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EXACT CONTROLLABILITY OF THE BOUSSINESQ EQUATION ON A BOUNDED DOMAIN

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Abstract. The purpose of this article is to study the exact boundary controllability of the classical Boussinesq equation. The control is applied to the first spatial derivative and then, to the second spatial derivative, both at the right endpoint. The exact controllability of the linearized problem is essentially proved by using the Hilbert Uniqueness Method. From this result, we deduce the exact boundary controllability for the nonlinear Boussinesq equations. The main improvements compared to existing results in the literature are the absence of restrictions on controllability time, the use of more regular spaces and the extension of the exact boundary controllability to the nonlinear Boussinesq equation.

1. Introduction

Classical Boussinesq equations are often used as models for the propagation of small amplitude, long waves on the water surface. They are perturbations of the linear wave equations and were discovered by Boussinesq [3]. In particular, they were the first models to take into account nonlinear and dispersive effects of waves propagation. They are often used to predict wave elevation in harbors [13] and wave interaction in the nearshore zone [11]. Boussinesq theory was the first one to provide a scientific explanation of the phenomenon of solitary waves, discovered by Scott-Russell. In this article, we consider an equation of Boussinesq type which may serve as model for nonlinear strings, see Bona and Sachs [2], named the "good Boussinesq equation,"

$$y_{tt} - y_{xx} + y_{xxxx} + (y^2)_{xx} = 0. \tag{1.1}$$

Recent controllability results have been obtained by B.Y. Zhang for a distributed control [16]. The boundary controllability of the linear Boussinesq

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equation has been investigated by J.L. Lions in [8]. Two kinds of boundary control have been considered. When $y_x(L,t)$ is assumed to be controlled, the following result is proved in [8].

Theorem 1. *Let $T > 2L$. For every (y^0, y^1) in $L^2(0,L) \times H^{-2}(0,L)$, there exists a control v in $L^2(0,T)$ so that the solution of*

$$\begin{cases} y_{tt} - y_{xx} + y_{xxxx} = 0, \\ y(0,t) = y_x(0,t) = y(L,t) = 0,\ y_x(L,t) = v(t),\ y_{|t=0} = y^0,\ y_{t|t=0} = y^1, \end{cases}$$

satisfies $y_{|t=T} = y_{t|t=T} = 0$.

Alternatively, when $y_{xx}(L,t)$ is controlled, the following result is also provided in [8].

Theorem 2. *Let $T > 2(L + \frac{1}{2\sqrt{\lambda_0}})$, where λ_0 denotes the first eigenvalue for the operator $-\Delta$ with the Dirichlet boundary conditions. Then, for every (y^0, y^1) in $L^2(0,L) \times H^{-1}(0,L)$, there exists a control (v_1, v_2) in $L^2(0,T) \times H^1(0,T)$ so that the solution of*

$$\begin{cases} y_{tt} - y_{xx} + y_{xxxx} = 0, \\ y(0,t) = y_{xx}(0,t) = 0,\ y(L,t) = v_1(t),\ y_{xx}(L,t) = v_2(t), \\ y_{|t=0} = y^0,\ y_{t|t=0} = y^1, \end{cases}$$

satisfies $y_{|t=T} = y_{t|t=T} = 0$.

First, we prove the exact controllability of the nonlinear Boussinesq equation with control on $y_x(L,t)$, for initial data taken in $H^2(0,L) \times L^2(0,L)$ and for every time $T > 0$. Then, we prove the exact controllability of the nonlinear Boussinesq equation with control on $y_{xx}(L,t)$ with initial data taken in $H^3(0,L) \times H_0^1(0,L)$ and for every time $T > 0$. The main changes compared to Theorem 1 and Theorem 2 are the following: there is no restriction on T and L, we use more regular spaces and we prove the exact controllability of the nonlinear Boussinesq equations. The main theorems are described below. First, we introduce two new spaces,

$$H_2 = \{v \in H^2(0,L) :\ v(0) = v(L) = v_x(0) = 0\},$$
$$H_3 = \{v \in H^3(0,L) :\ v(0) = v(L) = v_{xx}(0) = 0\}.$$

Theorem 3. *Let $L > 0$ and $T > 0$. There exists $r_0 > 0$ so that, for every*

$$(y^0, y^1),\ (y_T^0, y_T^1) \text{ in } H_2 \times L^2(0,L) \tag{1.2}$$

with

$$\|(y^0, y^1)\|_{H^2(0,L)\times L^2(0,L)} < r_0 \quad \text{and} \quad \|(y_T^0, y_T^1)\|_{H^2(0,L)\times L^2(0,L)} < r_0, \tag{1.3}$$

there exist $v \in H^1(0,T)$ *and* $y \in C([0,T], H^2(0,L)) \cap C^1([0,T], L^2(0,L))$ *satisfying*

$$\begin{cases} y_{tt} = y_{xx} - (y^2 + y_{xx})_{xx}, \\ y(0,t) = y(L,t) = 0, \quad y_x(0,t) = 0, \\ y_x(L,t) = v(t), \quad y(x,0) = y^0(x) \text{ and } y_t(x,0) = y^1(x), \end{cases}$$

and so that $y(.,T) = y_T^0$ *and* $y_t(.,T) = y_T^1$.

Theorem 4. *Let* $L > 0$ *and* $T > 0$. *There exists* $r_0 > 0$ *so that, for every* (y^0, y^1), (y_T^0, y_T^1) *in* $H_3 \times H_0^1(0,L)$ *with*

$$\|(y^0, y^1)\|_{H^3(0,L)\times H_0^1(0,L)} < r_0 \text{ and } \|(y_T^0, y_T^1)\|_{H^3(0,L)\times H_0^1(0,L)} < r_0, \tag{1.4}$$

there exist $v \in H^1(0,T)$ *and* $y \in C([0,T], H^3(0,L)) \cap C^1([0,T], H_0^1(0,L))$ *satisfying*

$$\begin{cases} y_{tt} = y_{xx} - (y^2 + y_{xx})_{xx}, \\ y(0,t) = y(L,t) = 0, \quad y_{xx}(0,t) = 0, \\ y_{xx}(L,t) = v(t), \quad y(x,0) = y^0(x) \text{ and } y_t(x,0) = y^1(x) \end{cases}$$

and so that $y(.,T) = y_T^0$ *and* $y_t(.,T) = y_T^1$.

The plan of the article is as follow. In Section 2, we study the problem with control on $y_x(L,t)$. In Subsection 2.2, we first show that the result in Theorem 1 is still valid for any $T > 0$. Then we prove the exact controllability with initial data taken in $H_2 \times L^2(0,L)$. In Subsection 2.3, we demonstrate the exact controllability of the nonlinear Boussinesq equation (Theorem 3). In Section 3, we study the problem with control on $y_{xx}(L,t)$. New energy inequalities are needed to prove (in Section 3.2) the controllability of the linear equation with initial data taken in $H_3 \times H_0^1(0,L)$. Finally, we prove in Subsection 3.3 the exact controllability of the nonlinear Boussinesq equation (Theorem 4).

2. Exact controllability with control on $y_x(L,t)$

Let L be a positive number. Since we aim to apply the Hilbert Uniqueness Method to the linearized problem, we first have to study the homogeneous linear problem.

2.1. **The homogeneous Cauchy problem.** We apply the theory of compact operators to the following Cauchy problem, with (ϕ^0, ϕ^1) in $H_0^2(0,L) \times L^2(0,L)$.

$$\phi_{tt} - \phi_{xx} + \phi_{xxxx} = 0, \tag{2.1}$$

$$\phi(0,t) = \phi(L,t) = 0,\ \phi_x(0,t) = \phi_x(L,t) = 0,\ \phi(.,0) = \phi^0,\ \phi_t(.,0) = \phi^1. \tag{2.2}$$

Let $V_0 = H_0^2(0,L)$ and $\|v\|_{V_0} = \|v_{xx}\|_{L^2(0,L)}$. We define the operator A as $A\phi = -\phi_{xx} + \phi_{xxxx}$, with domain $D(A) = \{\phi \in V_0;\ A\phi \in L^2(0,L)\} \subset L^2(0,L)$; that is, $D(A) = \{\phi \in H^4(0,L), \phi(0) = \phi(L) = \phi_x(0) = \phi_x(L) = 0\}$. It is easy to check that A is a closed operator with dense domain in $L^2(0,L)$. Let $K = A^{-1}$. It is obvious that K is a (bounded) selfadjoint and compact operator from $L^2(0,L)$ into itself. We then deduce that $L^2(0,L)$ has an orthonormal basis $(\phi_n)_{n\in\mathbb{N}^*}$ of eigenvectors of K, i.e., for each $n \geq 1$, $K\phi_n = \mu_n\phi_n$ for some real number μ_n. We have $\mu_n > 0$, $\mu_{n+1} \leq \mu_n$ and $\mu_n \to 0$ as $n \to +\infty$. Therefore, setting $\lambda_n = 1/\mu_n$, we have $A\phi_n = \lambda_n\phi_n$ and $\lambda_n \to +\infty$ as $n \to +\infty$. We then deduce, by the variational theory ([9], Chapter 3, Theorem 8.2), that for any $(\phi^0,\phi^1) \in H_0^2(0,L) \times L^2(0,L)$ the homogeneous problem (2.1)-(2.2) has a unique solution $\phi \in C(\mathbb{R}; H_0^2(0,L)) \cap C^1(\mathbb{R}; L^2(0,L))$, and the solution depends continuously on the initial data (ϕ^0,ϕ^1) for the usual topology on $H_0^2(0,L) \times L^2(0,L)$. More precisely, we have the following result:

Proposition 1. *For every $(\phi^0,\phi^1) \in H_0^2(0,L)\times L^2(0,L)$, the solution of the homogeneous problem* (2.1)-(2.2) *is decomposed in Fourier series as*

$$\phi(x,t) = \sum_{k=1}^{+\infty}(\phi_k^0\cos(\omega_k t) + \frac{\phi_k^1}{\omega_k}\sin(\omega_k t))\phi_k,\ \ \phi_k^0 = (\phi^0,\phi_k)_{L^2},\ \phi_k^1 = (\phi^1,\phi_k)_{L^2},$$

with $\omega_k = \sqrt{\lambda_k}$.

2.2. **Observability of the homogeneous problem and controllability of the linear equation.** In this section, we first apply the classical Hilbert Uniqueness Method to prove that the result in Theorem 1 is valid for every $T > 0$. Then, we apply a method due to José Urquiza (see [14]) to prove the exact controllability of the linear Boussinesq problem in $H_2 \times L^2(0,L)$. We first study the sequences of eigenvalues and eigenvectors of A, in order to obtain an observability inequality.

It is easy to check that the eigenvectors of A may be written as

$$\begin{aligned}\phi(x) &= a\cos\big(\alpha(x-\frac{L}{2})\big) + b\sin\big(\alpha(x-\frac{L}{2})\big)\\ &\quad + c\cosh\big(\beta(x-\frac{L}{2})\big) + d\sinh\big(\beta(x-\frac{L}{2})\big),\\ \phi(0) &= \phi(L) = \phi_x(0) = \phi_x(L) = 0,\end{aligned}$$

with

$$\alpha = \sqrt{\frac{\sqrt{1+4\lambda}-1}{2}} \quad \text{and} \quad \beta = \sqrt{\frac{\sqrt{1+4\lambda}+1}{2}}, \tag{2.3}$$

λ being the corresponding eigenvalue. After some computations, we get the two following sequences of eigenvalues $(\lambda_{1,n})_{n\geq 1}$ and $(\lambda_{2,n})_{n\geq 0}$.

- $(\lambda_{1,n})_{n\geq 1}$ corresponds to the positive solutions of

$$\alpha \tan\big(\frac{\alpha L}{2}\big) = -\beta \tanh\big(\frac{\beta L}{2}\big) \tag{2.4}$$

and the associated eigenvectors take the form

$$\phi_{1,n}(x) = f_{1,n}\big[\cosh\big(\frac{\beta L}{2}\big)\cos\big(\alpha(x-\frac{L}{2})\big) - \cos\big(\frac{\alpha L}{2}\big)\cosh\big(\beta(x-\frac{L}{2})\big)\big]. \tag{2.5}$$

The coefficients $(f_{1,n})$ are chosen in such a way that the $\phi_{1,n}$'s are normalized in $L^2(0,L)$.

The proof of the following lemma is left to the reader.

Lemma 1. *For every $n \geq 1$, (2.4) has a unique solution $\alpha_{1,n}$ so that*

$$\frac{\alpha_{1,n}L}{2} \in \big(-\frac{\pi}{2}+n\pi, \frac{\pi}{2}+n\pi\big).$$

Notice that for $n = 0$, the solution $\alpha_{1,0}$ would be negative, which is excluded (see (2.3)).

- $(\lambda_{2,n})_{n\geq 0}$ corresponds to the positive solutions of

$$\beta \tan\big(\frac{\alpha L}{2}\big) = \alpha \tanh\big(\frac{\beta L}{2}\big) \tag{2.6}$$

and the associated eigenvectors take the form

$$\phi_{2,n}(x) = f_{2,n}\big[\sinh\big(\frac{\beta L}{2}\big)\sin\big(\alpha(x-\frac{L}{2})\big) - \sin\big(\frac{\alpha L}{2}\big)\sinh\big(\beta(x-\frac{L}{2})\big)\big]. \tag{2.7}$$

The coefficients $(f_{2,n})$ are chosen in such a way that the $\phi_{2,n}$'s are normalized in $L^2(0,L)$.

Lemma 2. *For every $n \geq 1$, (2.6) has a unique solution $\alpha_{2,n}$ so that $\frac{\alpha_{2,n}L}{2} \in \big(-\frac{\pi}{2}+n\pi, \frac{\pi}{2}+n\pi\big)$,*

Notice that for $n = 0$, the solution would be 0, which is excluded ($\lambda_n = 1/\mu_n$).

Proof. Let $L > 0$, we aim to prove that

$$f : \alpha \longmapsto \sqrt{\alpha^2+1}\tan\big(\frac{\alpha L}{2}\big) - \alpha \tanh\big(\frac{\sqrt{\alpha^2+1}L}{2}\big)$$

is increasing as long as $\tan\left(\frac{\alpha L}{2}\right) > 0$.

$$\begin{aligned} f'(\alpha) =& \frac{\alpha}{\sqrt{\alpha^2+1}}\tan\big(\frac{\alpha L}{2}\big) + \frac{L}{2}\sqrt{\alpha^2+1}\big(1+\tan^2(\frac{\alpha L}{2})\big) \\ & - \tanh\big(\frac{\sqrt{\alpha^2+1}L}{2}\big) - \frac{L}{2}\frac{\alpha^2}{\sqrt{\alpha^2+1}}\big(1-\tanh^2(\frac{\sqrt{\alpha^2+1}L}{2})\big). \end{aligned}$$

As $\tan(\frac{\alpha L}{2}) > 0$,

$$\begin{aligned} & f'(\alpha) \geq \frac{L}{2}\sqrt{\alpha^2+1} + \frac{L}{2}\frac{\alpha^2}{\sqrt{\alpha^2+1}}\tanh^2\big(\frac{\sqrt{\alpha^2+1}L}{2}\big) \\ & - \frac{L}{2}\frac{\alpha^2}{\sqrt{\alpha^2+1}} - \tanh\big(\frac{\sqrt{\alpha^2+1}L}{2}\big) \\ & \geq \frac{L}{2\sqrt{\alpha^2+1}}\Big(1+\alpha^2\tanh^2\big(\frac{\sqrt{\alpha^2+1}L}{2}\big) - \frac{2\sqrt{\alpha^2+1}}{L}\tanh\big(\frac{\sqrt{\alpha^2+1}L}{2}\big)\Big) \\ & \geq \frac{L}{2\sqrt{\alpha^2+1}}\big[1-\tanh^2\big(\frac{\sqrt{\alpha^2+1}L}{2}\big)\big] \\ & + \frac{2}{L\sqrt{\alpha^2+1}}\Big[\frac{L^2}{4}(\alpha^2+1)\tanh^2\big(\frac{\sqrt{\alpha^2+1}L}{2}\big) - \frac{L}{2}\sqrt{\alpha^2+1}\tanh\big(\frac{\sqrt{\alpha^2+1}L}{2}\big)\Big]. \end{aligned}$$

But $1-\tanh^2(\frac{\sqrt{\alpha^2+1}L}{2}) > 0$ for every $\alpha \geq 0$. Let $y = \frac{L}{2}\sqrt{\alpha^2+1}$, then

$$\begin{aligned} & \frac{L^2}{4}(\alpha^2+1)\tanh^2\big(\frac{\sqrt{\alpha^2+1}L}{2}\big) \\ & \qquad - \frac{L}{2}\sqrt{\alpha^2+1}\tanh\big(\frac{\sqrt{\alpha^2+1}L}{2}\big) = y^2\tanh^2(y) - y\tanh(y). \end{aligned}$$

For $y > 1.2$, $y^2\tanh^2(y) - y\tanh(y) > 0$; hence, f is increasing on each interval where $\tan(\frac{\alpha L}{2}) > 0$ and $\frac{\alpha L}{2} > 1.2$.

If $0 < \frac{\alpha L}{2} < \frac{\pi}{2}$, then $\tan(\frac{\alpha L}{2}) \geq \frac{\alpha L}{2}$, $\tanh(\frac{\sqrt{\alpha^2+1}L}{2}) < \frac{\sqrt{\alpha^2+1}L}{2}$. Thus,

$$\frac{\sqrt{\alpha^2+1}L}{2}\tan(\frac{\alpha L}{2}) - \frac{\alpha L}{2}\tanh(\frac{\sqrt{\alpha^2+1}L}{2}) > 0,$$

which completes the proof of Lemma 2.

We study in more details both sequences of eigenvalues and the associated eigenvectors.

Lemma 3. *For every $n \in \mathbb{N}^*$, $\lambda_{1,n} < \lambda_{2,n}$.*

Proof. We first notice that for every $n > 0$,

$$\lambda_{1,n}, \lambda_{2,n} \in \big(\frac{[2(-\frac{\pi}{L}+2n\frac{\pi}{L})^2+1]^2-1}{4}, \frac{[2(\frac{\pi}{L}+2n\frac{\pi}{L})^2+1]^2-1}{4}\big).$$

Let $n \in \mathbb{N}^*$ and α, β be the constants associated with $\lambda_{2,n}$, $\tan(\frac{\alpha L}{2}) = \frac{\alpha}{\beta}\tanh(\frac{\beta L}{2})$. Hence, $\alpha\tan(\frac{\alpha L}{2}) + \beta\tanh(\frac{\beta L}{2}) = \frac{\alpha^2+\beta^2}{\beta}\tanh(\frac{\beta L}{2}) > 0$. As the function $\lambda \longmapsto \alpha\tan(\frac{\alpha L}{2}) + \beta\tanh(\frac{\beta L}{2})$ is increasing on the interval in question, we can conclude that $\lambda_{1,n} < \lambda_{2,n}$.

We bring together the two sequences in one and only one by letting, $\lambda_{2k-1} = \lambda_{1,k}$, $\lambda_{2k} = \lambda_{2,k}$, for any $k \geq 1$ and by letting $\phi_{2k-1} = \phi_{1,k}$, $\phi_{2k} = \phi_{2,k}$, for any $k \geq 1$.

Let $n \in \mathbb{N}^*$, we denote by α_n and β_n the two numbers associated with λ_n according (2.3). First of all, we observe that $\lim_{n\to+\infty} \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} = +\infty$. Indeed, we already know that $\lim_{n\to+\infty} \lambda_n = +\infty$, hence, $\lim_{n\to+\infty} \tanh(\frac{\beta_n L}{2}) = 1$. But $\lim_{n\to+\infty} \frac{\alpha_n}{\beta_n} = 1$, so $\lim_{k\to+\infty} \tan(\frac{\alpha_{2k}L}{2}) = 1$ and $\lim_{k\to+\infty} \tan(\frac{\alpha_{2k-1}L}{2}) = -1$. We deduce

$$\lim_{k\to+\infty} \frac{\alpha_{2k}L}{2} - \big(\frac{\pi}{4} + k\pi\big) = 0, \quad \lim_{k\to+\infty} \frac{\alpha_{2k+1}L}{2} - \big(\frac{3\pi}{4} + k\pi\big) = 0,$$

Thus

$$\lim_{n\to+\infty} \sqrt{\lambda_{n+1}} - \sqrt{\lambda_n} = +\infty. \tag{2.8}$$

Moreover, we have

Lemma 4. *For every $n \in \mathbb{N}^*$, $\phi_{nxx}(L) \neq 0$, and $\lim_{n\to+\infty} \big|\frac{\phi_{nxx}(L)}{\omega_n}\big| = \frac{2}{\sqrt{L}}$ with $\omega_n = \sqrt{\lambda_n}$.*

Proof. Let k in $\mathbb{N}^*$,

$$(\phi_{2k})_{xx}(L) = -f_{2,k}(\alpha_{2,k}^2 + \beta_{2,k}^2)\sinh\big(\frac{\beta_{2,k}L}{2}\big)\sin\big(\frac{\alpha_{2,k}L}{2}\big).$$

Thanks to equation (2.6), $(\phi_{2k})_{xx}(L) \neq 0$. On the other hand,

$$(\phi_{2k-1})_{xx}(L) = -f_{1,k}(\alpha_{1,k}^2 + \beta_{1,k}^2)\cosh\big(\frac{\beta_{1,k}L}{2}\big)\cos\big(\frac{\alpha_{1,k}L}{2}\big),$$

hence, thanks to equation (2.4), we get $(\phi_{2k-1})_{xx}(L) \neq 0$. By integration, we get

$$f_{2k}^2 = \frac{1}{\sinh^2(\frac{\beta L}{2})(\frac{L}{2} - \frac{1}{\alpha}\sin(\frac{\alpha L}{2})\cos(\frac{\alpha L}{2})) + \sin^2(\frac{\alpha L}{2})(\frac{1}{\beta}\sinh(\frac{\beta L}{2})\cosh(\frac{\beta L}{2}) - \frac{L}{2})},$$

$$f_{2k-1}^2 = \frac{1}{\cosh^2(\frac{\beta L}{2})(\frac{L}{2}+\frac{1}{\alpha}\sin(\frac{\alpha L}{2})\cos(\frac{\alpha L}{2}))+\cos^2(\frac{\alpha L}{2})(\frac{1}{\beta}\sinh(\frac{\beta L}{2})\cosh(\frac{\beta L}{2})+\frac{L}{2})},$$

so we easily get Lemma 4.

Remark 1. $\sum_{k=1}^{+\infty}(\omega_k\phi_k^0)^2$ is the square of a norm equivalent to $\|\phi^0\|_{H^2(0,L)}$, $\sum_{k=1}^{+\infty}(\phi_k^1)^2 = \|\phi^1\|^2_{L^2(0,L)}$.

We can apply the following Ingham inequality (see for example [4], [1]) to our problem.

Lemma 5. *Let $(\mu_k) \in \mathbb{R}^{\mathbb{Z}}$ be a sequence of pairwise distinct real numbers so that $\lim_{|k|\to\infty}(\mu_{k+1}-\mu_k) = +\infty$ and let $T > 0$. Then there exist two strictly positive constants c_1 and c_2 so that for every sequence $(a_k)_k \in l^2(\mathbb{Z})$, the series $f(t) = \sum_{-\infty}^{+\infty} a_k e^{i\mu_k t}$ converges in $L^2(0,T)$ and it satisfies*

$$c_1 \sum_{k=-\infty}^{+\infty}|a_k|^2 \le \int_0^T |\sum_{k=-\infty}^{+\infty} a_k e^{i\mu_k t}|^2 dt \le c_2 \sum_{k=-\infty}^{+\infty}|a_k|^2.$$

Let $(\phi_0,\phi_1) \in V_0 \times L^2(0,L)$ and ϕ be the solution of (2.1)-(2.2). We apply Lemma 5 to $f(t) = \phi_{xx}(L,t)+\alpha$, with α real, $a_0 = \alpha$, $\mu_0 = 0$ and for $k \ge 1$

$$\mu_k = -\mu_{-k} = \omega_k, \quad 2a_k = 2\overline{a_{-k}} = (\phi_k^0 - i\frac{\phi_k^1}{\omega_k})(\phi_{kxx}(L)).$$

Thanks to (2.8), we get for every $T > 0$ two constants c and d so that

$$c\Big(|\alpha|^2 + \sum_{k=1}^{+\infty}\big[(\phi_k^0)^2 + (\frac{\phi_k^1}{\omega_k})^2\big]\phi_{kxx}(L)^2\Big) \le \int_0^T |f(t)|^2 dt$$
$$\le d\Big(|\alpha|^2 + \sum_{k=1}^{+\infty}\big[(\phi_k^0)^2 + (\frac{\phi_k^1}{\omega_k})^2\big]\phi_{kxx}^2(L)\Big).$$

Thanks to Lemma 4, there exist two constants $\bar{c_1}, \bar{c_2} > 0$ so that

$$\bar{c_1}\Big(\alpha^2 + \sum_{k=1}^{+\infty}(\omega_k\phi_k^0)^2 + (\phi_k^1)^2\Big) \le \int_0^T |f(t)|^2 dt \le \bar{c_2}\Big(\alpha^2 + \sum_{k=1}^{+\infty}(\omega_k\phi_k^0)^2 + (\phi_k^1)^2\Big).$$

So using Remark 1, we get the following observability result.

Proposition 2. *Let $T > 0$. There exist two strictly positive constants $\bar{c_1}$ and $\bar{c_2}$ so that, $\forall(\phi^0,\phi^1) \in H_0^2(0,L)\times L^2(0,L)$, $\forall\alpha \in \mathbb{R}$,*

$$\bar{c_1}\left(\|(\phi^0,\phi^1)\|^2_{V_0\times L^2(0,L)} + \alpha^2\right) \le \int_0^T (\phi_{xx}(L,t)+\alpha)^2 dt$$

$$\leq \bar{c}_2(\|(\phi^0,\phi^1)\|^2_{V_0\times L^2(0,L)} + \alpha^2).$$

On the other hand, we have the following proposition (see also Komornik [5], Theorem 2.5),

Proposition 3. *Let v in $L^2(0,T)$, (y^0,y^1) in $L^2(0,L)\times H^{-2}(0,L)$. Then the nonhomogeneous problem*

$$y_{tt} - y_{xx} + y_{xxxx} = 0, \tag{2.9}$$

$$y(0,t) = y(L,t) = 0,\ y_x(0,t) = 0,\ y_x(L,t) = v(t),\ y(.,0) = y^0,\ y_t(.,0) = y^1, \tag{2.10}$$

has a unique solution y (defined by transposition) fulfilling

$$y \in C([0,T];L^2(0,L)) \cap C^1([0,T];H^{-2}(0,L)). \tag{2.11}$$

Proof. Suppose that the solution y of (2.9)-(2.10) is regular enough. Let ϕ be a solution of the homogeneous problem (2.1)-(2.2), with $(\phi^0,\phi^1) \in H_0^2(0,L)\times L^2(0,L)$. Multiplying (2.1) by y and integrating by parts, we get for any $S\in[0,T]$

$$\int_0^L (\phi_t(x,S)y(x,S) - \phi(x,S)y_t(x,S))dx \tag{2.12}$$
$$= \int_0^S \phi_{xx}(L,t)v(t) + \langle(-y^1,y^0),(\phi^0,\phi^1)\rangle_{H^{-2}(0,L)\times L^2(0,L),H_0^2(0,L)\times L^2(0,L)}.$$

Let

$$L_S(\phi(.,S),\phi_t(.,S))$$
$$= \int_0^S \phi_{xx}(L,t)v(t) + \langle(-y^1,y^0),(\phi^0,\phi^1)\rangle_{H^{-2}(0,L)\times L^2(0,L),H_0^2(0,L)\times L^2(0,L)}.$$

The linear map, $(\phi(.,S),\phi_t(.,S)) \longmapsto (\phi^0,\phi^1)$ is an isomorphism from $H_0^2(0,L)\times L^2(0,L)$ into itself. Hence, we deduce from Proposition 2 that $(\phi(.,S),\phi_t(.,S))\longmapsto L_S(\phi(.,S),\phi_t(.,S))$ is continuous on $H_0^2(0,L)\times L^2(0,L)$. Therefore, for any $S\in[0,T]$, equation (2.12) defines $(y(.,S),y_t(.,S))$ as a unique element in $L^2(0,L)\times H^{-2}(0,L)$. We conclude as usual that if $(y^0,y^1)\in L^2(0,L)\times V_0'$ and $v\in C^\infty([0,T])$, then

$$y \in C([0,T];L^2(0,L)) \cap C^1([0,T];H^{-2}(0,L)).$$

The proof of Proposition 3 is completed by a density argument.

The result of controllability is the following one.

Proposition 4. *Let $T > 0$ and $(y^0, y^1) \in L^2(0, L) \times H^{-2}(0, L)$. Then there exists a control $v \in L^2(0, T)$ so that the solution of* (2.9)-(2.10) *satisfies $y(x, T) = y_t(x, T) = 0$.*

Proof. We apply the Hilbert Uniqueness Method due to J.L.Lions [7]. We consider the map

$$\Lambda_{0,T} : V_0 \times L^2(0, L) \to V_0' \times L^2(0, L), \quad (\phi^0, \phi^1) \longmapsto (\psi_t(.,0), -\psi(.,0)),$$

where ϕ is the solution of the homogeneous problem (2.1)-(2.2), and ψ is the solution of the backward problem

$$\psi_{tt} - \psi_{xx} + \psi_{xxxx} = 0, \tag{2.13}$$

$$\psi(0,t) = \psi(L,t) = \psi_x(0,t) = 0,\ \psi_x(L,t) = \phi_{xx}(L,t),\ \psi(x,T) = \psi_t(x,T) = 0. \tag{2.14}$$

Thanks to the time reversibility of (2.13)-(2.14) and Proposition 3, $\Lambda_{0,T}$ is a well-defined continuous map from $F = H_0^2(0, L) \times L^2(0, L)$ into its dual F'. On the other hand,

$$\langle \Lambda_{0,T}(\phi^0, \phi^1), (\phi^0, \phi^1) \rangle_{F',F} = \int_0^T (\phi_{xx}(L,t))^2 dt.$$

Thanks to Proposition 2, this expression is the square of a norm on F equivalent to the norm considered. Consequently, $\Lambda_{0,T}$ is an isomorphism from F onto F', and we conclude that the linear Boussinesq equation is exactly controllable in $L^2(0, L) \times H^{-2}(0, L)$.

We now prove the exact controllability of the linear Boussinesq equation with initial data taken in $H_2 \times L^2(0, L)$ and with a more regular control ($H^1(0, T)$ instead of $L^2(0, T)$), by means of a method due to José Urquiza [14].

Proposition 5. *Let $T > 0$ and $(y^0, y^1) \in H_2 \times L^2(0, L)$. Then there exists a control $v \in H^1(0, T)$ so that the solution of* (2.9)-(2.10) *satisfies $y(x, T) = y_t(x, T) = 0$.*

Proof. Given initial data $(y^0, y^1) \in H_2 \times L^2(0, L)$ and a function $v \in H^1(0, T)$, with $v(0) = y_x^0(L)$, we look at the following problem

$$y_{tt} - y_{xx} + y_{xxxx} = 0, \tag{2.15}$$

$$y(0,.) = y(L,.) = 0,\ y_x(0,.) = 0,\ y_x(L,.) = v,\ y(.,0) = y^0,\ y_t(.,0) = y^1. \tag{2.16}$$

and we search v in such a way that $y(.,T) = y_t(.,T) = 0$. Letting $y_t = z$, we get the following problem,

$$z_{tt} - z_{xx} + z_{xxxx} = 0, \tag{2.17}$$

$$z(0,.) = z(L,.) = 0,\ z_x(0,.) = 0,\ z_x(L,.) = u,\ z(.,0) = z^0,\ z_t(.,0) = z^1, \tag{2.18}$$

where $u = v_t \in L^2(0,T)$, and $(z^0, z^1) = (y^1, -Ay^0) \in L^2(0,L) \times H^{-2}(0,L)$ (recall that A is defined by $Ay = -y_{xx} + y_{xxxx}$, see Subsection 2.1). According to Proposition 3, $y_t = z \in C([0,T]; L^2(0,L))$, $z_t \in C([0,T]; H^{-2}(0,L))$ and y is a solution of the following elliptic problem, with t viewed as a parameter,

$$-y_{xx} + y_{xxxx} = -z_t, \tag{2.19}$$

$$y(0,t) = y(L,t) = 0,\ y_x(0,t) = 0,\ y_x(L,t) = v(t). \tag{2.20}$$

Thanks to the continuity of the data in relation to t (namely, z_t and v), we deduce that $y \in C([0,T], H_2) \cap C^1([0,T]; L^2(0,L))$. The problem is to find a function $u \in L^2(0,T)$ so that the solution (z, v) of

$$z_{tt} - z_{xx} + z_{xxxx} = 0, \tag{2.21}$$

$$z(0,t) = z(L,t) = 0,\ z_x(0,t) = 0,\ z_x(L,t) = u(t),\ z(.,0) = z^0,\ z_t(.,0) = z^1,$$

$$v_t(t) = u(t),\ v(0) = y_x^0(L), \tag{2.22}$$

satisfies $z(.,T) = z_t(.,T) = 0$ and $v(T) = 0$. We look at the application Λ : $V_0 \times L^2(0,L) \times \mathbb{R} \to V_0' \times L^2(0,L) \times \mathbb{R}$, $(\phi^0, \phi^1, \alpha^0) \mapsto (\psi_t(.,0), -\psi(.,0), -\omega(0))$, where (ϕ, α) is the solution of the homogeneous problem

$$\begin{cases} \phi_{tt} - \phi_{xx} + \phi_{xxxx} = 0, \\ \phi(0,t) = \phi(L,t) = 0,\ \phi_x(0,t) = 0, \\ \phi_x(L,t) = 0,\ \phi(.,0) = \phi^0,\ \phi_t(.,0) = \phi^1, \end{cases}$$

$$\begin{cases} \alpha_t(t) = 0, \\ \alpha(0) = \alpha^0, \end{cases}$$

and (ψ, ω) is the solution of the backward problem

$$\begin{cases} \psi_{tt} - \psi_{xx} + \psi_{xxxx} = 0, \\ \psi(0,t) = \psi(L,t) = 0,\ \psi_x(0,t) = 0, \\ \psi_x(L,t) = \phi_{xx}(L,t) + \alpha(t),\ \psi(.,T) = 0,\ \psi_t(.,T) = 0, \end{cases}$$

$$\begin{cases} \omega_t = \phi_{xx}(L,t) + \alpha(t), \\ \omega(T) = 0. \end{cases}$$

Λ is a linear continuous operator from $F = H_0^2(0,L) \times L^2(0,L) \times \mathbb{R}$ into its dual space. As

$$\langle \Lambda(\phi^0, \phi^1, \alpha^0), (\phi^0, \phi^1, \alpha^0) \rangle_{F',F} = \int_0^T (\phi_{xx}(L,t) + \alpha^0)^2 dt,$$

we deduce from Proposition 2 that it is an isomorphism.

For every initial conditions $(y^0, y^1) \in H_2 \times L^2(0,L)$, let $(z^0, z^1) = (y^1, -Ay^0) \in L^2(0,L) \times H^{-2}(0,L)$, $(\phi^0, \phi^1, \alpha) = \Lambda^{-1}(z^1, -z^0, -y_x^0(L))$, $u(t) = \phi_{xx}(L,t) + \alpha^0$ and $v(t) = \int_0^t u(s)ds + y_x^0(L)$. The solution y of (2.19)-(2.20) satisfies (2.15)-(2.16) and the control v belongs to $H^1(0,T)$ and satisfies $v(0) = y_x^0(L)$ and $v(T) = 0$.

2.3. **Exact controllability of the nonlinear equation.** We now prove Theorem 3, i.e., the fact that the following boundary-control system is exactly controllable in a neighborhood of the null state

$$y_{tt} - y_{xx} + y_{xxxx} + (y^2)_{xx} = 0, \tag{2.23}$$

$$y(0,.) = y(L,.) = 0,\ y_x(0,.) = 0,\ y_x(L,.) = v, \tag{2.24}$$

$$y(.,0) = y^0, y_t(.,0) = y^1,\ y(.,T) = y_T^0,\ y_t(.,T) = y_T^1, \tag{2.25}$$

with (y^0, y^1), $(y_T^0, y_T^1) \in H_2 \times L^2(0,L)$ and $v \in H^1(0,T)$.

The strategy is as usual to find a good Hilbert space, and to apply the Banach fixed point theorem in order to find a solution for the nonlinear problem (see for example Rosier [12], Section 4).

Let $y^0, y^1 \in H_2 \times L^2(0,L)$, $v \in H^1(0,T)$ and $f \in L^1(0,T; L^2(0,L))$. In order to solve (2.23)-(2.25), we write $y = S^0(.)(y^0, y^1) + y_1 + y_2$, where $S = (S^0, S^1)$ is the unitary group associated with the problem (2.1)-(2.2) of Section 2.1, and y_1, y_2 are defined by

$$y_{1tt} - y_{1xx} + y_{1xxxx} = 0, \tag{2.26}$$

$$y_1(0,t) = y_1(L,t) = 0,\ y_{1x}(0,t) = 0, \tag{2.27}$$

$$y_{1x}(L,.) = v,\ y_1(.,0) = 0,\ y_{1t}(.,0) = 0, \tag{2.28}$$

$$y_{2tt} - y_{2xx} + y_{2xxxx} = f, \tag{2.29}$$

$$y_2(0,t) = y_2(L,t) = 0,\ y_{2x}(0,t) = y_{2x}(L,t) = 0, \tag{2.30}$$

$$y_2(.,0) = 0,\ y_{2t}(.,0) = 0. \tag{2.31}$$

Proposition 6. *The map*

$$\phi \in L^2(0,T,H^2(0,L)) \longmapsto (\phi^2)_{xx} \in L^1(0,T,L^2(0,L)$$

is well-defined and continuous.

Proof. Let $y, z \in L^2(0,T,H^2(0,L))$,

$$\|(y^2)_{xx} - (z^2)_{xx}\|_{L^1(0,T,L^2(0,L))} = 2\int_0^T \|yy_{xx} + (y_x)^2 - zz_{xx} - (z_x)^2\|_{L^2(0,L)}dt$$
$$\le 2\int_0^T \|yy_{xx} - zz_{xx}\|_{L^2(0,L)}dt + 2\int_0^T \|(y_x)^2 - (z_x)^2\|_{L^2(0,L)}dt.$$

We have

$$\int_0^T \|yy_{xx} - zz_{xx}\|_{L^2(0,L)}dt \le \int_0^T \|(y-z)y_{xx}\|_{L^2(0,L)} + \|z(y_{xx} - z_{xx})\|_{L^2(0,L)},$$
$$\le \int_0^T \|y-z\|_{L^\infty}\|y_{xx}\|_{L^2(0,L)} + \|z\|_{L^\infty}\|y_{xx} - z_{xx}\|_{L^2(0,L)}, \tag{2.32}$$
$$\le C_2(\int_0^T \|y-z\|_{H^2(0,L)}\|y\|_{H^2(0,L)} + \int_0^T \|z\|_{H^2(0,L)}\|y-z\|_{H^2(0,L)}),$$
$$\le C_2\left(\|y\|_{L^2(0,T,H^2(0,L))} + \|z\|_{L^2(0,T,H^2(0,L))}\right)\|y-z\|_{L^2(0,T,H^2(0,L))}.$$

Hence, with $z = 0$, we get $yy_{xx} \in L^1(0,T,L^2(0,L))$ and letting $z \to y$, we readily get the continuity of the map $y \mapsto yy_{xx}$. On the other hand,

$$\int_0^T \|(y_x)^2 - (z_x)^2\|_{L^2(0,L)} = \int_0^T \|(y_x - z_x)(y_x + z_x)\|_{L^2(0,L)}, \tag{2.33}$$
$$\le \int_0^T \|y_x - z_x\|_{L^2(0,L)}\|y_x + z_x\|_{L^2(0,L)},$$
$$\le \int_0^T \|y-z\|_{H^2(0,L)}\|y+z\|_{H^2(0,L)},$$
$$\le \|y-z\|_{L^2(0,T,H^2(0,L))}\|y+z\|_{L^2(0,T,H^2(0,L))}.$$

Thus, $(y_x)^2 \in L^1(0,T,L^2(0,L))$ and the map $y \mapsto (y_x)^2$ is continuous. So we get the continuity of the map $\phi \in L^2(0,T,H^2(0,L)) \mapsto (\phi^2)_{xx} \in L^1(0,T,L^2(0,L))$.

We now study the solutions y_1 and y_2 designed above. Let $\psi_1 : v \in H^1(0,T) \longmapsto y_1 \in C([0,T],H_2) \subset C([0,T],L^2(0,L)) \cap L^2(0,T,H^2(0,L))$ be the map which associates with v the weak solution of (2.26)-(2.28). The map ψ_1 is linear and continuous.

Proposition 7. *For $f \in L^1(0,T,L^2(0,L))$ the mild solution y_2 of* (2.29)-(2.31) *fulfills $(y_2, y_{2t}) \in C^0([0,T],H_0^2(0,L) \times L^2(0,L))$. Moreover, the map $\psi_2 : f \longmapsto (y_2, y_{2t})$ is continuous.*

We now introduce a map whose the fixed point is the solution of the problem. Assume that $y^0 = y^1 = 0$. Using Proposition 5 and the reversibility of (2.15)-(2.16) we obtain a continuous map $\Gamma : (y_T^0, y_T^1) \in H_2 \times L^2(0,L) \mapsto v \in H^1(0,T)$ so that the solution y of (2.15)-(2.16) with $y^0 = y^1 = 0$ fulfills $y(.,T) = y_T^0$ and $y_t(.,T) = y_T^1$. Let F be the following map: $F : y \in L^2(0,T,H^2(0,L)) \mapsto F(y)$ with

$$\begin{aligned} F(y) := S^0(.)(y^0,y^1) + \psi_1 \circ \Gamma\Big(&(y_T^0, y_T^1) - S(T)(y^0,y^1) \\ &+ (\psi_2((y^2)_{xx})(.,T), (\psi_2((y^2)_{xx}))_t(.,T)\Big) + \psi_2(-(y^2)_{xx}), \end{aligned}$$

$F(y) \in L^2(0,T,H^2(0,L))$. F is well defined and continuous and each fixed point y of F satisfies the nonlinear equation (2.23)-(2.25) and $(y(T,.), y_t(T,.)) = (y_T^0, y_T^1)$. For proving the existence of some fixed point for F, we apply the Banach fixed point theorem to the restriction of F to some closed ball $\bar{B}(0,R)$ in $L^2(0,T,H^2(0,L))$. We need to show that

$$F(\bar{B}(0,R)) \subset \bar{B}(0,R) \tag{2.34}$$

and $\exists C_3 \in (0,1), \forall y,z \in \bar{B}(0,R), |||F(y) - F(z)||| \leq C_3|||y-z|||$, where $|||.|||$ denotes the norm in $L^2(0,T,H^2(0,L))$. Let $\|.\|$ denote the norm in $H^2(0,L) \times L^2(0,L)$. Let K_1, (resp. K_2 ,K_2') denote the norm of ψ_1 (resp. ψ_2, ψ_2) as a map from $H^1(0,T)$ (resp. $L^1(0,T,L^2(0,L))$) into $L^2(0,T,H^2(0,L))$ (resp. $L^2(0,T,H^2(0,L))$, $C([0,T],H_0^2(0,L) \times L^2(0,L))$). And let K denote the norm of Γ as a map from $H_2 \times L^2(0,L)$ into $H^1(0,T)$.

Assume that $\|(y^0,y^1)\| \leq r$, $\|(y_T^0,y_T^1)\| \leq r$, (r will be precised afterwards). Let $y,z \in L^2(0,T,H^2(0,L))$, and suppose $|||y||| \leq R$ and $|||z||| \leq R$. Then, by (2.33)

$$\begin{aligned} &|||F(y)||| \leq C_0\|(y_0,y_1)\| + KK_1(\|(y_T^0,y_T^1)\| + \|(y^0,y^1)\| \\ &+ K_2'C_1\|y\|^2) + K_2C_1\|y\|^2 \leq C_0r + 2K_1Kr + (K_1KK_2'C_1 + K_2C_1)R^2, \end{aligned}$$

where r is a real so that $\|(y_T^0,y_T^1)\| \leq r$ and $\|(y^0,y^1)\| \leq r$ and $C_1 = 2(1+C_2)$ with C_2 defined in (2.32). Hence, (2.34) holds if $(C_0+2K_1K)r+(K_1KK_2'C_1+K_2C_1)R^2 \leq R$. But,

$$\begin{aligned} F(y) - F(z) = \psi_1 \circ \Gamma(&\psi_2((y^2)_{xx} - (z^2)_{xx})(T,.), \psi_{2t}((y^2)_{xx} - (z^2)_{xx})(T,.)) \\ &+ \psi_2((z^2)_{xx} - (y^2)_{xx}). \end{aligned}$$

Hence,

$$\begin{aligned} |||F(y) - F(z)||| &\leq 2(K_1KK_2'C_1R|||y-z||| + C_1K_2R|||y-z|||) \\ &\leq 2C_1R(K_1KK_2' + K_2)|||y-z|||. \end{aligned}$$

We want F to be a contraction so we impose

$$(K_1 K K_2' C_1 + C_1 K_2) R < \frac{1}{2}. \tag{2.35}$$

Let R be some positive number satisfying (2.35). We then take

$$r = \frac{R}{2(C_0 + 2K_1 K)}.$$

The proof of Theorem 3 is complete.

3. Exact controllability with control on $y_{xx}(L,t)$

In this section, we apply as often as possible the same methods as in Section 2. Hence, most of the proofs will be omitted for an easier lecture, except if there are changes from Section 2, and most of the notations of this section are coming from Section 2. We apply the Hilbert Uniqueness Method to the problem, hence we first study the homogeneous Cauchy problem (subsection 3.1). In the second subsection, we prove some observability inequalities. In Subsection 3.3, we shall prove the controllability on $H_0^1(0,L) \times H^{-1}(0,L)$, and then we prove the exact controllability on $H_3 \times H_0^1(0,L)$, for every $T > 0$. Finally, we prove the exact boundary controllability of the nonlinear equation by means of the Banach fixed point theorem, as above.

3.1. **The homogeneous Cauchy problem.** We study the following Cauchy problem,

$$\begin{cases} \phi_{tt} - \phi_{xx} + \phi_{xxxx} = 0, \\ \phi(0,t) = \phi(L,t) = 0, \quad \phi_{xx}(0,t) = \phi_{xx}(L,t) = 0, \\ \phi(.,0) = \phi^0, \quad \phi_t(.,0) = \phi^1. \end{cases}$$

We first apply the theory of closed operators. Let us recall that $H_3 = \{\phi \in H^3(0,L); \phi(0) = \phi(L) = \phi_{xx}(0) = 0\}$ and $\|\phi\|_{H_3} = \|\phi_{xxx}\|_{L^2(0,L)}$. We define the operator A by $A\phi = -\phi_{xx} + \phi_{xxxx}$, with $D(A) = \{\phi \in H_3; \phi_{xx} \in H^2(0,L)\}$. $D(A) = \{\phi \in H^4(0,L), \phi(0) = \phi(L) = \phi_{xx}(0) = \phi_{xx}(L) = 0\}$. It is easy to see that A is linear continuous from $D(A)$ with the H^4-topology into $L^2(0,L)$. With the same arguments as in Section 2, we can prove that A admits a sequence of eigenvectors which is an orthonormal basis of $L^2(0,L)$.

Let λ be an eigenvalue of the operator $-\Delta$ in the space $H_0^1(0,L)$ and v the eigenvector associated. Then

$$v_{xxxx} - v_{xx} = \lambda(\lambda + 1)v, \quad v(0) = v(L) = v_{xx}(0) = v_{xx}(L) = 0.$$

Hence, each eigenvector v of $-\Delta_{|H_0^1(0,L)}$ is also an eigenvector of A and is associated with the eigenvalue $\lambda(\lambda+1)$. But the space generated by the eigenvectors of $-\Delta_{|H_0^1(0,L)}$ is dense in $L^2(0,L)$, hence A cannot have another eigenvectors. Consequently, the eigenvalues of A are written $\lambda_n = \frac{\pi^2 n^2}{L^2}(\frac{\pi^2 n^2}{L^2}+1)$ and are associated with the eigenvectors $\phi_n(x) = \sqrt{\frac{2}{L}}\sin(\frac{\pi n}{L}x)$. Let $\omega_n = \sqrt{\lambda_n}$. Let us consider the homogeneous problem

$$(*) \qquad \begin{cases} \phi_{tt} - \phi_{xx} + \phi_{xxxx} = 0, \\ \phi(0,t) = \phi(L,t) = 0,\ \phi_{xx}(0,t) = \phi_{xx}(L,t) = 0, \\ \phi(.,0) = \phi^0,\ \phi_t(.,0) = \phi^1. \end{cases}$$

By the variational theory, we obtain that if $(\phi^0,\phi^1) \in H_0^1(0,L)\times H^{-1}(0,L)$, then $(*)$ has a unique solution $\phi \in C(\mathbb{R}; H_0^1(0,L)) \cap C^1(\mathbb{R}; H^{-1}(0,L))$, and this solution depends continuously on the initial data (ϕ^0,ϕ^1) for the usual topology on $H_0^1(0,L)\times H^{-1}(0,L)$.

We get the following proposition.

Proposition 8. *For every $(\phi^0,\phi^1) \in H_0^1(0,L)\times H^{-1}(0,L)$, the solution of the homogeneous problem $(*)$ is written in Fourier series as*

$$\phi(x,t) = \sum_{k=1}^{+\infty}(\phi_k^0\cos(\omega_k t) + \frac{\phi_k^1}{\omega_k}\sin(\omega_k t))\phi_k,$$
$$\phi_k^0 = (\phi^0,\phi_k)_{L^2},\ \phi_k^1 = (\phi^1,\phi_k)_{L^2},$$

and we get $\phi \in C^0([0,T], H_0^1(0,L)) \cap C^1([0,T], H^{-1}(0,L))$.

3.2. **Observability of the homogeneous problem.** In this subsection, we adopt a quite different proof from the one given by Lions in [8], for proving the inequality of observability and the exact controllability in $H_0^1(0,L)\times H^{-1}(0,L)$. As we have the expression of the eigenvalues and eigenvectors, the proofs are easier than in Section 2.

The energy $E(t)$ of the solution of the homogeneous problem $(*)$ is written

$$E(t) = \frac{1}{2}\int_0^L (|\phi_t(t)|^2 + |\phi_x(t)|^2 + |\phi_{xx}(t)|^2)dx.$$

We readily get that $E(t) = E(0) = E_0$ for every t in $[0,T]$ and for ϕ satisfying $(*)$. Let $(\phi_0,\phi_1) \in H_0^1(0,L)\times H^{-1}(0,L)$ and let ϕ be the solution of the corresponding homogeneous problem $(*)$. As $\lim_{n\to+\infty}\omega_{n+1}-\omega_n = +\infty$ and $\omega_{n+1} > \omega_n$ we can apply Lemma 5 to $f(t) = \phi_x(L,t)+\alpha$, with α real,

$a_0 = \alpha$, $\mu_0 = 0$ and for $k \geq 1$

$$\begin{cases} \mu_k = -\mu_{-k} = \omega_k, \\ 2a_k = 2\overline{a_{-k}} = (\phi_k^0 - i\frac{\phi_k^1}{\omega_k})(\phi_{kx}(L)). \end{cases}$$

Then we get for every $T > 0$ two constants c_1 and c_2 so that

$$c_1\Big(|\alpha|^2 + \sum_{k=1}^{+\infty}\big[(\phi_k^0)^2 + (\frac{\phi_k^1}{\omega_k})^2\big]\phi_{kx}(L)^2\Big) \leq \int_0^T |f(t)|^2 dt$$
$$\leq c_2\Big(|\alpha|^2 + \sum_{k=1}^{+\infty}\big[(\phi_k^0)^2 + (\frac{\phi_k^1}{\omega_k})\big]^2\phi_{kx}(L)^2\Big).$$

But $\phi_{kx}(L) = (-1)^k \frac{\pi k}{L}\sqrt{\frac{2}{L}}$ hence, we get

$$c_1\Big(|\alpha|^2 + \|\phi^0\|^2_{H_0^1(0,L)} + \sum_{k=1}^{+\infty}\frac{2\pi^2k^2}{L^3\omega_k^2}(\phi_k^1)^2\Big) \leq \int_0^T |f(t)|^2 dt$$
$$\leq c_2\Big(|\alpha|^2 + \|\phi^0\|^2_{H_0^1(0,L)} + \sum_{k=1}^{+\infty}\frac{2\pi^2k^2}{L^3\omega_k^2}(\phi_k^1)^2\Big).$$

But $\phi^1 \in H^{-1}(0,L)$, hence

$$\bar{c_1}(\|\phi^0\|^2_{H_0^1(0,L)} + \|\phi^1\|^2_{H^{-1}(0,L)} + \alpha^2) \leq \int_0^T |f(t)|^2 dt$$
$$\leq \bar{c_2}(\|\phi^0\|^2_{H_0^1(0,L)} + \|\phi^1\|^2_{H^{-1}(0,L)} + \alpha^2).$$

So we get the following proposition

Proposition 9. *Let $T > 0$, there exist two strictly positive constants $\bar{c_1}$ and $\bar{c_2}$ so that,* $\forall(\phi^0, \phi^1) \in H_0^1(0,L) \times H^{-1}(0,L)$, $\forall \alpha \in \mathbb{R}$,

$$\bar{c_1}(\|(\phi^0, \phi^1)\|^2_{H_0^1(0,L)\times H^{-1}(0,L)} + \alpha^2) \leq \int_0^T (\phi_x(L,t) + \alpha)^2 dt$$
$$\leq \bar{c_2}(\|(\phi^0, \phi^1)\|^2_{H_0^1(0,L)\times H^{-1}(0,L)} + \alpha^2).$$

3.3. **Exact controllability of the linear equation.** We consider the non-homogeneous problem

$$y_{tt} - y_{xx} + y_{xxxx} = 0, \tag{3.1}$$
$$y(0,t) = y(L,t) = y_{xx}(0,t) = 0, \tag{3.2}$$
$$y_{xx}(L,t) = v(t), \tag{3.3}$$

$$y(.,0) = y^0,\ y_t(.,0) = y^1. \tag{3.4}$$

It is exactly controllable at the time T if for every initial conditions (y^0, y^1) taken in a certain space, there exists a control v so that the solution y of the problem (3.1)-(3.4) satisfies $y(x,T) = y_t(x,T) = 0$.

We aim to define a mild solution of (3.1)-(3.4) by transposition,(See Komornik [5], Theorem 2.5).

Proposition 10. *If $v \in L^2(0,T)$ and $(y^0, y^1) \in H_0^1(0,L) \times H^{-1}(0,L)$, then the nonhomogeneous problem (3.1)-(3.4) admits a unique solution y defined by transposition and so that $y \in C([0,T]; H_0^1(0,L)) \cap C^1([0,T]; H^{-1}(0,L))$.*

Proof. Let ϕ be the solution of the homogeneous problem $(*)$ with $(\phi^0, \phi^1) \in H_0^1(0,L) \times H^{-1}(0,L)$. Let y be a function in $C^4([0,L] \times [0,T])$ satisfying (3.1)-(3.4). Multiplying equation (3.1) by ϕ and integrating by parts on $[0,L] \times [0,S]$, with $0 \le S \le T$, we get

$$\int_0^L (\phi_t(x,S)y(x,S) - \phi(x,S)y_t(x,S))dx = -\int_0^S \phi_x(L,t)v(t)dt$$
$$+ < (-y^1, y^0), (\phi^0, \phi^1) >_{H^{-1}(0,L)\times H_0^1(0,L), H_0^1(0,L)\times H^{-1}(0,L)} .$$

Let L_S be the following linear operator

$$L_S(\phi^0, \phi^1) = -\int_0^S \phi_x(L,t)v(t)dt$$
$$+ < (-y^1, y^0), (\phi^0, \phi^1) >_{H^{-1}(0,L)\times H_0^1(0,L), H_0^1(0,L)\times H^{-1}(0,L)} .$$

We have

$$L_S(\phi^0, \phi^1) = \int_0^1 (\phi_t(S)y(S) - \phi(S)y_t(S))dx.$$

The linear application $(\phi(S), \phi_t(S)) \longmapsto (\phi^0, \phi^1)$ is an isomorphism from $H_0^1(0,L) \times H^{-1}(0,L)$ into itself, hence we deduce that L_S is continuous on $H_0^1(0,L)\times H^{-1}(0,L)$. Thanks to Proposition 9, there exists a constant $C > 0$ independent of y^0, y^1, v so that

$$\|(y(S), y_t(S))\|_{H_0^1(0,L)\times H^{-1}(0,L)} \le C(\|v\|_{L^2(0,L)} + \|(y^0, y^1)\|_{H_0^1(0,L)\times H^{-1}(0,L)}),$$

for every $S \in [0,T]$. More else, if $(y^0, y^1) \in H_0^1(0,L) \times H^{-1}(0,L)$ and $v \in C^\infty([0,T])$ with $v(0) = 0$, then $y \in C([0,T]; H_0^1) \cap C^1([0,T]; H^{-1}(0,L))$. We deduce the result of regularity by density.

We can now prove the controllability of the problem.

Proposition 11. *Let* $T > 0$ *and* $(y^0, y^1) \in H^{-1}(0,L) \times H_0^1(0,L)$*. There exists a control* $v \in L^2(0,T)$ *so that the solution of* (3.1)-(3.4) *satisfies* $y(x,T) = y_t(x,T) = 0$*, for every* x *in* $[0,L]$*.*

Proof. We apply H.U.M. That is, we consider the application $\Lambda_{0,T} : H_0^1(0,L) \times H^{-1}(0,L) \to H^{-1}(0,L) \times H_0^1(0,L)$ $(\phi^0, \phi^1) \to (\psi_t(.,0), -\psi(.,0))$, where ϕ is the solution of the homogeneous problem $(*)$ and ψ the solution of the backward problem

$$\begin{cases} \psi_{tt} - \psi_{xx} + \psi_{xxxx} = 0, \\ \psi(0,t) = \psi(L,t) = \psi_{xx}(0,t) = 0, \quad \psi_{xx}(L,t) = -\phi_x(L,t), \\ \psi(x,T) = 0, \quad \psi_t(x,T) = 0. \end{cases}$$

Thanks to time reversibility of the backward problem and Proposition 10, $\Lambda_{0,T}$ is a well defined continuous map from $F = H_0^1(0,L) \times H^{-1}(0,L)$ into its dual F'. On the other hand,

$$\langle \Lambda_{0,T}(\phi^0, \phi^1), (\phi^0, \phi^1) \rangle_{F',F} = \int_0^T (\phi_x(L,t))^2 dt.$$

Thanks to Proposition 9, this expression is the square of a norm on F equivalent to the norm considered. Hence, $\Lambda_{0,T}$ is an isomorphism from F onto F', and the system is controllable.

It can be interesting to get a control in $H^1(0,T)$, with initial conditions taken in $H_3 \times H_0^1(0,L)$ with $H_3 = \{y \in H^3(0,L),\ y(0) = y(L) = y_{xx}(0) = 0\}$. We still apply the method due to José Urquiza.

Proposition 12. *Let* $T > 0$ *and* $(y^0, y^1) \in H_3 \times H_0^1(0,L)$*. There exists a control* $v \in H^1(0,T)$ *so that the solution of* (3.1)-(3.4) *satisfies* $y(x,T) = y_t(x,T) = 0$*, for every* x *in* $[0,L]$*.*

Proof. Given $(y^0, y^1) \in H_3 \times H_0^1(0,L)$, we consider the following problem

$$y_{tt} - y_{xx} + y_{xxxx} = 0, \tag{3.5}$$

$$y(0,t) = y_{xx}(0,t) = y(L,t) = 0, \tag{3.6}$$

$$y_{xx}(L,t) = v(t), \tag{3.7}$$

$$y(.,0) = y^0,\ y_t(.,0) = y^1, \tag{3.8}$$

and we search v so that $y(.,T) = 0$, $y_t(.,T) = 0$. Letting $y_t = z$, we get the following problem

$$\begin{cases} z_{tt} - z_{xx} + z_{xxxx} = 0, \\ z(0,.) = z_{xx}(0,.) = z(L,.) = 0, \quad z_{xx}(L,.) = u, \\ z(.,0) = z^0, \quad z_t(.,0) = z^1, \end{cases}$$

with $u = v_t \in L^2(0,T)$, $(z^0, z^1) = (y^1, -Ay^0) \in H_0^1(0,L) \times H^{-1}(0,L)$, and A being the operator of Subsection 3.1. According to Proposition 11, $y_t = z \in C([0,T], H_0^1(0,L))$, and y is the solution of the following elliptic problem, with t viewed as a parameter and $z_t \in C([0,T], H^{-1}(0,L))$.

$$\begin{cases} -y_{xx} + y_{xxxx} = -z_t, \\ y(0,.) = y_{xx}(0,.) = y(L,.) = 0, \quad y_{xx}(L,.) = v. \end{cases}$$

From the regularity and the continuity of the data in relation to t, we deduce that $y \in C([0,T], H_3)$. The problem of controllability lies in the existence of a function $u \in L^2(0,T)$ so that the solution (z, v) of

$$\begin{cases} z_{tt} - z_{xx} + z_{xxxx} = 0, \\ z(0,.) = z_{xx}(0,.) = z(L,.) = 0, \\ z_{xx}(L,.) = u, \quad z(.,0) = z^0, \; z_t(.,0) = z^1, \end{cases}$$

$$v_t = u, \quad v(0) = y^0_{xx}(L),$$

satisfies $z(.,T) = z_t(.,T) = 0$ and $v(T) = 0$. We consider the application $\Lambda_T : H_0^1(0,L) \times H^{-1}(0,L) \times \mathbb{R} \to H^{-1}(0,L) \times H_0^1(0,L) \times \mathbb{R}$, $(\phi^0, \phi^1, \alpha^0) \mapsto (\psi_t(.,0), -\psi(.,0), -\omega(0))$, where (ϕ, α) is the solution of the following homogeneous problem

$$\begin{cases} \phi_{tt} - \phi_{xx} + \phi_{xxxx} = 0, \\ \phi(0,.) = \phi_{xx}(0,.) = \phi(L,.) = 0, \\ \phi_{xx}(L,.) = 0, \quad \phi(.,0) = \phi^0, \; \phi_t(.,0) = \phi^1, \end{cases}$$

$$\alpha_t = 0, \quad \alpha(0) = \alpha^0,$$

and (ψ, ω) is the solution of the backward problem

$$\begin{cases} \psi_{tt} - \psi_{xx} + \psi_{xxxx} = 0, \\ \psi(0,.) = \psi_{xx}(0,.) = \psi(L,.) = 0, \\ \psi_{xx}(L,.) = -\phi_x(L,.) + \alpha, \quad \psi(.,T) = 0, \; \psi_t(.,T) = 0, \end{cases}$$

$$\omega_t = -\phi_x(L,.) + \alpha, \quad \omega(T) = 0.$$

We then get

$$(\Lambda(\phi^0, \phi^1, \alpha^0), (\phi^0, \phi^1, \alpha^0)) = -\int_0^T \phi_x(L)\psi_{xx}(L)dt - \alpha^0\omega(0).$$

Hence,

$$(\Lambda(\phi^0, \phi^1, \alpha^0), (\phi^0, \phi^1, \alpha^0)) = \int_0^T (\phi_x(L) - \alpha^0)^2.$$

We get then the controllability on $H_3 \times H_0^1(0,L)$ with Proposition 9.

3.4. **Exact controllability of the nonlinear equation.** We prove Theorem 4, i.e., the fact that the following boundary-control system is exactly controllable in a neighborhood of the null state.

$$y_{tt} - y_{xx} + y_{xxxx} + (y^2)_{xx} = 0, \tag{3.9}$$

$$y(0,t) = y(L,t) = 0, \tag{3.10}$$

$$y_{xx}(0,t) = 0, \tag{3.11}$$

$$y_{xx}(L,t) = v(t), \tag{3.12}$$

$$y(x,0) = y^0(x), y_t(x,0) = y^1(x), \tag{3.13}$$

where $(y^0, y^1) \in H_3 \times H_0^1(0,L)$ and $v \in H^1(0,T)$. The strategy is the same as in Section 2, i.e., to find a good Hilbert space, and to apply the fixed point theorem in order to find a solution for the nonlinear problem. In order to solve (3.9)-(3.12), we write $y = \bar{S^0}(t)(y^0, y^1) + y_1 + y_2$, where $\bar{S} = (\bar{S^0}, \bar{S^1})$ is the unitary group associated with the operator A of Section 3.1 and y_1 and y_2 are the respective solutions of the two nonhomogeneous problems

$$y_{1tt} - y_{1xx} + y_{1xxxx} = 0, \tag{3.14}$$

$$y_1(0,t) = y_1(L,t) = 0, \quad y_{1xx}(0,t) = 0, \tag{3.15}$$

$$y_{1xx}(L,t) = v(t), \quad y_1(x,0) = 0, \quad y_{1t}(x,0) = 0. \tag{3.16}$$

$$y_{2tt} - y_{2xx} + y_{2xxxx} = f, \tag{3.17}$$

$$y_2(0,t) = y_2(L,t) = 0, \quad y_{2xx}(0,t) = y_{2xx}(L,t) = 0, \tag{3.18}$$

$$y_2(x,0) = 0, \quad y_{2t}(x,0) = 0, \tag{3.19}$$

Proposition 13. *The map* $\phi \in L^2([0,T], H_3) \to (\phi^2)_{xx} \in L^1([0,T], H^1(0,L))$ *is well-defined and continuous.*

Proof. Let $y, z \in L^2([0,T], H_3)$,

$$\|(y^2)_{xx} - (z^2)_{xx}\|_{L^1(0,T,H^1(0,L))} = 2\int_0^T \|yy_{xx} + (y_x)^2 - zz_{xx} - (z_x)^2\|_{H^1(0,L)} dt$$
$$\leq 2\int_0^T \|yy_{xx} - zz_{xx}\|_{H^1(0,L)} dt + 2\int_0^T \|(y_x)^2 - (z_x)^2\|_{H^1(0,L)} dt.$$

We have

$$\int_0^T \|yy_{xx} - zz_{xx}\|_{H^1(0,L)} dt$$
$$\leq \int_0^T \|(y-z)y_{xx}\|_{H^1(0,L)} + \|z(y_{xx} - z_{xx})\|_{H^1(0,L)}$$

$$\le \int_0^T \|y-z\|_{H^1(0,L)}\|y_{xx}\|_{H^1(0,L)} + \|z\|_{H^1(0,L)}\|y_{xx}-z_{xx}\|_{H^1(0,L)}$$
$$\le Const\big(\|y\|_{L^2(0,T,H_3)} + \|z\|_{L^2(0,T,H_3)}\big)\|y-z\|_{L^2(0,T,H_3)}.$$

Hence, with $z=0$, we get $yy_{xx} \in L^1(0,T,H^1(0,L))$ and letting $z \to y$, we readily get the continuity of the map $y \mapsto yy_{xx}$. On the other hand,

$$\begin{aligned}\int_0^T \|(y_x)^2-(z_x)^2\|_{H^1(0,L)} &= \int_0^T \|(y_x-z_x)(y_x+z_x)\|_{H^1(0,L)}\\ &\le \int_0^T \|y_x-z_x\|_{H^1(0,L)}\|y_x+z_x\|_{H^1(0,L)}\\ &\le \int_0^T \|y-z\|_{H_3(0,L)}\|y+z\|_{H_3(0,L)} \le \|y-z\|_{L^2(0,T,H_3)}\|y+z\|_{L^2(0,T,H_3)}.\end{aligned} \tag{3.20}$$

Thus, $(y_x)^2 \in L^1(0,T,H^1(0,L))$ and the map $y \mapsto (y_x)^2$ is continuous. So we get the continuity of the map $\phi \in L^2(0,T,H_3) \mapsto (\phi^2)_{xx} \in L^1(0,T,H^1(0,L))$. We now study the solutions y_1 and y_2 designed above. Let $\psi_1 : v \in H^1(0,T) \longmapsto y_1 \in C([0,T],H_3) \subset L^2([0,T],H_3)$ be the map which associates with v the weak solution of (2.26)-(2.28). The map ψ_1 is linear and continuous.

Proposition 14. *For $f \in L^1(0,T,H^1(0,L))$ the mild solution y_2 of* (2.29)-(2.31) *fulfills $(y_2,y_{2t}) \in C^0([0,T],H_3\times H^1(0,L))$. Moreover, the map $\psi_2 : f \longmapsto (y_2,y_{2t})$ is continuous.*

We now introduce a map whose fixed point is the solution of the problem. Assume that $y^0=y^1=0$. Using Proposition 12 and the reversibility of (3.5)-(3.8) we obtain a continuous map $\Gamma : (y_T^0,y_T^1) \in H_3 \times H_0^1(0,L) \mapsto v \in H^1(0,T)$ so that the solution y of (3.5)-(3.8) with $y^0=y^1=0$ fulfills $y(.,T)=y_T^0$ and $y_t(.,T)=y_T^1$.

Let F be the map: $F : y \in L^2(0,T,H_3) \mapsto F(y)$ with

$$\begin{aligned}F(y) := \bar{S}^0(.)(y^0,y^1) + \psi_1\circ\Gamma\Big((y_T^0,y_T^1) - \bar{S}(T)(y^0,y^1)\\ + (\psi_2((y^2)_{xx})(.,T),(\psi_2((y^2)_{xx}))_t(.,T)\Big) + \psi_2(-(y^2)_{xx}).\end{aligned}$$

$F(y) \in L^2(0,T,H_3)$. F is well defined and continuous and each fixed point y of F satisfies the nonlinear equation (2.23)-(2.25) and $(y(T,.),y_t(T,.)) = (y_T^0,y_T^1)$. For proving the existence of some fixed point for F, we apply the Banach fixed point theorem to the restriction of F to some closed ball $\bar{B}(0,R)$ in $L^2(0,T,H_3)$.

We need to show that

$$F(\bar{B}(0,R)) \subset \bar{B}(0,R) \tag{3.21}$$

and $\exists C_3 \in (0,1), \forall y,z \in \bar{B}(0,R), |||F(y)-F(z)||| \leq C_3|||y-z|||$, where $|||.|||$ denote the norm in $L^2(0,T,H_3)$. Let $\|.\|$ denote the norm in $H_3 \times H^1(0,L)$. Let K_1, (resp. K_2 ,K_2') denote the norm of ψ_1 (resp. ψ_2, ψ_2) as a map from $H^1(0,T)$ (resp. $L^1(0,T,H^1(0,L))$) into $L^2(0,T,H_3)$ (resp. $L^2(0,T,H_3)$, $C([0,T],H_3 \times H_0^1(0,L))$). And let K denote the norm of Γ as a map from $H_3 \times H_0^1(0,L)$ into $H^1(0,T)$. Assume that $\|y^0,y^1\| \leq r$, $\|y_T^0,y_T^1\| \leq r$, (r will be precised afterwards). Let $y,z \in L^2(0,T,H_3)$, and suppose $|||y||| \leq R$ and $|||z||| \leq R$. Then, by (3.20)

$$\begin{aligned}|||F(y)||| \leq C_0\|(y_0,y_1)\| &+ KK_1(\|(y_T^0,y_T^1)\| \\ &+ \|(y^0,y^1)\| + K_2'C_1\|y\|^2) + K_2C_1\|y\|^2 \\ &\leq C_0 r + 2K_1Kr + (K_1KK_2'C_1 + K_2C_1)R^2,\end{aligned}$$

where r is a real so that $\|(y_T^0,y_T^1)\| \leq r$ and $\|(y^0,y^1)\| \leq r$. Hence, (3.21) holds if $(C_0+2K_1K)r + (K_1KK_2'C_1+K_2C_1)R^2 \leq R$. But,

$$\begin{aligned}F(y)-F(z) = \psi_1 \circ \Gamma\Big(&\psi_2((y^2)_{xx}-(z^2)_{xx})(T,.), \psi_{2t}((y^2)_{xx}-(z^2)_{xx})(T,.)\Big) \\ &+ \psi_2((z^2)_{xx}-(y^2)_{xx}).\end{aligned}$$

Hence,

$$\begin{aligned}|||F(y)-F(z)||| &\leq 2(K_1KK_2'C_1R|||y-z||| + C_1K_2R|||y-z|||) \\ &\leq 2C_1(K_1KK_2'+K_2)R|||y-z|||\end{aligned}$$

We want F to be a contraction so we impose

$$(K_1KK_2'C_1 + C_1K_2)R < \tfrac{1}{2}. \tag{3.22}$$

Let R be some positive number satisfying (3.22). We then take

$$r = \frac{R}{2(C_0+2K_1K)}.$$

The proof of Theorem 4 is complete.

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