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ON A NONLINEAR DIRICHLET PROBLEM WITH A SINGULARITY ALONG THE BOUNDARY

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Abstract. Let Ω be a $C^{1,1}$ and bounded domain in R^n, with $n \geq 2$. Let K and f be two nonnegative functions on Ω belonging to $L^p(\Omega)$ for some $p > n/2$ and let $\alpha > 0$. In this paper we prove existence and uniqueness of a strong solution $u \in W^{2,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ for the elliptic Dirichlet problem $-\Delta u + \lambda u = Ku^{-\alpha} + f$ in Ω, $u > 0$ in Ω, $u = 0$ in $\partial\Omega$. Moreover, for the case $\lambda = 0$, $f = 0$, we prove, under the weaker hypothesis $p > \frac{(\alpha^2+1)n}{2\alpha^2+n}$, the existence of a solution $u \in W^{2,p}_{loc}(\Omega)$ for the above problem that satisfies, in a suitable extended sense, the boundary condition $u = 0$ on $\partial\Omega$.

1. Introduction

Let Ω be a bounded domain in R^n with $n \geq 2$, let α be a positive real number and let K be a real valued function defined on Ω. The existence of positive solutions for the Dirichlet problem

$$-\Delta u = Ku^{-\alpha} \text{ on } \Omega, \quad u = 0 \text{ in } \partial\Omega,$$

has been widely studied by several authors in [2], [3], [5], [6] and [8]. In [2] it is proved, for C^3 domains and for K strictly positive and Holder continuous on $\overline{\Omega}$, the existence of solutions $u \in C^{2+\gamma}(\Omega) \cap C(\overline{\Omega})$ for some $0 < \gamma < 1$ and estimates for u near the boundary are given. In [6], a simplified proof of results contained in [2] is given and cases corresponding to some domains with corners are studied. In [5] the existence of positive solutions is proved, under the assumption that $0 \leq K(x) \leq c\ dist(x, \partial\Omega)^q$ for some constants $c > 0$, $q \geq \alpha - 1$ and all $x \in \Omega$. In [8], del Pino proves existence and uniqueness

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of positive solutions under the hypothesis $K \geq 0$, $K \in L^\infty(\Omega)$ and K strictly positive on a subset with positive measure. The related problem

$$-\Delta u = Ku^{-\alpha} + f \quad \text{on } \Omega, \qquad u = 0 \quad \text{in } \partial\Omega,$$

is studied in [3], for regular domains and for K and f regular enough and positive on $\overline{\Omega}$.

Our aim in this paper is to prove, for a bounded and $C^{1,1}$ domain Ω, existence and uniqueness of strong solutions $u \in W^{2,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ for the problem

$$-\Delta u + \lambda u = Ku^{-\alpha} + f \quad \text{on } \Omega \quad u = 0 \text{ in } \partial\Omega, \tag{1.1}$$

where λ and α are non negative real numbers, under the assumptions $K \geq 0$, $f \geq 0$ and $K, f \in L^p(\Omega)$ for some $p > \frac{n}{2}$.

Moreover, for the case $\lambda = 0$, $f = 0$, we prove, under the weaker assumption $p > \frac{(\alpha^2+1)n}{2\alpha^2+n}$, the existence of a solution $u \in W^{2,p}_{loc}(\Omega)$ for the above problem, that satisfies the boundary condition in a suitable extended sense.

2. Auxiliary results

Let Ω be a $C^{1,1}$ and bounded domain in R^n, fixed from now on, with $n \geq 2$ and let α be a positive real number. We fix, from now on, two strictly decreasing sequences of positive real numbers $\{\delta_j\}_{j=1}^\infty$ and $\{\widetilde{\delta}_j\}_{j=1}^\infty$ such that $\lim_{j\to\infty} \delta_j = \lim_{j\to\infty} \widetilde{\delta}_j = 0$ and satisfying $\delta_{j+1} < \widetilde{\delta}_j < \delta_j$ for all $j \in N$. Given these two sequences, it is easy to check that there exists a sequence of functions $\{b_j\}_{j=1}^\infty$, with $b_j \in C^\infty(R)$, $j \in N$ satisfying

$$b_j(s) = s^{-\alpha} \text{ if } s \geq \delta_j, \quad b_j(s) = (\widetilde{\delta}_j)^{-\alpha} \text{ if } s \leq \widetilde{\delta}_j$$

and such that $b_j(s) \geq b_j(t)$ if $s \leq t$. Moreover, the sequence $\{b_j\}_{j=1}^\infty$ can be chosen satisfying, in addition that

$$b_j(s) \leq b_{j+1}(s), \qquad s \in R.$$

We fix, from now on, such a sequence $\{b_j\}_{j=1}^\infty$. If u is a real function defined on Ω, we will write $b_j(u)$ and $b_j'(u)$ for the functions defined by $b_j(u)(x) = b_j(u(x))$ and $b_j'(u)(x) = b_j'(u(x))$ respectively.

Remark 2.1. A standard argument using the properties of the functions b_j gives the following comparison principle: If $\lambda \geq 0$, $\alpha \geq 0$, if $K \in L^p(\Omega)$ with $p > \frac{n}{2}$ and if $u, v \in W^{2,p}(\Omega)$ satisfy

$$-\Delta u + \lambda u - Kb_j(u) \leq -\Delta v + \lambda v - Kb_j(v) \text{ on } \Omega, \qquad u \leq v \text{ in } \partial\Omega, \tag{2.1}$$

then $u \leq v$ in Ω. Indeed, let $m : \Omega \to R$ be defined by

$$m(x) = -\frac{b_j(u(x)) - b_j(v(x))}{u(x) - v(x)} \quad \text{if} \quad u(x) \neq v(x)$$
$$m(x) = -b_j'(u(x)) \quad \text{if} \quad u(x) = v(x).$$

The properties of the sequence $\{b_j\}_{j=1}^{\infty}$ imply that $m \geq 0$ and that $m \in L^\infty(\Omega)$ and (2.1) gives

$$-\Delta(u - v) + (\lambda + Km)(u - v) \leq 0 \quad \text{on} \quad \Omega, \qquad u \leq v \quad \text{in } \partial\Omega.$$

Now, $p > \frac{n}{2}$ implies $W^{2,p}(\Omega) \subset H^1(\Omega)$ and the maximum principle (as stated e.g. in [4], Theorem 8.1) gives $u \leq v$ in Ω.

Let us start with the following:

Lemma 2.2. *Let K, f be two non negative functions belonging to $L^p(\Omega)$ for some $p > \frac{n}{2}$, let $\alpha, \lambda \in R$ with $\alpha > 0$ and $\lambda \geq 0$. Then for all $\varepsilon > 0$ and $j \in N$, there exists a unique $u_{j,\varepsilon} \in W^{2,p}(\Omega)$ satisfying*

$$-\Delta u_{j,\varepsilon} + \lambda u_{j,\varepsilon} = K b_j(u_{j,\varepsilon}) + f \quad \textit{on} \quad \Omega \qquad u_{j,\varepsilon} = \varepsilon \quad \textit{in} \quad \partial\Omega. \tag{2.2}$$

Proof. For fixed $j \in N$ and $\varepsilon > 0$, consider the continuous map $F : W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \to L^p(\Omega)$ defined by

$$F(u) = -\Delta u + \lambda(u + \varepsilon) - K b_j(u + \varepsilon),$$

where $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ is provided with the Banach space structure given by the $W^{2,p}$ norm. Observe that F is a continuously differentiable map with differential given by

$$dF(u)(v) = -\Delta v + (\lambda - K b_j'(u + \varepsilon))v, \quad u, v \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

Also, $\lambda - K b_j'(u + \varepsilon) \in L^p(\Omega)$ and, since $b_j'(u + \varepsilon) \leq 0$ and $K \geq 0$, we have $\lambda - K b_j'(u + \varepsilon) \geq 0$. Thus for all $f \in L^p(\Omega)$ the problem $dF(u)(v) = f$ has a unique solution $u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see e.g. [4], Theorem 9.15) and so F is a locally invertible map.

We claim that F is proper, i.e., if $\mathcal{K}$ is a compact subset of $L^p(\Omega)$, then $F^{-1}(\mathcal{K})$ is a compact subset of $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. Indeed, let $\{u_l\}_{l=1}^{\infty}$ be a sequence in $F^{-1}(\mathcal{K})$ and let $f_l = F(u_l)$. After passing to a subsequence, we can assume that $\{f_l\}_{l=1}^{\infty}$ converges to some f in $L^p(\Omega)$. Now, $\{f_l\}_{l=1}^{\infty}$ is bounded in $L^p(\Omega)$, $K \in L^p(\Omega)$, $b_j \in L^\infty(\Omega)$ and

$$-\Delta u_l + \lambda u_l = K b_j(u_l + \varepsilon) + f_l - \lambda\varepsilon, \qquad u_l \in W^{2,p}(\Omega) \tag{2.3}$$

so, the $W^{2,p}$ estimates for elliptic problems (as stated e.g. in [4], Lemma 9.17) given that $\{u_l\}_{l=1}^{\infty}$ is bounded in $W^{2,p}(\Omega)$. Thus the Rellich Kondrachov's

Theorem (see e.g. [4], Theorem 7.26) gives a subsequence $\{u_{l_r}\}_{r=1}^{\infty}$ that converges to some u in $C(\overline{\Omega})$. Then $\{Kb_j(u_{l_r}+\varepsilon)+f_{l_r}\}_{r=1}^{\infty}$ converges to $Kb_j(u+\varepsilon)+f$ in $L^p(\Omega)$ and so (2.3) and the $W^{2,p}$ estimates imply that $\{u_{l_r}\}_{r=1}^{\infty}$ converges to u in $W^{2,p}(\Omega)$ and the claim follows.

Then F is a homeomorphism from $W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$ onto $L^p(\Omega)$ (see e.g. [1], Theorem 1.8, p. 41) and so for all $f\in L^p(\Omega)$ the problem

$$-\Delta u+\lambda(u+\varepsilon)-Kb_j(u+\varepsilon)=f$$

has at least a solution $u\in W^{2,p}(\Omega)\cap W_0^{1,p}(\Omega)$. Taking $u_{j,\varepsilon}=u+\varepsilon$ the existence follows.

On the other hand, if $u_{j,\varepsilon}$ and $v_{j,\varepsilon}$ are solutions in $W^{2,p}(\Omega)$ of the problem (2.2), then

$$-\Delta(u_{j,\varepsilon}-v_{j,\varepsilon})+(\lambda+Km)(u_{j,\varepsilon}-v_{j,\varepsilon})=0 \text{ on } \Omega,\ \ u_{j,\varepsilon}-v_{j,\varepsilon}=0 \text{ in } \partial\Omega, \quad (2.4)$$

where $m=m(x)$ is the function defined by

$$\begin{aligned} m(x)&=-\frac{b_j(u_{j,\varepsilon}(x))-b_j(v_{j,\varepsilon}(x))}{u_{j,\varepsilon}(x)-v_{j,\varepsilon}(x)} && \text{if } u_{j,\varepsilon}(x)\neq v_{j,\varepsilon}(x)\\ m(x)&=-b_j'(u_{j,\varepsilon}(x)) && \text{if } u_{j,\varepsilon}(x)=v_{j,\varepsilon}(x). \end{aligned}$$

The properties of the sequence $\{b_j\}_{j=1}^{\infty}$ imply that $m\geq 0$ and that $m\in L^p(\Omega)$. So, (2.4) and the maximum principle ([4], Theorem 8.1) imply $u_{j,\varepsilon}=v_{j,\varepsilon}$ on Ω.

Lemma 2.3. *Assume the hypothesis of Lemma* 2.2. *Then for each* $\varepsilon>0$ *the problem*

$$-\Delta u_\varepsilon+\lambda u_\varepsilon=Ku_\varepsilon^{-\alpha}+f \text{ on } \Omega \quad u_\varepsilon=\varepsilon \text{ in } \partial\Omega \qquad (2.5)$$

has a unique solution in $W^{2,p}(\Omega)$. *Moreover,* $\min_{x\in\overline{\Omega}}u_\varepsilon(x)>0$.

Proof. For $j\in N$, let $u_{j,\varepsilon}$ be a solution in $W^{2,p}(\Omega)$ (and so in $C(\overline{\Omega})$, because $p>\frac{n}{2}$) of the problem (2.2). Note also that, since $p>\frac{n}{2}$ and $n\geq 2$, we have $W^{2,p}(\Omega)\subset H^1(\Omega)$. Also, $K\geq 0$ and $b_{j+1}\geq b_j$ imply

$$\begin{aligned} -\Delta u_{j,\varepsilon}+\lambda u_{j,\varepsilon}-Kb_{j+1}(u_{j,\varepsilon}) &\leq -\Delta u_{j,\varepsilon}+\lambda u_{j,\varepsilon}-Kb_j(u_{j,\varepsilon})\\ &= f=-\Delta u_{j+1,\varepsilon}+\lambda u_{j+1,\varepsilon}-Kb_{j+1}(u_{j+1,\varepsilon}). \end{aligned}$$

Then

$$-\Delta u_{j,\varepsilon}+\lambda u_{j,\varepsilon}-Kb_{j+1}(u_{j,\varepsilon})\leq -\Delta u_{j+1,\varepsilon}+\lambda u_{j+1,\varepsilon}-Kb_{j+1}(u_{j+1,\varepsilon}).$$

Since

$$u_{j,\varepsilon}-u_{j+1,\varepsilon}=0 \text{ in } \partial\Omega,$$

the comparison principle, as stated in Remark 2.1, gives

$$u_{j,\varepsilon} \le u_{j+1,\varepsilon} \quad \text{on} \quad \Omega.$$

Observe also that $-\Delta u_{j,\varepsilon} + \lambda u_{j,\varepsilon} \ge 0$ on Ω and that $u_{j,\varepsilon} \ge 0$ in $\partial\Omega$. Then the maximum principle gives $\min_{\overline{\Omega}} u_{j,\varepsilon} \ge 0$. Now, if $u_{j,\varepsilon}(x_0) = 0$ for some $x_0 \in \Omega$, then the strong maximum principle ([4], Theorem 8.19) implies that u vanishes identically on Ω, contradicting $u = \varepsilon$ in $\partial\Omega$. Thus, each $u_{j,\varepsilon}$ is strictly positive on $\overline{\Omega}$. Let $c_{1,\varepsilon} = \min_{\overline{\Omega}} u_{1,\varepsilon}$. Then $u_{j,\varepsilon}(x) \ge u_{1,\varepsilon}(x) \ge c_{1,\varepsilon} > 0$ for all $j \in N$, $x \in \overline{\Omega}$ and $\varepsilon > 0$.

Now, take j large enough such that $\delta_j < c_{1,\varepsilon}$. Since $b_j(s) = s^{-\alpha}$ for $s \ge \delta_j$, it follows that $u_{j,\varepsilon}$ is the desired solution of (2.5). The uniqueness follows proceeding as in Lemma 2.1.

Lemma 2.4. *Assume the hypothesis of Lemma* 2.2 *and for* $\varepsilon > 0$, *let* u_ε *be the solution in* $W^{2,p}(\Omega)$ *of the problem* (2.5). *Then*

1) $\varepsilon_1 < \varepsilon_2$ *implies that* $u_{\varepsilon_1}(x) < u_{\varepsilon_2}(x)$ *for all* $x \in \overline{\Omega}$.

2) *There exists a* $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ *satisfying* $w(x) > 0$ *for all* $x \in \Omega$ *and such that* $u_\varepsilon(x) \ge w(x)$ *for all* $\varepsilon > 0$ *and* $x \in \Omega$.

Proof. Suppose $0 < \varepsilon_0 < \varepsilon_1 \le 1$. We take $j \in N$ large enough such that $\delta_j \le \min_{\overline{\Omega}}(u_{\varepsilon_i})$, $i = 0, 1$. Then u_{ε_0} and u_{ε_1} are two solutions in $W^{2,p}(\Omega)$ of

$$-\Delta u + \lambda u = K b_j(u) + f \quad \text{on } \Omega,$$

and $u_{\varepsilon_0} < u_{\varepsilon_1}$ on $\partial\Omega$. So, the comparison principle implies that $u_{\varepsilon_0} \le u_{\varepsilon_1}$ on Ω. Let w be the solution in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ of the problem

$$\begin{aligned} -\Delta w + \lambda w &= K u_1^{-\alpha} \quad \text{in } \Omega, \\ w &= 0 \qquad\quad\ \text{in } \partial\Omega. \end{aligned}$$

The strong maximum principle implies $w(x) > 0$ for all $x \in \Omega$. Also, for $j \in N$, $\varepsilon > 0$, we have

$$-\Delta u_\varepsilon + \lambda u_\varepsilon \ge -\Delta u_\varepsilon + \lambda u_\varepsilon - f = K u_\varepsilon^{-\alpha} \ge K u_1^{-\alpha} = -\Delta w + \lambda w \quad \text{on } \Omega$$

and $u_\varepsilon > w$ on $\partial\Omega$. Then the maximum principle gives $u_\varepsilon \ge w$ on Ω. □

Remark 2.5. Let w be the function provided by Lemma 2.4. Then $w \in C(\overline{\Omega})$. Moreover, if Ω' is a subdomain of Ω such that $\overline{\Omega'} \subset \Omega$, then the maximum principle implies that $\min_{x \in \overline{\Omega'}} w(x) > 0$.

If Ω' is a subdomain of Ω, we write $\Omega' \subset\subset \Omega$ to mean $\overline{\Omega'} \subset \Omega$.

Remark 2.6. Let U be an arbitrary bounded domain in R^n and let $u \in W_{loc}^{2,p}(U) \cap C(\overline{U})$ for some $p > \frac{n}{2}$. Then $-\Delta u \ge 0$ on U and $u = 0$ in ∂U imply $u \ge 0$ in U. Indeed, it is easy to construct, for each $\varepsilon > 0$, a regular (C^∞)

domain $U_\varepsilon \subset\subset U$ such that $dist(\partial U, \partial U_\varepsilon) \leq \varepsilon$ (for $i = (i_1, ..., i_n) \in Z^n$, let $Q_i = \prod_{j=1}^n [i_j \frac{\varepsilon}{2}, (1+i_j)\frac{\varepsilon}{2}]$ and $K = \bigcup_{i \in Z^n : Q_i \subset U} Q_i$, take now $\overset{\circ}{K}$. Smoothing the corners that appear in the boundary of $\overset{\circ}{K}$ such a U_ε is obtained). Now, given $\eta > 0$, since $u \in C(\overline{U})$ and $u = 0$ in ∂U, there exists $\varepsilon(\eta) > 0$ such that for all positive $\varepsilon < \varepsilon(\eta)$ and all $x \in \partial U_\varepsilon$, we have $|u(x)| \leq \eta$. Then $-\Delta(u+\eta) \geq 0$ on U_ε and $u + \eta \geq 0$ in ∂U_e and so, the usual maximum principle gives $u \geq -\eta$ in U_ε. This implies $u \geq 0$ on U.

3. The main results

Theorem 3.1. *Let Ω be a $C^{1,1}$ and bounded domain in R^n with $n \geq 2$. Let $\alpha, \lambda \in R$ with $\alpha > 0$ and $\lambda \geq 0$. Let K, f be two non negative functions belonging to $L^p(\Omega)$ for some $p > \frac{n}{2}$. Then there exists a unique $u \in W^{2,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ such that*

$$-\Delta u + \lambda u = K u^{-\alpha} + f \quad on\ \Omega \qquad u = 0 \quad in\ \partial\Omega. \tag{3.1}$$

Proof. Let u_ε be the solution in $W^{2,p}(\Omega)$ of the problem (2.5). Let $u = \lim_{\varepsilon \to 0^+} u_\varepsilon(x)$, $x \in \overline{\Omega}$. Lemma 2.4 says that this limit exists and that it is positive for all $x \in \overline{\Omega}$. On the other hand, if Ω' and Ω'' are subdomains of Ω such that $\Omega' \subset\subset \Omega'' \subset\subset \Omega$, then taking into account that $u_\varepsilon < u_1$ for $0 < \varepsilon < 1$, from (2.5) and from the inner $W^{2,p}$ estimates for linear elliptic problems (as stated e.g. in [4], Theorem 9.13) it follows that there exists a positive constant $c = c(\Omega', \Omega'', p, \lambda)$ such that for all $0 < \varepsilon < 1$

$$\|u_\varepsilon - \varepsilon\|_{W^{2,p}(\Omega'')} \leq c(\Omega', \Omega'', p, \lambda)(\|u_1\|_{L^p(\Omega')} + \|K u_\varepsilon^{-\alpha} + f\|_{L^p(\Omega')}).$$

Let w be the function provided by Lemma 2.4. Since $u_\varepsilon \geq w$, we have, for $0 < \varepsilon \leq 1$, that

$$\|u_\varepsilon - \varepsilon\|_{W^{2,p}(\Omega'')} \leq c(\Omega', \Omega'', p, \lambda)(\|u_1\|_{L^p(\Omega')} + \|K w^{-\alpha} + f\|_{L^p(\Omega')}).$$

From this inequality a standard argument gives that if $\{\varepsilon_j\}_{j=1}^\infty$ is a sequence such that $\varepsilon_j > 0$ and $\lim_{j\to\infty} \varepsilon_j = 0$, then there exists a subsequence $\{\varepsilon_{j_l}\}_{l=1}^\infty$ such that for all $\Omega' \subset\subset \Omega$ the sequences $\{u_{\varepsilon_{j_l}}\}_{l=1}^\infty$, $\{\frac{\partial u_{\varepsilon_{j_l}}}{\partial x_i}\}_{l=1}^\infty$ and $\{\frac{\partial^2 u_{\varepsilon_{j_l}}}{\partial x_i \partial x_s} u_{\varepsilon_{j_l}}\}_{l=1}^\infty$ converge weakly in $L^p(\Omega')$ for each $1 \leq i, s \leq n$. So, going to the limit in (2.5), we get that $u \in W^{2,p}_{loc}(\Omega)$ and that u satisfies (3.1).

Moreover, $u \in C(\overline{\Omega})$. Indeed, we know that $u \in C(\Omega)$. Let $x_0 \in \partial\Omega$ and $\eta > 0$. Let ε_0 be such that $0 < \varepsilon_0 < \frac{\eta}{2}$ and $\delta_0 > 0$ such that $x \in \Omega$ and

$|x-x_0| \le \delta_0$ imply $u_{\varepsilon_0} - \varepsilon_0 < \frac{\eta}{2}$. For $0 < \varepsilon < \varepsilon_0$ if $x \in \Omega$ and $|x - x_0| \le \delta_0$, we have

$$u(x) \le u_\varepsilon(x) \le u_{\varepsilon_0}(x) \le \varepsilon_0 + \frac{\eta}{2} < \eta.$$

So u is continuous also on $\partial\Omega$.

Suppose now that $u, v \in W^{2,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ are solutions of the problem (3.1). Then

$$-\Delta(u-v) = k(u^{-\alpha} - v^{-\alpha}).$$

Let $\Omega^+ = \{x \in \Omega : u(x) - v(x) > 0\}$. So Ω^+ is open. Suppose $\Omega^+ \ne \emptyset$. Then $u = v$ on $\partial\Omega^+$. Also, $-\Delta(u-v) \le 0$ on Ω^+. Thus (by Remark 2.6) $u \le v$ on Ω^+, which is a contradiction. Similarly, $\{x \in \Omega : u(x) - v(x) < 0\} = \emptyset$. Thus, $u = v$ on Ω. □

Remark 3.2. Let us turn to the proof of Theorem 3.1. There it is proved (under the hypothesis of the theorem) that if $\{\varepsilon_l\}_{l=1}^\infty$ is a sequence of positive numbers satisfying $\lim_{l\to\infty} \varepsilon_l = 0$, then there exists a subsequence $\{\varepsilon_{l_s}\}_{s=1}^\infty$ such that u_{l_s} converges to a strong solution $u \in W^{2,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ of (3.1).

Consider now a sequence $\{K_r\}_{r=1}^\infty$ of non negative functions belonging to $L^p(\Omega)$ for some $p > \frac{n}{2}$ and assume that Ω and α and f are as in Theorem 3.1. For $\varepsilon > 0$ and $r \in N$, let $v_{r,\varepsilon}$ be the solution in $W^{2,p}(\Omega)$ of the problem

$$-\Delta v_{r,\varepsilon} + \lambda = K_r v_{r,\varepsilon}^{-\alpha} + f \quad \text{on } \Omega, v_{r,\varepsilon} = \varepsilon \quad \text{in } \partial\Omega. \tag{3.2}$$

A standard argument (which involves a diagonal process) gives the existence of a sequence $\{\varepsilon'_l\}_{l=1}^\infty$ with $\varepsilon'_l > 0$, $\lim_{l\to\infty} \varepsilon'_l = 0$ and such that for all r, $\lim_{l\to\infty} v_{r,\varepsilon'_l}$ is a solution in $W^{2,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ of the problem

$$-\Delta u + \lambda u = K_r u^{-\alpha} + f \quad \text{in } \Omega, \qquad u = 0 \quad \text{in } \partial\Omega.$$

Remark 3.3. Let us recall the following version of the Hopf's lemma (see [3], Lemma 1). Let λ_1 be the principal eigenvalue for $-\Delta$ on Ω with the homogeneous Dirichlet boundary condition and let Φ_1 be a positive principal eigenfunction associated, normalized, e.g., by requiring that $\|\Phi_1\|_\infty = 1$. For $f \in L^1(\Omega)$, let $\langle f, \Phi_1 \rangle = \int_\Omega f\phi_1$. Then there exists a positive constant $c = c(\Omega)$ such that if f is a non-negative function in $L^1(\Omega)$ and if $u \in W^{2,1}(\Omega)$ satisfies $-\Delta u = f$ on Ω and $u = 0$ in $\partial\Omega$, then $u \ge c(\Omega) \langle f, \Phi_1 \rangle \Phi_1$ on Ω.

Let K be a nonnegative function in $L^p(\Omega)$ for some $p \ge 1$. For $r \in N$, we set

$$K_r(x) = \min\{r, K(x)\}. \tag{3.3}$$

For $\varepsilon > 0$ and $r \in N$, let $v_{r,\varepsilon}$ be the solution in $W^{2,p}(\Omega)$ of $-\Delta v_{r,\varepsilon} = K_r {v_{r,\varepsilon}}^{-\alpha}$ on Ω, $v_{r,\varepsilon} = \varepsilon$ in $\partial\Omega$ (i.e., the function given by (3.2) taking there $\lambda = 0$ and

$f=0$) and let w_r be the solution in $W^{2,p}(\Omega)$ of the problem

$$-\Delta w_r = K_r v_{r,1}^{-\alpha} \text{ in } \Omega, w_r = 0 \text{ in } \partial\Omega. \tag{3.4}$$

With this notations we have:

Lemma 3.4. *Let Ω be a $C^{1,1}$ and bounded domain in R^n with $n \geq 2$, let α be a positive real number, let K be a non-negative function belonging to $L^p(\Omega)$ for some $p > \frac{(\alpha^2+1)n}{2\alpha^2+n}$ and for $r \in N$, let w_r be defined by (3.4) and let Φ_1 be a positive eigenfunction for $-\Delta$ on Ω with the Dirichlet homogeneous boundary condition. Then there exists a positive constant c such that $w_r \geq c\Phi_1$ on Ω for all $r \in N$.*

Proof. We have, (see Remark 3.3), for some positive constant $\widetilde{c} = \widetilde{c}(\Omega)$ and for all $r \in N$, $x \in \Omega$

$$w_r(x) \geq \widetilde{c}\langle k_r v_{r,1}^{-\alpha}, \Phi_1\rangle \Phi_1(x), \quad x \in \Omega.$$

Thus,

$$w_r^{-\alpha}(x) \leq \widetilde{c}^{-\alpha}\Big(\int_\Omega K_r v_{r,1}^{-\alpha}\Phi_1\Big)^{-\alpha}\Phi_1(x)^{-\alpha}, \quad x \in \Omega.$$

Since $t \to t^{-\alpha}$ is convex on $(0,\infty)$, the Jensen's inequality gives, for $x \in \Omega$, that

$$\begin{aligned}\langle k_r v_{r,1}^{-\alpha}, \Phi_1\rangle^{-\alpha} &\leq \Big(\int_\Omega K_r\Phi_1\Big)^{-\alpha-1}\Big(\int_\Omega K_r v_{r,1}^{\alpha^2}\Phi_1\Big)\\ &\leq \Big(\int_\Omega K_1\Phi_1\Big)^{-\alpha-1}\|\Phi_1\|_\infty\|K\|_p\|v_{r,1}\|_{\alpha^2 p'}^{\alpha^2} = c'(K,\Omega)\|v_{r,1}\|_{\alpha^2 p'}^{\alpha^2}.\end{aligned}$$

Now, $p > \frac{(\alpha^2+1)n}{2\alpha^2+n}$ implies $\alpha^2 p' < p^{**}$, where p^{**} is defined by $\frac{1}{p^{**}} = \frac{1}{p} - \frac{2}{n}$ if $p < \frac{n}{2}$ and $p^{**} = \infty$ if $p = \frac{n}{2}$. Thus

$$\big\|v_{r,1}\big\|_{\alpha^2 p'} \leq \big\|v_{r,1}\big\|_{W^{2,p}(\Omega)} \leq const.\big\|K_r v_{r,1}^{-\alpha}\big\|_p \leq const.\|K\|_p\big\|v_{1,1}^{-\alpha}\big\|_{p'}.$$

Then, we have, for some positive constant c'' and all r, that

$$w_r^{-\alpha}(x) \leq c''\Phi_1^{-\alpha}(x)$$

and the lemma follows. □

For $\alpha > 0$, let $M_\alpha = \{u \in W_{loc}^{2,p}(\Omega) : 0 \leq u \leq w^{\frac{1}{\alpha+1}}$ for some nonnegative $w \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)\}$. So, the functions belonging to M_α vanish in some sense on the boundary. With this notation we have the following:

Theorem 3.5. *Let Ω be a $C^{1,1}$ and bounded domain in R^n with $n \geq 2$. Let $\alpha > 0$ and K be a non-negative function belonging to $L^p(\Omega)$ for some*

$p > \frac{(\alpha^2+1)n}{2\alpha^2+n}$. *Then there exists* $u \in W^{2,p}_{loc}(\Omega) \cap M_\alpha$ *satisfying*

$$-\Delta u = K u^{-\alpha} \text{ on } \Omega. \tag{3.5}$$

Proof. For $r \in N$, $\varepsilon > 0$, let K_r be defined by (3.3) and $v_{r,\varepsilon}$ be defined by (3.2) taking there $\lambda = 0$ and $f = 0$. So, each $K_r \in L^\infty(\Omega)$. Let $\{\varepsilon_l\}_{l=1}^\infty$ be a decreasing sequence (provided by Remark 3.2) with $\lim_{l\to\infty} \varepsilon_l = 0$ and such that for all $r \in N$ the function v_r defined by $v_r = \lim_{l\to\infty} v_{r,\varepsilon_l}$ is a solution in $W^{2,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ for the problem

$$-\Delta v_r = K_r v_r^{-\alpha} \quad \text{on} \quad \Omega, v_r = 0 \quad \text{in} \quad \partial\Omega. \tag{3.6}$$

It is easy to see, using the maximum principle as in Lemma 2.4 that $v_{r,\varepsilon_1} \leq v_{r,\varepsilon_2}$ for $r \in N$, $0 < \varepsilon_1 < \varepsilon_2$. Also, for $\varepsilon > 0$, we have

$$-\Delta(v_{r+1,\varepsilon} - v_{r,\varepsilon}) \geq K_r(v_{r+1,\varepsilon}^{-\alpha} - v_{r,\varepsilon}^{-\alpha})$$

and so $(-\Delta + m)(v_{r+1,\varepsilon} - v_{r,\varepsilon}) \geq 0$, where m is defined by

$$m = -\frac{v_{r+1,\varepsilon}^{-\alpha} - v_{r,\varepsilon}^{-\alpha}}{v_{r+1,\varepsilon} - v_{r,\varepsilon}}$$

since m belongs to $L^\infty(\Omega)$ and $m \geq 0$, the maximum principle implies that $v_{r+1,\varepsilon} \geq v_{r,\varepsilon}$.

Now, from $-\Delta v_{r,1} = K_r v_{r,1}^{-\alpha}$ on Ω, $v_{r,1} = 1$ in $\partial\Omega$ and $-\Delta w_r = K_r v_{r,1}^{-\alpha}$ on Ω, $w_r = 0$ in $\partial\Omega$ the maximum principle gives $v_{r,1} \geq w_r$ on Ω. Thus, $v_{r,1} \geq c\Phi_1$ on Ω for some $c > 0$ and all r. Let now $w_{r,\varepsilon} \in W^{2,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ be the solution of $-\Delta w_{r,\varepsilon} = K_r v_{r,\varepsilon}^{-\alpha}$ on Ω, $w_{r,\varepsilon} = 0$ in $\partial\Omega$. Then $-\Delta w_{r,\varepsilon} = K_r v_{r,\varepsilon}^{-\alpha} \geq K_r v_{r,1}^{-\alpha} = -\Delta w_r$ on Ω and $w_{r,\varepsilon} = w_r$ in $\partial\Omega$, then the maximum principle gives $w_{r,\varepsilon} \geq w_r$ on Ω, thus, for some positive c and all r we have $w_{r,\varepsilon} \geq c\Phi_1$ on Ω. Now, taking into account the equations satisfied by $v_{r,\varepsilon}$ and $w_{r,\varepsilon}$ and the boundary conditions, the maximum principle gives, for all $\varepsilon > 0$ and $r \in N$, that $v_{r,\varepsilon} \geq w_{r,\varepsilon} \geq c\Phi_1$ on Ω.

Then, for $\Omega' \subset\subset \Omega'' \subset\subset \Omega$ we have, for some positive constants c, c', c'' and c''' (independent of r and ε) that

$$\begin{aligned}
&\|v_{r,\varepsilon} - \varepsilon\|_{W^{2,p}(\Omega')} \leq c\big(\|v_{r,\varepsilon}\|_{L^p(\Omega'')} + \|K_r v_{r,\varepsilon}^{-\alpha}\|_{L^p(\Omega'')}\big)\\
&\leq c\big(\|v_{r,1}\|_{L^p(\Omega'')} + \|K_r v_{r,\varepsilon}^{-\alpha}\|_{L^p(\Omega'')}\big)\\
&\leq c(\|(v_{r,1} - 1) + 1\|_{L^p(\Omega)} + \|K_r v_{r,\varepsilon}^{-\alpha}\|_{L^p(\Omega'')})\\
&\leq c'\big(c'' + \|K v_{r,1}^{-\alpha}\|_{L^p(\Omega)} + \|K_r v_{r,\varepsilon}^{-\alpha}\|_{L^p(\Omega'')}\big) \leq c'''\big(c'' + \|K_r \Phi_1^{-\alpha}\|_{L^p(\Omega'')}\big).
\end{aligned}$$

So $\{\|v_{r,\varepsilon}\|_{W^{2,p}(\Omega')}\}_{r=1}^{\infty}$ is bounded in $W^{2,p}(\Omega')$ for all $\Omega' \subset\subset \Omega$, with bounds independent of ε. Let $u = \lim_{r\to\infty} v_r$. Now, taking into account the above inequalities and proceeding as in the proof of Theorem 3.1, we get that u is well defined, that $u \in W^{2,p}_{loc}(\Omega)$ and that u satisfies $-\Delta u = Ku^{-\alpha}$ in Ω.

On the other hand, for each $r \in N$ and $\varepsilon > 0$, we have that $v_{r,\varepsilon} \in W^{2,q}(\Omega)$ for all $q \geq 1$ and that $v_{r,\varepsilon}$ is strictly positive on $\overline{\Omega}$. So $v^{\alpha}_{r,\varepsilon} \in W^{2,q}(\Omega)$ for all $q \geq 1$, and for all $\beta \geq 1$, we have

$$-\Delta(v^{\beta}_{r,\varepsilon}) = -\beta v^{\beta-1}_{r,\varepsilon}\Delta v_{r,\varepsilon} - \beta(\beta-1)v^{\beta-2}_{r,\varepsilon}|\nabla v_{r,\varepsilon}|^2 \leq \beta K_r v^{-\alpha+\beta-1}_{r,\varepsilon}.$$

Now we take $\beta = \alpha + 1$ to obtain

$$-\Delta(v^{\alpha+1}_{r,\varepsilon}) \leq (\alpha+1)K_r.$$

Let $\psi_{r,\varepsilon} \in W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ be the solution of the problem $-\Delta\psi_{r,\varepsilon} = K_r$, $\psi_{r,\varepsilon} = \varepsilon$ on $\partial\Omega$. Then

$$-\Delta(v^{\alpha+1}_{r,\varepsilon} - (\alpha+1)\psi_{r,\varepsilon}) \leq 0 \quad \text{on } \Omega$$

and

$$v^{\alpha+1}_{r,\varepsilon} - (\alpha+1)\psi_{r,\varepsilon} = \varepsilon^{\alpha+1} - (\alpha+1)\varepsilon \quad \text{on } \partial\omega.$$

Thus, $v^{\alpha+1}_{r,\varepsilon} - (\alpha+1)\psi_{r,\varepsilon} \leq 0$ on $\partial\Omega$ if ε is small enough. Thus, the maximum principle implies that $v_{r,\varepsilon} \leq (\alpha+1)^{\frac{1}{\alpha+1}}\psi^{\frac{1}{\alpha+1}}_{r,\varepsilon}$ on Ω. From this inequality it follows that the above constructed solution u for the problem (3.5) satisfies $u \leq (\alpha+1)^{\frac{1}{\alpha+1}}\psi^{\frac{1}{\alpha+1}}$, where ψ is the solution in $W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ for the problem $-\Delta\psi = K$ on Ω, $\psi = 0$ in $\partial\Omega$. □

Remark 3.6. In some cases certain information about the behavior at the boundary of the solutions of the problem $-\Delta u = Ku^{-\alpha}$ on Ω, $u = 0$ in $\partial\Omega$ is available. Indeed, suppose that $\alpha > 1$ and that for some $0 < \varepsilon < \frac{2}{n}$ and for some positive constants c_1 and c_2 it holds that

$$c_1\Phi_1^{-\frac{2}{n}+\varepsilon} \leq K \leq c_2\Phi_1^{-\frac{2}{n}+\varepsilon} \text{ on } \Omega,$$

where Φ_1 is as in Remark 3.3. Then $K \in L^p(\Omega)$ for some $p > \frac{n}{2}$ and so, by Theorem 3.1, there exists a unique solution $u \in W^{2,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ for the problem $-\Delta u = Ku^{-\alpha}$ on Ω, $u = 0$ in $\partial\Omega$. Let us show that there exist positive constants $\widetilde{c}_1$ and $\widetilde{c}_2$ such that

$$\widetilde{c}_1\Phi_1^{\frac{2-\frac{2}{n}+\varepsilon}{1+\alpha}} \leq u \leq \widetilde{c}_2\Phi_1^{\frac{2-\frac{2}{n}+\varepsilon}{1+\alpha}} \quad \text{on } \Omega.$$

Indeed, Let $\beta = \frac{2-\frac{2}{n}+\varepsilon}{1+\alpha}$ and let $v = \Phi_1^{\beta}$. A computation gives that $-\Delta v = \widetilde{K} v^{-\alpha}$, where $\widetilde{K}$ is the function defined by

$$\widetilde{K}(x) = \beta\lambda_1 \Phi_1^{2-\frac{2}{n}+\varepsilon} - \beta(\beta-1)|\nabla\Phi_1|^2 \Phi_1^{-\frac{2}{n}+\varepsilon}.$$

Moreover, $0 < \beta < 1$, so well known properties of the principal eigenfunction Φ_1 imply that there exist positive constants c_1' and c_2' such that

$$c_1' \Phi_1^{-\frac{2}{n}+\varepsilon} \leq \widetilde{K} \leq c_2' \Phi_1^{-\frac{2}{n}+\varepsilon} \qquad \text{on } \Omega.$$

Thus, there exist positive constants c_1'' and c_2'' such that $c_1''\widetilde{K} \leq K \leq c_2''\widetilde{K}$ on Ω. Moreover, $\widetilde{K} \in L^p(\Omega)$ for some $p > \frac{n}{2}$ and, since $v \in W^{2,p}_{loc}(\Omega) \cap C(\overline{\Omega})$ satisfies $-\Delta v = \widetilde{K} v^{-\alpha}$ on Ω, $v = 0$ in $\partial\Omega$, we have

$$-\Delta((c_1'')^{\frac{1}{\alpha+1}} v) = c_1''\widetilde{K}((c_1'')^{\frac{1}{\alpha+1}} v)^{-\alpha} \leq K((c_1'')^{\frac{1}{\alpha+1}} v)^{-\alpha} \text{ on } \Omega.$$

Also $-\Delta u = K u^{-\alpha}$ on Ω, $u = 0$ in $\partial\Omega$. Then $-\Delta((c_1'')^{\frac{1}{\alpha+1}} v - u) \leq 0$ in Ω and $(c_1'')^{\frac{1}{\alpha+1}} v - u = 0$ in $\partial\Omega$ and so, the maximum principle gives that

$$u \geq (c_1'')^{\frac{1}{\alpha+1}} \Phi_1^{\frac{2-\frac{2}{n}+\varepsilon}{1+\alpha}} \qquad \text{on } \Omega.$$

Similarly, we obtain that

$$u \leq (c_2'')^{\frac{1}{\alpha+1}} \Phi_1^{\frac{2-\frac{2}{n}+\varepsilon}{1+\alpha}}.$$

Observe that the limit case $\varepsilon = \frac{2}{n}$ corresponds to the case studied in [6] (for K Holder continuous and strictly positive on $\overline{\Omega}$). The above estimates for u should be compared with those involved in the global estimates for $|\nabla u|$ obtained in [5] for the case

$$0 \leq K(x) \leq c\Phi_1^q, \qquad q > \alpha - 1 \tag{3.7}$$

and its extension, given in [8], for the case $K \in L^\infty(\Omega)$, $K \geq 0$ and strictly positive on a subset of Ω of positive measure and satisfying some growth condition that generalizes (3.7). See also the Holder estimates up to the boundary for u obtained in [7] for a nonnegative and essentially bounded K.

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