SOME REMARKS ABOUT THE FUCIK SPECTRUM AND APPLICATION TO EQUATIONS WITH JUMPING NONLINEARITIES

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Abstract. Let \( L \) be a selfadjoint operator with compact resolvent and \( \lambda \) an eigenvalue of \( L \). When \( \lambda \) is simple, it is well known that the Fucik spectrum \( \Sigma \) near \( \lambda \) consists of two nonincreasing curves. In this paper, we show that when \( \lambda \) is not simple, \( \Sigma \) contains two nonincreasing curves such that all points above or under both curves are not in \( \Sigma \). After that, we give some existence results of solutions of the equation \( Lu = \alpha u^+ - \beta u^- + g(., u) \) where \( u^\pm = \max(0, \pm u) \).

1. Introduction

Let \( \Omega \) be an open domain in \( \mathbb{R}^n \) and let \( L : D(L) \subset L^2(\Omega) \to L^2(\Omega) \) be a selfadjoint operator with compact resolvent. Hence, \( L \) is closed and the set of eigenfunctions of \( L \) is an orthogonal basis of \( L^2(\Omega) \). Let \( \lambda \) be an eigenvalue of \( L \). It is shown by T. Gallouët and O. Kavian [1], B. Ruf [8] that when \( m = 1 \), the Fuşik spectrum \( \Sigma \) (i.e., the set of \( (\alpha, \beta) \in \mathbb{R}^2 \) such that the equation \( Lu = \alpha u^+ - \beta u^- \) has a nontrivial solution) consists in \( I \times I \) of two nonincreasing curves which may coincide.

When \( m \geq 2 \), N.P. Cac [2] has shown that for the Laplacian operator with Dirichlet boundary conditions, \( \Sigma \) contains two curves in \( I \times I \). M. Schechter [3] and O. Kavian [4] obtain the same result for any operator \( L \) such that \( \sigma(L) \subset \mathbb{R}^{+\ast} \).

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In the third section of this paper, we show that when $\sigma(L)$ is not limited at left and $m \geq 2$, the Fučík spectrum in $I \times I$ contains two nonincreasing curves $\Gamma_1$ and $\Gamma_2$ passing through the point $(\lambda, \lambda)$ such that all points above or under $\Gamma_1$ and $\Gamma_2$ are not in $\Sigma$. The curves $\Gamma_1$ and $\Gamma_2$ are given by

$$\Gamma_1 = \{(\alpha, \beta) \in I \times I \text{ such that } \sigma(\alpha, \beta) = 0\}$$

$$\Gamma_2 = \{(\alpha, \beta) \in I \times I \text{ such that } \tau(\alpha, \beta) = 0\}$$

where

$$\sigma(\alpha, \beta) = \min_{\sum_{i=1}^{m} t_i = 1} \sum_{i=1}^{m} C_i(\alpha, \beta, t_i)$$

$$\tau(\alpha, \beta) = \max_{\sum_{i=1}^{m} t_i = 1} \sum_{i=1}^{m} C_i(\alpha, \beta, t_i)$$

and the coefficients $C_i(\alpha, \beta, t)$ verify

$$\begin{cases} Lw = \alpha w^+ - \beta w^- + \sum_{i=1}^{m} C_i(\alpha, \beta, t) \varphi_i \\ \int w \varphi_i = t_i \text{ for } i = 1, \ldots, m \end{cases}$$

$(\varphi_i)_{1 \leq i \leq m}$ are the eigenfunctions corresponding to $\lambda$.

In the fourth section, we study the equation

$$Lu = \alpha u^+ - \beta u^- + g(x, u) \quad (E)$$

in different cases of $g$ and different positions of $(\alpha, \beta)$ toward the Fučík spectrum. First we show that when $(\alpha, \beta)$ is above or under $\Gamma_1$ and $\Gamma_2$ and $g$ verifies $|g(x,s)| \leq h^{1-\zeta}(x)|s|^{\zeta} + k(x)$, where $(h,k) \in L^2(\Omega) \times L^2(\Omega)$ and $0 \leq \zeta < 1$, then $(E)$ has at least one solution.

After that, we prove that if $(\alpha_0, \beta_0) \in \Gamma_1$, $\lambda_p + \eta \leq a(x) \leq \alpha_0$ and $\lambda_p + \eta \leq b(x) \leq \beta_0$ a.e. in $\Omega$ for some $\eta > 0$ and

$$\int (\alpha_0 - a(x))(w^+)^2 + \int (\beta_0 - b(x))(w^-)^2 > 0$$

for each nontrivial solution of $Lw = \alpha_0 w^+ - \beta_0 w^-$, then the equation $Lu = \alpha u^+ - \beta u^-$ has only the trivial solution.

We use the last result for proving that if $(\alpha_0, \beta_0) \in \Gamma_1$, $g(x,s) = sh(x,s) + k(x,s)$,

$$\lambda_p + \eta \leq \alpha + \inf_{s \geq 0} h(x,s) \leq \alpha + \sup_{s \geq 0} h(x,s) < \alpha_0$$

$$\lambda_p + \eta \leq \beta + \inf_{s \leq 0} h(x,s) \leq \beta + \sup_{s \leq 0} h(x,s) < \beta_0$$
we suppose that for each element \( \phi \) of \( N(L - \lambda I) \), we have \( \| \phi^+ \| \| \phi^- \| \neq 0 \).

If, in addition, \( \sup_{s \in \mathbb{R}} |g(x, s)| \leq \eta \), \( \lim_{s \to \pm \infty} g(x, s) = g_\pm(x) \) and for each nontrivial solution of \( Lw = \alpha w^+ - \beta w^- \), we have:

- \( \text{meas} \{ x \in \Omega / w(x) = 0 \} = 0 \)
- \( \int g_+ w^+ - g_- w^- < 0 \) if \((\alpha, \beta) \in \Gamma_1\) and \( \int g_+ w^+ - g_- w^- > 0 \) if \((\alpha, \beta) \in \Gamma_2\), then \( (E) \) has at least one solution.

Finally, we study the equation \( (E) \) when \((\alpha, \beta) \) is on \( \Gamma_1 \) or \( \Gamma_2 \). For that, we suppose that for each element \( \varphi \) of \( N(L - \lambda I) \), we have \( \| \varphi^+ \| \| \varphi^- \| \neq 0 \).

2. Preliminaries

We denote \( V = N(L - \lambda I), V_1 = \oplus_{r<\lambda} N(L - rI) \cap D(L) \) and \( V_2 = \oplus_{r>\lambda} N(L - rI) \cap D(L) \).

Let \( h \in L^2(\Omega), \phi \in V \) and \((a, b) \in L^\infty(\Omega) \times L^\infty(\Omega) \) be such that \( \lambda_p + \eta \leq a(x), b(x) \leq \lambda_q - \eta \) a.e. in \( \Omega \) for some \( \eta > 0 \). Consider the function \( J_h(a, b, .) : D(L) \rightarrow \mathbb{R} \) such that

\[
J_h(a, b, u) = \frac{1}{2} \langle Lu, u \rangle - \frac{1}{2} \int a(u^+)^2 - \frac{1}{2} \int b(u^-)^2 - \int hu
\]

and the function \( K_{h, \phi}(a, b, v) : V_1 \times V_2 \rightarrow \mathbb{R} \) such that \( K_{h, \phi}(a, b, v_1, v_2) = J_h(a, b, v_1 + \phi + v_2) \).

Lemma 1. ([5]) The function \( J_h(a, b, .) \) belongs to \( C^1(D(L), \mathbb{R}) \) and its differential is given by

\[
(DJ_h(a, b, u), v) = \langle Lu, v \rangle - \int au^+v + \int bv^-v - \int hv.
\]

For each \( v_2 \in V_2 \), the function \( K_{h, \phi}(a, b, v_2) \) is strictly concave on \( V_1 \). For each \( v_1 \in V_1 \), the function \( K_{h, \phi}(a, v_1, .) \) is strictly convex on \( V_2 \).

Theorem 1. ([1]) There exists a unique \( v \in V_1 \oplus V_2 \) such that

\[
Lv = P_{V_1} [a(v + \phi)^+ - b(v + \phi)^- + h].
\]

Remark 1. ([1]) a) There exists a constant \( c > 0 \) depending on \( a, b, \lambda_p, \lambda, \lambda_q, \eta \) such that \( \| v \|_{D(L)} \leq c(\| \phi \| + \| h \|) \), where \( v \) is a solution of (2.1).

b) If we write \( v \) as \( v = v_1^0 + v_2^0 \), where \( v_1^0 \in V_1 \) and \( v_2^0 \in V_2 \), then

\[
K_{h, \phi}(a, b, v_1^0, v_2^0) = \max_{v_1 \in V_1} \min_{v_2 \in V_2} K_{h, \phi}(a, b, v_1, v_2) = \min_{v_2 \in V_2} \max_{v_1 \in V_1} K_{h, \phi}(a, b, v_1, v_2)
\]

and for \( v_1 \neq v_1^0, v_2 \neq v_2^0 \) we have

\[
K_{h, \phi}(a, b, v_1, v_2) < K_{h, \phi}(a, b, v_1^0, v_2^0) < K_{h, \phi}(a, b, v_1^0, v_2).\]
Theorem 2. a) For each \( z_2 \in V \oplus V_2 \), there exists a unique \( v_1^0 \in V_1 \) such that \( J_h(a,b,v_1^0 + z_2) = \max_{v_1 \in V_1} J_h(a,b,v_1 + z_2) \). The function \( v_1^0 \) verifies

\[
\langle Lv_1^0, v_1 \rangle = \int a(v_1^0 + z_2)^+ v - \int b(v_1^0 + z_2)^- v + \int hv_1 \quad \forall v_1 \in V_1
\]

and there exists \( c_1 > 0 \) such that \( \|v_1^0\|_{D(L)} \leq c_1 (\|z_2\| + \|h\|) \).

b) For each \( z_1 \in V_1 \oplus V \), there exists a unique \( v_2^0 \in V_2 \) such that \( J_h(a,b,z_1 + v_2^0) = \min_{v_2 \in V_2} J_h(a,b,z_1 + v_2) \). Then \( v_2^0 \) verifies

\[
\langle Lv_2^0, v_2 \rangle = \int a(z_1 + v_2^0)^+ v - \int b(z_1 + v_2^0)^- v + \int hv_2 \quad \forall v_2 \in V_2
\]

and there exists \( c_2 > 0 \) such that \( \|v_2^0\|_{D(L)} \leq c_2 (\|z_1\| + \|h\|) \).

Let the function \( \theta_1 : V \oplus V_2 \to V_1 \) defined by \( J_0(a,b,\theta_1(z_2) + z_2) = \max_{v_1 \in V_1} J_0(a,b,v_1 + z_2) \) and the function \( \theta_2 : V_1 \oplus V \to V_2 \) defined by \( J_0(a,b,z_1 + \theta_2(z_1)) = \min_{v_2 \in V_2} J_0(a,b,z_1 + v_2) \). \( \theta_1 \) and \( \theta_2 \) are positively homogeneous and there exists \( c_1 > 0, c_2 > 0 \) such that \( \|\theta_1(z_2)\|_{D(L)} \leq c_1 \|z_2\| \)

and \( \|\theta_2(z_1)\|_{D(L)} \leq c_2 \|z_1\| \). Finally, if \((z_{2n})_{n \geq 0} \) (resp. \((z_{1n})_{n \geq 0} \)) converges weakly in \( D(L) \) to \( z_2 \) (resp. \( z_1 \)), then \((\theta_1(z_{2n}))_{n \geq 0} \) (resp. \((\theta_2(z_{1n}))_{n \geq 0} \)) converges weakly in \( D(L) \) to \( \theta_1(z_2) \) (resp. \( \theta_2(z_1) \)).

3. The Fučík spectrum

We suppose, without loosing of generality, that \( m = 2 \).

Theorem 2. For each \((\alpha, \beta) \in I \times I \) and \((t_1,t_2) \in \mathbb{R} \times \mathbb{R} \), there exists a unique \((u,d_1,d_2) \in D(L) \times \mathbb{R} \times \mathbb{R} \) such that

\[
\begin{cases}
Lu = \alpha u^+ - \beta u^- + d_1 \varphi_1 + d_2 \varphi_2 \\
\int u \varphi_1 = t_1, \int u \varphi_2 = t_2.
\end{cases}
\]

Proof. Let \( u = v + t_1 \varphi_1 + t_2 \varphi_2 \), then (3.1) is equivalent to

\[
\begin{cases}
Lv = P_{V^+} \left[ \alpha (v + t_1 \varphi_1 + t_2 \varphi_2)^+ - \beta (v + t_1 \varphi_1 + t_2 \varphi_2)^- \right] \\
d_1 = \lambda_1 - \int \left[ \alpha (v + t_1 \varphi_1 + t_2 \varphi_2)^+ - \beta (v + t_1 \varphi_1 + t_2 \varphi_2)^- \right] \varphi_1 \\
d_2 = \lambda_2 - \int \left[ \alpha (v + t_1 \varphi_1 + t_2 \varphi_2)^+ - \beta (v + t_1 \varphi_1 + t_2 \varphi_2)^- \right] \varphi_2.
\end{cases}
\]

We conclude by Theorem 1 that \( v \) exists and it is unique. So (3.1) has a unique solution \((u,d_1,d_2) \) in \( D(L) \times \mathbb{R} \times \mathbb{R} \).

Remark 3. There exists a constant \( c > 0 \) depending on \( \min(\alpha, \beta) - \lambda_p \) and \( \lambda_g - \max(\alpha, \beta) \) such that \( \|u\|_{D(L)} \leq c \|t_1 \varphi_1 + t_2 \varphi_2\| \) and we deduce from
Remark 1.b that
\[ t_1 d_1 + t_2 d_2 = 2 \max_{v_1 \in V_1} \min_{v_2 \in V_2} J_0(\alpha, \beta, v_1 + t_1 \varphi_1 + t_2 \varphi_2 + v_2) \]
\[ = 2 \min_{v_2 \in V_2} \max_{v_1 \in V_1} J_0(\alpha, \beta, v_1 + t_1 \varphi_1 + t_2 \varphi_2 + v_2) = 2J_0(\alpha, \beta, u). \]

For \( i = 1, 2 \), we denote by \( D_i : I \times I \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) the function such that \( D_i(\alpha, \beta, t_1, t_2) = d_i \).

**Theorem 3.** a) For \( r \geq 0 \) and \( i = 0, 1 \), we have \( D_i(\alpha, \beta, rt_1, rt_2) = rD_i(\alpha, \beta, t_1, t_2) \).

b) For \( i = 0, 1 \), we have \( D_i(\alpha, \alpha, t_1, t_2) = (\lambda - \alpha)t_i \).

c) \( D_1 \) and \( D_2 \) are continuous on \( I \times I \times \mathbb{R} \times \mathbb{R} \).

**Proof.** For the proof of a) we use the definition of \( D_i \). For b) we let \( u = t_1 \varphi_1 + t_2 \varphi_2 \) so
\[ Lu = \lambda u = \alpha u + (\lambda - \alpha)t_1 \varphi_1 + (\lambda - \alpha)t_2 \varphi_2. \]

For c) we consider a sequence \( (\alpha_n, \beta_n, t_{1n}, t_{2n})_{n \geq 0} \) which converges to \( (\alpha, \beta, t_1, t_2) \). Let
\[ d_{1n} = D_1(\alpha_n, \beta_n, t_{1n}, t_{2n}) \]
\[ = \lambda t_{1n} - \int [\alpha_n(v_n + t_{1n} \varphi_1 + t_{2n} \varphi_2)^+ - \beta_n(v_n + t_{1n} \varphi_1 + t_{2n} \varphi_2)^-] \varphi_1, \]
\[ d_{2n} = D_2(\alpha_n, \beta_n, t_{1n}, t_{2n}) \]
\[ = \lambda t_{2n} - \int [\alpha_n(v_n + t_{1n} \varphi_1 + t_{2n} \varphi_2)^+ - \beta_n(v_n + t_{1n} \varphi_1 + t_{2n} \varphi_2)^-] \varphi_2, \]
where \( v_n \in V_1 \oplus V_2 \) is the unique solution of the equation
\[ Lv_n = P_{V_1} \left[ \alpha_n(v_n + t_{1n} \varphi_1 + t_{2n} \varphi_2)^+ - \beta_n(v_n + t_{1n} \varphi_1 + t_{2n} \varphi_2)^- \right]. \]
The sequence \( (v_n)_{n \geq 0} \) is bounded because
\[ \|v_n\|_{D(L)} \leq c_n \|t_{1n} \varphi_1 + t_{2n} \varphi_2\| \]
and \( (c_n)_{n \geq 0} \) is bounded (Remark 1.3). We can then extract from \( (v_n)_{n \geq 0} \) a subsequence \( (v_{nk}) \) which converges weakly in \( D(L) \) to \( v \) such that
\[ Lv = P_{V_1} \left[ \alpha(v + t_1 \varphi_1 + t_2 \varphi_2)^+ - \beta(v + t_1 \varphi_1 + t_2 \varphi_2)^- \right] \]
hence, \( (d_{1nk}) \) tends to
\[ \lambda t_1 - \int [\alpha(v + t_1 \varphi_1 + t_2 \varphi_2)^+ - \beta(v + t_1 \varphi_1 + t_2 \varphi_2)^-] \varphi_1 = D_1(\alpha, \beta, t_1, t_2). \]
and \((d_{2n_k})\) tends to
\[\lambda t_2 - \int \left[ \alpha(v + t_1 \varphi_1 + t_2 \varphi_2) - \beta(v + t_1 \varphi_1 + t_2 \varphi_2) \right] \varphi_2 = D_2(\alpha, \beta, t_1, t_2).\]

The last limits do not depend on the subsequence considered, so \((d_{1n})_{n \geq 0}\) (resp. \((d_{2n})_{n \geq 0}\)) converges to \(D_1(\alpha, \beta, t_1, t_2)\) (resp. \(D_2(\alpha, \beta, t_1, t_2)\)). We have then proved the continuity of \(D_1\) and \(D_2\). \(\square\)

We introduce now the functions \(C_1, C_2 : I \times I \times [0, 2\pi] \to \mathbb{R}\) defined by
\[C_1(\alpha, \beta, \theta) = D_1(\alpha, \beta, \cos(\theta), \sin(\theta)),\]
\[C_2(\alpha, \beta, \theta) = D_2(\alpha, \beta, \cos(\theta), \sin(\theta)).\]

**Assumption \(\Theta\).** We suppose that
\[||\cos(\theta)\varphi_1 + \sin(\theta)\varphi_2|| - \neq 0\]
and
\[||\cos(\theta)\varphi_1 + \sin(\theta)\varphi_2|| + \neq 0\]
for each \(\theta \in [0, 2\pi]\).

For the case \(\sigma(L) \subset \mathbb{R}^+\), M. Schechter [3] supposed that the first eigenvalue of \(L\) is simple and that the corresponding eigenfunction is positive. By orthogonality between eigenfunctions, we deduce that \(\Theta\) is verified.

**Theorem 4.** If \(\Theta\) is verified, then for each \(\theta \in [0, 2\pi]\), the function
\[C(\alpha, \beta, \theta) = C_1(\alpha, \beta, \theta) \cos(\theta) + C_2(\alpha, \beta, \theta) \sin(\theta)\]
is decreasing in each variable \(\alpha\) and \(\beta\).

**Proof.** Let \(\beta_1 > \beta\) and \((u_1, u_2) \in D(L) \times D(L)\) be such that
\[
\begin{align*}
Lu_1 &= \alpha u_1^+ - \beta u_1^- + C_1(\alpha, \beta, \theta) \varphi_1 + C_2(\alpha, \beta, \theta) \varphi_2 \\
\int u_1 \varphi_1 &= \cos(\theta), \quad \int u_1 \varphi_2 = \sin(\theta)
\end{align*}
\]
\[
\begin{align*}
Lu_2 &= \alpha u_2^+ - \beta_1 u_2^- + C_1(\alpha, \beta_1, \theta) \varphi_1 + C_2(\alpha, \beta_1, \theta) \varphi_2 \\
\int u_2 \varphi_1 &= \cos(\theta), \quad \int u_2 \varphi_2 = \sin(\theta)
\end{align*}
\]
We have
a) \(||u_1^-|| \neq ||u_1^+|| \neq 0\) and \(||u_2^-|| \neq ||u_2^+|| \neq 0\). Suppose that \(||u_1^-|| = 0\), if we write \(u_1\) as \(u_1 = v + \cos(\theta) \varphi_1 + \sin(\theta) \varphi_2\), where \(v \in V_1 \oplus V_2\), then \(Lv = \alpha v\). Hence, \(v = 0\) because \(\alpha \in I\). So \(u_1 = \cos(\theta) \varphi_1 + \sin(\theta) \varphi_2 \geq 0\), which is in contradiction with \(\Theta\).

b) If we write \(u_1\) and \(u_2\) as \(u_1 = v_1 + \cos(\theta) \varphi_1 + \sin(\theta) \varphi_2 + v_2\) and \(u_2 = w_1 + \cos(\theta) \varphi_1 + \sin(\theta) \varphi_2 + v_2\), where \((v_1, v_2) \in V_1 \times V_2\) and \((w_1, w_2) \in V_1 \times V_2\), then we have \(v_1 + v_2 \neq w_1 + w_2\). Suppose that \(v_1 + v_2 = w_1 + w_2\), then \(u_1 = u_2\) and
\[
Lu_1 = \alpha u_1^+ - \beta u_1^- + C_1(\alpha, \beta, \theta) \varphi_1 + C_2(\alpha, \beta, \theta) \varphi_2
\]
\[ \sigma = \alpha u_1^+ - \beta_1 u_1^- + C_1(\alpha, \beta_1, \theta) \varphi_1 + C_2(\alpha, \beta_1, \theta) \varphi_2 \]

hence,

\[ 0 \leq (\beta_1 - \beta) u_1^- = (C_1(\alpha, \beta_1, \theta) - C_1(\alpha, \beta, \theta)) \varphi_1 + (C_2(\alpha, \beta_1, \theta) - C_2(\alpha, \beta, \theta)) \varphi_2 \]

which is in contradiction with \( \Theta \).

\[ \text{proof} \]

Suppose that \( \sigma \) and \( \tau \) are continuous on \( I \times I \).

c) Since \( v_1 + v_2 \neq w_1 + w_2 \), we suppose for example that \( v_1 \neq w_1 \), then

\[ C(\alpha, \beta, \theta) = C_1(\alpha, \beta, \theta) \cos(\theta) + C_2(\alpha, \beta, \theta) \sin(\theta) \]

\[ = 2J_0(\alpha, \beta, v_1 + \cos(\theta) \varphi_1 + \sin(\theta) \varphi_2 + v_2) \]

\[ > 2J_0(\alpha, \beta, w_1 + \cos(\theta) \varphi_1 + \sin(\theta) \varphi_2 + v_2) \]

and

\[ C(\alpha, \beta_1, \theta) = C_1(\alpha, \beta_1, \theta) \cos(\theta) + C_2(\alpha, \beta_1, \theta) \sin(\theta) \]

\[ = 2J_0(\alpha, \beta_1, w_1 + \cos(\theta) \varphi_1 + \sin(\theta) \varphi_2 + w_2) \]

\[ \leq 2J_0(\alpha, \beta, w_1 + \cos(\theta) \varphi_1 + \sin(\theta) \varphi_2 + v_2) \]

\[ \leq 2J_0(\alpha, \beta, w_1 + \cos(\theta) \varphi_1 + \sin(\theta) \varphi_2 + v_2) \]

so \( C(\alpha, \beta, \theta) > C(\alpha, \beta_1, \theta) \).

**Remark 4.** If \( \Theta \) is not verified, then for each \( \theta \in [0, 2\pi] \), the function \( C(\alpha, \beta, \theta) \) is nonincreasing in each variable \( \alpha, \beta \).

Let \( \sigma, \tau : I \times I \rightarrow \mathbb{R} \) be such that

\[ \sigma(\alpha, \beta) = \min_{\theta \in [0, 2\pi]} C(\alpha, \beta, \theta), \quad \tau(\alpha, \beta) = \max_{\theta \in [0, 2\pi]} C(\alpha, \beta, \theta). \]

**Remark 5.** a) \( \sigma(\alpha, \alpha) = \tau(\alpha, \alpha) = \lambda - \alpha \).

b) \( \sigma \) and \( \tau \) are continuous on \( I \times I \).

c) If \( \Theta \) is verified, then \( \sigma \) and \( \tau \) are decreasing in each variable \( \alpha \) and \( \beta \). Otherwise, \( \sigma \) and \( \tau \) are nonincreasing in each variable \( \alpha \) and \( \beta \).

d) If \( \sigma(\alpha, \beta) > 0 \) or \( \tau(\alpha, \beta) < 0 \), then \( (\alpha, \beta) \notin \Sigma \).

e) If \( \sigma(\alpha, \beta) = 0 \), then \( \sigma(\beta, \alpha) = 0 \). If \( \tau(\alpha, \beta) = 0 \), then \( \tau(\beta, \alpha) = 0 \).

**Theorem 5.** a) If \( \sigma(\alpha, \beta) = 0 \), then \( (\alpha, \beta) \in \Sigma \). b) If \( \tau(\alpha, \beta) = 0 \), then \( (\alpha, \beta) \in \Sigma \).

**Proof.** Suppose that \( \sigma(\alpha, \beta) = 0 \). Since the function \( C(\alpha, \beta, \theta) \) is continuous on \( [0, 2\pi] \), there exists \( \theta_0 \in [0, 2\pi] \) such that \( C(\alpha, \beta, \theta_0) = \sigma(\alpha, \beta) = 0 \). Let \( u_0 \in D(L) \) be such that

\[ \begin{cases} 
Lu_0 = \alpha u_0^+ - \beta u_0^- + C_1(\alpha, \beta, \theta_0) \varphi_1 + C_2(\alpha, \beta, \theta_0) \varphi_2 \\
\int u_0 \varphi_1 = \cos(\theta_0), \quad \int u_0 \varphi_2 = \sin(\theta_0).
\end{cases} \]
Let $u_0 = v_1^0 + z_2^0$, where $v_1^0 \in V_1$ and $z_2^0 = \cos(\theta_0)\varphi_1 + \sin(\theta_0)\varphi_2 + v_2^0$ with $v_2^0 \in V_2$. We conclude from (3.2) and Remark 2 that $v_1^0 = \theta_1(z_2^0)$ and from Remark 3 that

$$0 = C_1(\alpha, \beta, \theta_0)\cos(\theta_0) + C_2(\alpha, \beta, \theta_0)\sin(\theta_0)$$

$$= 2 \min_{v_2 \in V_2} \max_{v_1 \in V_1} J_0(\alpha, \beta, v_1 + \cos(\theta_0)\varphi_1 + \sin(\theta_0)\varphi_2 + v_2) = 2J_0(\alpha, \beta, u_0).$$

Since $\sigma(\alpha, \beta) = 0$, we have

$$\min_{(t_1, t_2) \in \mathbb{R}} \min_{v_2 \in V_2} \max_{v_1 \in V_1} J_0(\alpha, \beta, v_1 + t_1\varphi_1 + t_2\varphi_2 + v_2) = 0$$

so

$$\min_{z_2 \in V \oplus V_2} \max_{v_1 \in V_1} J_0(v_1 + z_2) = 0.$$
we conclude that
\[
\langle Lu_0, v \rangle = \alpha \int u_0^+ v - \beta \int u_0^- v
\]
for each \( v \in D(L) \). So \( Lu_0 = \alpha u_0^+ - \beta u_0^- \), because \( D(L) \) is dense in \( L^2(\Omega) \). Hence, \((\alpha, \beta) \in \Sigma\). For the proof of the second part of Theorem 6, we proceed as for the first part.

**Remark 6.** Let
\[
\Gamma_1 = \{ (\alpha, \beta) \in I \times I \text{ such that } \sigma(\alpha, \beta) = 0 \},
\]
\[
\Gamma_2 = \{ (\alpha, \beta) \in I \times I \text{ such that } \tau(\alpha, \beta) = 0 \}.
\]
We have proved that \( \Gamma_1 \cup \Gamma_2 \subset \Sigma \). If \( \Gamma_1 \neq \Gamma_2 \), then \( \Gamma_2 \) is above \( \Gamma_1 \) and if \( \Gamma_1 = \Gamma_2 \), then the Fučik spectrum consists in \( I \times I \) of one curve.

The problem of determining completely the Fučik spectrum remains open.

### 4. Existence results

In this section, we study the equation
\[
Lu = \alpha u^+ - \beta u^- + g(\cdot, u),
\] (4.1)
where \( g : \Omega \times \mathbb{R} \to \mathbb{R} \) is a caratheodory function and \((\alpha, \beta) \in I \times I\). Let \((\mu, \nu) \in \mathbb{R} \times \mathbb{R} \) be such that \( \lambda_p < \alpha + \mu, \beta + \nu < \lambda_g \). For each \( s \in [0, 1] \), we consider the equation
\[
Lu = \alpha u^+ - \beta u^- + sg(\cdot, u) + (1 - s)(\mu u^+ - \nu u^-).
\] (4.2)
A function \( u \) is a solution of (4.2) if and only if \( u = T(s, u) \), where \( T : [0, 1] \times L^2(\Omega) \to L^2(\Omega) \) is the completely continuous operator defined by
\[
T(s, u) = (L - \gamma I)^{-1} [\alpha u^+ - \beta u^- + sg(\cdot, u) + (1 - s)(\mu u^+ - \nu u^-) - \gamma u]
\]
The scalar \( \gamma \) is such that \( (L - \gamma I)^{-1} \) is compact.

**Theorem 6.** Suppose that \(|g(x, s)| \leq h^{1-\zeta}(x)|s|^\zeta + k(x)\), where \( h \) and \( k \) are in \( L^2(\Omega) \) and \( 0 \leq \zeta < 1 \). If \( \sigma(\alpha, \beta) > 0 \) or \( \tau(\alpha, \beta) < 0 \), then (4.1) has at least one solution.

**Proposition 1.** Suppose that \( \sigma(\alpha, \beta) > 0 \) and \( g \) as in Theorem 6. Consider \( \mu = \rho - \alpha, \nu = \rho - \beta \), where \( \rho \in (\lambda_p, \min(\alpha, \beta)) \), then there exists \( R > 0 \) such that for each \( s \in [0, 1] \) and \( u \) a solution of (4.2), we have \( |u|_{D(L)} < R \).
Proof. (by contradiction) \( \forall n > 0, \exists s_n \in [0, 1], \exists u_n \in D(L) \) such that 
\[
\alpha_n = \|u_n\|_{D(L)} \geq n
\]
and
\[
Lu_n = \alpha u_n^+ - \beta u_n^- + s_n g(\cdot, u_n) + (1 - s_n)(\mu u_n^+ - \nu u_n^-).
\]
Since \((s_n)_{n \geq 0}\) and \((v_n = u_n/\alpha_n)_{n \geq 0}\) are bounded, we can extract from them subsequences denoted in the same manner such that \((s_n)_{n \geq 0}\) tends to \(s\) in \(\mathbb{R}\) and \((v_n)_{n \geq 0}\) tends to \(v\) weakly in \(D(L)\). Since \(\|g(\cdot, u_n)\| \leq \|h\| t \xi \|u_n\| \xi + \|k\|\), then \(v\) satisfies
\[
Lv = (\alpha + (1 - s)\mu)v^+ - (\beta + (1 - s)\nu)v^-
\]
and \((v_n)_{n \geq 0}\) converges to \(v\) in \(D(L)\). We conclude that \(v = 0\) because \(\sigma(\alpha + (1 - s)\mu, \beta + (1 - s)\nu) \geq \sigma(\alpha, \beta) > 0\). In contradiction with \(\|v\|_{D(L)} = 1\).

Proof of Theorem 6. Since \(u \neq T(s, u)\) for each \(s \in [0, 1]\) and \(u \in \partial B(R)\) \((R\) is given by Proposition 4.1), the invariance by homotopy of topological degree of Leray-Schauder allows us to conclude that \(D(I - T(1, \cdot), B(R), 0) = D(I - T(0, \cdot), B(R), 0) = \pm 1\). Hence, (4.1) has at least one solution. \(\square\)

Now, we give some remarks about the nonexistence of nontrivial solutions of the equation \(Lu = a(x)u^+ - b(x)u^-\).

Theorem 7. Suppose that \(\sigma(\alpha_0, \beta_0) = 0\), \(\lambda_p + \eta \leq a(x) \leq \alpha_0\) and \(\lambda_p + \eta \leq b(x) \leq \beta_0\) a.e. in \(\Omega\) for some \(\eta > 0\). If in addition
\[
\int (\alpha_0 - a(x))(w^+)^2 + \int (\beta_0 - b(x))(w^-)^2 > 0
\]
for each nontrivial solution of \(Lw = \alpha_0 w^+ - \beta_0 w^-\), then the equation \(Lu = a u^+ - b u^-\) has only the trivial solution.

Proof. (by contradiction) Let \(u \in D(L)\) be such that \(\|u\| \neq 0\) and \(Lu = a u^+ - b u^-\).

a) \(\int (w\varphi_1)^2 + (w\varphi_2)^2 \neq 0\) because if not, the function \(u\) is then orthogonal to \(V\) and verifies \(Lu = P_{V^\perp}(a u^+ - b u^-)\). By Theorem 1, we deduce that \(u = 0\).

b) Let \(\xi \geq 0\) be such that \(\xi^2 = \int (w\varphi_1)^2 + (w\varphi_2)^2\) and put \(v = u/\xi\), \(\cos(\theta_0) = \int w\varphi_1/\xi\) and \(\sin(\theta_0) = \int w\varphi_2/\xi\). The function \(v\) verifies
\[
\begin{cases}
Lu = a v^+ - b v^- \\
v\varphi_1 = \cos(\theta_0), \ v\varphi_2 = \sin(\theta_0).
\end{cases}
\]
If we write \(v\) as \(v = v_0^0 + \phi_0 + v_0^0\), where \(\phi_0 = \cos(\theta_0)\varphi_1 + \sin(\theta_0)\varphi_2\), then we have
\[
0 = K_{0, \phi_0}(a, b, v_0^0, v_0^0) = \max_{v_1 \in V_1} K_{0, \phi_0}(a, b, v_1, v_0^0)
\]
for each nontrivial solution of $Lw$

Suppose that

Remark 7.

which is impossible. □

Let $w = v_1 + \phi_0 + v_0$, this function satisfies

\[
\begin{cases}
Lw = \alpha_0 w^+ - \beta_0 w^- \\
\int w \varphi_1 = \cos(\theta_0), \int w \varphi_2 = \sin(\theta_0).
\end{cases}
\]

c) $v_1 \neq v_0$ because if not, $v = w$. So $Lw = \alpha_0 w^+ - \beta_0 w^- = aw^+ - bw^-$ which implies that

\[
\int (\alpha_0 - a(x))(w^+)^2 + \int (\beta_0 - b(x))(w^-)^2 = 0
\]

which is in contradiction with assumptions. Since $v_1 \neq v_0$, we have

\[
0 = K_{0,\phi_0}(a, b, v_1, v_2) = \max_{v_1 \in V_1} K_{0,\phi_0}(a, b, v_1, v_2)
\]

\[
> K_{0,\phi_0}(a, b, v_1^1, v_2^1) > K_{0,\phi_0}(a, \beta_0, v_1^0, v_2^0) = 0
\]

which is impossible.

Remark 7. Suppose that $\tau(\alpha_0, \beta_0) = 0$, $\alpha_0 \leq a(x) \leq \lambda_g - \eta$ and $\beta_0 \leq b(x) \leq \lambda_g - \eta$ a.e. in $\Omega$ for some $\eta > 0$. If in addition

\[
\int (a(x) - \alpha_0)(w^+)^2 + \int (b(x) - \beta_0)(w^-)^2 > 0
\]

for each nontrivial solution of $Lw = \alpha_0 w^+ - \beta_0 w^-$, then the equation $Lu = au^+ - bu^-$ has only the trivial solution.

The next result is a consequence of Theorem 7.

Theorem 8. Suppose that $g(x, s) = sh(x, s) + k(x, s)$ and $\Omega$ is bounded. If $\sigma(\alpha_0, \beta_0) = 0$ and

\[
\lambda_p + \eta \leq \alpha + \inf_{s \geq 0} h(x, s) \leq \alpha + \sup_{s \geq 0} h(x, s) < \alpha_0
\]

\[
\lambda_p + \eta \leq \beta + \inf_{s \leq 0} h(x, s) \leq \beta + \sup_{s \leq 0} h(x, s) < \beta_0
\]
a.e. in $\Omega$ for some $\eta > 0$

$$\sup_{s \in \mathbb{R}} |k(x, s)| \in L^2(\Omega),$$

then (4.1) has at least one solution.

**Proposition 2.** Let $\mu = \rho - \alpha$ and $\nu = \rho - \beta$, where $\rho \in (\lambda_p, \min(\alpha_0, \beta_0))$.
Under the assumptions of Theorem 8, there exists $R > 0$ such that for each $s \in [0, 1]$ and $u$ a solution of (4.2), we have $\|u\|_{D(L)} < R$.

**Proof.** (by contradiction) $\forall n > 0$, $\exists s_n \in [0, 1]$, $\exists u_n \in D(L)$ such that $\alpha_n = \|u_n\|_{D(L)} \geq n$ and

$$Lu_n = \alpha u_n^+ - \beta u_n^- + s_n g(\cdot, u_n) + (1 - s_n)(\mu u_n^+ - \nu u_n^-)
$$

which is equivalent to

$$Lu_n = \left[\alpha + s_n h(\cdot, u_n^+) + (1 - s_n)\mu\right] u_n^+ - \left[\beta + s_n h(\cdot, -u_n^-) + (1 - s_n)\nu\right] u_n^- + s_n k(\cdot, u_n).$$

We can extract from $(s_n)_{n \geq 0}$, $(v_n = u_n/\alpha_n)_{n \geq 0}$, $(h(\cdot, u_n^+))_{n \geq 0}$, $(h(\cdot, -u_n^-))_{n \geq 0}$ subsequences denoted in the same manner such that $(s_n)_{n \geq 0}$ tends to $s$ in $\mathbb{R}$, $(v_n)_{n \geq 0}$ tends to $v$ weakly in $D(L)$, $(h(\cdot, u_n^+))_{n \geq 0}$ (resp. $(h(\cdot, -u_n^-))_{n \geq 0}$) tends to $h^+_*(\text{resp. } h^-_*)$ weakly star in $L^\infty(\Omega)$ (and weakly in $L^2(\Omega)$). The function $v$ is then a solution of the equation

$$Lv = (\alpha + sh^+_* + (1 - s)\mu)v^+ - (\beta + sh^-_* + (1 - s)\nu)v^-.$$

Since

$$\min(\lambda_p + \eta, \rho) \leq \alpha + sh^+_* + (1 - s)\mu < \alpha_0$$

$$\min(\lambda_p + \eta, \rho) \leq \beta + sh^-_* + (1 - s)\nu < \beta_0$$

we conclude by Theorem 7 that $v = 0$. So $(v_n)_{n \geq 0}$ tends to $v$ in $D(L)$ and $\|v\|_{D(L)} = 1$. We have then a contradiction.

For the proof of Theorem 8, we use the invariance of topological degree of Leray-Schauder and Proposition 2.

**Remark 8.** Suppose that $g(x, s) = sh(x, s) + k(x, s)$ and $\Omega$ is bounded. If $\tau(\alpha_0, \beta_0) = 0$ and

$$\alpha_0 < \alpha + \inf_{s \geq 0} h(x, s) \leq \alpha + \sup_{s \geq 0} h(x, s) \leq \lambda_g - \eta$$

$$\beta_0 < \beta + \inf_{s \leq 0} h(x, s) \leq \beta + \sup_{s \leq 0} h(x, s) \leq \lambda_g - \eta$$

then (4.1) has at least one solution.
then (4.1) has at least one solution.

Suppose that

\begin{align*}
\sigma_n &= \alpha_n u_n^+ - \beta u_n^- + s_n g(., u_n) + (1 - s_n)(\mu u_n^+ - \nu u_n^-). 
\end{align*}

We can extract from \((s_n)_{n \geq 0}\) and \((v_n = u_n / \alpha_n)_{n \geq 0}\) subsequences denoted in the same manner such that \((s_n)_{n \geq 0}\) tends to \(s\) in \(\mathbb{R}\) and \((v_n)_{n \geq 0}\) tends to \(v\) weakly in \(D(L)\). The function \(v\) is then a solution of the equation

\begin{align*}
Lv &= (\alpha + (1 - s)s)\mu v^+ - (\beta + (1 - s)s)\nu v^- 
\end{align*}

and it is easy to prove that \((v_n)_{n \geq 0}\) converges to \(v\) in \(D(L)\). If \(s \neq 1\), then

\begin{align*}
\sigma(\alpha + (1 - s)s, \beta + (1 - s)s) > \sigma(\alpha, \beta) = 0,
\end{align*}

so \(v = 0\) which is in contradiction with \(\|v\|_{D(L)} = 1\). Hence, \(v\) is a nontrivial solution of \(Lv = \alpha v^+ - \beta v^-\). Let \(a_n, a\) be nonnegative scalars such that

\begin{align*}
a_n^2 &= \int (v_n \phi_1)^2 + \int (v_n \phi_2)^2 \quad \text{and} \quad a^2 = \int (v \phi_1)^2 + \int (v \phi_2)^2.
\end{align*}

The constant \(a\) is positive because if \(a = 0\), then \(v = 0\) (Theorem 1). Since \((a_n)_{n \geq 0}\) converges to \(a\), there exists \(N > 0\) such that for \(n \geq N\), the constant
\( a_n > 0 \). For \( n \geq N \), we define \( \theta_n \in [0, 2\pi] \) by \( \cos(\theta_n) = \int (v_n \varphi_1)/a_n \) and \( \sin(\theta_n) = \int (v_n \varphi_2)/a_n \). The sequence \((u_{\theta_n})_{n \geq 0}\) converges to \( u_{\theta_0} \), where \( v = au_{\theta_0} \). Let \( z_n = v_n - a_n u_{\theta_n} \), then \( z_n \) is orthogonal to \( V \) and \((\|z_n\|_{D(L)})_{n \geq 0}\) tends to 0. Multiplying (4.4) by \( u_{\theta_n} \), we obtain

\[
\langle Lu_n, u_{\theta_n} \rangle = \int (\alpha u_n^+ - \beta u_n^-)u_{\theta_n} + s_n \int g(\cdot, u_n)u_{\theta_n} + (1 - s_n) \int (\mu u_n^+ - \nu u_n^-)u_{\theta_n} = \langle u_n, Lu_{\theta_n} \rangle
\]

\[
= \alpha_n a_n (C_1(\alpha, \beta, \theta_n) \cos(\theta_n) + C_2(\alpha, \beta, \theta_n) \sin(\theta_n)) + \int (\alpha u_n^+ - \beta u_n^-)u_{\theta_n},
\]

then

\[
s_n \int g(\cdot, u_n)u_{\theta_n} + (1 - s_n) \int (\mu u_n^+ - \nu u_n^-)u_{\theta_n} - \alpha_n a_n (C_1(\alpha, \beta, \theta_n) \cos(\theta_n) + C_2(\alpha, \beta, \theta_n) \sin(\theta_n)) = \alpha_n E_n,
\]

where

\[
E_n = (\alpha - \beta) \int (v_n^+ u_{\theta_n}^+ - v_n^- u_{\theta_n}^-).
\]

Let \( n \geq N \), if \( v_n(x) \geq 0 \) and \( u_{\theta_n}(x) \leq 0 \), then \( v_n(x) \leq z_n(x) \) and \( u_{\theta_n}(x) \geq -z_n(x)/a_n \), so \( \int v_n^+ u_{\theta_n}^- \leq \|z_n\|^2/a_n \). Hence, \( |E_n| \leq |\alpha - \beta| (2/a_n) \|z_n\|_{D(L)}^2 \).

We claim that

there exists \( M > 0 \) such that for each \( n \in \mathbb{N} \), we have \( \alpha_n \|z_n\|_{D(L)} \leq M \)

\((4.6)\)

so

\[
s_n \int g(\cdot, u_n)u_{\theta_n} + (1 - s_n) \int (\mu u_n^+ - \nu u_n^-)u_{\theta_n} - \alpha_n a_n (C_1(\alpha, \beta, \theta_n) \cos(\theta_n) + C_2(\alpha, \beta, \theta_n) \sin(\theta_n))
\]

\[
+ C_2(\alpha, \beta, \theta_n) \sin(\theta_n)) \geq -|\alpha - \beta| (2/a_n) M \|z_n\|_{D(L)}
\]

since \( \lim \int (\mu u_n^+ - \nu u_n^-)u_{\theta_n} < 0 \) because \( \mu < 0 \) and \( \nu < 0 \) and since

\( C_1(\alpha, \beta, \theta_n) \cos(\theta_n) + C_2(\alpha, \beta, \theta_n) \sin(\theta_n) \geq 0 \)

because \( \sigma(\alpha, \beta) = 0 \), we conclude that \( \lim s_n \int g(\cdot, u_n)u_{\theta_n} \geq 0 \) and, by using Lebesgue theorem, that \( \int g_+ u_{\theta_0}^+ - g_- u_{\theta_0}^- \geq 0 \), which is in contradiction with assumptions. The proof will be complete once we have established (4.6), for that we proceed by contradiction. Suppose that \( \lim \alpha_n \|z_n\|_{D(L)} = \infty \) and put \( c_n = \|z_n\|_{D(L)} \) and \( y_n = z_n/c_n \). We can extract from \((y_n)_{n \geq 0}\) a subsequence such that \((y_n)_{n \geq 0}\) tends to \( y \) weakly in \( D(L) \) and there exists
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\[ l \in L^2(\Omega) \text{ verifying } |y_n(x)| \leq l(x) \text{ a.e. in } \Omega. \] Dividing (4.5) by \( \alpha_n c_n \), we obtain

\[ \frac{(1 - s_n)}{c_n} \int (\mu v^+_n - \nu v^-_n) u_{\theta_n} - \frac{a_n}{c_n} (C_1(\alpha, \beta, \theta_n) \cos(\theta_n) + C_2(\alpha, \beta, \theta_n) \sin(\theta_n)) \]

\[ = \frac{1}{c_n} E_n - \frac{s_n}{\alpha_n c_n} \int g(., u_n) u_{\theta_n}, \]

when \( n \) goes to \( \infty \), we deduce that \( \lim (1 - s_n)/c_n = 0 \) and that

\[ \lim \frac{1}{c_n} (C_1(\alpha, \beta, \theta_n) \cos \theta_n + C_2(\alpha, \beta, \theta_n) \sin(\theta_n)) = 0 \]

because \( \sigma(\alpha, \beta) = 0, \mu < 0 \) and \( \nu < 0 \). On the other hand, the function \( y_n \) verifies

\[ Ly_n = \alpha \left[ (y_n + \frac{a_n}{c_n} u_{\theta_n})^+ - \frac{a_n}{c_n} u_{\theta_n}^+ \right] - \beta \left[ (y_n + \frac{a_n}{c_n} u_{\theta_n})^- - \frac{a_n}{c_n} u_{\theta_n}^- \right] \]

\[ + \frac{s_n}{\alpha_n c_n} g(., u_n) + \frac{(1 - s_n)}{c_n} (\mu v^+_n - \nu v^-_n) \]

\[ - \frac{a_n}{c_n} (C_1(\alpha, \beta, \theta_n) \varphi_1 + C_2(\alpha, \beta, \theta_n) \varphi_2). \]

Since \( y_n \) is orthogonal to \( V \) and \( L \) is a selfadjoint operator, \( y_n \) verifies also

\[ Ly_n = P_{V^\perp} \left\{ \alpha \left[ (y_n + \frac{a_n}{c_n} u_{\theta_n})^+ - \frac{a_n}{c_n} u_{\theta_n}^+ \right] - \beta \left[ (y_n + \frac{a_n}{c_n} u_{\theta_n})^- - \frac{a_n}{c_n} u_{\theta_n}^- \right] \right. \]

\[ + \frac{s_n}{\alpha_n c_n} g(., u_n) + \left. \frac{(1 - s_n)}{c_n} (\mu v^+_n - \nu v^-_n) \right\}. \]

Put

\[ \omega_n = \alpha \left[ (y_n + \frac{a_n}{c_n} u_{\theta_n})^+ - \frac{a_n}{c_n} u_{\theta_n}^+ \right] - \beta \left[ (y_n + \frac{a_n}{c_n} u_{\theta_n})^- - \frac{a_n}{c_n} u_{\theta_n}^- \right] \]

by using the inequality \( |(p + q)^+ - p^\pm| \leq |q| \) for each \( (p, q) \in \mathbb{R} \times \mathbb{R} \), we deduce that \( \omega_n(x) \leq (|\alpha| + |\beta|) l(x) \text{ a.e. in } \Omega \). Moreover, if \( u_{\theta_n}(x) > 0 \), then \( \omega_n(x) \) tends to \( \alpha y(x) \) and if \( u_{\theta_n}(x) < 0 \), then \( \omega_n(x) \) tends to \( \beta y(x) \). Since \( \text{meas } \{ x \in \Omega / u_{\theta_n}(x) = 0 \} = 0 \), we conclude by Lebesgue theorem that \( (\omega_n) \) converges in \( L^2(\Omega) \) to \( \chi y \) where

\[ \chi(x) = \begin{cases} \alpha & \text{if } u_{\theta_n}(x) > 0 \\ \beta & \text{if } u_{\theta_n}(x) < 0. \end{cases} \]

In addition \( \frac{s_n}{\alpha_n c_n} g(., u_n) + \frac{(1 - s_n)}{c_n} (\mu v^+_n - \nu v^-_n) \) tends to 0 in \( L^2(\Omega) \), so \((Ly_n)\) converges in \( L^2(\Omega) \) to \( Ly = P_{V^\perp}(\chi y) \) and \( \|y\|_{D(L)} = 1 \). If we write \( y \) as
\( y = y_2 + y_1 \), where \( y_2 \in V_2 \) and \( y_1 \in V_1 \), then we have
\[
\langle L(y_2 + y_1), y_2 - y_1 \rangle = \langle \chi(y_2 + y_1), y_2 - y_1 \rangle
\]
which is equivalent to
\[
\langle Ly_2, y_2 \rangle - \langle \chi y_2, y_2 \rangle = \langle Ly_1, y_1 \rangle - \langle \chi y_1, y_1 \rangle
\]
so
\[
0 \leq \langle \lambda g - \chi y_2, y_2 \rangle \leq \langle \lambda p - \chi y_1, y_1 \rangle \leq 0,
\]
thus \( y_1 = y_2 = 0 \), which is in contradiction with \( \|y\|_{D(L)} = 1 \).
\[\square\]

For the proof of Theorem 9, we use the invariance by homotopy of the topological degree of Leray-Schauder.

**Remark 9.** When \( \tau(\alpha, \beta) = 0 \) and \( \Theta \) is verified, if \( \lim_{s \to \mp \infty} g(x, s) \) does not exist and if we suppose that each nontrivial solution of \( Lw = \alpha w^+ - \beta w^- \) verifies
\[
\text{meas}\{x \in \Omega / w(x) = 0\} = 0
\]
\[
\int \liminf_{s \to \infty} g(\cdot, s)w^+ - \limsup_{s \to -\infty} g(\cdot, s)w^- > 0,
\]
then (4.1) has at least one solution.

**References**


