{8} \|\nabla w_\varepsilon(t)\|^2_H + \frac{1}{16} \left\| w_\varepsilon' \right\|^2_{L^2(0,t;H)} \right) + c, \end{aligned} \tag{4.11}$$

and similarly,

$$\begin{aligned} |I_6(t)| &= \left| \int_0^t \langle (\varepsilon H_\varepsilon' + H_\varepsilon)(s), w_\varepsilon(s) \rangle \, ds \right| \\ &\le \int_0^t \left\| (\varepsilon H_\varepsilon' + H_\varepsilon)(s) \right\|_{V'} \|\nabla w_\varepsilon(s)\|_H \, ds \\ &+ \frac{1}{16} \left\| w_\varepsilon' \right\|^2_{L^2(0,t;H)} + c \left\| \varepsilon H_\varepsilon' + H_\varepsilon \right\|^2_{L^2(0,T;V')} \\ &\le \frac{1}{16} \left\| w_\varepsilon' \right\|^2_{L^2(0,t;H)} + \int_0^t \left\| (\varepsilon H_\varepsilon' + H_\varepsilon)(s) \right\|_{V'} \|\nabla w_\varepsilon(s)\|_H \, ds + c. \end{aligned} \tag{4.12}$$

Besides, observe that (4.4) ensures also that $\int_0^t \|(\varepsilon H_\varepsilon' + H_\varepsilon)(s)\|_{V'} \, ds \le c$.

Therefore, by collecting (4.6)-(4.12) and recalling (3.6), from (4.5) we obtain

$$\begin{aligned} &\varepsilon \left\| w_\varepsilon'(t) \right\|^2_H + \left\| w_\varepsilon' \right\|^2_{L^2(0,t;H)} + \|\nabla w_\varepsilon(t)\|^2_H \\ &\le c + \frac{1}{4} \left\| \chi_{1\varepsilon}' \right\|^2_{L^2(0,t;H)} + c \left(\|\chi_{1\varepsilon}\|^2_{L^2(0,t;H)} + \varepsilon \left\| w_\varepsilon' \right\|^2_{L^2(0,t;H)} \right. \\ &\left. + \int_0^t \|F_\varepsilon(s)\|_H \left\| \varepsilon^{1/2} w_\varepsilon'(s) \right\|_H ds + \int_0^t \left\| (\varepsilon H_\varepsilon' + H_\varepsilon)(s) \right\|_{V'} \|\nabla w_\varepsilon(s)\|_H \, ds \right). \end{aligned} \tag{4.13}$$

Some of the next a priori estimates are only formal. Actually, to make the arguments rigorous, we should regularize ∂I_K by its Yosida approximation $(\partial I_K)_\sigma$ (cf., for instance, [2]), prove the estimates for the solutions of the regularized equations, and then pass to the limit as σ tends to 0. Nonetheless, we are omitting such details and refer to the procedure presented in [3, Lemma 8.1], where analogous estimates are performed for a general class of parabolic multivalued equations of type (3.18).

Second a priori estimate. We test (3.18) by $(\chi_{1\varepsilon}, \chi_{2\varepsilon})$ and then integrate over $(0,t)$ with the help of (2.50). Owing to the monotonicity of ∂I_K and the fact that $\binom{0}{0} \in \partial I_K(0,0)$, the sum $\sum_{i=1}^2 \int_0^t \int_\Omega \xi_{i\varepsilon}\chi_{i\varepsilon}$ turns out to be positive. Thus, properties (2.20), (2.55), and (3.6) ensure that the following inequality holds

$$\begin{aligned}
&\sum_{i=1}^{2}\left(\frac{1}{2}\left\|\chi_{i\varepsilon}(t)\right\|_H^2 - \frac{1}{2}\left\|\chi_{i\varepsilon}^0\right\|_H^2 + \eta(\varepsilon)\left\|\nabla\chi_{i\varepsilon}\right\|_{L^2(0,t;H)}^2\right)\\
&\le \frac{1}{\theta^*}\int_0^t \left\|w_\varepsilon'(s) - \theta^*\right\|_H \left\|\chi_{1\varepsilon}(s)\right\|_H \, ds\\
&+ \int_0^t \left\|\alpha(w_\varepsilon'(s)) \operatorname{div}\mathbf{u}_\varepsilon(s)\right\|_H \left\|\chi_{2\varepsilon}(s)\right\|_H \, ds\\
&\le c\sum_{i=1}^{2}\left\|\chi_{i\varepsilon}\right\|_{L^2(0,t;H)}^2 + \frac{1}{4}\left\|w_\varepsilon'\right\|_{L^2(0,t;H)}^2 + c.
\end{aligned} \tag{4.14}$$

Let us stress that the constant c in the left hand side of (4.14) does not depend on ε because of (2.55) and (3.5). Indeed, we have that for all $\varepsilon \in (0, \widetilde{\varepsilon})$

$$\left\|\operatorname{div}\mathbf{u}_\varepsilon\right\|_{L^\infty(\Omega\times(0,T))} \le \widehat{c}_3, \tag{4.15}$$

where $\widehat{c}_3$ is a constant independent of $\left\|\mathbf{g}_\varepsilon\right\|_{L^\infty(0,T;L^2(\Gamma_1)^3)}$ and depending only on $\|\alpha\|_{W^{1,\infty}(\mathbf{R})}$, Ω, c_4, C, C_K and on the Sobolev embedding $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$.

Third a priori estimate. Test (3.18) by $(\chi_{1\varepsilon}, \chi_{2\varepsilon})'$ and then integrate over $(0,t)$. Proceeding similarly as in the previous estimate and observing that the assumption $(\chi_{1\varepsilon}^0, \chi_{2\varepsilon}^0) \in K$ (cf. (2.45)) yields $\sum_{i=1}^2 \int_0^t \int_\Omega \xi_i \chi_{i\varepsilon}' \ge 0$, we can obtain

$$\begin{aligned}
&\sum_{i=1}^{2}\left(\frac{1}{2}\left\|\chi_{i\varepsilon}'\right\|_{L^2(0,t;H)}^2 + \frac{\eta(\varepsilon)}{2}\left\|\nabla\chi_{i\varepsilon}(t)\right\|_H^2 - \frac{\eta(\varepsilon)}{2}\left\|\nabla\chi_{i\varepsilon}^0\right\|_H^2\right)\\
&\le \frac{1}{4}\left\|w_\varepsilon'\right\|_{L^2(0,t;H)}^2 + c.
\end{aligned} \tag{4.16}$$

Note that, due to (3.6), the term $\eta(\varepsilon)\left\|\nabla\chi_{i\varepsilon}^0\right\|_H^2$ is bounded independently of ε.

Fourth a priori estimate. Now, we formally test (3.18) by $(\xi_{1\varepsilon}, \xi_{2\varepsilon})$ and, arguing as for (4.14) and (4.16), we can estimate the term

$$\sum_{i=1}^{2} \left(\int_\Omega \chi'_{i\varepsilon}(t)\xi_{i\varepsilon}(t) + \eta(\varepsilon)\langle A\chi_{i\varepsilon}(t), \xi_{i\varepsilon}(t)\rangle + \int_\Omega |\xi_{i\varepsilon}(t)|^2 \right)$$
$$= \sum_{i=1}^{2} \int_\Omega \left(\frac{1}{\theta^*}(w'_\varepsilon - \theta^*)\xi_{1\varepsilon} - \alpha(w'_\varepsilon)(\operatorname{div} \mathbf{u}_\varepsilon)\xi_{2\varepsilon} \right).$$

Indeed, integrating over $(0,t)$ and noting that the integrand $\langle A\chi_{i\varepsilon}(t), \xi_{i\varepsilon}(t)\rangle$ is non negative because of the monotonicity of ∂I_K (see [3, Appendix] for rigorous computations), we easily obtain

$$\sum_{i=1}^{2} \|\xi_{i\varepsilon}\|^2_{L^2(0,t;H)} \leq \ c + c \left\| w'_\varepsilon \right\|^2_{L^2(0,t;H)}. \tag{4.17}$$

Now, taking the sum of (4.13), (4.14), (4.16), (4.17) and using (3.6) lead to the following inequality

$$\begin{aligned} &\varepsilon \left\| w'_\varepsilon(t) \right\|^2_H + \left\| w'_\varepsilon \right\|^2_{L^2(0,t;H)} + \|\nabla w_\varepsilon(t)\|^2_H \\ &+ \sum_{i=1}^{2} \left(\|\chi_{i\varepsilon}(t)\|^2_H + \eta(\varepsilon) \|\nabla\chi_{i\varepsilon}(t)\|^2_H + \left\|\chi'_{i\varepsilon}\right\|^2_{L^2(0,t;H)} + \|\xi_{i\varepsilon}\|^2_{L^2(0,t:H)} \right) \\ &\leq c + c \left(\varepsilon \left\| w'_\varepsilon \right\|^2_{L^2(0,t;H)} + \int_0^t \|F_\varepsilon(s)\|_H \left\| \varepsilon^{1/2} w'_\varepsilon(s) \right\|_H ds \right. \\ &\left. + \int_0^t \left\| (\varepsilon H'_\varepsilon + H_\varepsilon)(s) \right\|_{V'} \|\nabla w_\varepsilon(s)\|_H \ ds + \sum_{i=1}^{2} \|\chi_{i\varepsilon}\|^2_{L^2(0,t;H)} \right). \end{aligned} \tag{4.18}$$

At this point, we apply a generalized version of the Gronwall lemma (see, e.g., [1, Theorem 2.1]). Hence, recalling also that $(\chi_{1\varepsilon}(t), \chi_{2\varepsilon}(t)) \in K$ for a.a. $t \in (0,T)$ by virtue of (2.54), from (4.18), (3.2), and (2.7) we finally infer that

$$\varepsilon^{1/2} \|w_\varepsilon\|_{W^{1,\infty}(0,T;H)} + \|w_\varepsilon\|_{H^1(0,T;H)\cap L^\infty(0,T;V)} \leq c \tag{4.19}$$

$$\sum_{i=1}^{2} \eta(\varepsilon)^{1/2} \|\chi_{i\varepsilon}\|_{L^\infty(0,T;V)} + \|\chi_{i\varepsilon}\|_{H^1(0,T;H)} + \|\chi_{i\varepsilon}\|_{L^\infty(Q)} \leq c \tag{4.20}$$

$$\sum_{i=1}^{2} \|\xi_{i\varepsilon}\|_{L^2(0,T;H)} \leq c. \tag{4.21}$$

Moreover, thanks to a comparison in (3.18), by (2.20), (4.15), and (4.19)-(4.21) one easily deduces

$$\sum_{i=1}^{2} \eta(\varepsilon) \|A\chi_{i\varepsilon}\|_{L^2(0,T;H)} \leq c. \tag{4.22}$$

Now, we set

$$v_\varepsilon = w_\varepsilon - (1 * \chi_{1\varepsilon}) - y_\varepsilon \tag{4.23}$$

where y_ε is defined as the unique solution of the boundary value problem

$$\begin{aligned} y_\varepsilon(t) - k_0 \Delta y_\varepsilon(t) &= \theta_\varepsilon^0 - \chi_{1\varepsilon}^0 \text{ a.e. in } \Omega, \\ k_0 \partial_n y_\varepsilon(t) &= (\varepsilon h_\varepsilon + 1 * h_\varepsilon)(t) \text{ a.e. in } \Gamma, \end{aligned} \tag{4.24}$$

for a.a. $t \in (0,T)$. Note that $y_\varepsilon + k_0 A y_\varepsilon = \theta_\varepsilon^0 - \chi_{1\varepsilon}^0 + \varepsilon H_\varepsilon + 1 * H_\varepsilon$ a.e. in $(0,T)$. Then, due to (2.42), (2.44)-(2.45), and the well-known regularity results, $y_\varepsilon \in W^{2,1}(0,T;V) \cap W^{1,1}(0,T;H^2(\Omega))$ for all $\varepsilon \in (0,\widehat{\varepsilon})$. Besides, it is a standard matter to verify that (3.4) and (3.6) entail

$$\|y_\varepsilon\|_{W^{1,1}(0,T;H^2(\Omega)) \cap W^{2,1}(0,T;V)} \leq c. \tag{4.25}$$

Henceforth, one could rewrite (4.2) in terms of the new unknown v_ε and infer that

$$\varepsilon v_\varepsilon'' + v_\varepsilon' + k_0 A v_\varepsilon = -k_0 A(1 * \chi_{1\varepsilon}) + \varepsilon F_\varepsilon + 1 * F_\varepsilon + Y_\varepsilon, \tag{4.26}$$

where Y_ε is specified by

$$\langle Y_\varepsilon(t), v\rangle = \int_\Omega (-\varepsilon y_\varepsilon''(t) - y_\varepsilon'(t) + y_\varepsilon(t))v, \quad v \in V, \tag{4.27}$$

for a.a. $t \in (0,T)$. In particular, let us observe that Y_ε belongs to $L^1(0,T;V)$ and that (4.25) and (4.27) imply

$$\|Y_\varepsilon\|_{L^1(0,T;V)} \leq c. \tag{4.28}$$

In the next step, we want to test (4.26) by $\varepsilon A v_\varepsilon'$ and then integrate over $(0,t)$. Owing to the linear structure of (4.26), let us perform this procedure formally. A justification will be provided by the computation itself. We note at once that the right hand side of (4.26) belongs to $W^{1,1}(0,T;H) + L^1(0,T;V)$ (cf. (2.46)) and (2.50), (3.20), and (4.23) yield

$$v_\varepsilon(0) = y_\varepsilon(0), \quad v_\varepsilon'(0) = \theta_\varepsilon^0 - \chi_{1\varepsilon}^0 - y_\varepsilon'(0).$$

Then, (4.25) and (3.6) along with (3.2) ensure that

$$\varepsilon^{1/2} \left\|v_\varepsilon'(0)\right\|_V + \|v_\varepsilon(0)\|_{H^2(\Omega)} \leq c. \tag{4.29}$$

Remark 4.1. From the estimate we are carrying on, we will obtain an information for $\|v_\varepsilon\|_{W^{2,1}(0,T;H)\cap W^{1,\infty}(0,T;V)}$, in particular. Then, in view of the regularity of the functions y_ε and $\chi_{1\varepsilon}$ (cf. Proposition 2.3) we can deduce that $w_\varepsilon \in W^{1,\infty}(0,T;V)\cap W^{2,1}(0,T;H)$, which is not enough to entail $\theta_\varepsilon \in H^1(0,T;H)\cap L^\infty(0,T;V)$ (see Proposition 2.3). Nonetheless, by exploiting the sixth a priori estimate, in the sequel we will check that $w_\varepsilon \in H^2(0,T;H)$ and this concludes the arguments outlined in Remark 2.4.

Fifth a priori estimate. Let us then start our testing of (4.26) by $\varepsilon A v_\varepsilon'$. Integrating by parts in time, we obtain

$$\begin{aligned}&\frac{\varepsilon^2}{2}\left\|\nabla v_\varepsilon'(t)\right\|_H^2 - \frac{\varepsilon^2}{2}\left\|\nabla v_\varepsilon'(0)\right\|_H^2 + \varepsilon\left\|\nabla v_\varepsilon'\right\|_{L^2(0,t;H)}^2\\ &+\frac{k_0\varepsilon}{2}\left\|Av_\varepsilon(t)\right\|_H^2 - \frac{k_0\varepsilon}{2}\left\|Av_\varepsilon(0)\right\|_H^2 \le \sum_{j=1}^5 I_j(t),\end{aligned} \tag{4.30}$$

where $I_j(t)$, for $j=1,...,5$, will be specified below in estimating the right hand side of (4.30). For the sake of simplicity, from now on we will denote by the same index j possibly different integral terms $I_j(t)$ if they clearly refer to different a priori estimates. By virtue of (3.2) and (4.22), we infer that

$$\begin{aligned}|I_1(t)| &= \left|\int_\Omega k_0\varepsilon A(1*\chi_{1\varepsilon}(t))Av_\varepsilon(t)\right|\\ &\le \frac{k_0\varepsilon}{8}\left\|Av_\varepsilon(t)\right\|_H^2 + 2k_0\varepsilon T\left\|A\chi_{1\varepsilon}\right\|_{L^2(0,T;H)}^2 \le \frac{k_0\varepsilon}{8}\left\|Av_\varepsilon(t)\right\|_H^2 + c,\end{aligned} \tag{4.31}$$

and analogously

$$\begin{aligned}|I_2(t)| &= \left|\int_0^t\int_\Omega k_0\varepsilon A\chi_{1\varepsilon}Av_\varepsilon\right|\\ &\le \frac{k_0\varepsilon}{2}\left\|Av_\varepsilon\right\|_{L^2(0,t;H)}^2 + \frac{k_0\varepsilon}{2}\left\|A\chi_{1\varepsilon}\right\|_{L^2(0,T;H)}^2 \le \frac{k_0\varepsilon}{2}\left\|Av_\varepsilon\right\|_{L^2(0,t;H)}^2 + c.\end{aligned} \tag{4.32}$$

Next, (4.3) ensures that

$$\begin{aligned}|I_3(t)| &= \left|\int_\Omega(\varepsilon F_\varepsilon(t) + 1*F_\varepsilon(t))\varepsilon Av_\varepsilon(t)\right|\\ &\le \frac{k_0\varepsilon}{8}\left\|Av_\varepsilon(t)\right\|_H^2 + \frac{2\varepsilon}{k_0}\left\|\varepsilon F_\varepsilon + 1*F_\varepsilon\right\|_{L^\infty(0,T;H)}^2 \le \frac{k_0\varepsilon}{8}\left\|Av_\varepsilon(t)\right\|_H^2 + c,\end{aligned} \tag{4.33}$$

and, moreover,

$$|I_4(t)| = \left| \int_0^t \int_\Omega (\varepsilon F'_\varepsilon + F_\varepsilon) \varepsilon A v_\varepsilon \right| \leq c \int_0^t \left\| \varepsilon F'_\varepsilon(s) + F_\varepsilon(s) \right\|_H \varepsilon^{1/2} \left\| A v_\varepsilon(s) \right\|_H \, ds. \tag{4.34}$$

Finally, owing to (2.11) we can write

$$|I_5(t)| = \left| \int_0^t \int_\Omega \nabla Y_\varepsilon \cdot \varepsilon \nabla v'_\varepsilon \right| \leq \int_0^t \left\| \nabla Y_\varepsilon(s) \right\|_H \varepsilon \left\| \nabla v'_\varepsilon(s) \right\|_H \, ds. \tag{4.35}$$

Let us stress that, by (4.3) and (4.28), the terms

$$\int_0^t \left\| \varepsilon F'_\varepsilon(s) + F_\varepsilon(s) \right\|_H \, ds \quad \text{and} \quad \int_0^t \left\| \nabla Y_\varepsilon(s) \right\|_H \, ds$$

turn out to be bounded independently of ε (and t, of course).

At this point, by combining (4.30) with (4.31)-(4.35), due to (4.29) we infer that

$$\begin{aligned} &\varepsilon^2 \left\| \nabla v'_\varepsilon(t) \right\|_H^2 + \varepsilon \left\| \nabla v'_\varepsilon \right\|_{L^2(0,t;H)}^2 + \varepsilon \left\| A v_\varepsilon(t) \right\|_H^2 \\ &\leq c + c \left(\varepsilon \left\| A v_\varepsilon \right\|_{L^2(0,t;H)}^2 + \int_0^t \left\| \varepsilon F'_\varepsilon(s) + F_\varepsilon(s) \right\|_H \varepsilon^{1/2} \left\| A v_\varepsilon(s) \right\|_H \, ds \right. \\ &\left. + \int_0^t \left\| \nabla Y_\varepsilon(s) \right\|_H \varepsilon \left\| \nabla v'_\varepsilon(s) \right\|_H \, ds \right), \end{aligned} \tag{4.36}$$

and thus, by applying the general version of the Gronwall lemma reported in [1], we are led to deduce that

$$\varepsilon \left\| v_\varepsilon \right\|_{W^{1,\infty}(0,T;V)} + \varepsilon^{1/2} \left\| v_\varepsilon \right\|_{H^1(0,T;V)} + \varepsilon^{1/2} \left\| A v_\varepsilon \right\|_{L^\infty(0,T;H)} \leq c. \tag{4.37}$$

Now, by recalling position (4.23) and estimates (4.20), (4.25), and (4.37) we have that

$$\varepsilon \left\| w_\varepsilon \right\|_{W^{1,\infty}(0,T;V)} + \varepsilon^{1/2} \left\| w_\varepsilon \right\|_{H^1(0,T;V)} \leq c. \tag{4.38}$$

Sixth a priori estimate. Let us test formally (4.2) by $\varepsilon w''_\varepsilon$ (cf. Remark 4.1). Integrating over $(0,t)$ with the help of (3.20) it is now a standard procedure for us to get

$$\varepsilon^2 \left\| w''_\varepsilon \right\|_{L^2(0,t;H)}^2 + \frac{\varepsilon}{2} \left\| w'_\varepsilon(t) \right\|_H^2 - \frac{\varepsilon}{2} \left\| \theta_\varepsilon^0 \right\|_H^2 \leq \sum_{j=1}^{8} I_j(t). \tag{4.39}$$

Proceeding as above, we make precise the right hand side of (4.39) in a moment. By (4.19) and (4.38) we obtain

$$|I_1(t)| = \left| \int_\Omega k_0 \varepsilon \nabla w_\varepsilon(t) \cdot \nabla w'_\varepsilon(t) \right| \\ \leq c\,\varepsilon^2 \left\| \nabla w'_\varepsilon \right\|^2_{L^\infty(0,T;H)} + c \left\| \nabla w_\varepsilon \right\|^2_{L^\infty(0,T;H)} \leq c. \tag{4.40}$$

Moreover, from (4.38) it follows also that

$$|I_2(t)| = \int_0^t \int_\Omega k_0 \varepsilon \left| \nabla w'_\varepsilon \right|^2 \leq k_0 \varepsilon \left\| \nabla w'_\varepsilon \right\|^2_{L^2(0,T;H)} \leq c. \tag{4.41}$$

Estimate (4.20) yields

$$|I_3(t)| = \left| \int_0^t \int_\Omega \varepsilon(\varepsilon \chi'_{1\varepsilon} + \chi_{1\varepsilon}) w''_\varepsilon \right| \tag{4.42}$$
$$\leq \frac{\varepsilon^2}{8} \left\| w''_\varepsilon \right\|^2_{L^2(0,t;H)} + 2 \left\| \varepsilon \chi'_{1\varepsilon} + \chi_{1\varepsilon} \right\|^2_{L^2(0,T;H)} \leq \frac{\varepsilon^2}{8} \left\| w''_\varepsilon \right\|^2_{L^2(0,t;H)} + c.$$

On account of (3.6), we have that

$$|I_4(t)| = \left| \int_0^t \int_\Omega \varepsilon(\theta^0_\varepsilon - \chi^0_{1\varepsilon}) w''_\varepsilon \right| \tag{4.43}$$
$$\leq \frac{\varepsilon^2}{8} \left\| w''_\varepsilon \right\|^2_{L^2(0,t;H)} + 2 \left\| \theta^0_\varepsilon - \chi^0_{1\varepsilon} \right\|^2_{L^2(0,T;H)} \leq \frac{\varepsilon^2}{8} \left\| w''_\varepsilon \right\|^2_{L^2(0,t;H)} + c.$$

Next, by virtue of (4.3) and (4.19), we infer that

$$|I_5(t)| = \left| \int_\Omega \varepsilon(\varepsilon F_\varepsilon(t) + 1 * F_\varepsilon(t)) w'_\varepsilon(t) \right| \\ \leq \frac{\varepsilon}{8} \left\| w'_\varepsilon(t) \right\|^2_H + 2\varepsilon \left\| \varepsilon F_\varepsilon + 1 * F_\varepsilon \right\|^2_{L^\infty(0,T;H)} \leq c, \tag{4.44}$$

and, similarly, that

$$|I_6(t)| = \left| \int_0^t \int_\Omega \varepsilon(\varepsilon F'_\varepsilon + F_\varepsilon) w'_\varepsilon \right| \leq \left\| \varepsilon F'_\varepsilon + F_\varepsilon \right\|_{L^1(0,T;H)} \varepsilon \left\| w'_\varepsilon \right\|_{L^\infty(0,T;H)} \leq c. \tag{4.45}$$

Finally, (4.4) and (4.38) ensure that

$$|I_7(t)| = \left| \langle \varepsilon H_\varepsilon(t) + 1 * H_\varepsilon(t), \varepsilon w'_\varepsilon(t) \rangle \right| \\ \leq c \left(\left\| H_\varepsilon \right\|^2_{L^\infty(0,T;V')} + \varepsilon^2 \left\| w'_\varepsilon \right\|^2_{L^\infty(0,T;V)} \right) \leq c, \tag{4.46}$$

and analogously,

$$|I_8(t)| = \left| \int_0^t \langle \varepsilon H'_\varepsilon + H_\varepsilon, \varepsilon w'_\varepsilon \rangle \right| \le \left\| \varepsilon H'_\varepsilon + H_\varepsilon \right\|_{L^1(0,T;V')} \varepsilon \left\| w'_\varepsilon \right\|_{L^\infty(0,T;V)} \le c. \tag{4.47}$$

Thus, on account of (4.40)-(4.47) and (3.6), from (4.39) we finally obtain (cf. also (4.19))

$$\varepsilon \left\| w''_\varepsilon \right\|_{L^2(0,T;H)} + \varepsilon^{1/2} \left\| w'_\varepsilon \right\|_{L^\infty(0,T;H)} \le c. \tag{4.48}$$

Seventh a priori estimate. Now, our aim is to point out suitable boundedness properties of $\mathbf{u}_\varepsilon$. Let us test (3.19) by $\mathbf{u}_\varepsilon$. Due to (2.5), we can write

$$\|\mathbf{u}_\varepsilon(t)\|^2_{\mathbf{W}} \le \frac{1}{C} \left(\left| \int_\Omega \alpha(w'_\varepsilon(t)) \chi_{2\varepsilon}(t) \operatorname{div} \mathbf{u}_\varepsilon(t) \right| + \left| \int_{\Gamma_1} \mathbf{g}_\varepsilon(t) \cdot \mathbf{u}_{\varepsilon|_\Gamma}(t) \right| \right) \tag{4.49}$$

and consequently, in view of (2.7), (2.20), (3.5), and (4.15), it is easy to derive the following estimate

$$\|\mathbf{u}_\varepsilon\|_{L^\infty(0,T;\mathbf{W})} \le c. \tag{4.50}$$

In conclusion, it is sufficient to combine (4.19)-(4.22), (4.38), (4.48), and (4.50) in order to obtain (3.21)-(3.23).

5. Convergences

In this section we prove convergences (3.24)-(3.27) and thus we complete the proof of Theorem 3.1. Our argumentation is based on the uniform boundedness estimates (3.21)-(3.23) we have just proved in the previous section. Indeed, thanks to well-known compactness results, (3.21)-(3.23) and (4.21) ensure that there exist w, χ_1, χ_2, ξ_1, ξ_2, and $\mathbf{u}$ such that, at least for some subsequence of $\varepsilon \searrow 0$ (which, by abuse of notation, we still denote by ε), the following weak ($\rightharpoonup$) and weak star ($\overset{*}{\rightharpoonup}$) convergences hold

$$w_\varepsilon \overset{*}{\rightharpoonup} w \quad \text{in } H^1(0,T;H) \cap L^\infty(0,T;V), \tag{5.1}$$

$$\varepsilon^{1/2} w_\varepsilon \overset{*}{\rightharpoonup} 0 \quad \text{in } W^{1,\infty}(0,T;H) \cap H^1(0,T;V), \tag{5.2}$$

$$\varepsilon w_\varepsilon \overset{*}{\rightharpoonup} 0 \quad \text{in } H^2(0,T;H) \cap W^{1,\infty}(0,T;V), \tag{5.3}$$

$$\chi_{i\varepsilon} \overset{*}{\rightharpoonup} \chi_i \quad \text{in } H^1(0,T;H) \cap L^\infty(Q), \quad i = 1,2, \tag{5.4}$$

$$\eta(\varepsilon)^{1/2} \chi_{i\varepsilon} \overset{*}{\rightharpoonup} 0 \quad \text{in } L^\infty(0,T;V), \quad i = 1,2, \tag{5.5}$$

$$\eta(\varepsilon) \chi_{i\varepsilon} \rightharpoonup 0 \quad \text{in } L^2(0,T;H^2(\Omega)), \quad i = 1,2, \tag{5.6}$$

$$\xi_{i\varepsilon} \rightharpoonup \xi_i \quad \text{in } L^2(0,T;H), \quad i=1,2, \tag{5.7}$$

$$\mathbf{u}_\varepsilon \overset{*}{\rightharpoonup} \mathbf{u} \quad \text{in } L^\infty(0,T;\mathbf{W}). \tag{5.8}$$

Now, we aim to show that the limit functions w, χ_1, χ_2, and $\mathbf{u}$ solve the problem (P). In the first instance, let us note that from (5.1) it follows

$$w_\varepsilon \to w \quad \text{in } C^0([0,T];H), \tag{5.9}$$

and that w and χ_1 solve the equation obtained by integrating (3.13) from 0 to t, or, analogously, by taking formally $\varepsilon = 0$ in (4.2), namely (cf. also (3.10)-(3.11))

$$w' + k_0 A w - \chi_1 = \theta^0 - \chi_1^0 + 1 * F + 1 * H. \tag{5.10}$$

Let us note that, from (5.4) we get $\chi_{i\varepsilon}(t) \rightharpoonup \chi_i(t)$ in H for all $t \in [0,T]$, $i = 1,2$. Thus, from (3.11) we can recover that $\chi_i(0) = \chi_i^0$ and, consequently, due to (2.17)-(2.18) from (5.10) we deduce that $w'(0) \in H$. Thus, if we differentiate (5.10) with respect to time, by virtue of (2.17), (2.21), (2.22), and (5.4), we can observe that w' solves an abstract parabolic equation of type $y' + Ay = N$ where $N \in L^1(0,T;H) + L^2(0,T;V')$ and $y(0) \in H$. in H. Thus, as a regularity result according to [1, Theorem 3.2], there holds

$$w' \in C^0([0,T];H) \cap L^2(0,T;V). \tag{5.11}$$

At this point, to pass to the limit in the other equations (3.18) and (3.19), we first need to prove strong convergence of w'_ε in order to deduce convergence for the term $\alpha(w'_\varepsilon)$ by virtue of (2.20). To this aim, let us test by $(w_\varepsilon - w)'$ the difference of equation (4.2) and its limit equation (5.10). Integrating over $(0,t)$ leads to

$$\begin{aligned} &\frac{\varepsilon}{2}\left\|w'_\varepsilon(t)\right\|_H^2 - \frac{\varepsilon}{2}\left\|\theta_\varepsilon^0\right\|_H^2 - \int_0^t \int_\Omega \varepsilon w''_\varepsilon w' \\ &+ \left\|(w_\varepsilon - w)'\right\|_{L^2(0,t;H)}^2 + \frac{k_0}{2}\left\|\nabla(w_\varepsilon - w)(t)\right\|_H^2 \le \sum_{j=1}^8 I_j(t). \end{aligned} \tag{5.12}$$

Now, by known procedures we can handle the right hand side of (5.12) and check the following inequalities. First, we consider

$$\begin{aligned} |I_1(t)| &= \left|\int_0^t \int_\Omega \varepsilon \chi'_{1\varepsilon}(w_\varepsilon - w)'\right| \le \frac{1}{16}\left\|(w_\varepsilon - w)'\right\|_{L^2(0,t;H)}^2 + 4\varepsilon^2\left\|\chi'_{1\varepsilon}\right\|_{L^2(0,T;H)}^2 \\ &= \frac{1}{16}\left\|(w_\varepsilon - w)'\right\|_{L^2(0,t;H)}^2 + Q_1(\varepsilon), \end{aligned} \tag{5.13}$$

where $Q_1(\varepsilon)$ denotes a quantity depending on ε that tends to zero as $\varepsilon \searrow 0$ (see (5.4)). Then, an integration by parts yields

$$\begin{aligned} |I_2(t)| &= \left| \int_\Omega (\chi_{1\varepsilon} - \chi_1)(t)(w_\varepsilon - w)(t) \right| \\ &\le \|\chi_{1\varepsilon} - \chi_1\|_{C^0([0,T];H)} \|w_\varepsilon - w\|_{C^0([0,T];H)} = Q_2(\varepsilon) \end{aligned} \tag{5.14}$$

and, moreover,

$$\begin{aligned} |I_3(t)| &= \left| \int_0^t \int_\Omega (\chi_{1\varepsilon} - \chi_{1\varepsilon})'(w_\varepsilon - w) \right| \\ &\le c \left\| (\chi_{1\varepsilon} - \chi_1)' \right\|_{L^2(0,T;H)} \|w_\varepsilon - w\|_{C^0([0,T];H)} = Q_3(\varepsilon). \end{aligned} \tag{5.15}$$

As to concerns the initial terms, we can estimate

$$\begin{aligned} |I_4(t)| &= \left| \int_0^t \int_\Omega (\theta_\varepsilon^0 - \theta^0 + \chi_1^0 - \chi_{1\varepsilon}^0)(w_\varepsilon - w)' \right| \\ &\le \frac{1}{16} \left\| (w_\varepsilon - w)' \right\|_{L^2(0,t;H)}^2 + c \left(\left\| \theta_\varepsilon^0 - \theta^0 \right\|_H^2 + \left\| \chi_{1\varepsilon}^0 - \chi_1^0 \right\|_H^2 \right) \\ &= \frac{1}{16} \left\| (w_\varepsilon - w)' \right\|_{L^2(0,t;H)}^2 + Q_4(\varepsilon). \end{aligned} \tag{5.16}$$

Analogously, due to well-known properties for convolution, we can write

$$|I_5(t)| = \left| \int_\Omega (\varepsilon F_\varepsilon(t) + 1 * (F_\varepsilon - F)(t))(w_\varepsilon - w)(t) \right| \tag{5.17}$$

$$\le c \left(\varepsilon \|F_\varepsilon\|_{L^\infty(0,T;H)} + \|F_\varepsilon - F\|_{L^1(0,T;H)} \right) \|w_\varepsilon - w\|_{C^0([0,T];H)} = Q_5(\varepsilon)$$

and, besides,

$$|I_6(t)| = \left| \int_0^t \int_\Omega (\varepsilon F_\varepsilon' + (F_\varepsilon - F))(w_\varepsilon - w) \right| \tag{5.18}$$

$$\le \left(\varepsilon \left\| F_\varepsilon' \right\|_{L^1(0,T;H)} + \|F_\varepsilon - F\|_{L^1(0,T;H)} \right) \|w_\varepsilon - w\|_{C^0([0,T];H)} = Q_6(\varepsilon).$$

Finally, thanks to a similar procedure, we have that

$$|I_7(t)| = |\langle \varepsilon H_\varepsilon(t) + 1 * (H_\varepsilon - H)(t), (w_\varepsilon - w)(t) \rangle| \tag{5.19}$$

$$\le c \left(\varepsilon \|H_\varepsilon\|_{L^\infty(0,T;V')} + \|H_\varepsilon - H\|_{L^1(0,T;V')} \right) \|w_\varepsilon - w\|_{L^\infty(0,T;V)} = Q_7(\varepsilon)$$

and

$$|I_8(t)| = \left| \int_0^t \langle \varepsilon H_\varepsilon'(s) + (H_\varepsilon - H)(s), (w_\varepsilon - w)(s) \rangle \, ds \right| \tag{5.20}$$

$$\leq \Big(\varepsilon \left\|H_\varepsilon'\right\|_{L^1(0,T;V')} + \|H_\varepsilon - H\|_{L^1(0,T;V')}\Big) \|w_\varepsilon - w\|_{L^\infty(0,T;V)} = Q_8(\varepsilon).$$

Now, let us note that (3.7) and (3.8) obviously imply

$$F_\varepsilon \to F \quad \text{in } L^1(0,T;H), \tag{5.21}$$

$$H_\varepsilon \to H \quad \text{in } L^1(0,T;V'). \tag{5.22}$$

Thus, by virtue of (3.10), (3.11), (4.3), (4.4), (5.1), (5.4), (5.9), (5.21), and (5.22), there holds

$$\lim_{\varepsilon\searrow 0} \sum_{j=1}^{8} Q_j(\varepsilon) = 0. \tag{5.23}$$

Hence, by comparing (5.12) with (5.13)-(5.20), we can write

$$\begin{aligned} &\left\|(w_\varepsilon - w)'\right\|^2_{L^2(0,t;H)} + \|\nabla(w_\varepsilon - w)(t)\|^2_H \\ &\leq\ c\Big(\varepsilon \left\|\theta_\varepsilon^0\right\|^2_H + \sum_{j=1}^{8} Q_j(\varepsilon)\Big) + R_\varepsilon(t), \end{aligned} \tag{5.24}$$

where R_ε is specified by

$$R_\varepsilon(t) = \int_0^t \int_\Omega \varepsilon w_\varepsilon'' w'.$$

Thanks to (3.21) and (5.11) it is straightforward to check that R_ε is bounded in $H^1(0,T)$ independently of ε. Moreover, thanks to (5.3) there holds $\lim_{\varepsilon\searrow 0} R_\varepsilon(t) = 0$ for a.e. $t \in (0,T)$ and thus by compactness we have also that

$$\lim_{\varepsilon\searrow 0} \|R_\varepsilon\|_{C^0([0,T])} = 0. \tag{5.25}$$

Finally, (3.10), (5.23), and (5.25) allow us to deduce from (5.24) that

$$\lim_{\varepsilon\searrow 0} \Big(\left\|(w_\varepsilon - w)'\right\|_{L^2(0,T;H)} + \|\nabla(w_\varepsilon - w)\|_{L^\infty(0,T;H)}\Big) = 0, \tag{5.26}$$

and consequently

$$w_\varepsilon \to w \quad \text{in } H^1(0,T;H) \cap L^\infty(0;T;V). \tag{5.27}$$

Hence, by (5.27) and (2.20) we infer that

$$\alpha(w_\varepsilon') \to \alpha(w') \quad \text{in } L^2(0,T;H), \quad \text{as } \varepsilon \searrow 0. \tag{5.28}$$

Now, we can take the limit in (3.19) with the help of (5.4), (5.8), and (3.9), since

$$\alpha(w_\varepsilon')\chi_{2\varepsilon} \rightharpoonup \alpha(w')\chi_2, \quad \text{in } L^2(0,T;H).$$

On the other hand, to pass to the limit in (3.18) we need strong convergence either for $\chi_{2\varepsilon}$ or for $\operatorname{div}\mathbf{u}_\varepsilon$. We are going to prove it for both $\chi_{1\varepsilon}$, $\chi_{2\varepsilon}$, and $\mathbf{u}_\varepsilon$. To this aim, we proceed by rewriting (3.18) in the equivalent formulation

$$\begin{aligned}
&\int_\Omega \sum_{i=1}^{2} \big((\chi_{i\varepsilon}-\chi_i)'(\chi_{i\varepsilon}-\gamma_i) + {\chi_i}'(\chi_{i\varepsilon}-\gamma_i) + \eta(\varepsilon)A\chi_{i\varepsilon}(\chi_{i\varepsilon}-\gamma_i)\big)\\
&+\int_\Omega \big((w_\varepsilon - w)'(\chi_{1\varepsilon}-\gamma_1)/\theta^* + (w'-\theta^*)(\chi_{1\varepsilon}-\gamma_1)/\theta^*\\
&+\alpha(w_\varepsilon')\operatorname{div}(\mathbf{u}_\varepsilon-\mathbf{u})(\chi_{2\varepsilon}-\gamma_2)\\
&+\alpha(w_\varepsilon')\operatorname{div}\mathbf{u}(\chi_{2\varepsilon}-\gamma_2)\big) \le 0,\ \forall\,(\gamma_1,\gamma_2)\in K,\ \text{ a.e. in } (0,T)
\end{aligned} \tag{5.29}$$

and then taking $(\gamma_1,\gamma_2)=(\chi_1,\chi_2)$. Observe that this choice is possible, since the property $(\chi_1(t),\chi_2(t))\in K$ for a.a. $t\in(0,T)$ follows from (5.4), the convexity of K and the analogous inclusion for the pair $(\chi_{1\varepsilon},\chi_{2\varepsilon})$. Now, let S_ε be specified by

$$\begin{aligned}
S_\varepsilon(t) := \sum_{i=1}^{2}\int_0^t\int_\Omega \Big(&-\chi_i'(\chi_{i\varepsilon}-\chi_i) - \eta(\varepsilon)A\chi_{i\varepsilon}\chi_i\\
&+\frac{1}{\theta^*}(w'-\theta^*)(\chi_{1\varepsilon}-\chi_1) + \alpha(w_\varepsilon')\operatorname{div}\mathbf{u}(\chi_{2\varepsilon}-\chi_2)\Big),
\end{aligned} \tag{5.30}$$

and observe that (5.4), (5.6), (2.33) and (5.28) imply that

$$\lim_{\varepsilon\searrow 0} S_\varepsilon(t) = 0,$$

for a.a. $t\in(0,T)$. Moreover, it is now a standard procedure (cf. (3.22) and (2.20)) to verify that S_ε is bounded in $H^1(0,T)$ independently of ε. Thus, by compactness we infer that

$$\lim_{\varepsilon\searrow 0}\|S_\varepsilon\|_{C^0([0,T])} = 0. \tag{5.31}$$

At this point, we integrate (5.29) over $(0,t)$ and, after recalling that S_ε is specified by (5.30), it is straightforward to obtain

$$\begin{aligned}
&\sum_{i=1}^{2}\left(\frac{1}{2}\|(\chi_{i\varepsilon}-\chi_i)(t)\|_H^2 + \eta(\varepsilon)\|\nabla\chi_{i\varepsilon}\|_{L^2(0,t;H)}^2\right)\\
&\le \frac{1}{2}\sum_{i=1}^{2}\left(\left\|\chi_{i\varepsilon}^0-\chi_i^0\right\|_H^2 + \|\chi_{i\varepsilon}-\chi_i\|_{L^2(0,t;H)}^2\right) + \frac{1}{2(\theta^*)^2}\left\|(w_\varepsilon-w)'\right\|_{L^2(0,T;H)}^2
\end{aligned}$$

$$+\frac{\|\alpha\|^2_{W^{1,\infty}(0,T)}}{2}\|\operatorname{div}(\mathbf{u}_\varepsilon-\mathbf{u})\|^2_{L^2(0,t;H)}+\|S_\varepsilon\|_{C^0([0,T])}\,. \tag{5.32}$$

To estimate the left hand side of (5.32) we try to control the term

$$\|\operatorname{div}(\mathbf{u}_\varepsilon-\mathbf{u})\|^2_{L^2(0,t;H)}\,.$$

Thus, we take the difference between (3.19) and (3.15), and test it by $(\mathbf{u}_\varepsilon-\mathbf{u})$. Integrating over $(0,t)$, by virtue of (2.5), we obtain

$$\begin{aligned}&\|\mathbf{u}_\varepsilon-\mathbf{u}\|^2_{L^2(0,t;\mathbf{W})}\le\frac{1}{C}\int_0^t\int_\Omega\big(\big|\alpha(w_\varepsilon')(\chi_{2\varepsilon}-\chi_2)\operatorname{div}(\mathbf{u}_\varepsilon-\mathbf{u})\big|\\&+\big|\chi_2(\alpha(w_\varepsilon')-\alpha(w'))\operatorname{div}(\mathbf{u}_\varepsilon-\mathbf{u})\big|\big)\\&+\left|\int_0^t{}_{\mathbf{W}'}\langle(\mathbf{G}_\varepsilon-\mathbf{G})(s),(\mathbf{u}_\varepsilon-\mathbf{u})(s)\rangle_{\mathbf{W}}\,ds\right|,\end{aligned} \tag{5.33}$$

from which, by recalling just (2.7) and (2.20), we deduce

$$\begin{aligned}&\|\mathbf{u}_\varepsilon-\mathbf{u}\|^2_{L^2(0,t;\mathbf{W})}\le c\Big(\big\|(w_\varepsilon-w)'\big\|^2_{L^2(0,t;H)}+\|\chi_{2\varepsilon}-\chi_2\|^2_{L^2(0,t;H)}\\&+\|\mathbf{G}_\varepsilon-\mathbf{G}\|^2_{L^2(0,T;\mathbf{W}')}\Big)\,.\end{aligned} \tag{5.34}$$

In particular, let us point out that (3.9) obviously implies

$$\mathbf{G}_\varepsilon\to\mathbf{G}\quad\text{in }L^2(0,T;\mathbf{W}'). \tag{5.35}$$

Thus, using (5.34) to estimate $\|\operatorname{div}(\mathbf{u}_\varepsilon-\mathbf{u})\|_{L^2(0,t;H)}$ in the right hand side of (5.32) and applying the Gronwall lemma we infer that

$$\begin{aligned}&\sum_{i=1}^2\|(\chi_{i\varepsilon}-\chi_i)(t)\|_H\le c\left(\sum_{i=1}^2\big\|\chi^0_{i\varepsilon}-\chi^0_i\big\|_H\right.\\&\left.+\|w_\varepsilon-w\|_{H^1(0,T;H)}+\|S_\varepsilon\|_{C^0([0,T])}+\|\mathbf{G}_\varepsilon-\mathbf{G}\|_{L^2(0,T;\mathbf{W}')}\right),\end{aligned} \tag{5.36}$$

and consequently, by virtue of (3.11), (5.27), (5.31), and (5.35), there holds

$$\lim_{\varepsilon\searrow0}\sum_{i=1}^2\|\chi_{i\varepsilon}-\chi\|_{C^0([0,T];H)}=0, \tag{5.37}$$

namely

$$\chi_{i\varepsilon}\to\chi_i\quad\text{in }C^0([0,T];H), \tag{5.38}$$

for $i=1,2$. Finally, from (5.34) it follows that

$$\mathbf{u}_\varepsilon\to\mathbf{u}\quad\text{in }L^2(0,T;\mathbf{W}), \tag{5.39}$$

which brings the proof of Theorem 3.1 to the end. Indeed, taking the limit in (3.18), we find (3.14). Moreover, the condition (2.32) is satisfied as well because (5.38) and (5.7) yield $\limsup_{\varepsilon\searrow 0}\int_\Omega \xi_{i\varepsilon}\chi_{i\varepsilon} = \int_\Omega \xi_i\chi_i$, $i = 1, 2$, and one may invoke, e.g., [2, Prop. 1.1, p. 42]. Finally, we note that thanks to the uniqueness result for the solution of the limit problem (P) (cf. Proposition 2.1), all the previous convergences hold for the whole net.

6. Error estimates

This section is devoted to the proof of Theorem 3.2. It appears quite convenient to continue arguing in terms of the functions w and w_ε instead of θ and θ_ε, and consequently refer to the equations (3.13)-(3.15) and (3.17)-(3.19). In the sequel, the value of c may also depend on the new constants specified in the statement, but not on ε. First, by virtue of (3.29)-(3.33) we can prove some further stability bounds to add to (3.21)-(3.23) by performing other a priori estimates.

Eighth a priori estimate. In the first instance, thanks to (3.29) we can differentiate (3.19) in time (cf. Proposition 2.3 and especially (2.59)) and test it by $\varepsilon^2\mathbf{u}'_\varepsilon$. Let us note that (3.33) yields

$$\varepsilon \left\|\mathbf{G}_\varepsilon\right\|_{H^1(0,T;\mathbf{W}')} \le c. \tag{6.1}$$

After an integration over $(0, t)$, owing to (2.5), (2.7), and (2.20) we have that

$$\begin{aligned} C\,\varepsilon^2\left\|\mathbf{u}'_\varepsilon\right\|^2_{L^2(0,t;\mathbf{W})} &\le \left|\int_0^t\int_\Omega \varepsilon^2(\alpha'(w'_\varepsilon)w''_\varepsilon\chi_{2\varepsilon} + \alpha(w'_\varepsilon)\chi'_{2\varepsilon})\operatorname{div}\mathbf{u}'_\varepsilon\right| \\ &+ \left|\int_0^t \varepsilon^2\,{}_{\mathbf{W}'}\langle\mathbf{G}'_\varepsilon(s),\mathbf{u}'_\varepsilon(s)\rangle_{\mathbf{W}}\,ds\right| \\ &\le c\left(\varepsilon\left\|w''_\varepsilon\right\|_{L^2(0,t;H)} + \left\|\chi'_{2\varepsilon}\right\|_{L^2(0,t;H)}\right)\varepsilon\left\|\operatorname{div}\mathbf{u}'_\varepsilon\right\|_{L^2(0,t;H)} \\ &+ \varepsilon\left\|\mathbf{G}'_\varepsilon\right\|_{L^2(0,t;\mathbf{W}')}\varepsilon\left\|\mathbf{u}'_\varepsilon\right\|_{L^2(0,t;\mathbf{W})}. \end{aligned} \tag{6.2}$$

Thus, owing to (3.21), (3.22), and (6.1) it is now a standard matter to deduce

$$\varepsilon\left\|\mathbf{u}'_\varepsilon\right\|_{L^2(0,T;\mathbf{W})} \le c. \tag{6.3}$$

Hence, we can argue as in Remark 2.2 and point out that (2.60), (3.21), (3.22), (6.1), and (6.3) imply

$$\varepsilon\left\|\operatorname{div}\mathbf{u}_\varepsilon\right\|_{H^1(0,T;H^2(\Omega))} \le c. \tag{6.4}$$

Ninth a priori estimate. Now, by virtue of (3.30) and Proposition 2.3, the regularity property (2.57) holds. Consequently, we can test (3.18) by $(A\chi_{1\varepsilon}, A\chi_{2\varepsilon})$ and integrate over $(0,t)$. Due to (2.20), (3.31), (3.32), and (4.15), we formally achieve

$$\sum_{i=1}^{2}\left(\frac{1}{2}\|\nabla\chi_{i\varepsilon}(t)\|_H^2 + d\varepsilon^{1/2}\|A\chi_{i\varepsilon}\|^2_{L^2(0,t;H)}\right) \le \sum_{i=1}^{2}\frac{1}{2}\left\|\nabla\chi^0_{i\varepsilon}\right\|_H^2 \tag{6.5}$$
$$+\left|\int_0^t\int_\Omega \frac{1}{\theta^*}\nabla(w'_\varepsilon-\theta^*)\cdot\nabla\chi_{1\varepsilon} + (\operatorname{div}\mathbf{u}_\varepsilon\alpha'(w'_\varepsilon)\nabla w'_\varepsilon + \alpha(w'_\varepsilon)\nabla\operatorname{div}\mathbf{u}_\varepsilon)\cdot\nabla\chi_{2\varepsilon}\right|$$
$$\le c\Big(\sum_{i=1}^{2}\|\nabla\chi_{i\varepsilon}\|^2_{L^2(0,t;H)} + \left\|\nabla w'_\varepsilon\right\|^2_{L^2(0,t;H)} + \|\nabla\operatorname{div}\mathbf{u}_\varepsilon\|^2_{L^2(0,t;H)}\Big) + c,$$

by referring to [3, Appendix] for a rigorous derivation of (6.5). Recalling now (3.23), it turns out that $\|\operatorname{div}\mathbf{u}_\varepsilon\|_{L^2(0,T;V)} \le c$, whence an application of the Gronwall lemma to (6.5) entails that

$$\sum_{i=1}^{2}\left(\|\nabla\chi_{i\varepsilon}(t)\|_H + \varepsilon^{1/4}\|A\chi_{i\varepsilon}\|_{L^2(0,t;H)}\right) \le c\left\|\nabla w'_\varepsilon\right\|_{L^2(0,T;H)} + c. \tag{6.6}$$

Tenth a priori estimate. In view of (2.57) again, we can perform a formal derivation of (3.18) with respect to time and test it by $\varepsilon(\chi_{1\varepsilon},\chi_{2\varepsilon})'$. The rigorous computation must be done on an approximated version of (3.18), (2.54) where ∂I_K is replaced by its Yosida approximation (see [3, Appendix]). So, if we assume that some Yosida approximation is considered, a comparison in (3.18) leads to

$$\sum_{i=1}^{2}\varepsilon^{1/2}\left\|\chi'_{i\varepsilon}(0)\right\|_H \le c \tag{6.7}$$

because of (2.45), (3.6), (3.31), and (3.32). Moreover, after integration over $(0,t)$ a standard procedure yields

$$\sum_{i=1}^{2}\left(\frac{\varepsilon}{2}\left\|\chi'_{i\varepsilon}(t)\right\|_H^2 - \frac{\varepsilon}{2}\left\|\chi'_{i\varepsilon}(0)\right\|_H^2 + d\varepsilon^{3/2}\left\|\nabla\chi'_{i\varepsilon}\right\|^2_{L^2(0,t;H)}\right)$$
$$\le \int_0^t\int_\Omega \frac{1}{\theta^*}\varepsilon\left|w''_\varepsilon\right|\left|\chi'_{1\varepsilon}\right| + \varepsilon(\left|\alpha'(w'_\varepsilon)w''_\varepsilon\right||\operatorname{div}\mathbf{u}_\varepsilon| + \left|\alpha(w'_\varepsilon)\right|\left|\operatorname{div}\mathbf{u}'_\varepsilon\right|)\left|\chi'_{2\varepsilon}\right|$$
$$\le c\left(\varepsilon^2\left\|w''_\varepsilon\right\|^2_{L^2(0,t;H)} + \sum_{i=1}^{2}\left\|\chi'_{i\varepsilon}\right\|^2_{L^2(0,t;H)} + \varepsilon^2\left\|\operatorname{div}\mathbf{u}'_\varepsilon\right\|^2_{L^2(0,t;H)}\right). \tag{6.8}$$

Now, estimates (3.21), (3.22), (6.4), and (6.7) ensure that

$$\varepsilon^{1/2} \left\|\chi'_{i\varepsilon}\right\|_{L^\infty(0,T;H)} + \varepsilon^{3/4} \left\|\nabla\chi'_{i\varepsilon}\right\|_{L^2(0,T;H)} \leq c. \tag{6.9}$$

Eleventh a priori estimate. Let us consider the function

$$z_\varepsilon = w_\varepsilon - y_\varepsilon, \tag{6.10}$$

where y_ε is defined by (4.24). We recall that $y_\varepsilon \in W^{2,1}(0,T;V) \cap W^{1,1}(0,T;H^2(\Omega))$ and moreover (4.25) still holds. Now, rewrite (4.2) in terms of the new unknown z_ε

$$\varepsilon z''_\varepsilon + z'_\varepsilon + k_0 A z_\varepsilon - \varepsilon\chi'_{1\varepsilon} - \chi_{1\varepsilon} = \varepsilon F_\varepsilon + 1 * F_\varepsilon + Y_\varepsilon, \tag{6.11}$$

where Y_ε is specified by (4.27). Obviously, by construction, z_ε turns out to be as regular as v_ε and due to (3.20), (3.6), and (4.25), it satisfies

$$\varepsilon^{1/2} \left\|z'_\varepsilon(0)\right\|_V + \|z_\varepsilon(0)\|_{H^2(\Omega)} \leq c. \tag{6.12}$$

We formally proceed by testing (6.11) by Az'_ε and then integrating over $(0,t)$. It is easy to check that

$$\begin{aligned} &\frac{\varepsilon}{2}\left\|\nabla z'_\varepsilon(t)\right\|_H^2 - \frac{\varepsilon}{2}\left\|\nabla z'_\varepsilon(0)\right\|_H^2 + \left\|\nabla z'_\varepsilon\right\|^2_{L^2(0,t;H)} \\ &\quad + \frac{k_0}{2}\|Az_\varepsilon(t)\|_H^2 - \frac{k_0}{2}\|Az_\varepsilon(0)\|_H^2 \leq \sum_{j=1}^4 I_j(t), \end{aligned} \tag{6.13}$$

where the right hand side of (6.13) can be handled just by recalling the hypotheses of Theorem 3.2 and the estimates we have previously proved. More precisely, by virtue of (6.9) we infer

$$\begin{aligned} |I_1(t)| &= \left|\int_0^t \int_\Omega \varepsilon\nabla\chi'_{1\varepsilon}\cdot\nabla z'_\varepsilon\right| \\ &\leq \frac{1}{2}\left\|\nabla z'_\varepsilon\right\|^2_{L^2(0,t;H)} + \frac{\varepsilon^2}{2}\left\|\nabla\chi'_{1\varepsilon}\right\|^2_{L^2(0,t;H)} \leq \frac{1}{2}\left\|\nabla z'_\varepsilon\right\|^2_{L^2(0,t;H)} + c. \end{aligned} \tag{6.14}$$

Besides, thanks to an integration by parts, by (3.22) we obtain

$$\begin{aligned} &|I_2(t)| \leq \left|\int_\Omega \chi_{1\varepsilon}(t)Az_\varepsilon(t)\right| + \left|\int_0^t\int_\Omega \chi'_{1\varepsilon}Az_\varepsilon\right| \\ &\leq \frac{k_0}{8}\|Az_\varepsilon(t)\|_H^2 + \frac{2}{k_0}\|\chi_{1\varepsilon}\|^2_{L^\infty(0,T;H)} + \frac{1}{2}\|Az_\varepsilon\|^2_{L^2(0,t;H)} + \frac{1}{2}\left\|\chi'_{1\varepsilon}\right\|^2_{L^2(0,T;H)} \\ &\leq \frac{k_0}{8}\|Az_\varepsilon(t)\|_H^2 + \frac{1}{2}\|Az_\varepsilon\|^2_{L^2(0,t;H)} + c. \end{aligned} \tag{6.15}$$

Concerning the right hand side of (6.11), in view of (4.3) there holds

$$\begin{aligned}|I_3(t)| &\le \left|\int_\Omega (\varepsilon F_\varepsilon + 1 * F_\varepsilon)(t) A z_\varepsilon(t)\right| + \left|\int_0^t \int_\Omega (\varepsilon F_\varepsilon' + F_\varepsilon) A z_\varepsilon\right| \\ &\le \frac{k_0}{8} \|A z_\varepsilon(t)\|_H^2 + \frac{2}{k_0} \|\varepsilon F_\varepsilon + 1 * F_\varepsilon\|_{L^\infty(0,T;H)}^2 \\ &+ \int_0^t \left\|(\varepsilon F_\varepsilon' + F_\varepsilon)(s)\right\|_H \|A z_\varepsilon(s)\|_H \, ds \\ &\le \frac{k_0}{8} \|A z_\varepsilon(t)\|_H^2 + \int_0^t \left\|(\varepsilon F_\varepsilon' + F_\varepsilon)(s)\right\|_H \|A z_\varepsilon(s)\|_H \, ds + c, \end{aligned} \tag{6.16}$$

and the term $\int_0^t \|(\varepsilon F_\varepsilon' + F_\varepsilon)(s)\|_H \, ds$ is bounded independently of ε. Moreover, owing to (4.25) and recalling that Y_ε is specified by (4.27) we have

$$\begin{aligned}|I_4(t)| &\le \left|\int_\Omega (-y_\varepsilon' + y_\varepsilon)(t) A z_\varepsilon(t)\right| + \left|\int_0^t \int_\Omega (-y_\varepsilon'' + y_\varepsilon') A z_\varepsilon\right| + \left|\int_0^t \int_\Omega \varepsilon \nabla y_\varepsilon'' \cdot \nabla z_\varepsilon'\right| \\ &\le \frac{k_0}{8} \|A z_\varepsilon(t)\|_H^2 + \|A z_\varepsilon\|_{L^2(0,t;H)}^2 + \frac{4}{k_0} \|y_\varepsilon\|_{W^{1,\infty}(0,T;H)}^2 \\ &+ \frac{1}{4} \left\|y_\varepsilon'\right\|_{L^2(0,T;H)}^2 + \int_0^t \|A z_\varepsilon(s)\|_H \left\|y_\varepsilon''(s)\right\|_H ds \\ &+ c \int_0^t \left\|\nabla y_\varepsilon''(s)\right\|_H \varepsilon^{1/2} \left\|\nabla z_\varepsilon'(s)\right\|_H ds \\ &\le \frac{k_0}{16} \|A z_\varepsilon(t)\|_H^2 + \|A z_\varepsilon\|_{L^2(0,t;H)}^2 + \int_0^t \|A z_\varepsilon(s)\|_H \left\|y_\varepsilon''(s)\right\|_H ds \\ &+ c \int_0^t \left\|\nabla y_\varepsilon''(s)_H\right\| \varepsilon^{1/2} \left\|\nabla z_\varepsilon'(s)\right\|_H ds + c, \end{aligned} \tag{6.17}$$

where $\int_0^t \|y_\varepsilon''(s)\|_H \, ds$ and $\int_0^t \|\nabla y_\varepsilon''(s)\|_H$ are bounded independently of ε (and t, of course).

Now, we can combine estimates (6.14)-(6.17) with (6.13). Due to (6.12), we deduce

$$\begin{aligned}&\varepsilon \left\|\nabla z_\varepsilon'(t)\right\|_H^2 + \left\|\nabla z_\varepsilon'\right\|_{L^2(0,t;H)}^2 + \|A z_\varepsilon(t)\|_H^2 \\ &\le c \left(\left\|\nabla z_\varepsilon'\right\|_{L^2(0,t;H)}^2 + \|A z_\varepsilon\|_{L^2(0,t;H)}^2 + \int_0^t \left\|(\varepsilon F_\varepsilon' + F_\varepsilon)(s)\right\|_H \|A z_\varepsilon(s)\|_H \, ds \right. \\ &\left. + \int_0^t \|A z_\varepsilon(s)\|_H \left\|y_\varepsilon''(s)\right\|_H ds + \int_0^t \left\|\nabla y_\varepsilon''(s)_H\right\| \varepsilon^{1/2} \left\|\nabla z_\varepsilon'(s)\right\|_H ds \right) + c. \end{aligned} \tag{6.18}$$

Consequently, the extended version of the Gronwall lemma presented in [1] allows us to conclude that

$$\varepsilon^{1/2}\left\|\nabla z'_\varepsilon\right\|_{L^\infty(0,T;H)}+\left\|\nabla z'_\varepsilon\right\|_{L^2(0,T;H)}+\|Az_\varepsilon\|_{L^\infty(0,T;H)}\le c. \tag{6.19}$$

Now, it is a standard matter to deduce analogous estimates on w_ε by (6.10) and (4.25). Thus, (6.19) ensures that

$$\varepsilon^{1/2}\left\|\nabla w'_\varepsilon\right\|_{L^\infty(0,T;H)}+\left\|\nabla w'_\varepsilon\right\|_{L^2(0,T;H)}+\|w_\varepsilon\|_{L^\infty(0,T;H^2(\Omega))}\le c. \tag{6.20}$$

Consequently, by (6.6) and (6.20) we infer that

$$\sum_{i=1}^{2}\|\chi_{i\varepsilon}\|_{L^\infty(0,T;V)}\le c, \tag{6.21}$$

$$\sum_{i=1}^{2}\varepsilon^{1/4}\|A\chi_{i\varepsilon}\|_{L^2(0,T;H)}\le c. \tag{6.22}$$

Remark 6.1. Let us point out that (6.21) implies, in particular, that there exists some subsequences of $\chi_{i\varepsilon}$ (still denoted by $\chi_{i\varepsilon}$) such that $\chi_{i\varepsilon}\overset{*}{\rightharpoonup}\chi_i$ in $L^\infty(0,T;V)$. Thus, we are allowed to recover for χ_i the regularity (2.37) in the case (3.28) holds (see Proposition 2.1 and Remark 2.2).

Twelfth a priori estimate. At this point, in order to treat w''_ε we can apply a similar procedure as that we have used to perform the sixth a priori estimate. However, owing to (6.20)-(6.22), now we can test (4.2) by $\varepsilon^{1/2}w''_\varepsilon$. Thus, after integrating by parts in time we obtain

$$\begin{aligned}
&\varepsilon^{3/2}\left\|w''_\varepsilon\right\|^2_{L^2(0,t;H)}+\frac{\varepsilon^{1/2}}{2}\left\|w'_\varepsilon(t)\right\|^2_H-\frac{\varepsilon^{1/2}}{2}\left\|\theta^0_\varepsilon\right\|^2_H \qquad (6.23)\\
&\le\left|\int_\Omega k_0\nabla w_\varepsilon(t)\cdot\varepsilon^{1/2}\nabla w'_\varepsilon(t)\right|+\varepsilon^{1/2}k_0\int_0^t\int_\Omega\left|\nabla w'_\varepsilon\right|^2+\left|\int_0^t\int_\Omega\chi'_{1\varepsilon}\varepsilon^{3/2}w''_\varepsilon\right|\\
&+\left|\int_\Omega\chi_{1\varepsilon}(t)\varepsilon^{1/2}w'_\varepsilon(t)\right|+\left|\int_0^t\int_\Omega\chi'_{1\varepsilon}\varepsilon^{1/2}w'_\varepsilon\right|+\left|\int_0^t\int_\Omega(\theta^0_\varepsilon-\chi^0_{1\varepsilon})\varepsilon^{1/2}w''_\varepsilon\right|\\
&\left|\int_0^t\int_\Omega F_\varepsilon\varepsilon^{3/2}w''_\varepsilon\right|+\left|\int_\Omega(1*F_\varepsilon)(t)\varepsilon^{1/2}w'_\varepsilon(t)\right|+\left|\int_0^t\int_\Omega F_\varepsilon(t)\varepsilon^{1/2}w'_\varepsilon(t)\right|\\
&+\langle(\varepsilon H_\varepsilon+1*H_\varepsilon)(t),\varepsilon^{1/2}w'_\varepsilon(t)\rangle+\left|\int_0^t\langle(\varepsilon H'_\varepsilon+H_\varepsilon)(s),\varepsilon^{1/2}w'_\varepsilon(s)\rangle\,ds\right|.
\end{aligned}$$

For the sake of synthesis, we refer to the detailed calculations of the sixth a priori estimate, even if here the test function $\varepsilon^{1/2}w''_\varepsilon$ differs from that we have used in the previous section by a factor of $\varepsilon^{1/2}$. Nonetheless, the reader can easily check that the procedure still fit well. The only difference is given by the term $\int_0^t \int_\Omega (\theta_\varepsilon^0 - \chi_{1\varepsilon}^0)\varepsilon^{1/2} w_\varepsilon''$, for which we use the property $\int_0^t \int_\Omega (\theta_\varepsilon^0 - \chi_{1\varepsilon}^0)\varepsilon^{1/2} w_\varepsilon'' = \int_\Omega (\theta_\varepsilon^0 - \chi_{1\varepsilon}^0)\varepsilon^{1/2}(w_\varepsilon'(t) - \theta_\varepsilon^0)$. Thus, by (3.6) it follows

$$\left| \int_0^t \int_\Omega (\theta_\varepsilon^0 - \chi_{1\varepsilon}^0)\varepsilon^{1/2} w_\varepsilon'' \right| \leq \frac{\varepsilon^{1/2}}{8} \left\| w_\varepsilon'(t) \right\|_H^2 + c. \tag{6.24}$$

At the end, it is easy to check that (recall also (3.6), (4.3), (4.4))

$$\varepsilon^{3/4} \left\| w_\varepsilon'' \right\|_{L^2(0,T;H)} + \varepsilon^{1/4} \left\| w_\varepsilon' \right\|_{L^\infty(0,T;H)} \leq c, \tag{6.25}$$

which concludes the proof of (3.38) (note that (6.25) improves the analogous result in the sixth a priori estimate (4.48) by a factor of $\varepsilon^{1/4}$).

Now, we are in a position of proving estimate (3.39). First of all, let us point out that assuming (3.35)-(3.37) is equivalent to require in our abstract formulation

$$\|F_\varepsilon - F\|_{L^1(0,T;H)} \leq c_5\, \varepsilon^{1/4} \tag{6.26}$$

$$\|H_\varepsilon - H\|_{L^1(0,T;V')} \leq c_5\, \varepsilon^{1/4} \tag{6.27}$$

$$\|\mathbf{G}_\varepsilon - \mathbf{G}\|_{L^2(0,T;\mathbf{W}')} \leq c_5\, \varepsilon^{1/4}. \tag{6.28}$$

We proceed by subtracting equation (5.10) (concerning w) from equation (4.2) (concerning w_ε) and testing their difference by $(w_\varepsilon - w)'$, which is rigorous thanks to (5.11). If we integrate over $(0,t)$, then we find

$$\begin{aligned}
&\varepsilon \int_0^t \int_\Omega w_\varepsilon''(w_\varepsilon - w)' + \left\|(w_\varepsilon - w)'\right\|_{L^2(0,t;H)}^2 + \frac{k_0}{2} \left\|\nabla(w_\varepsilon - w)(t)\right\|_H^2 \\
&- \varepsilon \int_0^t \int_\Omega \chi_{1\varepsilon}'(w_\varepsilon - w)' - \int_0^t \int_\Omega (\chi_{1\varepsilon} - \chi_1)(w_\varepsilon - w)' \\
&= \int_0^t \int_\Omega (\theta_\varepsilon^0 - \theta^0)(w_\varepsilon - w)' + \int_0^t \int_\Omega (\chi_{1\varepsilon}^0 - \chi_1^0)(w_\varepsilon - w)' \\
&+ \int_0^t \int_\Omega 1 * (F_\varepsilon - F)(w_\varepsilon - w)' + \varepsilon \int_0^t \int_\Omega F_\varepsilon (w_\varepsilon - w)' \\
&+ \int_0^t \langle 1 * (H_\varepsilon - H)(s), (w_\varepsilon - w)'(s)\rangle\, ds \\
&+ \varepsilon \int_0^t \langle H_\varepsilon(s), (w_\varepsilon - w)'(s)\rangle\, ds.
\end{aligned} \tag{6.29}$$

First we note that, due to (6.25), the following inequality holds

$$\left|\varepsilon \int_0^t \int_\Omega w_\varepsilon''(w_\varepsilon - w)'\right| \le 4\varepsilon^2 \left\|w_\varepsilon''\right\|^2_{L^2(0,T;H)} + \frac{1}{16}\left\|(w_\varepsilon - w)'\right\|^2_{L^2(0,t;H)}$$
$$\le \frac{1}{16}\left\|(w_\varepsilon - w)'\right\|^2_{L^2(0,t;H)} + c\varepsilon^{1/2}. \tag{6.30}$$

Moreover, (3.22) ensures

$$\left|\varepsilon \int_0^t \int_\Omega \chi_{1\varepsilon}'(w_\varepsilon - w)'\right| \le 4\varepsilon^2 \left\|\chi_{1\varepsilon}'\right\|^2_{L^2(0,t;H)} + \frac{1}{16}\left\|(w_\varepsilon - w)'\right\|^2_{L^2(0,t;H)}$$
$$\le \frac{1}{16}\left\|(w_\varepsilon - w)'\right\|^2_{L^2(0,t;H)} + c\varepsilon^{1/2}, \tag{6.31}$$

and, thanks to Young's inequality, there holds

$$\left|\int_0^t \int_\Omega (\chi_{1\varepsilon} - \chi_1)(w_\varepsilon - w)'\right| \le \frac{1}{16}\left\|(w_\varepsilon - w)'\right\|^2_{L^2(0,t;H)} + 4\left\|\chi_{1\varepsilon} - \chi_1\right\|^2_{L^2(0,t;H)}. \tag{6.32}$$

Next, by referring to (3.34) we treat the two integrands concerning the initial data

$$\left|\int_0^t \int_\Omega (\theta_\varepsilon^0 - \theta^0)(w_\varepsilon - w)' + \int_0^t \int_\Omega (\chi_{1\varepsilon}^0 - \chi_1^0)(w_\varepsilon - w)'\right|$$
$$\le \frac{1}{16}\left\|(w_\varepsilon - w)'\right\|^2_{L^2(0,t;H)} + c\left(\left\|\theta_\varepsilon^0 - \theta^0\right\|^2_H + \left\|\chi_{1\varepsilon}^0 - \chi_1^0\right\|^2_H\right)$$
$$\le \frac{1}{16}\left\|(w_\varepsilon - w)'\right\|^2_{L^2(0,t;H)} + c\varepsilon^{1/2}. \tag{6.33}$$

Hence, (4.3) and (6.26) implies

$$\left|\varepsilon \int_0^t \int_\Omega F_\varepsilon(w_\varepsilon - w)'\right| + \left|\int_0^t \int_\Omega 1 * (F_\varepsilon - F)(w_\varepsilon - w)'\right|$$
$$\le c\left(\varepsilon^2 \left\|F_\varepsilon\right\|_{L^\infty(0,T;H)} \left\|F_\varepsilon\right\|_{L^1(0,T;H)} + \left\|F_\varepsilon - F\right\|^2_{L^1(0,T;H)}\right)$$
$$+ \frac{1}{16}\left\|(w_\varepsilon - w)'\right\|^2_{L^2(0,T;H)} \le \frac{1}{16}\left\|(w_\varepsilon - w)'\right\|^2_{L^2(0,T;H)} + c\varepsilon^{1/2}. \tag{6.34}$$

Finally, as $\|(w_\varepsilon - w)(t)\|_H \le \|(w_\varepsilon - w)'\|_{L^1(0,t;H)}$, thanks to (4.4), (6.20), and (6.27), after an integration by parts in time, we are allowed to deduce

$$\left|\int_0^t \langle(\varepsilon H_\varepsilon + 1 * (H_\varepsilon - H))(s), (w_\varepsilon - w)'(s)\rangle\, ds\right|$$
$$\le \left|\langle(\varepsilon H_\varepsilon + 1 * (H_\varepsilon - H))(t), (w_\varepsilon - w)(t)\rangle\right|$$

$$
\begin{aligned}
&+ \left| \int_0^t \langle (\varepsilon H_\varepsilon' + (H_\varepsilon - H))(s), (w_\varepsilon - w)(s) \rangle \, ds \right| \\
&\le \| (\varepsilon H_\varepsilon + 1 * (H_\varepsilon - H))(t) \|_{V'} \, \| (w_\varepsilon - w)(t) \|_V \\
&+ \int_0^t \left\| (\varepsilon H_\varepsilon' + (H_\varepsilon - H))(s) \right\|_{V'} \| (w_\varepsilon - w)(s) \|_V \, ds \\
&\le \frac{1}{16} \left\| (w_\varepsilon - w)' \right\|^2_{L^2(0,t;H)} + \frac{k(0)}{8} \| \nabla (w_\varepsilon - w)(t) \|^2_H \\
&+ c \Big(\| \varepsilon H_\varepsilon \|_{W^{1,1}(0,T;V')} + \| 1 * (H_\varepsilon - H) \|^2_{W^{1,1}(0,T;V')} \\
&+ \int_0^t \left\| (\varepsilon H_\varepsilon' + (H_\varepsilon - H))(s) \right\|_V' \| \nabla (w_\varepsilon - w)(s) \|_H \, ds \Big) \\
&\le \frac{1}{16} \left\| (w_\varepsilon - w)' \right\|^2_{L^2(0,t;H)} + \frac{k(0)}{8} \| \nabla (w_\varepsilon - w)(t) \|^2_H \\
&+ \int_0^t \left\| (\varepsilon H_\varepsilon' + (H_\varepsilon - H))(s) \right\|_{V'} \| \nabla (w_\varepsilon - w)(s) \|_H \, ds + c\varepsilon^{1/2}. \qquad (6.35)
\end{aligned}
$$

In particular, note that (4.4) and (6.27) ensure that

$$
\left\| \varepsilon H_\varepsilon' + H_\varepsilon - H \right\|_{L^1(0,T;V')} \le c \varepsilon^{1/4}.
$$

Thus, if we combine (6.29) with (6.30)-(6.35), we can write

$$
\begin{aligned}
&\left\| (w_\varepsilon - w)' \right\|^2_{L^2(0,t;H)} + \| \nabla (w_\varepsilon - w)(t) \|^2_H \le c \, \| \chi_{1\varepsilon} - \chi_1 \|^2_{L^2(0,t;H)} \\
&+ \int_0^t \left\| (\varepsilon H_\varepsilon' + (H_\varepsilon - H))(s) \right\|_{V'} \| \nabla (w_\varepsilon - w)(s) \|_H \, ds + c\varepsilon^{1/2}. \qquad (6.36)
\end{aligned}
$$

Analogously, let us now consider the difference of the equations (3.18) and (3.14) (concerning $(\chi_{1\varepsilon}, \chi_{2\varepsilon})$ and (χ_1, χ_2), respectively) and test it by $(\chi_{1\varepsilon} - \chi_1, \chi_{2\varepsilon} - \chi_2)$. After an integration over $(0, t)$, it is now a standard matter to get

$$
\begin{aligned}
&\sum_{i=1}^{2} \left(\frac{1}{2} \| (\chi_{i\varepsilon} - \chi_i)(t) \|^2_H - \frac{1}{2} \left\| \chi^0_{i\varepsilon} - \chi^0_i \right\|^2_H + \eta(\varepsilon) \int_0^t \int_\Omega A \chi_{i\varepsilon} (\chi_{i\varepsilon} - \chi_i) \right) \\
&\le c \int_0^t \left\| (w_\varepsilon - w)'(s) \right\|_H \| (\chi_{1\varepsilon} - \chi_1)(s) \|_H \, ds \\
&+ \max\{ \| \alpha \|_{W^{1,\infty}(0,T)} , \| \operatorname{div} \mathbf{u}_\varepsilon \|_{L^\infty(Q)} \} \int_0^t \int_\Omega \big(\left\| (w_\varepsilon - w)'(s) \right\|_H \\
&+ \| \operatorname{div} (\mathbf{u}_\varepsilon - \mathbf{u})(s) \|_H \big) \, \| (\chi_{2\varepsilon} - \chi_2)(s) \|_H \, ds \le c \Big(\left\| (w_\varepsilon - w)' \right\|^2_{L^2(0,t;H)}
\end{aligned}
$$

$$+ \|\mathrm{div}(\mathbf{u}_\varepsilon - \mathbf{u})\|^2_{L^2(0,t;H)} + \sum_{i=1}^{2} \|\chi_{i\varepsilon} - \chi_i\|^2_{L^2(0,t;H)} \Bigg) . \tag{6.37}$$

Let us point out that due to (3.31) and (6.22), there holds

$$\left|\eta(\varepsilon)\int_0^t \int_\Omega A\chi_{i\varepsilon}(\chi_{i\varepsilon} - \chi_i)\right| \le c\frac{\varepsilon}{4}\|A\chi_{1\varepsilon}\|^2_{L^2(0,T;H)} + \|\chi_{1\varepsilon} - \chi_1\|^2_{L^2(0,t;H)}$$
$$\le \|\chi_{i\varepsilon} - \chi_i\|^2_{L^2(0,t;H)} + c\varepsilon^{1/2}, \tag{6.38}$$

for $i = 1, 2$, and thus (6.37) reduces to (cf. also (3.34))

$$\sum_{i=1}^{2} \|(\chi_{i\varepsilon} - \chi_i)(t)\|^2_H \le c\Bigg(\big\|(w_\varepsilon - w)'\big\|^2_{L^2(0,t;H)} + \|\mathrm{div}(\mathbf{u}_\varepsilon - \mathbf{u})\|^2_{L^2(0,t;H)}$$
$$+ \sum_{i=1}^{2} \|\chi_{i\varepsilon} - \chi_i\|^2_{L^2(0,t;H)}\Bigg) + c\varepsilon^{1/2}. \tag{6.39}$$

Finally, let us consider the equations concerning $\mathbf{u}_\varepsilon$, (3.19), and $\mathbf{u}$, (3.15), take their difference, and test it by $(\mathbf{u}_\varepsilon - \mathbf{u})$. Integrating in time over $(0, t)$ and performing a standard procedure lead us to the following inequality

$$\|\mathbf{u}_\varepsilon - \mathbf{u}\|^2_{L^2(0,t;\mathbf{W})} \le c\Big(\big\|(w_\varepsilon - w)'\big\|^2_{L^2(0,t;H)}$$
$$+ \|\chi_{2\varepsilon} - \chi_2\|^2_{L^2(0,t;H)} + \|\mathbf{G}_\varepsilon - \mathbf{G}\|^2_{L^2(0,T;\mathbf{W}')}\Big) . \tag{6.40}$$

At this point, after a multiplication by suitable constants we can add together (6.36), (6.39), and (6.40) in order to get at the end

$$\big\|(w_\varepsilon - w)'\big\|^2_{L^2(0,t;H)} + \|\nabla(w_\varepsilon - w)(t)\|^2_H + \sum_{i=1}^{2} \|(\chi_{i\varepsilon} - \chi_i)(t)\|^2_H$$
$$\|\mathbf{u}_\varepsilon - \mathbf{u}\|^2_{L^2(0,t;\mathbf{W})} \le c\Bigg(\sum_{i=1}^{2} \|\chi_{i\varepsilon} - \chi_i\|^2_{L^2(0,t;H)} + \|\mathbf{G}_\varepsilon - \mathbf{G}\|^2_{L^2(0,T;\mathbf{W}')}$$
$$+ \int_0^t \big\|(\varepsilon H'_\varepsilon + (H_\varepsilon - H))(s)\big\|_{V'} \|\nabla(w_\varepsilon - w)(s)\|_H \, ds\Bigg) + c\varepsilon^{1/2}. \tag{6.41}$$

Now, we can apply the generalized version of the Gronwall lemma introduced in [1] to handle the left hand side of (6.41). Indeed, by virtue of (3.34), (4.4), (6.27) , and (6.28), we are allowed to deduce that

$$\big\|(w_\varepsilon - w)'\big\|_{L^2(0,T;H)} + \|\nabla(w_\varepsilon - w)\|_{C^0([0,T];H)}$$

$$+\sum_{i=1}^{2}\|\chi_{i\varepsilon}-\chi_i\|_{C^0([0,T];H)}+\|\mathbf{u}_\varepsilon-\mathbf{u}\|_{L^2(0,T;\mathbf{W})}\le c\varepsilon^{1/4}, \tag{6.42}$$

which concludes the proof of (3.39).

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