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SMALL DATA GLOBAL SOLUTIONS FOR DIRAC–KLEIN–GORDON EQUATION

Shuji Machihara
Department of Mathematics, Hokkaido University, Sapporo 060-0810, Japan

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Abstract. In this paper we study the global existence for Dirac–Klein–Gordon equations with small data. We apply the contraction mapping principle and use the Strichartz estimate.

1. Introduction

We consider the problem of global existence with small data for the Dirac–Klein–Gordon equations coupled by Yukawa–like interactions (DKG):

$$\begin{gathered}\partial_t\psi + \alpha\nabla\psi + i\beta\psi = v(\beta\psi|\psi)^{\gamma-1}\psi, \\ \partial_t^2 v - \Delta v + v = (\beta\psi|\psi)^{\kappa},\end{gathered} \tag{1.1}$$

where ψ and v are functions from $\mathbb{R}^4$ to $\mathbb{C}^4$ and from $\mathbb{R}^4$ to $\mathbb{C}$, respectively, of the variables $(x,t) \in \mathbb{R}^3 \times \mathbb{R}$ with $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $\partial_t = \partial/\partial t$, $\alpha\nabla = \sum_{j=1}^3 \alpha_j\partial_j$, $\partial_j = \partial/\partial x_j$ and $\gamma, \kappa \geq 1$. We follow the standard notation of relativistic quantum mechanics. The α_j's and β are 4×4 Hermitian matrices satisfying anticommutation relations, i.e., $\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I$, where δ_{jk} is Kronecker's delta and I is the 4×4 unit matrix, $\alpha_j\beta + \beta\alpha_j = 0$, and $\beta^2 = I$. $(\cdot|\cdot)$ denotes the Hermitian product in $\mathbb{C}^4$, Δ denotes Laplacian $\sum_{j=1}^3 \partial_j^2$. We prescribe the initial data at time $t = 0$: $\psi(x,0) = \psi_0(x)$, $v(x,0) = v_0(x)$, $\partial_t v(x,0) = v_1(x)$. This type of Dirac–Klein–Gordon equations can be found in [6], [16]. Especially the problem in the case of $\gamma = \kappa = 1$ have been studied in a lot of literature, for instance [1], [3], [4], [7], [19].

In this paper, we study the global existence of DKG. Before stating our results, we shall give a scaling approach on this problem. For instance let us consider the massless case of DKG

$$\begin{gathered}\partial_t\psi + \alpha\nabla\psi = v(\beta\psi|\psi)^{\gamma-1}\psi, \\ \partial_t^2 v - \Delta v = (\beta\psi|\psi)^{\kappa},\end{gathered} \tag{1.2}$$

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with initial data $\psi(x,0) = \psi_0(x)$, $v(x,0) = v_0(x)$, $\partial_t v(x,0) = 0$. We scale the unknown functions ψ, v in the form

$$\psi_\lambda(t,x) = \lambda^{3/2-\mu}\psi(\lambda t, \lambda x), \quad v_\lambda(t,x) = \lambda^{3/2-\nu} v(\lambda t, \lambda x), \tag{1.3}$$

with $\lambda > 0$. Then massless DKG is equivalent to following problem,

$$\begin{aligned} \lambda^{\mu-5/2}\partial_t\psi_\lambda + \lambda^{\mu-5/2}\alpha\nabla\psi_\lambda &= \lambda^{(2\gamma-1)(\mu-3/2)+\nu-3/2} v_\lambda(\beta\psi_\lambda|\psi_\lambda)^{\gamma-1}\psi_\lambda, \\ \lambda^{\nu-7/2}\partial_t^2 v_\lambda - \lambda^{\nu-7/2}\Delta v_\lambda &= \lambda^{2\kappa(\mu-3/2)}(\beta\psi_\lambda|\psi_\lambda)^\kappa, \end{aligned} \tag{1.4}$$

with initial data

$$\psi_\lambda^0(x,0) = \lambda^{3/2-\mu}\psi_0(\lambda x), \quad v_\lambda^0(x,0) = \lambda^{3/2-\nu} v_0(\lambda x), \quad \partial_t v_\lambda^0(x,0) = 0.$$

We take the initial data belonging to the homogeneous Sobolev space $\dot{H}^s$ introduced below in such a way that the following invariance holds:

$$\|\psi_\lambda^0\|_{\dot{H}^\mu} = \|\psi_0\|_{\dot{H}^\mu}, \quad \|v_\lambda^0\|_{\dot{H}^\nu} = \|v_0\|_{\dot{H}^\nu}.$$

Therefore, we may think the following case is critical for (1.2):

$$\mu = 3/2 - 3/(2\gamma + 2\kappa - 2), \quad \nu = 7/2 - 3\kappa/(\gamma + \kappa - 1). \tag{1.5}$$

Below, we show the existence of the global solutions with small data for DKG in subcritical framework of inhomogeneous Sobolev space. Our basic tool is the Strichartz estimate for Klein–Gordon equation which enables us to deal with the problem in "almost" critical argument. In the critical case, the difficulty there consists in the lack of Strichartz estimates on $L_t^2 L^\infty$. That reminds us the ill–posedness results in [15] for nonlinear wave equations.

To state the main results precisely, we introduce the following notation. For any r with $1 \le r \le \infty$, $L^r = L^r(\mathbb{R}^n)$ denotes the Lebesgue space on $\mathbb{R}^n$. For any $s \in \mathbb{R}$ and any r with $1 < r < \infty$, H_r^s, [resp. $\dot{H}_r^s$] denotes the inhomogeneous [resp. homogeneous] Sobolev space. For any $s \in \mathbb{R}$ and any r, m with $1 \le r$, $m \le \infty$, $B_{r,m}^s$, [resp. $\dot{B}_{r,m}^s$] denotes the inhomogeneous [resp. homogeneous] Besov space. We refer to [2] for notation and general information on these spaces. We shall work in the three dimensional space $\mathbb{R}^3$. We make abbreviations such as $H^s = H_2^s$, $\dot{H}^s = \dot{H}_2^s$, $B_r^s = B_{r,2}^s$, $\dot{B}_r^s = \dot{B}_{r,2}^s$. Occasionally we use $\lesssim$ which means $\le C$, where C is a positive constant.

We define

$$X^s = L^\infty(\mathbb{R}; H^s) \cap L^{q_0}(\mathbb{R}; B_{r_0}^{s-\sigma_0}), \tag{1.6}$$

where $1/q_0 = 1/2 - 1/r_0 = \sigma_0/2$ and $q_0 > 2$ is sufficiently close to 2.

Now we give our result.

Theorem 1. *Let γ, κ, μ, ν satisfy $\gamma \geq 2$, $\kappa \geq 1$, $\gamma + \kappa \geq 4$, $\mu = 3/2 - 3/(2\gamma + 2\kappa - 2) + \varepsilon_1$, $\nu = 7/2 - 3\kappa/(\gamma + \kappa - 1) - \varepsilon_2$ for $\varepsilon_1 > 0$ and sufficiently small $\varepsilon_2 \geq 0$. If $\gamma - 1$ [resp. κ] is not an integer, in addition assume that $\mu < \gamma - 1$ [resp. $\nu < \kappa$]. Let $\psi_0 \in H^\mu, v_0 \in H^\nu, v_1 \in H^{\nu-1}$ and let $\|\psi_0\|_{H^\mu}, \|v_0\|_{H^\nu}, \|v_1\|_{H^{\nu-1}}$ be sufficiently small. Then there exist unique solutions (ψ, v) for (1.1) such that*

$$\psi \in C(\mathbb{R}; H^\mu) \cap X^\mu, \;\; v \in C(\mathbb{R}; H^\nu) \cap X^\nu. \tag{1.7}$$

Remark. We are faced with the restriction $\varepsilon_1 > 0$ in our proof below. ε_2 depends on ε_1. Examples of $(\gamma, \kappa, \mu, \nu)$ are $(2, 2, 1+\varepsilon_1, 3/2-\varepsilon_2), (9/2, 7/2, 2, 2)$. We consider the problems with fractional power nonlinearities or initial data in Sobolev space of fractional order in virtue of the estimates on Besov space.

2. Proof

We apply the contraction mapping principle to obtain global solutions with small data. The solutions of DKG satisfy the following system of integral equations,

$$\psi(t) = U(t)\psi_0 + \int_0^t U(t-s)F(s)ds, \tag{2.1}$$

$$v(t) = V_1(t)v_0 + V_2(t)v_1 + \int_0^t V_2(t-s)G(s)ds, \tag{2.2}$$

where

$$U(t) = I\cos t(1-\Delta)^{1/2} - (\alpha\nabla + i\beta)(1-\Delta)^{-1/2}\sin t(1-\Delta)^{1/2},$$

$$V_1(t) = \cos t(1-\Delta)^{1/2}, \;\; V_2(t) = (1-\Delta)^{-1/2}\sin t(1-\Delta)^{1/2},$$

$$F(t) = v(t)(\beta\psi(t)|\psi(t))^{\gamma-1}\psi(t), \;\; G(t) = (\beta\psi(t)|\psi(t))^\kappa.$$

Corresponding norm for X^s is given by

$$\|u\|_{X^s} = \|u\|_{L^\infty(\mathbb{R};H^s)} + \|u\|_{L^{q_0}(\mathbb{R};B^{s-\sigma_0}_{r_0})}.$$

We note the interpolation estimates

$$\|u\|_{L^q(\mathbb{R};B^{s-\sigma}_r)} \leq C\|u\|_{X^s}, \tag{2.3}$$

where $q_0 \leq q \leq \infty$, $1/q = 1/2 - 1/r = \sigma/2$ and C is independent of q.

In addition, we set the complete metric space $X^{\mu,\nu}_M$ as

$$X^{\mu,\nu}_M = \{(\psi, v) \in X^\mu \oplus X^\nu : \|(\psi, v)\|_{X^{\mu,\nu}_M} \leq M\} \tag{2.4}$$

with $\|(\psi,v)\|_{X_M^{\mu,\nu}} = \|\psi\|_{X^\mu} + \|v\|_{X^\nu}$. We show that the map Φ given by

$$\Phi(\psi,v)=\Big(U(t)\psi_0+\int_0^t U(t-s)F(s)ds,\ V_1(t)v_0+V_2(t)v_1+\int_0^t V_2(t-s)G(s)ds\Big) \tag{2.5}$$

is a contraction on $X_M^{\mu,\nu}$.

We use Strichartz's estimate for Klein–Gordon equation. We investigate the operator $K_\pm(t) = e^{\pm it(1-\Delta)^{1/2}}$, which is an essential part of $U(t), V_1(t)$ and $V_2(t)$.

Lemma 2. (see [5], [8], [9], [11], [13], [18]). *Let the space dimension $n = 3$. Then*

$$\|K_\pm(t)u\|_{L_t^{q_1}B_{r_1}^{-\sigma_1}} \lesssim \|u\|_{L^2}, \tag{2.6}$$

$$\Big\|\int_{t'<t} K_\pm(t-t')f(t')dt'\Big\|_{L_t^{q_2}B_{r_2}^{-\sigma_2}} \lesssim \|f\|_{L_t^{q_3'}B_{r_3'}^{\sigma_3}}. \tag{2.7}$$

Here $1/q_j = 1/2 - 1/r_j,\ \sigma_j = 2/q_j,\ 2 < q_j \le \infty,\ j = 1,2,3$ and p' denotes the dual exponent to p defined by $1/p + 1/p' = 1$.

We estimate (2.5) by (2.6), (2.7),

$$\|\Phi(\psi,v)\|_{X_M^{\mu,\nu}} \lesssim \|\psi_0\|_{H^\mu} + \|F\|_{L_t^1H^\mu} + \|v_0\|_{H^\nu} + \|v_1\|_{H^{\nu-1}} + \|G\|_{L_t^{q'}B_{r'}^{\nu-1+\sigma}}. \tag{2.8}$$

We employ the following lemmas for the estimate on nonlinear terms F and G.

Lemma 3. (see [8], [12]). *Let γ and s satisfy $1 \le \gamma < \infty$, $0 \le s \le \gamma$. Let p_0, p_1, p_2 satisfy $1 < p_0 \le p_1 < \infty$, $1 < p_2 \le \infty$, $1/p_0 = 1/p_1 + (2\gamma-1)/p_2$.*

(1) *The case of s is not an integer. Assume in addition that $2 \le p_1, p_2$ and $s < \gamma$. Then*

$$\|(\beta\psi|\psi)^\gamma\|_{\dot B_{p_0}^s} \lesssim \begin{cases} \|\psi\|_{\dot B_{p_1}^s}\|\psi\|_{\dot B_{p_2}^0}^{2\gamma-1} & p_2 < \infty, \\ \|\psi\|_{\dot B_{p_1}^s}(\|\psi\|_{\dot B_\infty^0} + \|\psi\|_{L^\infty})^{2\gamma-1} & p_2 = \infty. \end{cases} \tag{2.9}$$

(2) *The case of s is an integer. Then*

$$\|(\beta\psi|\psi)^\gamma\|_{\dot H_{p_0}^s} \lesssim \|\psi\|_{\dot H_{p_1}^s}\|\psi\|_{L^{p_2}}^{2\gamma-1}. \tag{2.10}$$

If γ is an integer, the conditions $s < \gamma$ in (1) and $s \le \gamma$ in (2) are superfluous.

Lemma 4. (see [10], [14]). *Let* $s \geq 0$. *Let* p_0, p_1, p_2, p_3, p_4, *satisfy* $1 < p_0, p_1, p_3 < \infty$, $1 \leq p_2$, $p_4 \leq \infty$, $1/p_0 = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$. *Then*

$$\|uv\|_{H^s_{p_0}} \lesssim \|u\|_{H^s_{p_1}} \|v\|_{L^{p_2}} + \|v\|_{H^s_{p_3}} \|u\|_{L^{p_4}}. \tag{2.11}$$

We use these lemmas for space variable norms and Hölder's inequality for time variable norms as follows

$$\begin{aligned} \|F\|_{L^1_t H^\mu} \lesssim \|v\|_{L^{q_1}_t H^\mu} \|\psi\|^{2\gamma-1}_{L^{q_2}_t L^\infty} + \|\psi\|_{L^\infty_t H^\mu} (\|\psi\|_{L^{q_3}_t L^\infty} \\ + \|\psi\|_{L^{q_3}_t \dot{B}^0_\infty})^{2\gamma-2} \|v\|_{L^{q_4}_t L^\infty}, \end{aligned} \tag{2.12}$$

with $1 = 1/q_1 + (2\gamma-1)/q_2,\ 1 = (2\gamma-2)/q_3 + 1/q_4$.

For the first term of RHS of (2.12), we estimate from the embedding theorem,

$$\begin{aligned} L^{q_1}_t B^{\nu-\sigma_1}_{r_1} &\hookrightarrow L^{q_1}_t H^\mu, \quad 3/2 - \mu \geq 3/r_1 - \nu + \sigma_1, \\ L^{q_2}_t B^{\mu-\sigma_2}_{r_2} &\hookrightarrow L^{q_2}_t L^\infty, \quad 0 > 3/r_2 - \mu + \sigma_2. \end{aligned}$$

For the second term,

$$\begin{aligned} L^{q_3}_t B^{\mu-\sigma_3}_{r_3} &\hookrightarrow L^{q_3}_t L^\infty \cap L^{q_3}_t \dot{B}^0_\infty, \quad 3/r_3 - \mu + \sigma_3 < 0, \\ L^{q_4}_t B^{\nu-\sigma_4}_{r_4} &\hookrightarrow \qquad L^{q_4}_t L^\infty, \qquad\quad 3/r_4 - \nu + \sigma_4 < 0. \end{aligned}$$

Therefore, we are restricted as follows

$$\mu > 1, \quad \nu - 1/2 > 2(\gamma-1)(3/2 - \mu). \tag{2.13}$$

We see that the assumptions of the theorem are sufficient for (2.13).

We estimate term G,

$$\|G\|_{L^{q'}_t B^{\nu-1+\sigma}_{r'}} \lesssim \|\psi\|^{2\kappa-1}_{L^{q_1}_t B^0_{p_1}} \|\psi\|_{L^{q_2}_t B^{\nu-1+\sigma}_{p_2}}, \tag{2.14}$$

with $1/q' = (2\kappa-1)/q_1 + 1/q_2,\ 1/r' = (2\kappa-1)/p_1 + 1/p_2$.

The embedding theorem implies

$$\begin{aligned} L^{q_1}_t B^{\mu-\sigma_1}_{r_1} &\hookrightarrow L^{q_1}_t B^0_{p_1}, \qquad\quad 3/p_1 \geq 3/r_1 - \mu + \sigma_1, \\ L^{q_2}_t B^{\mu-\sigma_2}_{r_2} &\hookrightarrow L^{q_2}_t B^{\nu-1+\sigma}_{p_2}, \quad 3/p_2 - \nu + 1 - \sigma \geq 3/r_2 - \mu + \sigma_2, \end{aligned}$$

which conclude the restriction

$$7/2 - \nu \geq 2\kappa(3/2 - \mu). \tag{2.15}$$

We see that the assumptions of the theorem are sufficient for (2.15). From (2.8), (2.12) and (2.14), there exists a constant C_1 such that

$$\|\Phi(\psi, v)\|_{X^{\mu,\nu}_M} \leq C_1(\|\psi_0\|_{H^\mu} + \|v_0\|_{H^\nu} + \|v_1\|_{H^{\nu-1}} + M^{2\gamma} + M^{2\kappa}). \tag{2.16}$$

Similarly we have

$$\begin{aligned}\|F(\psi_1,v_1)-F(\psi_2,v_2)\|_{X^\mu} \lesssim\ & \|v_1-v_2\|_{X^\nu}(\|\psi_1\|_{X^\mu}+\|\psi_2\|_{X^\mu})^{2\gamma-1}\\ &+\|v_2\|_{X^\nu}\|\psi_1-\psi_2\|_{X^\mu}(\|\psi_1\|_{X^\mu}+\|\psi_2\|_{X^\mu})^{2\gamma-2}, \qquad (2.17)\end{aligned}$$

$$\|G(\psi_1)-G(\psi_2)\|_{X^\nu} \lesssim\ (\|\psi_1\|_{X^\mu}+\|\psi_2\|_{X^\mu})^{2\kappa-1}\|\psi_1-\psi_2\|_{X^\mu}. \qquad (2.18)$$

Therefore, there exists a constant C_2,

$$\|\Phi(\psi_1,v_1)-\Phi(\psi_2,v_2)\|_{X_M^{\mu,\nu}} \le C_2(M^{2\gamma-1}+M^{2\kappa-1})\|(\psi_1,v_1)-(\psi_2,v_2)\|_{X_M^{\mu,\nu}}. \qquad (2.19)$$

If the norms of initial data are so small that M is chosen to satisfies

$$C_1(\|\psi_0\|_{H^\mu}+\|v_0\|_{H^\nu}+\|v_1\|_{H^{\nu-1}}+M^{2\gamma}+M^{2\kappa}) \le M,$$
$$C_2(M^{2\gamma-1}+M^{2\kappa-1}) < 1,$$

then Φ is a contraction on $X_M^{\mu,\nu}$ to have a unique fixed point. This completes the proof.

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