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THE CARLEMAN INEQUALITY FOR LINEAR PARABOLIC EQUATIONS IN L^q NORM

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Abstract. The Carleman inequality for linear parabolic equations (see [5]) is extended to $L^q, 1 \leq q \leq 2$ norm. Two applications pertaining maximum principle for the Bolza problem and stabilization of the semilinear heat equation are given.

1. Introduction

Consider the backward linear parabolic equation

$$\begin{array}{ll} (p_t + \Delta p + ap + \text{ div } (bp))(x,t) = g(x,t) & \text{in } \; Q = \Omega \times (0,T) \\ p = 0 & \text{on } \; \Sigma = \Omega \times (0,T) \\ p(T) \in L^q(\Omega) & \end{array} \tag{1.1}$$

where $a \in L^\infty(Q)$, $b \in C^1(\overline{Q}; R^n)$, $g \in L^2(Q)$ and Ω is a bounded and open set of R^n with a sufficently smooth boundary $\partial\Omega$.

Let ω be any open subset of Ω and let $\psi \in C^2(\bar{\Omega})$ be such that $\psi > 0$ in Ω, $\psi = 0$ on $\partial\Omega$ and $\{x \in \Omega;\ |\nabla\psi(x)| = 0\} \subset \omega_0$, where ω_0 is an open subset of ω such that $\overline{\omega}_0 \subset \omega$. We set

$$\begin{array}{l} \alpha(x,t) = (t(T-t))^{-1}(e^{\lambda\psi(x)} - e^{2\lambda\|\psi\|_{C(\bar{\Omega})}}) \\ \varphi(x,t) = (t(T-t))^{-1}e^{\lambda\psi(x)},\ x \in \Omega,\ t \in (0,T) \end{array} \tag{1.2}$$

Then there is $\lambda_0 > 0$ and $\mu = \mu(\lambda) > 0$ such that for $\lambda \geq \lambda_0$ and $s \geq s_0(\lambda)$ the solutions p to (1.1) satisfy the estimate (see [5])

$$\begin{aligned} &\int_Q e^{2s\alpha}((\varphi s)^{-1}(p_t^2 + |\Delta p|^2) + s\varphi|\nabla p|^2 + s^3\varphi^3 p^2)dx\,dt \\ &\leq C_\lambda\Big(s^3\int_{Q_\omega} e^{2s\alpha}\varphi^3 p^2 dx\,dt + \int_Q e^{2s\alpha} g^2 dx\,dt\Big) \end{aligned} \tag{1.3}$$

along with the observability inequality

$$\int_Q p^2(x,0)dx \leq C_\lambda e^{2\mu s}\Big(\int_{Q_\omega} e^{2s\alpha}\varphi^3 p^2 dx\,dt + \int_Q g^2 dx\,dt\Big). \tag{1.4}$$

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Here $Q_\omega = \Omega \times (0,T)$ and $\mu = \mu(\lambda)$.

These inequalities, which are true for general second order parabolic equations with smooth coefficients (see also [6] for an extension to parabolic equations with nonsmooth first order coefficients), are relevant in the exact null controllability of the forward Cauchy problem

$$\begin{aligned}&(y_t - \Delta y - ay - b\cdot\nabla y)(x,t) = m(x)u(x,t) + fe^{2s\alpha}\varphi^3(x,t)\\&y(x,0) = y_0(x) \text{ in } \Omega;\ \ y = 0 \text{ on } \Sigma.\end{aligned} \tag{1.5}$$

Here m is the characteristic function of ω. As a matter of fact, as we shall see below, the Carleman inequality (1.2) is equivalent with the exact null controllability of (1.5) in a certain controller energetic space.

The aim of this work is to sharpen the Carleman inequality (1.3) to the L^q–norm $1 \le q \le 2$ (see Theorem 1 below). A related observability result was proved in [4] in connection with the exact controllability of superlinear heat equations. On these lines see also [3]. A such a result is also relevant in optimal control theory of linear parabolic equations with L^∞ state constraints as well as to exponential stabilizability of semilinear heat equations (see Section 4).

Throughout in the sequel we shall denote by $|\cdot|_q$ the norm in $L^q(\Omega)$ for $1 \le q \le \infty$ and by $W^{k,p}(\Omega)$ and $W^{2,1}_p(Q), H^{2,1}(Q) = W^{2,1}_p(Q)$ the usual Sobolev spaces on Ω and Q, respectively. We refer to [7] for other notations and standard results on theory of linear parabolic equations in Sobolev spaces to be used in this work.

2. The main result

Theorem 1. *Let $a \in L^\infty(Q)$. Then there are $s_0 = s_0(\lambda), \lambda_0 > 0$ and $\mu = \mu(\lambda) > 0$ such that for $\lambda \ge \lambda_0$ and $s \ge s_0(\lambda)$,$s_0 < \delta \le s$ the following inequalities*

$$\Big(\int_Q e^{2s\alpha}\varphi^3 p^2 dx\,dt\Big)^{1/2} \le Cs^3 \int_{Q_\omega} e^{(s(1-\eta)-\delta(1+\eta))\alpha}|p|dx\,dt \tag{2.1}$$

$$|p(0)|_\infty \le Ce^{\mu s}\int_{Q_\omega} e^{(s(1-\eta)-\delta(1+\eta))\alpha}|p|dx\,dt \tag{2.2}$$

hold for all weak solutions $p \in L^1(Q)$ to equation (1.1) with $g \equiv 0$. Here $C = C^\lambda_\delta$ is independent of s, p and $\eta(\lambda) = e^{-\lambda\|\psi\|_{C(\bar\Omega)}}$.

If combine inequalities (2.1) and (1.3) we obtain the estimate

$$\begin{aligned}&\Big(\int_Q e^{2s\alpha}((\varphi s)^{-1}(p_t^2+|\Delta p|^2)+s\varphi|\nabla p|^2+s^3\varphi^3p^2dx\,dt)\Big)^{1/2}\\&\le C_\delta s^3\int_{Q_\omega}e^{(s(1-\eta)-\delta(1+\eta))\alpha}|p|dx\,dt\end{aligned}\tag{2.3}$$

for all $s\ge s_0(\lambda),\ \lambda\ge\lambda_0$ and $s_0(\lambda)<\delta<s$.

In particular, it follows by (2.1) that

$$\Big(\int_Q e^{qs\alpha}\varphi^{\frac{3q}{2}}|p|^q dx\,dt\Big)^{1/q}\le Cs^3\Big(\int_{Q_\omega}e^{q(s(1-\eta)-\delta(1+\eta))\alpha}|p|^q dx\,dt\Big)^{1/q}\tag{2.4}$$

for $1\le q\le 2$.

As regards the Carleman inequality for the nonhomogeneous equation (1.1), we have the following partial result:

Corollary 1. *Let $g\in L^1(Q)$ and $1\le q\le\min(2,\frac{n+2}{n})\cdot$ Then under the conditions of Theorem 1, we have*

$$\int_Q e^{qs\alpha}\varphi^{\frac{3q}{2}}|p|^q dx\,dt\le Cs^{3q}\int_{Q_\omega}e^{q(s(1-\eta)-\delta(1+\eta))\alpha}|p|^q dx\,dt+C\int_Q|g|dx\,dt\tag{2.5}$$

$$|p(0)|_1\le C_s\Big(\int_{Q_\omega}e^{(s(1-\eta)-\delta(1+\eta))\alpha}|p|+\int_Q|g|dx\,dt\Big).\tag{2.6}$$

for all solutions p to equation (1.1).

Proof. We write $p=p_1+p_2$ where p_1 is a solution to the homogeneous equation (1.1) and p_2 is the solution to (1.1) with the final condition $p_2(T)=0$. We have

$$p_2(t)=-\int_t^T S(\tau-t)(ap_2(\tau)+\text{ div }(bp_2)(\tau)-g(\tau))d\tau\tag{2.7}$$

where $S(t)$ is the semigroup generated by the Laplace operator with null Dirichlet boundary value conditions. Recalling that

$$|S(t)z|_q\le Ct^{-\frac{n}{2}(p^{-1}-q^{-1})}|z|_p\tag{2.8}$$

$$|S(t)z_{x_i}|_q\le Ct^{-\frac{n}{2}(p^{-1}-q^{-1})-\frac{1}{2}}|z|_p,\tag{2.9}$$

after some calculation involving Gronwall's lemma we get the estimate

$$|p_2(t)|_q\le C\int_t^T|\tau-t|^{-\frac{n}{2}(1-q^{-1})}|g(\tau)|_1d\tau.$$

Then one applies inequality (2.4) to p_1 along with the previous estimate which implies (2.5). As regards (2.6) it follows for $q = 1$ from (2.5) and the inequality

$$|p(0)|_1 \le C(|p(t)|_1 + |g(t)|_1) \quad \forall\, t \in (0,T)$$

which follows by (2.7)-(2.9).

It should be emphasized that inequality (2.1) is true for all mild solutions p to homogeneous equation (1.1) with $p(T) \in L^1(\Omega)$ or more generally, with $p(T) \in M(\Omega)$ (the space of all Borelian bounded measures on Ω). However, in both cases we have

$$p \in C([0,T]; H_0^1(\Omega)),\ p_t \in L^2(0, T-\eta; L^2(\Omega)), \forall\, \eta \in (0,T)$$

and $p(t) \in L^\infty(\Omega)$, $\forall\, t \in [0,T)$.

The idea of the proof , already used in [5] in a different context, relies on the following observation: if system (1.5) is exactly null controllable with controllers $u \in X$, then the dual equation satisfies the Carleman (observability) inequality in X'-norm. (Here X is a suitable function space on Q.)

As explicitly seen from the proof, Theorem 1 extends to general second order elliptic differential operators with smooth coefficients as well as for equations (1.1) with flux boundary value conditions.

3. Proof of Theorem 1

Lemma 1 below is the main ingredient of the proof.

Lemma 1. *Let $y_0 \in L^2(\Omega)$ and $f \in L^2_{\text{loc}}(Q)$ be such that $fe^{s\alpha}\varphi^{3/2} \in L^2(Q)$. Then for $s \ge s_0(\lambda)$, $\lambda \ge \lambda_0$, $\delta \in (s_0, s)$ large enough (but independent of y_0 and f) there is $u^* \in L^\infty(Q)$ such that $y^*(T) \equiv 0$ and*

$$\begin{aligned} I(u^*) &= \int_Q e^{-2s\alpha}\varphi^{-3}|u^*|^2 dx\, dt + \left\| e^{-(s(1-\eta)+\delta(1+\eta))\alpha} u^* \right\|^2_{L^\infty(Q)} \\ &\le C_\delta^\lambda \Big(e^{\mu s}|y_0|_2^2 + s^6 \int_Q e^{2s\alpha}\varphi^3 f^2 dx\, dt \Big). \end{aligned} \tag{3.1}$$

If $y_0 \in L^1(\Omega)$, then

$$I(u^*) \le C_\delta^\lambda e^{\mu s}\Big(|y_0|_1^2 + \int_Q e^{2s\alpha}\varphi^3 f^2 dx\, dt \Big) \tag{3.1$'$}$$

where $\mu = \mu(\lambda) > 0$ and $\eta = e^{-\lambda\|\psi\|_{C(\bar\Omega))}}$.

Here $y^* \in C([0,T]; L^1(\Omega)) \cap C(]0,T]; H_0^1(\Omega)) \cap W^{1,2}(]0,T]; L^2(\Omega))$ is the weak ("mild") solution to equation (1.5) where $u = u^*$.

Proof. We shall assume first that $y_0 \in L^2(\Omega)$ and we will argue as in [3]. Namely, consider the optimal control problem

$$\text{Minimize} \int_Q e^{-2s\alpha}\varphi^{-3}u^2 dx\, dt + \varepsilon^{-1}\int_\Omega y^2(x,T)dx \tag{3.2}$$

subject to (1.5).

Let $(y_\varepsilon, u_\varepsilon)$ be the optimal pair of (3.2). Then, by the Pontriaghin maximum principle,we have

$$u_\varepsilon = mp_\varepsilon e^{2s\alpha}\varphi^3 \text{ in } \ Q, \tag{3.3}$$

where

$$\begin{array}{ll} (p_\varepsilon)_t + \Delta p_\varepsilon + ap_\varepsilon + \text{ div } (bp) = 0 & \text{in } \ Q \\ p_\varepsilon(x,T) = -\varepsilon^{-1}y_\varepsilon(x,T); \ p_\varepsilon = 0 & \text{on } \ \Sigma. \end{array} \tag{3.4}$$

By (3.2), (3.3) and (3.4) it follows that

$$\int_{Q_\omega} p_\varepsilon^2 e^{2s\alpha}\varphi^3 dx\, dt + \int_Q f e^{2s\alpha}\varphi^3 p_\varepsilon dx\, dt + \varepsilon^{-1}\int_\Omega y_\varepsilon^2(x,T)dx$$
$$= -\int_\Omega y_0(x)p_\varepsilon(x,0)dx.$$

Then (1.3) yields

$$\int_{Q_\omega} p_\varepsilon^2 e^{2s\alpha}\varphi^3 dx\, dt \le C\Big(e^{2\mu s}|y_0|_2^2 + \int_Q e^{2s\alpha}\varphi^3 f^2 dx\, dt\Big). \tag{3.5}$$

Equivalently,

$$\int_Q e^{-2s\alpha}\varphi^{-3}u_\varepsilon^2 dx\, dt \le C\Big(e^{2\mu s}|y_0|_2^2 + \int_Q e^{2s\alpha}\varphi^3 f^2 dx\, dt\Big). \tag{3.6}$$

We shall prove by a "bootstrap" argument that u_ε satisfies an estimate of the form (3.1). Let $\delta \in (s_0, s)$ be arbitrarily small but fixed.

We set

$$\alpha_0(t) = \frac{1 - e^{2\lambda\|\psi\|_{C(\overline{\Omega})}}}{t(T-t)}, \ \varphi_0(t) = \frac{1}{t(T-t)}$$

and note that

$$\alpha_0(t) \le \alpha(x,t) \le \alpha_0(t)e^{\lambda\|\psi\|_{C(\overline{\Omega})}}(1 + e^{\lambda\|\psi\|_{C(\overline{\Omega})}})^{-1}, \ \ \forall\,(x,t) \in Q. \tag{3.7}$$

and $\varphi_0 \le \varphi \le e^{\lambda\|\psi\|_{C(\overline{\Omega})}}\varphi_o$. Let $\{\delta_j\}$ be an increasing sequence such that $0 < \delta_j < \delta\ \forall\, j$. For each j we set $v_j(x,t) = e^{(s+\delta_j)\alpha_0(t)}\varphi_0^3(t)p_j(x,t)$. We have

$$\begin{array}{ll} (v_j)_t + \Delta v_j + av_j + \text{ div } (bp_j) = g_j & \text{in } Q \\ v_j = 0 & \text{on } \Sigma \\ v_j(x,0) = v_j(x,T) = 0 & \text{in } \Omega \end{array} \tag{3.8}$$

where $g_j = p_\varepsilon(e^{(s+\delta_j)\alpha_0}\varphi_0^3)_t$. By (1.3) and (3.7) we see that

$$\int_Q ((g_1)_t^2 + g_1^2 + |\nabla g_1|^2)dx\,dt \tag{3.9}$$
$$\leq C_1 \int_Q e^{2(s+\delta_1)\alpha}\varphi^8((s+\delta_1)^4 t p_\varepsilon^2 + |\nabla p_\varepsilon|^2) + (s+\delta_1)^2(p_\varepsilon)_t^2)dx\,dt$$
$$\leq C_2 s^3 \int_Q e^{2s\alpha}(s\varphi^3 p_\varepsilon^2 + s\varphi|\nabla p_\varepsilon|^2 + (s\varphi)^{-1}|p_\varepsilon|_t^2) \leq C_3 s^6 \int_{Q_\omega} e^{2\alpha}\varphi^3 p_\varepsilon^2 dx\,dt$$

and by (3.6) we have

$$\|g_1\|_{H^1(Q)} \leq C_4\Big(e^{2\mu s}|y_0|_2^2 + \int_Q e^{2s\alpha}\varphi^3 f^2 dx\,dt\Big)$$

and so by the Sobolev imbedding theorem

$$\|g_1\|_{L^{p_1}(Q)} \leq C_5\Big(e^{\mu s}|y_0|_2 + s^3\Big(\int_Q e^{2s\alpha}\varphi^3 f^2 dx\,dt\Big)^{1/2}\Big)$$

for $p_1 = \dfrac{2(n+1)}{n-1}$. Then by the parabolic regularity we have

$$\|v_1\|_{W_{p_1}^{2,1}(Q)} \leq C\Big(e^{\mu s}|y_0|_2 + s^3\Big(\int_Q e^{2s\alpha}\varphi^3 f^2 dx\,dt\Big)^{1/2}\Big). \tag{3.10}$$

Recalling that (see e.g. [7], Lemma 3.3, Chap.II) $W_p^{2,1}(Q) \subset L^q(Q)$ for $\frac{1}{p} - \frac{1}{q} \leq \frac{2}{n+2}$ we obtain by (3.10) that

$$\|v_1\|_{L^{p_2}(Q)} \leq C\Big(e^{\mu s}|y_0|_2 + s^3\Big(\int_Q e^{2s\alpha}\varphi^3 f^2 dx\,dt\Big)^{1/2}\Big),$$

where $p_2 = p_1 + \frac{2p_1^2}{n+2-2p_1}$ (if $n+2-2p_1 \leq 0$ then $W_{p_1}^{2,1}(Q) \subset L^\infty(Q)$). This implies that

$$\|g_2\|_{L^{p_2}(Q)} \leq C\Big(e^{\mu s}|y_0|_2 + s^3\Big(\int_Q e^{2s\alpha}\varphi^3 f^2 dx\,dt\Big)^{1/2}\Big)$$

and therefore,

$$\|v_2\|_{W_{p_2}^{2,1}(Q)} \leq C\Big(e^{\mu s}|y_0|_2 + s^3\Big(\int_Q e^{2s\alpha}\varphi^3 f^2 dx\,dt\Big)^{1/2}\Big).$$

In general, we have

$$\|v_j\|_{W_{p_j}^{2,1}(Q)} \leq C\Big(e^{\mu s}|y_0|_2 + s^3\Big(\int_Q e^{2s\alpha}\varphi^3 f^2 dx\,dt\Big)^{1/2}\Big), \tag{3.11}$$

where $p_j = p_{j-1} + \frac{2p_{j-1}^2}{n+2-2p_{j-1}}$. Thus there is N such that $N + 2 - 2p_N \leq 0$ and therefore $W_{p_N}^{2,1}(Q) \subset C(\overline{Q})$. Then from (3.11) we have

$$\|v_N\|_{L^\infty(Q)} \leq C\Big(e^{\mu s}|y_o|_2 + s^3\Big(\int_Q e^{2s\alpha}\varphi^3 f^2 dx\, dt\Big)^{1/2}\Big).$$

This yields

$$\Big\|e^{(s+\delta)\alpha_0}\varphi_0^3 p_\varepsilon\Big\|_{L^\infty(Q)} \leq C\Big(e^{\mu s}|y_o|_2 + s^3\Big(\int_Q e^{2s\alpha}\varphi^3 f^2 dx\, dt\Big)^{1/2}\Big)$$

and by (3.7) it follows that

$$\|e^{(s+\delta)(1+\eta)\alpha}\varphi^3 p_\varepsilon\|_{L^\infty(Q)}^2 \leq Cs^6\Big(e^{2\mu s}|y_0|_2^2 + \int_Q e^{2s\alpha}\varphi^3 f^2 dx\, dt\Big).$$

Then by (3.7), we obtain

$$\Big\|e^{-(s(1-\eta)+\delta(1+\eta))\alpha}u_\varepsilon\Big\|_{L^\infty(Q)}^2 \leq C\Big(e^{2\mu s}|y_0|_2^2 + s^6\int_Q e^{2s\alpha}\varphi^3 f^2 dx\, dt\Big) \quad (3.12)$$

for all $s \geq s_0(\lambda),\ \lambda \geq \lambda_0$ and $s_0(\lambda) \leq \delta < s$.

By (3.6) and (3.12) it follows that on a subsequence $\{\varepsilon\} \longrightarrow 0$ we have

$$\begin{aligned} u_\varepsilon &\longrightarrow \bar{u} \quad \text{weak star in } L^\infty(Q) \\ y_\varepsilon &\longrightarrow \bar{y} \quad \text{weakly in } H^{2,1}(Q),\ \text{strongly in } C([0,T];L^2(\Omega)). \end{aligned}$$

Clearly, $(\bar{y},\bar{u})$ satisfy (1.5) and $\bar{y}(T) \equiv 0$. Moreover, by (3.6) and (3.12) we obtain the estimate

$$\begin{aligned} &\int_Q e^{-2s\alpha}\varphi^{-3}|\bar{u}|^2 dx\, dt + \Big\|e^{-(s-\delta)\alpha}\bar{u}\Big\|_{L^\infty(Q)}^2 \\ &\quad \leq C\Big(e^{2\mu s}|y_0|_2^2 + s^6\int_Q e^{2s\alpha}\varphi^3 f^2 dx\, dt\Big) \end{aligned} \quad (3.13)$$

for $s \geq s_0(\lambda),\ \lambda \geq \lambda_0$ as claimed.

Now assume that $y_0 \in L^1(\Omega)$. Then, the equation

$$\begin{aligned} &y_t - \Delta y - ay - b\cdot\nabla y = fe^{2s\alpha}\varphi^3 \qquad \text{in } \Omega\times(0,\tau) \\ &y(0) = y_0 \text{ in } \Omega;\ y = 0 \text{ on } \partial\Omega\times(0,\tau) \end{aligned}$$

has a unique weak solution $y \in C([0,\eta];L^1(\Omega)) \cap C(]0,\tau];H_0^1(\Omega))$. Moreover, by a little calculations involving (2.8), (2.9), we see that

$$|y(\tau)|_2 \leq C_\tau|y_0|_1 + \Big(\int_Q f^2 e^{4s\alpha}\varphi^6 dx\, dt\Big)^{1/2}$$

where $\tau \in (0,T)$ is arbitrary but fixed. By the first part of the proof (see (3.13)) there are $(\widetilde{y},\widetilde{u})$ which satisfy (1.5) on $Q_\tau = \Omega \times (\tau, T)$ and such that $\widetilde{y}(\tau) = y(\tau)$, $\widetilde{y}(T) = 0$,

$$\begin{aligned}
&\int_\tau^T \int_\Omega e^{-2s\alpha}\varphi^{-3}|\widetilde{u}|^2 dx\, dt + \left\| e^{-(s(1-\eta)-\delta(1+\eta))\alpha}\bar{u}\right\|^2_{L^\infty(Q_\tau)} \\
&\le Cs^6\Big(e^{2\mu s}|y(\tau)|_2^2 + \int_Q e^{2s\alpha}\varphi^3 f^2 dx\, dt\Big) \\
&\le Cs^6 e^{2\mu s}\Big(|y_0|_1^2 + \int_Q e^{2s\alpha}\varphi^3 f^2 dx\, dt\Big).
\end{aligned}$$

If we define

$$u^*(t) = \begin{cases} 0 & \text{if } 0 < t \le \tau \\ \widetilde{u} & \text{if } \tau < t \le T \end{cases},$$

then clearly u^* satisfy estimate (3.1) as claimed.(In the above calculation we have redefined several times the function $\mu(\lambda)$.)

Proof of Theorem 1. (continued) Let (y^*, u^*) be as in Lemma 1 and let p be any solution to

$$\begin{aligned}
&p_t + \Delta p + ap + \text{ div } (bp) = 0 \text{ in } Q \\
&p = 0 \text{ on } \Sigma;\ p(T) \in L^\infty(\Omega).
\end{aligned} \tag{3.14}$$

By the maximum principle for parabolic equations we know that $p \in L^\infty(Q)$ and $p(0) \in L^\infty(\Omega)$. By (1.5) (where $u = u^*$) and by (3.14) we obtain

$$\int_\Omega y_0(x)p(x,0)dx + \int_{Q_\omega} pu^* dx\, dt + \int_Q e^{2s\alpha}\varphi^3 fp\, dx\, dt = 0. \tag{3.15}$$

Recalling estimate (3.1), we get for $y_0 = 0$ that

$$\begin{aligned}
&\int_Q e^{2s\alpha}\varphi^3 fp\, dx\, dt \le \left\| u^* e^{-(s(1-\eta)+\delta(1+\eta))\alpha}\right\|_{L^\infty(Q)} \int_{Q_\omega} e^{(s(1-\eta)-\delta(1+\eta))\alpha}|p|dx\, dt \\
&\le Cs^3\Big(\int_Q e^{2s\alpha}\varphi^3 f^2 dx\, dt\Big)^{1/2} \int_{Q_\omega} e^{(s(1-\eta)-\delta(1+\eta))\alpha}|p|dx\, dt.
\end{aligned}$$

For $f = p$ the latter yields

$$\int_Q e^{2s\alpha}\varphi^3 p^2 dx\, dt \le Cs^3\Big(\int_Q e^{2s\alpha}\varphi^3 p^2 dx\, dt\Big)^{1/2} \int_{Q_\omega} e^{(s(1-\eta)-\delta(1+\eta))\alpha}|p|dx\, dt.$$

Finally,

$$\Big(\int_Q e^{2s\alpha}\varphi^3 p^2 dx\, dt\Big)^{1/2} \le Cs^3 \int_{Q_\omega} e^{(s(1-\eta)-\delta(1+\eta))\alpha}|p|dx\, dt. \tag{3.16}$$

Next we take in (3.15), $f = 0$ and $y_0 \in L^1(\Omega)$. By $(3.1)'$, we obtain

$$\int_\Omega y_0(x)p(x,0)dx \leq Ce^{\mu s}|y_0|_1 \int_{Q_\omega} e^{(s(1-\eta)-\delta(1+\eta))\alpha}|p|\,dx\,dt,\ \forall\, y_0 \in L^1(\Omega).$$

Hence,

$$|p(0)|_\infty \leq C_\delta^\lambda e^{\mu s} \int_{Q_\omega} e^{(s(1-\eta)-\delta(1+\eta))\alpha}|p|\,dx\,dt. \tag{3.17}$$

By density estimates (3.16), (3.17) extend to all solutions $p \in C([0,T]; L^1(\Omega))$ to (3.14) and so we get (3.1) as claimed. This completes the proof.

Remark 3.1. As communicated to us by C. Lefter inequality (2.3) extends by the above duality arguments to $q \geq 2$.

4. Some examples in control theory

1. The maximum principle for the state constraint Bolza problem. Consider the problem

$$\text{Minimize} \int_Q h(u(x,t))dx\,dt + l(y(0), y(T)) \tag{4.1}$$

subject to

$$\begin{array}{ll} y_t - \Delta y - ay = mu & \text{in } Q \\ y = 0 & \text{on } \Sigma. \end{array} \tag{4.2}$$

Here $h : R \to \overline{R} = (-\infty, +\infty]$ is a lower semicontinuous convex function, $l : L^2(\Omega) \times L^2(\Omega) \longrightarrow \overline{R}$ is convex and lower semicontinuous, $a \in L^\infty(Q)$ and m is the characteristic function of an open subset ω of Ω.

Optimal control problems with endpoint constraint $(y(0), y(T)) \in K \subset L^2(\Omega) \times L^2(\Omega)$ (in particular, periodic optimal control problems) can be written in the form (4.1) if one define

$$l(y_1,y_2) = \begin{cases} 0 & \text{if } (y_1,y_2) \in K \\ +\infty & \text{otherwise.} \end{cases}$$

As an immediate consequence of the Carleman estimate (2.1) we shall derive here a Pontriaghin maximum principle for problem (4.1).

Corollary 2. *Let us further assume that* $\operatorname{int} D(h) \neq \emptyset$ *and that there is* $\bar{u} \in C(\overline{Q})$ *such that* $\bar{u}(x,t) \in \operatorname{int} D(h)$, $\forall\,(x,t) \in \overline{Q}$ *and* $(\bar{y}(0), \bar{y}(T)) \in D(l)$. *Let* $(y^*, u^*) \in C([0,T]; L^2(\Omega)) \times L^2(Q)$ *be optimal in* (4.1). *Then*

$$u^*(x,t) \in \partial h^*(m(x)p(x,t))\ ,\text{a.e. } (x,t) \in Q \tag{4.3}$$

where

$$\begin{aligned} &p_t + \Delta p + ap = 0 \quad \text{in } Q \\ &p = 0 \qquad\qquad\qquad \text{on } \Sigma \end{aligned} \tag{4.4}$$

$$e^{s\alpha}\varphi^{3/2}p \in L^2(Q) \text{ for } s \geq s_0, \lambda \geq \lambda_0. \tag{4.5}$$

Here $h^*(v) = \sup\{vu - h(u);\ u \in R\}$ is the conjugate function of h, $D(h) = \{u \in R;\ h(u) < +\infty\}$; ∂h^* is the subdifferential of h^* and $\bar{y}$ is the solution to (4.2) for $u = \bar{u}$. In particular, if $h(u) = 0$ for $|u| \leq \rho$ and $h(u) = \infty$ for $|u| > \rho$, (4.3) can be rewritten as

$$u^*(x,t) = \rho \text{ sgn } p(x,t), \text{ a.e. } (x,t) \in Q_\omega$$

and therefore, $u^* = \rho$ on a dense subset of Q_ω.

Proof. Consider the approximating control problem

$$\begin{aligned} &\text{Minimize } \int_Q (h(u) + 2^{-1}|u - u^*|_2^2) dx\, dt \\ &+ l_\varepsilon(y(0), y(T)) + \tfrac{1}{2}(|y(0) - y^*(0)|_2^2 + |y(T) - y^*(T)|_2^2) \end{aligned} \tag{4.6}$$

subject to (4.2). Here

$$\begin{aligned} &l_\varepsilon(y_1, y_2) = \inf\{(2\varepsilon)^{-1}(|y_1 - z_1|_2^2 + |y_2 - z_2|_2^2) + l(z_1, z_2) \\ &(z_1, z_2) \in L^2(\Omega) \times L^2(\Omega)\}. \end{aligned}$$

Let $(y_\varepsilon, u_\varepsilon)$ be optimal in problem (4.6). It follows that for $\varepsilon \longrightarrow 0$

$$\begin{array}{llll} u_\varepsilon & \longrightarrow & u^* & \text{strongly in } L^2(Q) \\ y_\varepsilon & \longrightarrow & y^* & \text{strongly in } C([0,T]; L^2(\Omega)). \end{array}$$

Moreover, we have

$$mp_\varepsilon \in \partial h(u_\varepsilon) + u_\varepsilon - u^* \text{ , a.e.in } Q \tag{4.7}$$

where

$$\begin{aligned} &(p_\varepsilon)_t + \Delta p_\varepsilon + ap_\varepsilon = 0 \text{ in } Q \text{ ; } p_\varepsilon = 0 \text{ on } \Sigma \\ &(p_\varepsilon(0) - (y_\varepsilon(0) - y^*(0)), -p_\varepsilon(T) - (y_\varepsilon(T) - y^*(T))) = \nabla l_\varepsilon(y_\varepsilon(0), y_\varepsilon(T)). \end{aligned}$$

Let $\rho > 0$ be such that $h(\bar{u} + \rho w) \leq C$, a.e. in Q for $|w| \leq 1$. If multiply (4.7) by $u_\varepsilon - \bar{u} - \rho w$, after some calculation, we get

$$\int_{Q_\omega} |p_\varepsilon| dx\, dt \leq C$$

and so by Theorem 1

$$\int_Q e^{2s\alpha}\varphi^3|p_\varepsilon|^2 dx\, dt \leq C, \ \forall\, \varepsilon > 0.$$

Then letting ε tend to zero into the above equations we get (4.3), (4.4), (4.5) as claimed.

In the case of periodic optimal control problem, i.e., $K = \{(y_1, y_2); y_1 = y_2\}$ we see that $p_\varepsilon(0) = p_\varepsilon(T)$ and so the function p arising in Corollary 2 is periodic, i.e., $p(0) = p(T)$.

2. Internal stabilization of semilinear heat equation. Consider the equation

$$\begin{aligned} & y_t - \Delta y - ay - b\cdot\nabla y + f(x,t,y) = mu && \text{in } \Omega\times R^+ \\ & y(x,0) = y_0(x), && x\in\Omega \\ & y = 0 && \text{on } \partial\Omega\times R^+ \end{aligned} \tag{4.8}$$

where m is the characteristic function of some open domain $\omega \subset \Omega$ and $a \in L^\infty(\Omega\times R^+)$, $b \in C^1(\overline{Q}; R^n)$, $f : \Omega\times R^+\times R \to R$ satisfy the following conditions

(i) $a \le \text{div} b$.

(ii) $f(x,t,y)$ is monotonically nondecreasing in y, continuous in (x,t),

$$f(x,t,0) = 0, |\frac{f(x,t,y)}{y}| \le C_R, \forall (x,t)\in\Omega\times R^+, |y|\le R, \forall R > 0.$$

Consider the feecback controller

$$u(x,t) = -y(x,t), \quad (x,t)\in\Omega\times R^+. \tag{4.9}$$

We have

Corollary 3. *For each $y_0 \in L^1(\Omega)$ the feedback controller (4.9) exponentially stabilizes system (4.8) in $L^1(\Omega)$, i.e.,*

$$|y(t)|_1 \le C\exp(-\gamma t)|y_0|_1, \ \forall\, t > 0, \tag{4.10}$$

where $\gamma > 0$ and y is the solution to closed loop system

$$\begin{aligned} & y_t - \Delta y - ay - b\cdot\nabla y + f(x,t,y) + my = 0, && \\ & y(x,0) = y_0(x) && \textit{in } \Omega \\ & y = 0 && \textit{on } \partial\Omega\times R^+. \end{aligned} \tag{4.11}$$

If a, b, f are independent of t and y_e is a steady–state solution to the uncontrolled system (4.8) it follows by the previous corollary that the feedback controller $u = -(y - y_e)$ exponentially stabilizes this solution in $L^1(\Omega)$.

It should be said that in general, under assumptions (i), (ii) the steady state solutions to uncontrolled system (4.8) are not asimptotically stable. (A sufficent condition for this is $\lambda_1 - a - \frac{1}{2}$ div $b > 0$ where λ_1 is the first eigenvalue of $-\Delta$.) Moreover, this result is not implied by the local exact controllability of system (4.8) since the latter usually implies the stability of the steady–state solutions only.

As regards the literature, there are very few results on internal stabilizability of parabolic semilinear equations. Previously such a result was obtained for system (4.8) with Neumann boundary conditions by Gh. Aniculăesei and S. Aniţa [1]. One might expect that this approach could be applicable to a broader class of nonlinear systems of dissipative type.

Proof. Assume first that $y_0 \in L^\infty(\Omega)$. Consider the equation

$$z_t - \Delta z - \lambda z - b\cdot\nabla z + mz = 0, z(x,0) = |y_0(x)|, z = 0 \text{ in } \partial\times R^+,$$

where $\lambda =$ ess supp a. Then by standard comparison techniques it follows by (i) and (ii) that $z \geq 0$, a.e. in $\Omega\times R^+$ and

$$|y(x,t)| \leq z(x,t), \text{ a.e.} \tag{4.12}$$

where y is the solution to (4.11).This follows by a standard device subtracting (4.11), (4.12) and multiplying the resulting equation by $((y-z)^+)$ and $(y+z)^+)$ respectively.) (The existence of a solution to problem (4.11) follows from the general theory of nonlinear parabolic equations with maximal monotone nonlinearities (see e.g. [2]).) Now multiplying equation (4.12) by sgn z and recalling that by Kato's inequality

$$-\Delta z \text{ sgn } z \geq -\Delta|z| \text{ in } \mathcal{D}'$$

we obtain that

$$\frac{d}{dt}|z(t)|_1 + |mz(t)|_1 \leq 0, \quad \text{a.e. } t>0.$$

This yields

$$|z(t+1)|_1 + \int_t^{t+1} |mu(s)|_1 ds \leq |z(t)|_1 \;\; \forall\, t\geq 0. \tag{4.13}$$

Then by the observability inequality (2.6) written on the interval $(t,t+1)$ for the forward equation (4.12) we se that

$$|z(t+1)|_1 \quad \leq C\int_t^{t+1} |mz(s)|_1 ds, \quad \forall\, t\geq 0$$

and together with (4.13) this yields

$$|z(t+1)|_1 \leq \gamma_0 |z(t)|_1 \;\; \forall\, t\geq 0 \tag{4.15}$$

where $\gamma_0 \leq 1$. By (4.14) we obtain (4.10). By density (4.10) extends to all $y_0\in L^1(\Omega)$ as claimed.

By Corollary 3 it follows by a standard argument that system (4.8) with boundary control is exponentially stabilizable in $L^1(\Omega)$.

Remark 1 Theorem 1 can be used to prove the existence in the following *data assimilation problem*

$$\text{Min}\ \{\int_{Q_\omega} |y - y_d|dx; y_t - \Delta y + ay = f \text{ in } \Omega, y = 0 \text{ on } \partial\Omega\}.$$

Here y_d and f are given functions on Q.

By Corollary 1 it is readily seen that this inverse problem has a solution on $\Omega \times (0, T)$. The details are left to the reader.

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