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BIFURCATING POSITIVE STABLE STEADY-STATES FOR A SYSTEM OF DAMPED WAVE EQUATIONS

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Abstract. A system of nonlinear damped wave equations with symmetric linear part is investigated. A positive steady-state bifurcates from the trivial solution as a parameter changes. The spectrum of the linearized operator is studied. Then the stability of the positive steady-state is considered as a solution of the nonlinear hyperbolic system. Asymptotic stability results are found for solutions in R^N, $N \geq 1$. Bifurcation methods are used to find the steady-states, and semigroup methods are used to study stability. Stability results are obtained although the semigroup is not analytic.

1. Introduction

This article considers the positive steady-state solutions and their stabilities for the following nonlinear system of damped wave equations:

$$\begin{cases} u_{tt} + \beta u_t = \Delta u + \lambda B(x)u + \lambda G(u)u & \text{for (x,t)} \in \Omega \times [0,\infty), \\ u = 0 & \text{for (x,t)} \in \partial\Omega \times [0,\infty). \end{cases} \tag{1.1}$$

Here, $u = \text{col.}(u_1,\dots,u_n), \beta > 0$, λ is a real parameter, $G = [g_{ij}]$ is an $n \times n$ matrix function with each entry in $C^2(R^n)$ and $g_{ij}(0,\dots,0) = 0$ for $i,j = 1,\dots,n$. Ω is a bounded domain in R^N, with boundary $\partial\Omega$ of class $C^{2+\mu}$, $0 < \mu < 1$. The $n \times n$ matrix $B(x) = [b_{ij}(x)]$ is assumed to satisfy

[$H1$] $B(x)$ is a real symmetric matrix, with each entry $b_{ij}(x)$ in $C^\mu(\bar{\Omega})$, $0 < \mu < 1$, and nonnegative in $\bar{\Omega}$.

[$H2$] There is a permutation $\{r_1, r_2, \dots, r_n\}$ of $\{1, 2, \dots, n\}$ such that $b_{r_1 r_2} \not\equiv 0$, $b_{r_2 r_3} \not\equiv 0, \dots, b_{r_{n-1} r_n} \not\equiv 0$, and $b_{r_n r_1} \not\equiv 0$ in $\bar{\Omega}$.

(The hypothesis [$H1$] will be weakened to [$H1^*$] below such that the diagonal entries of $B(x)$ may change sign). Nonlinear hyperbolic systems of similar nature arise in the study of physics and mechanics as the coupled

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sine-Gordon equations (see e.g. [11]). In [10], a recent study is made by energy method concerning the decay of the solutions for similar damped systems. This article can be considered as an extension of the scalar results in [12], where both $n = N = 1$, to systems with symmetric linear part.

In Section 2, we show that a positive steady-state bifurcates from the trivial solution as the parameter λ changes. Moreover, the corresponding elliptic system is linearized at the positive steady-state and the spectrum of the linear operator is investigated under appropriate conditions on the nonlinear terms. In Section 3, the stability of the positive steady-state is considered as a solution of hyperbolic problem (1.1). Note that the corresponding semigroup here is not analytic, and the stability theorem in [4] does not apply. The results in Section 3 concerning stability are shown to be applicable to solutions in R^1 by means of Morrey's inequality, and to solutions in R^3 or R^4 by means of Gagliardo-Nirenberg-Sobolev inequality. In Section 4, positive steady-state is found for $\lambda = 1$ under further assumptions.

For convenience, we will adopt the following conventions. Let $E := \{w = \text{col.}(w_1, \ldots, w_n) : w_i \in C^1(\bar{\Omega}),\ w_i = 0 \text{ on } \partial\Omega,\ i = 1, \ldots, n\}$, $P := \{w = \text{col.}(w_1, \ldots, w_n) \in E : w_i \geq 0 \text{ in } \bar{\Omega}, i = 1, \ldots, n\}$, and $Y := \{w = \text{col.}(w_1, \ldots, w_n) : w_i \in C^{2+\alpha}(\bar{\Omega}), w_i = 0 \text{ on } \partial\Omega, i = 1, \ldots, n\}$.

2. Bifurcation and spectrum of linearized equation

We first show that as the parameter λ passes a certain eigenvalue of the linearized system, a positive steady-state bifurcates from the trivial solution. Then, we will analyze the spectrum of the elliptic part of the operator equation linearized at the positive steady-state.

For convenience, we define the operator L_q with n components as: $L_q \equiv (-\Delta + q_1(x), \ldots, -\Delta + q_n(x))$, where $q_i(x), i = 1, \ldots, n$ are any nonnegative functions in $C^\mu(\bar{\Omega})$.

Theorem 2.1. *Under hypotheses [H1] and [H2], there exists* $(\lambda_0, u^0) \in R \times Y$, *such that*

$$L_q[u^0] = \lambda_0 B u^0 \text{ in } \Omega,\ \ u^0 = 0 \text{ on } \partial\Omega, \tag{2.1}$$

with $\lambda_0 > 0$, *and each component* $u_i^0 > 0$ *in* $\Omega, \partial u_i^0/\partial\nu < 0$ *on* $\partial\Omega$ *for* $i = 1, ...n$. *Furthermore,* $1/\lambda_0$ *is a simple eigenvalue of the operator* $L_q^{-1}B : [C^1(\bar{\Omega})]^n \to [C^1(\bar{\Omega})]^n$ *(i.e., eigenfunctions corresponding to this eigenvalue is unique up to a multiple). Also, the number* $\lambda = \lambda_0$ *is the unique positive number so that the problem* $u = \lambda L_q^{-1} B u$ *has a nontrivial nonnegative solution for* $u \in P$.

Proof. The operator $L_q^{-1}B$ is completely continuous and positive with respect to the cone P. By means of Theorem 2.5 in [5] we can obtain a nontrivial $u^0 \in P$ such that $L_q^{-1}Bu^0 = \rho_0 u^0$ for some $\rho_0 > 0$ (i.e., (2.1), with $\lambda_0 = 1/\rho_0$). By the nonnegativity of all entries of B and hypothesis $[H2]$, we show with maximum principle that $u_i^0 > 0$ in Ω and $\partial u_i^0/\partial\nu < 0$ on $\partial\Omega$ for each $i = 1, \ldots, n$. By using a comparison principle for systems (Lemma 2.1 in [8]), we can show in the same way as Theorem 2.1 in [8] that $1/\lambda_0$ is a simple eigenvalue, and λ_0 is the unique positive number with a nonnegative eigenfunction as described above.

For a more general situation, we will let the diagonal entries of $B(x)$ to change sign. For $i = 1, \ldots, n$, let $b_{ii}(x) = b_{ii}^+(x) + b_{ii}^-(x)$, where $b_{ii}^+(x) = \max\{b_{ii}(x), 0\}$ and $b_{ii}^-(x) = \min\{b_{ii}(x), 0\}$. We introduce following hypothesis:

$[H1^*]$ $B(x)$ is a real symmetric matrix with each entry $b_{ij}(x)$ in $C^\mu(\bar{\Omega})$. For all $i \neq j$, b_{ij} is nonnegative in $\bar{\Omega}$, and there exists an integer $k, 1 \leq k \leq n$, such that $b_{kk}^+(x) \not\equiv 0$ in $\bar{\Omega}$.

The following is an extension for Theorem 2.1.

Theorem 2.2. *Under hypotheses $[H1^*]$ and $[H2]$, there exists $(\hat{\lambda}_0, v^0) \in R \times Y$ such that*

$$-\Delta v^0 = \hat{\lambda}_0 B(x) v^0 \text{ in } \Omega, \;\; v^0 = 0 \text{ on } \partial\Omega, \tag{2.2}$$

with $\hat{\lambda}_0 > 0$, each component $v_i^0 > 0$ in Ω and $\partial v_i^0/\partial\nu < 0$ on $\partial\Omega$ for $i = 1, \ldots, n$. Furthermore, $1/\hat{\lambda}_0$ is a simple eigenvalue of the operator $(-\Delta)^{-1}B : [C^1(\bar{\Omega})]^n \to [C^1(\bar{\Omega})]^n$.

Proof. We will use Theorem 2.1 to prove this theorem. For convenience, define $\tilde{B} = \{\tilde{b}_{ij}(x)\}$ to be the $n \times n$ matrix function on $\bar{\Omega}$ as follows:

$$\tilde{b}_{ij}(x) = b_{ij}(x) \;\text{ if }\; i \neq j, \quad \tilde{b}_{ii}(x) = b_{ii}^+(x)$$

for $i, j = 1, \ldots, n$, $x \in \bar{\Omega}$. For each $\lambda \geq 0$, define the n component vector operator

$$\tilde{L}_\lambda \equiv (-\Delta - \lambda b_{11}^-(x), -\Delta - \lambda b_{22}^-(x), \ldots, -\Delta - \lambda b_{nn}^-(x))$$

and consider the eigenvalue problem

$$\tilde{L}_\lambda u = \rho \tilde{B} u \;\text{ in }\; \Omega, \quad u\,|_{\partial\Omega} = 0, \tag{2.3}$$

with eigenvalue ρ. Since $\tilde{B}$ satisfies the conditions in Theorem 2.1, the problem (2.3) has a unique positive solution $\rho = \hat{\rho}(\lambda)$, with corresponding eigenfunction u_λ whose components are all positive in Ω.

To proceed with the proof, we need the following two lemmas.

Lemma 2.1. *Under the hypotheses of Theorem* 2.2, *the function* $\hat{\rho}(\lambda)$ *is bounded for all* $\lambda \in [0, \infty)$.

Proof. By hypothesis $[H1^*]$, there is an open set D in Ω with its closure in Ω, such that $b_{kk} = b_{kk}^+$ in D for some k. Let Φ be a nontrivial C^∞ function with compact support contained in D. We clearly have $\int_D b_{kk}\Phi^2 dx > 0$. Let u_λ be as described above, and set $w_\lambda(x) = \ln(u_\lambda)_k(x)$ for $x \in \Omega$. Thus, we have in D that

$$\begin{aligned} &- \Delta w_\lambda - \sum_{i=1}^{N} (\partial(w_\lambda)/\partial x_i)^2 = [1/(u_\lambda)_k][-\Delta(u_\lambda)_k - b_{kk}^-(x)(u_\lambda)_k] \\ &= [\hat{\rho}(\lambda)/(u_\lambda)_k] \sum_{j=1}^{n} \tilde{b}_{kj}(x)(u_\lambda)_j \geq b_{kk}(x)\hat{\rho}(\lambda). \end{aligned}$$

Multiplying by Φ^2 and integrating over D by parts on the left, we obtain

$$\int_D < \Phi\nabla w_\lambda, 2\nabla\Phi - \Phi\nabla w_\lambda > dx \geq \hat{\rho}(\lambda) \int_D b_{kk}\Phi^2 dx.$$

From this we deduce

$$\Big[\int_D < \nabla\Phi, \nabla\Phi > dx\Big]\Big[\int_D b_{kk}\Phi^2 dx\Big]^{-1} \geq \hat{\rho}(\lambda) > 0$$

for all $\lambda \in [0, \infty)$.

Lemma 2.2. *Under the hypotheses of Theorem* 2.2, *the function* $\hat{\rho}(\lambda)$ *is continuous for* $\lambda \in [0, \infty)$.

The proof of the two lemmas above are similar to that for Lemmas 2.2 and 2.3 in [8]. Further details will thus be omitted.

To complete the proof of Theorem 2.2, we solve the equation $\hat{\rho}(\lambda) - \lambda = 0$ for $\lambda = \hat{\lambda}_0$ as described in (2.2). The solution of the equation must exist due to the properties of $\hat{\rho}(\lambda)$ described in Lemmas 2.1 and 2.2. The simplicity of the eigenvalue $1/\hat{\lambda}_0$ follows from (2.3) and Theorem 2.1.

Note that the pair $(\hat{\lambda}_0, v^0)$ in Theorem 2.2 also satisfies

$$-\Delta v^0 = \hat{\lambda}_0 B^T(x)v^0 \tag{2.4}$$

since $B = B^T$. For convenience, we define an operator $F : R^+ \times E \to E$ by

$$F(\lambda, u) := u - \lambda(-\Delta)^{-1}[B + G(u)]u \quad \text{for } (\lambda, u) \in R^+ \times E. \tag{2.5}$$

The steady-state solution of (1.1) can be written as

$$F(\lambda, u) = 0. \tag{2.6}$$

Defining operators

$$L_0 : E \to E \qquad \text{by} \qquad L_0 := I - \hat{\lambda}_0(-\Delta)^{-1}B,$$
$$L_1 : E \to E \qquad \text{by} \qquad L_1 := \Delta^{-1}B, \quad \text{and}$$
$$\tilde{G} : R^+ \times E \to E \qquad \text{by} \qquad \tilde{G}(\lambda, u) := \lambda\Delta^{-1}[G(u)u],$$

equation (2.6) becomes

$$L_0 u + (\lambda - \hat{\lambda}_0)L_1 u + \tilde{G}(\lambda, u) = 0, \text{ for } (\lambda, u) \in R^+ \times E. \tag{2.7}$$

As in [8], we readily obtain the following.

Lemma 2.3. *Under the hypotheses of Theorem* 2.2, *the null space and range of* L_0, *denoted respectively by* $N(L_0)$ *and* $R(L_0)$, *satisfy:*
(*i*) $N(L_0)$ *is one-dimensional, spanned by* v^0*;*
(*ii*) $\dim[E/R(L_0)] = 1$;
(*iii*) $L_1 v^0 \notin R(L_0)$.

Applying the bifurcation theorem in [1], we obtain the following.

Theorem 2.3. *Assume hypotheses* $[H1^*], [H2]$ *and each entry of* G *is in* $C^2(R^n)$. *Then the point* $(\hat{\lambda}_0, 0)$ *is a bifurcation point for the problem* (2.6). *Moreover, there exists a* $\delta > 0$ *and a* C^1*-curve* $(\hat{\lambda}(s), \hat{\phi}(s)) : (-\delta, \delta) \to R \times E$ *with* $\hat{\lambda}(0) = \hat{\lambda}_0, \hat{\phi}(0) = 0$, *such that in a neighborhood of* $(\hat{\lambda}_0, 0)$, *any solution of* (2.6) *is either of the form* $(\lambda, 0)$ *or on the curve* $(\hat{\lambda}(s), s[v^0 + \hat{\phi}(s)])$ *for* $| s |< \delta$, *where* $s[v^0 + \hat{\phi}(s)] > 0$ *in* Ω.

Further applying the theory in [1], we can assert that there exists $\delta_1 \in (0, \delta)$ and two functions $(\sigma(.), z(.)) : (\hat{\lambda}_0 - \delta_1, \hat{\lambda}_0 + \delta_1) \to R \times E$, and $(\eta(.), h(.)) : [0, \delta_1) \to R \times E$ with $(\sigma(\hat{\lambda}_0), z(\hat{\lambda}_0)) = (\eta(0), h(0)) = (0, v^0)$ such that

$$D_2F(\lambda, 0)z(\lambda) = \sigma(\lambda)\Delta^{-1}(z(\lambda)), \text{ and}$$
$$D_2F(\hat{\lambda}(s), s[v^0 + \hat{\phi}(s)])h(s) = \eta(s)\Delta^{-1}(h(s)).$$

Here, $\sigma(\lambda)$ and $\eta(s)$ are respectively Δ^{-1}-simple eigenvalues with eigenfunctions $z(\lambda)$ and $h(s)$. The theory in [1] leads to the following.

Lemma 2.4. *Assume all the hypotheses in Theorem* 2.3. *There exists* $\rho > 0$ *such that for each* $s \in [0, \delta_1)$, *there is a unique* (*real*) *eigenvalue* $\eta(s)$ *for the linear operator*

$$F_s^* := \Delta D_2F(\hat{\lambda}(s), s(v^0 + \hat{\phi}(s)) : Y \to [C^\mu(\bar{\Omega})]^n \tag{2.8}$$

satisfying $|\eta(s)| < \rho$ *with eigenfunction* $h(s) \in Y$. *That is,*

$$F_s^* h(s) \equiv \Delta[h(s)] + \hat{\lambda}(s)[B + G(u_s) + G_u(u_s)u_s]h(s) = \eta(s)h(s), \tag{2.9}$$

where $u_s := s(v^0 + \hat{\phi}(s))$. *Here,* $G_u(u_s)u_s$ *denotes the* $n \times n$ *matrix whose* i*-th column is* $\{\partial G/\partial u_i\}(u_s)u_s$.

In order to determine the signs of the real part of the eigenvalues we need additional assumptions.

Lemma 2.5. *Assume all the hypotheses of Theorem* 2.3. *Suppose further that*

$[H3]$ $\{\partial g_{ij}/\partial u_k\}(0) \le 0$ *for all* $i, j, k = 1, ..., n$, *with at least one inequality being strict. Then the function* $\hat{\lambda}(s)$ *satisfies* $\hat{\lambda}'(0) > 0$.

Proof. Theorem 2.3 asserts that $\hat{\lambda}'(0)$ exists; moreover, for $s \in [0, \delta)$, we have

$$\Delta s(v^0 + \hat{\phi}(s)) + \hat{\lambda}(s)B(x)s(v^0 + \hat{\phi}(s)) + \hat{\lambda}(s)G(s(v^0 + \hat{\phi}(s))s(v^0 + \hat{\phi}(s)) = 0.$$

Dividing by s, then differentiating with respect to s and setting $s = 0$, we obtain

$$\Delta\hat{\phi}'(0) + \hat{\lambda}'(0)B(x)v^0 + \hat{\lambda}_0 B(x)\hat{\phi}'(0) + \hat{\lambda}_0 \frac{d}{ds} G(s(v^0 + \hat{\phi}(s)) \mid_{s=0} v^0 = 0.$$

Multiplying by $(v^0)^T$ and integrating by parts over Ω, we find

$$\begin{aligned} \hat{\lambda}'(0) &= \frac{-\hat{\lambda}_0 \int_\Omega (v^0)^T \frac{d}{ds} G[s(v^0 + \hat{\phi}(s))] \mid_{s=0} v^0 dx}{\int_\Omega (v^0)^T B v^0 dx} \\ &= \frac{-\hat{\lambda}_0 \int_\Omega (v^0)^T [\sum_{i=1}^n v_i^0 \frac{\partial G}{\partial u_i}(0)] v^0 dx}{(1/\lambda_0) \int_\Omega \mid \nabla v^0 \mid^2 dx} > 0. \end{aligned}$$

The last inequality is due to hypothesis $[H3]$.

Lemma 2.6. *Under all the hypotheses of Theorem* 2.3. *The function* $\sigma(\lambda)$ *satisfies* $\sigma'(\hat{\lambda}_0) > 0$.

The proof of the last lemma is similar to that for Lemma 3.3 in [8], while using the additional fact that $B = B^T$ here. The details will be omitted.

Lemma 2.7. *Under all the hypotheses of Theorem* 2.3, *and* $[H3]$, *there exists* $\delta_2 \in (0, \delta_1)$ *such that* $\eta(s) < 0$ *for all* $s \in (0, \delta_2)$.

Proof. From Theorem 1.16 in [1], we find $-s\hat{\lambda}'(s)\sigma'(\hat{\lambda}_0)$ and $\eta(s)$ have the same sign for $s > 0$ near 0. The conclusion follows from Lemmas 2.5 and 2.6.

The linearized eigenvalue problem for (2.6) at the bifurcating solution $u = s(v^0 + \hat{\phi}(s))$ is precisely (2.9). When $s = 0$, $\lambda = \hat{\lambda}(0) = \hat{\lambda}_0$, the eigenvalue problem corresponding to (2.9) becomes

$$\Delta[h] + \hat{\lambda}_0 Bh = \eta h \qquad h \in Y \tag{2.10}$$

where η is the eigenvalue. Under hypotheses $[H1^*]$ and $[H2]$, Theorem 2.2 asserts that $\eta = 0$ is an eigenvalue for (2.10) with positive eigenfunction. Using this property and the fact that the off-diagonal terms of B are all nonnegative, we can show the following as in [7].

Lemma 2.8. *Under hypotheses* $[H1^*]$ *and* $[H2]$*, all eigenvalues in equation* (2.10) *except* $\eta = 0$ *satisfies* $Re(\eta) < -r$ *for some positive number* r.

Theorem 2.4. *Under the hypotheses of Theorem* 2.3 *and* $[H3]$*, there exists a number* $\delta^* \in (0, \delta)$ *and a positive function* $\hat{\eta}(s)$ *for* $s \in (0, \delta^*)$ *such that the real parts of all the numbers in the point spectrum of the linear operator* F_s^* *are contained in the interval* $(-\infty, -\hat{\eta}(s))$*, for* $s \in (0, \delta^*)$*.* (*Here,* δ *is described in Theorem* 2.3 *and* F_s^* *is described in* (2.8) *in Lemma* 2.4).

The proof of Lemma 2.8 is the same as the proof of Lemma 2.8 in [7]; and the proof of Theorem 2.4, using the assertions in Lemma 2.4 to 2.8 and perturbation arguments, is the same as Theorem 2.2 in [7]. The details are thus omitted. More details of proof of lemmas and theorems similar to those in this section can also be found in [8].

3. Stability of positive steady-states

For each $s \in (0, \delta^*)$, the function $u_s := s[v^0 + \hat{\phi}(s)]$ described in Theorem 2.3 and Lemma 2.4 can be considered as a steady-state solution of (1.1) with $\lambda = \hat{\lambda}(s)$. We now consider the time asymptotic stability of this steady state as a solution of the system of hyperbolic equations (1.1). We convert (1.1) into a first order system by letting

$$\begin{cases} \partial u_i/\partial t = v_i \qquad i = 1, \ldots, n, \\ \partial v_i/\partial t = \Delta u_i - \beta v_i + \lambda \sum_{j=1}^{n} [b_{ij}(x) + g_{ij}(u_1, \ldots, u_n)] u_j. \end{cases} \tag{3.1}$$

Let $J(u_s(x))$ be the linearization of $G(u)u$ (i.e., $\partial[G(u)u]/\partial u$) evaluated at $u = u_s$. For convenience, define $\bar{B}(u_s(x)) := B(x) + J(u_s(x))$ and let $\bar{b}_{ij}$ denotes the ij-th entry of $\bar{B}(u_s(x))$. The system (3.1) linearized at u_s and $\lambda = \hat{\lambda}(s)$ can be written in the form

$$\partial \xi/\partial t = A_s \xi, \tag{3.2}$$

where $\xi = \text{col.}(\hat{u}_1, \hat{v}_1, \ldots, \hat{u}_n, \hat{v}_n)$, and A_s is the differential linear operator on the $2n$ components of ξ as follows:

$$A_s = \begin{bmatrix} A_{11}^s \cdots\cdots A_{1n}^s \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ A_{n1}^s \cdots\cdots A_{nn}^s \end{bmatrix}, \text{ where } A_{ij}^s \text{ are } 2 \times 2 \text{ blocks}$$

$$A_{ii}^s = \begin{bmatrix} 0 & 1 \\ \Delta + \hat{\lambda}(s)\bar{b}_{ii} & -\beta \end{bmatrix}, \ A_{ij}^s = \begin{bmatrix} 0 & 0 \\ \hat{\lambda}(s)\bar{b}_{ij} & 0 \end{bmatrix} \text{ for } i \neq j, \ i, j = 1, \ldots, n.$$

Here, each A_{ij}^s can be considered as an operator from $[H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega)$ into $H_0^1(\Omega) \times L^2(\Omega)$. Direct calculation shows that if θ is an eigenvalue of A_s, then $\theta^2 + \beta\theta$ is an eigenvalue of F_s^*. That is, if $A_s\bar{\xi} = \theta\bar{\xi}$ for some $\bar{\xi} \in ([H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega))^n$, then

$$\Delta\bar{u} + \hat{\lambda}(s)\bar{B}(u_s)\bar{u} = (\theta^2 + \beta\theta)\bar{u},$$

where $\bar{\xi} = \mathrm{col}(\bar{u}_1, \bar{v}_1, \ldots, \bar{u}_n, \bar{v}_n)$ and $\bar{u} = \mathrm{col}(\bar{u}_1, \ldots, \bar{u}_n)$. In order to obtain the stability of the steady-state u_s, we will impose an additional assumption:

$[H4]$ $\bar{B}(u_s(x))$ is symmetric in $\bar{\Omega}$ for each $s \in (0, \delta^*)$.

By Theorem 2.4 and hypotheses $[H4]$, as an operator on Y, the eigenvalues of $F_s^* = \Delta + \hat{\lambda}(s)\bar{B}(u_s)$ are all strictly negative for $s \in (0, \delta^*)$. However, the eigenvalues of $\Delta + \hat{\lambda}(s)\bar{B}(u_s)$ are the same, as an operator on Y or $[H^2(\Omega) \cap H_0^1(\Omega)]^n$. Consequently, the eigenvalues θ of A_s, $s \in (0, \delta^*)$, satisfies

$$\theta^2 + \beta\theta - \eta = 0,$$

where η are negative real numbers. Thus, the real parts of θ are all strictly negative. The operator A_s generates a strongly continuous semigroup $T_s(t)$ on $X := [H_0^1(\Omega) \times L^2(\Omega)]^n$ with domain $D(A_s) := ([H^2(\Omega) \cap H_0^1(\Omega)] \times H_0^1(\Omega))^n$. Using the fact that the real parts of θ are negative, we now show the following stability result for the linearized system (3.2).

Theorem 3.1. *Assume hypotheses $[H1^*], [H2], [H3]$ and each entry of G is in $C^2(R^n)$. Suppose further that $\bar{B}(u_s(x))$ is symmetric as described in $[H4]$, then the semigroup of bounded linear operator $T_s(t)$, $0 \le t < \infty$, on X satisfies*

$$\|T_s(t)\| \le Me^{-kt}$$

for some positive constants M and k, which may depend on s.

Proof. Let p be a large enough positive constant such that

$$y^T[\hat{\lambda}(s)\bar{B}(u_s(x)) - pI]y \le -\omega(y^T.y)$$

for all $s \in (0, \delta^*)$, $x \in \bar{\Omega}$, where ω is a positive constant. Define the operator $S := -(\Delta + \hat{\lambda}\bar{B} - pI)^{-1}$ from $[L^2(\Omega)]^n$ into $[L^2(\Omega)]^n$ as follows. For each $f \in [L^2(\Omega)]^n$, let $g := S[f]$, where $g \in [H_0^1(\Omega)]^n \cap [H^2(\Omega)]^n$, and

$$<< \nabla g, \nabla h >> + \int_\Omega h^T(-\hat{\lambda}\bar{B} + pI)g\,dx = \int_\Omega h^T f\,dx$$

for all $h \in [H_0^1(\Omega)]^n$. Here $<< >>$ is the real inner product in $[L^2(\Omega)]^{nN}$. The operator $S : [L^2(\Omega)]^n \to [L^2(\Omega)]^n$ is well-defined and compact by means of the Lax-Milgram Theorem and Sobolev imbedding (cf. [3]). From the symmetric property of $\bar{B}$, we can readily verify that the operator S is symmetric as follows. Let $f, h \in [L^2(\Omega)]^n$, $g = S[f], q = S[h]$, and $< >$ denotes the real inner product in $[L^2(\Omega)]^n$. Then we have $< Sf, h > = < g, -(\Delta + \hat{\lambda}\bar{B} - pI)q > = < -\Delta g, q > + < g, -(\hat{\lambda}\bar{B} - pI)q > = < -\Delta g, q > + < -(\hat{\lambda}\bar{B} - pI)g, q > = < f, q > = < f, Sh >$. From the theory of compact symmetric operators, we assert that there exists a countable orthonormal basis of $[L^2(\Omega)]^n$ consisting of eigenfunctions of S. Direct computation shows that if η is an eigenvalue of $(\Delta + \hat{\lambda}\bar{B})$, then $1/(p - \eta)$ is an eigenvalue of S. Theorem 2.4 and the additional assumption that $\bar{B}$ is symmetric thus imply that all the eigenvalues of S are of the form $1/(p + \alpha_i)$, with $0 < \alpha_1 < \alpha_2 < \dots$, (Note that α_i depends on s). There exist corresponding eigenfunctions $\{\phi_m\}_{m=1}^{\infty}$, which form an orthonormal basis in $[L^2(\Omega)]^n$.

Let $[\tilde{H}_0^1(\Omega)]^n$ denotes the real Hilbert space of functions in $[H_0^1(\Omega)]^n$ with inner product

$$\tilde{J}[g, q] := << \nabla g, \nabla q >> + \int_\Omega q^T[-\hat{\lambda}\bar{B} + pI]g\,dx.$$

We verify that

$$\tilde{J}[\phi_j, \phi_k] = < -\Delta\phi_j, \phi_k > + \int_\Omega \phi_k^T[-\hat{\lambda}\bar{B} + pI]\phi_j\,dx = < (\alpha_j + p)\phi_j, \phi_k > .$$

Thus, $\{\phi_m/(\alpha_m + p)^{1/2}\}_{m=1}^{\infty}$ form an orthonormal set in $[\tilde{H}_0^1(\Omega)]^n$. Further, the identity

$$\tilde{J}[\phi_j, g] = \int_\Omega g^T[-\Delta\phi_j - \hat{\lambda}\bar{B} + pI]\phi_j\,dx \ = < (\alpha_j + p)\phi_j, g >$$

implies that if $\tilde{J}[\phi_j, g] = 0$ for each $j = 1, 2 \dots$, then we also have $< \phi_j, g > = 0$. This implies that $\{\phi_m/(\alpha_m + p)^{1/2}\}_{m=1}^{\infty}$ form an orthonormal basis in $[\tilde{H}_0^1(\Omega)]^n$. If $\hat{u} \in [\tilde{H}_0^1(\Omega)]^n$, we can assert that the series

$$\hat{u} = \sum_{m=1}^{\infty} < \hat{u}, \phi_m > \phi_m \qquad \text{converges in } [L^2(\Omega)]^n,$$

and further

$$\hat{u} = \sum_{m=1}^{\infty} \tilde{J}[\hat{u}, \phi_m/(\alpha_m + p)^{1/2}]\{\phi_m/(\alpha_m + p)^{1/2}\}$$
$$= \sum_{m=1}^{\infty} (\alpha_m + p)^{1/2} < \hat{u}, \phi_m > \{\phi_m/(\alpha_m + p)^{1/2}\} = \sum_{m=1}^{\infty} < \hat{u}, \phi_m > \phi_m$$

actually converges in $[\tilde{H}_0^1(\Omega)]^n$.

For $\phi \in D(A_s)$, let $\hat{w} = \hat{w}(t;\phi) := T_s(t)\phi \in D(A_s)$, we know from semigroup theory that $d\hat{w}/dt \in X$ for $t > 0$. That is, if we let $\hat{w} = (\hat{u}_1, \hat{v}_1, \ldots, \hat{u}_n, \hat{v}_n)$, and consider $\hat{u}_i$ as functions from $[0,\infty)$ into X, then $d\hat{u}_i/dt \in H_1^0(\Omega)$, and $d^2\hat{u}_i/dt^2 \in L^2$. This implies that if we let $\hat{u} =$ col.$(\hat{u}_1, \ldots, \hat{u}_n)$, $\hat{v} = \text{col}(\hat{v}_1, \ldots, \hat{v}_n)$ and $r_m(t) :=< \hat{u}(t), \phi_m >$, then the series

$$d\hat{u}/dt = \sum_{m=1}^{\infty} r'_m(t)\phi_m \qquad \text{converges in} \quad [H_1^0(\Omega)]^n, \text{ and}$$

$$d^2\hat{u}/dt^2 = \sum_{m=1}^{\infty} r''_m(t)\phi_m \qquad \text{converges in } [L^2(\Omega)]^n,$$

for $t > 0$, where $r'_m(t)$ and $r''_m(t)$ are the first and second derivatives of $r_m(t)$. From semigroup theory and the structure of the operator A_s, we find that $\hat{u}(t)$ satisfies

$$d^2\hat{u}/dt^2 + \beta d\hat{u}/dt = [\Delta + \hat{\lambda}(s)\bar{B}(u_s)]\hat{u}, \quad \text{for } t > 0.$$

Taking inner product with ϕ_m, we obtain

$$r''_m(t) + \beta r'_m(t) = -\alpha_m r_m(t) \quad \text{for } t > 0,\ m = 1, 2, \ldots \tag{3.3}$$

Denote $\phi := \text{col.}(g_1, h_1, \ldots, g_n, h_n) \in D(A_s) = [(H^2 \cap H_0^1) \times H_0^1]^n$, with

$$\begin{aligned} g &= \sum_{m=1}^{\infty} < g, \phi_m > \phi_m \\ &= \sum_{m=1}^{\infty} (\alpha_m + p)^{1/2} < g, \phi_m > [\phi_m/(\alpha_m + p)^{1/2}] \quad \text{in } [H_0^1(\Omega)]^n, \\ h &= \sum_{m=1}^{\infty} < h, \phi_m > \phi_m \qquad\qquad \text{in } [L^2(\Omega)]^n. \end{aligned}$$

Let $T_s(t)\phi :=$ col $(\hat{u}_1, \hat{v}_1, \ldots, \hat{u}_n, \hat{v}_n)$. From (3.3), we find

$$\begin{aligned} r_m(t) &=< g, \phi_m > y_m(t) + < h, \phi_m > z_m(t) \\ r'_m(t) &=< g, \phi_m > y'_m(t) + < h, \phi_m > z'_m(t) \end{aligned}$$

where $y_m(t)$ and $z_m(t)$ satisfy the same equations for $r_m(t)$ in (3.3), with initial conditions:

$$y_m(0) = 1, \quad y'_m(0) = 0 \quad \text{ and } \quad z_m(0) = 0, \quad z'_m(0) = 1.$$

For those m with $4\alpha_m > \beta^2$, we have

$$\begin{aligned} y_m(t) &= e^{-\beta t/2}\cos(4\alpha_m - \beta^2)^{1/2} t/2, \\ z_m(t) &= 2(4\alpha_m - \beta^2)^{-1/2} e^{-\beta t/2}\sin(4\alpha_m - \beta^2)^{1/2} t/2. \end{aligned} \tag{3.4}$$

Expanding $\hat{u}$ in $[\tilde{H}_0^1(\Omega)]^n$, and using (3.4) for all large m, we find

$$\begin{aligned} \|\hat{u}\|_{\tilde{H}^1} &= \sum_{m=1}^{\infty} [< g, \phi_m > y_m(t) + < h, \phi_m > z_m(t)]^2 (\alpha_m + p) \\ &\leq \hat{K} e^{-\epsilon t} \{ \sum_{m=1}^{\infty} < g, \phi_m >^2 (\alpha_m + p) + \sum_{m=1}^{\infty} < h, \phi_m >^2 \}, \\ \|\hat{v}\|_{L^2} &= \sum_{m=1}^{\infty} [< g, \phi_m > y_m'(t) + < h, \phi_m > z_m'(t)]^2 \\ &\leq \hat{K} e^{-\epsilon t} \{ \sum_{m=1}^{\infty} < g, \phi_m >^2 (\alpha_m + p) + \sum_{m=1}^{\infty} < h, \phi_m >^2 \}. \end{aligned}$$

Since $[\tilde{H}_0^1(\Omega)]^n$ and $[H_0^1(\Omega)]^n$ are equivalent, we obtain

$$\|T_s(t)\phi\|_X \leq M e^{-\epsilon t} \|\phi\|_X$$

for all ϕ in $D(A_s)$. Since $D(A_s)$ is dense in X, the proof of Theorem 3.1 is complete.

If $(u_1(t), v_1(t), \ldots, u_n(t), v_n(t))$ is a solution of (3.1), then its difference with the steady-state, i.e., $\mathrm{col}(w_1(t), w_2(t), \ldots, w_{2n}(t)) := \mathrm{col}(u_1(t), v_1(t), \ldots, u_n(t), v_n(t)) - \mathrm{col}((u_s)_1, 0, \ldots, (u_s)_n, 0)$, can be interpreted as a solution of:

$$\frac{dw}{dt} = A_s w + \hat{\lambda}(s)\gamma(w), \quad w = \mathrm{col}(w_1, w_2, \ldots, w_{2n}), \tag{3.5}$$

where

$$\begin{aligned} \gamma(w) : &= \mathrm{col}(0, z_1(\tilde{w}), 0, z_2(\tilde{w}), \ldots, 0, z_n(\tilde{w})), \\ \tilde{w} &= \mathrm{col}(\tilde{w}_1, \tilde{w}_2, \ldots, \tilde{w}_n) := \mathrm{col}(w_1, w_3, \ldots, w_{2n-1}), \\ z &= \mathrm{col}(z_1(\tilde{w}), z_2(\tilde{w}), \ldots, z_n(\tilde{w})) \\ &:= [G(u_s + \tilde{w})(u_s + \tilde{w}) - G(u_s)u_s - J(u_s)\tilde{w}]. \end{aligned}$$

We now clarify some terminologies and hypotheses which we will be using. A function $h : X \to X$ is said to satisfy a local Lipschitz condition if for every constant $c \geq 0$, there is a constant $L(c)$ such that

$$\|h(w^*) - h(w^{**})\|_X \leq L(c)\|w^* - w^{**}\|_X$$

holds for all $w^*, w^{**} \in X$ with $\|w^*\|_X \le c$ and $\|w^{**}\|_X \le c$. The function $\gamma(w)$ described above is said to satisfy property $[P]$ if

$[P]$ $\|z(\tilde{w})\|_{[L^2(\Omega)]^n} = o(\|\tilde{w}\|_{[H^1(\Omega)]^n})$, as $\|\tilde{w}\|_{[H^1(\Omega)]^n}$ tends to 0.

Remark 3.1. Since the norm in X uses the L^2-norm of the even components and the entries of $G(u)$ is in $C^2(R^n)$, it can be readily verified by using Sobolev imbedding or Morrey's inequality that the function $\gamma : X \to X$ satisfies a local Lipschitz condition, in the case $N = 1$. By Theorem 1.4 of Chapter 6 in [9], equation (3.5) has a unique mild solution w on $[0, t_{\max})$, for every given initial condition: $w(0) = w_0 \in X$. Moreover, if $t_{\max} < \infty$, then $\|w(t)\|_X \to \infty$, as $t \to \infty$.

Remark 3.2. Using the assumption that the entries of G are in C^2 and Sobolev imbedding again, we can readily verify that in the case $N = 1$, the function $\gamma(w)$ described above satisfies property $[P]$.

Remark 3.3. The even components of $\gamma(w)$ are expressed by $z(\tilde{w})$, where $\tilde{w}$ is an n-vector function consisting of the odd components of w. Careful calculations shows that the j-th component, $1 \le j \le n$, of $z(\tilde{w})$ can be written as:

$$\begin{aligned}
&\sum_{k=1}^{n} (u_s)_k < \nabla g_{jk}(u_s + \tau^*_{jk}\tilde{w}) - \nabla g_{jk}(u_s), \tilde{w} > \\
&+ \sum_{k=1}^{n} \tilde{w}_k (g_{jk}(u_s + \tilde{w}) - g_{jk}(u_s)) \\
&= \sum_{k=1}^{n} (u_s)_k \sum_{i=1}^{n} < (\nabla(\partial g_{jk}/\partial u_i)(u_s + \tau^{**}_{jk}\tilde{w}), \tau^*_{jk}\tilde{w} > \tilde{w}_i \\
&+ \sum_{k=1}^{n} \tilde{w}_k < (\nabla g_{jk})(u_s + \hat{\tau}_{jk}\tilde{w}), \tilde{w} >,
\end{aligned}$$

where $0 < \tau^{**}_{jk} < \tau^*_{jk} < 1, 0 < \hat{\tau}_{jk} < 1$. When the space dimension N is 3 or 4, Gagliardo-Nirenberg-Sobolev inequality asserts that the L^4 norm of a function in $\Omega \subset R^N$ is bounded by a constant multiple of its $H^1(\Omega)$ norm. If we assume all the first and second partial derivatives of g_{jk}, $1 \le j, k \le n$, are bounded in R^n, then from the above formula, we can readily obtain

$$\|\gamma(w)\|_X = \|z(\tilde{w})\|_{[L^2(\Omega)]^n} \le K(\|\tilde{w}\|_{[L^4(\Omega)]^n})^2$$

for all $w \in X = [H^1_0(\Omega) \times L^2(\Omega)]^n$. Thus, property $[P]$ is satisfied. The local Lipschitz condition of $\gamma : X \to X$ can be verified similarly. In short, if all the entries of $G(u)$ are in $C^2(R^n)$ and has bounded first and second partial derivatives in R^n, then the function γ satisfies the local Lipschitz condition and property $[P]$ described above, provided the space dimension N of Ω is 3 or 4.

Using Gronwall inequality type argument, we then use semigroup theory as in [4] or [9] to prove the following asymptotic stability theorem by means of Theorem 3.1.

Theorem 3.2. *Assume hypotheses* $[H1^*], [H2], [H3], G$ *is in* $C^2(R^n)$, *and* $\bar{B}(u_s)$ *is symmetric for each* $s \in (0, \delta^*)$ *as described in* $[H4]$. *Suppose that in equation* (3.5), *the function* γ *satisfies a local Lipschitz condition and property* $[P]$, *then the trivial steady-state solution* $w = 0$ *of equation* (3.5) *is locally asymptotically stable. That is, there exist positive constants* $\epsilon, \hat{M}, \hat{k}$ *such that if the initial condition* $w(0) := (w_1(0), w_2(0), \ldots, w_{2n}(0))$ *satisfies* $\|w(0)\|_X \leq \epsilon$, *then the unique mild solution of* (3.5) *satisfies*

$$\|w(t)\|_X \leq \hat{M} e^{-\hat{k}t} \|w(0)\|_X. \tag{3.6}$$

Proof. The local Lipschitz condition of γ insures the existence of a unique mild solution $w(t) \in X$ of (3.5) as described in Theorem 1.4 in Chapter 6 in [9]. The mild solution $w(t)$ can be expressed by:

$$w(t) = T_s(t)w(0) + \int_0^t T_s(t-\tau)\hat{\lambda}(s)\gamma(w(\tau))d\tau, \;\; w(0) \in X, \tag{3.7}$$

where T_s is the strongly continuous semigroup generated by A_s. Let M and k be the positive constants given by Theorem 3.1 for T_s, we obtain for $t > 0$ inside the interval of existence of $w(t)$:

$$\|w(t)\|_X \leq M e^{-kt} \|w(0)\|_X + \int_0^t M e^{-k(t-\tau)} \|\hat{\lambda}(s)\gamma(w(\tau))\|_X d\tau. \tag{3.8}$$

Without loss of generality, we may assume $M > 1$. By property $[P]$ for γ, there exist $\rho > 0$ so small such that

$$\|\hat{\lambda}(s)\gamma(w)\|_X \leq [k/(2M)] \|\tilde{w}\|_{[H^1]^n} \leq [k/(2M)] \|w\|_X \tag{3.9}$$

as long as $\|w\|_X \leq \rho$. Let $\|w(0)\|_X \leq \rho/M$, the solution will satisfy $\|w(t)\|_X \leq \rho$ for sufficiently small $t > 0$. For such small $t > 0$, (3.8) and (3.9) lead to

$$e^{kt} \|w(t)\|_X \leq M \|w(0)\|_X + (k/2) \int_0^t e^{k\tau} \|w(\tau)\|_X d\tau. \tag{3.10}$$

Thus, Gronwall's inequality gives

$$\|w(t)\|_X \leq M \|w(0)\|_X e^{-(k/2)t} \leq \rho e^{-(k/2)t} \tag{3.11}$$

as long as $\|w(t)\|_X \leq \rho$. Thus, $w(t)$ exists for all $t > 0$, if we choose $w(0)$ to satisfy $\|w(0)\|_X \leq \rho/M$; and (3.11) is valid for all $t > 0$ in such cases. Consequently, Theorem 3.2 is valid by choosing $\epsilon = \rho/M$, $\hat{M} = M$, and $\hat{k} = k/2$.

We now return to equation (3.1), which is the first order system form of (1.1). Equation (3.1) can be written as:

$$\frac{\partial y}{\partial t} = \tilde{A}_s y + \hat{\lambda}(s) q(y), \tag{3.12}$$

where $y = \text{col}(y_1, y_2, \ldots, y_{2n}) = \text{col}(u_1, v_1, \ldots, u_n, v_n)$, $\tilde{A}_s$ is the same as A_s with all $\bar{b}_{ij}$ replaced by the corresponding b_{ij}, and $q(y) := \text{col}(0, \hat{q}_1(\hat{y}), \ldots, 0, \hat{q}_n(\hat{y})), \hat{y} = \text{col}(y_1, y_3, \ldots, y_{2n-1})$, col $(\hat{q}_1(\hat{y}), \ldots, \hat{q}_n(\hat{y})) := G(\hat{y})\hat{y}$. We will express the results in Theorem 3.2 in terms of the mild solutions in X of the semigroup form of (3.12), that is,

$$\frac{dy}{dt} = \tilde{A}_s y + \hat{\lambda}(s) q(y), \quad y(t) \in X, t \geq 0. \tag{3.13}$$

The constant function $\eta(t) :=$ col $((u_s)_1, 0, \ldots, (u_s)_n, 0)$ is a strong solution of

$$\frac{d\eta}{dt} = A_s \eta + \hat{\lambda}(s) h(u_s), \tag{3.14}$$

where $h(u_s) =$ col $(0, \hat{h}_1(u_s), \ldots, 0, \hat{h}_n(u_s))$, and col $(\hat{h}_1(u_s), \ldots, \hat{h}_n(u_s)) = G(u_s)u_s - J(u_s)u_s$. Thus, $\eta(t)$ satisfies

$$\eta(t) = T_s(t)\eta(0) + \int_0^t T_s(t-\tau)\hat{\lambda}(s)h(u_s)d\tau.$$

If $w(t)$ is a mild solution of (3.5), then $w(t) + \eta$ satisfies

$$w(t) + \eta = T_s(t)[w(0) + \eta] + \int_0^t T_s(t-\tau)\hat{\lambda}(s)\{\gamma(w(\tau)) + h(u_s)\}d\tau. \tag{3.15}$$

If $w(0) \in D(A_s)$, then $y = w(t) + \eta$ is a strong solution of

$$\frac{dy}{dt} = A_s y + \hat{\lambda}(s)[\gamma(w) + h(u_s)] = \tilde{A}_s y + \hat{\lambda}(s)[\gamma(w) + h(u_s) + r(y)],$$

where $r(y) = col.(0, \hat{r}_1(\hat{y}), \ldots, 0, \hat{r}_n(\hat{y}))$, col $(\hat{r}_1(\hat{y}), \ldots, \hat{r}_n(\hat{y})) = J(u_s)\hat{y}$. Thus, if $w(0) \in D(A_s)$, we also have

$$\begin{aligned} w(t) + \eta &= \tilde{T}_s(t)[w(0) + \eta] \\ &\quad + \int_0^t \tilde{T}_s(t-\tau)\hat{\lambda}(s)\{\gamma(w(\tau)) + h(u_s) + r(w(\tau) + \eta)\}d\tau, \end{aligned} \tag{3.16}$$

where $\tilde{T}_s$ is continuous semigroup generated by $\tilde{A}_s$. By the density of $D(A_s)$ in X and the strong continuity of T_s and $\tilde{T}_s$, the right-hand sides of both (3.15) and (3.16) are still equal for all cases with $w(0) \in$ X. Since $\gamma(w) + h(u_s) + r(w+\eta) = q(w+\eta)$, we obtain from (3.16) that $w(t)+\eta$ is a mild solution of equation (3.13).

Theorem 3.2 and the above arguments lead to the following conclusion concerning solutions of (1.1) or (3.1).

Theorem 3.3. *Assume all the hypotheses of Theorem* 3.2 (*including those concerning* γ). *Then if the initial condition*

$$(y_1(0), y_2(0), \ldots, y_{2n}(0)) := (u_1(0), v_1(0), \ldots, u_n(0), v_n(0))$$

is given sufficiently close to $((u_s)_1, 0, \ldots, (u_s)_n, 0)$ *in* X, *the unique mild solution* $y(t) = (u_1(t), v_1(t), \ldots, u_n(t), v_n(t))$ *of the initial value problem corresponding to* (3.13) *exists for all* $t > 0$. *Moreover, there exist positive constants* $\epsilon, \hat{M}, \hat{k}$ *such that if*

$$\|(u_1(0), v_1(0), \ldots, u_n(0), v_n(0)) - ((u_s)_1, 0, \ldots, (u_s)_n, 0)\|_X \le \epsilon,$$

then the mild solution of (3.13) *satisfies*

$$\|(u_1(t), v_1(t), \ldots, u_n(t), v_n(t)) - ((u_s)_1, 0, \ldots, (u_s)_n, 0)\|_X \le$$

$$\hat{M}e^{-\hat{k}t}\|(u_1(0), v_1(0), \ldots, u_n(0), v_n(0)) - ((u_s)_1, 0, \ldots, (u_s)_n, 0)\|_X$$

for all $t > 0$. *Recall that* (3.13) *is the semigroup form of* (1.1) *or* (3.1).

For applications of the theorems above, consider the following examples.

Example 3.1. Let $\Omega = \{x = (x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 < 1\}$. Consider (1.1), with $\beta = 1, u = \text{col } (u_1, u_2, u_3)$,

$$B(x) = \begin{bmatrix} \sin 2\pi(x_1^2+x_2^2+x_3^2) & 1 & 2 \\ 1 & \cos 2\pi(x_1^2+x_2^2+x_3^2) & 3 \\ 2 & 3 & -(x_1^2+x_2^2+x_3^2) \end{bmatrix},$$

$$G(u)u = \begin{bmatrix} -u_1 & -u_1 & 0 \\ -u_1/2 & -\sin u_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -u_1^2 - u_1u_2 \\ -(1/2)u_1^2 - u_2 \sin u_2 \\ 0 \end{bmatrix}.$$

It is clear that $B(x)$ satisfies $[H1^*]$ and $[H2], G(u)$ satisfies $[H3]$ with every entry in C^2, and

$$\bar{B}(u_s(x)) = B(x) + \begin{bmatrix} -2(u_s)_1 - (u_s)_2 & -(u_s)_1 & 0 \\ -(u_s)_1 & -(u_s)_2\cos(u_s)_2 - \sin(u_s)_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is symmetric as specified in $[H4]$. Since the first and second partial derivatives of $G(u)$ are bounded, by Remark 3.3 the function γ satisfies the local Lipschitz condition and property $[P]$. Thus, all assumptions of Theorem 3.3 are satisfied, and the steady-state $((u_s)_1, 0, (u_s)_2, 0, (u_s)_3, 0)$ is locally asymptotically stable in the sense described by Theorem 3.3 for the equation (3.13), which is the semigroup form for (1.1). Here, we are considering those $\lambda = \hat{\lambda}(s)$ close to the right of $\hat{\lambda}_0$, $s \in (0, \delta^*)$, as described in Theorem 2.4.

Example 3.2. Let $\Omega = [0, \pi]$, consider (1.1) with $\beta = 1, u = \text{col}\,(u_1, u_2, u_3)$,

$$B(x) = \begin{bmatrix} 5 & 6 & 7 \\ 6 & 10 & 4 \\ 7 & 4 & -1 \end{bmatrix}$$

$$G(u)u = \begin{bmatrix} -u_1 & -\sin(u_1u_2) & 0 \\ -\sin(u_1u_2) & -u_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -u_1^2 - u_2\sin(u_1u_2) \\ -u_1\sin(u_1u_2) - u_2^3 \\ 0 \end{bmatrix}.$$

It can be verified readily that $\bar{B}(u_s(x))$ is symmetric and all the hypotheses of Theorem 3.3 are satisfied. Here, $N = 1$, by Remarks 3.1 and 3.2 we find γ satisfies the local Lipschitz condition and $[P]$, without requiring the boundedness of the first and second partial derivatives of G.

4. Global bifurcation and example

In this section, we consider the global behavior of the bifurcation curve of positive solution of equation (2.6). We find conditions so that equation (1.1) has a positive steady-state solution with $\lambda = 1$. We now introduce a few notation and hypotheses. Let $\mathcal{G} := \{(\lambda, w) \in R^+ \times P : F(\lambda, w) = 0, \lambda > 0$ and $w \in P\backslash\{0\}\}$, $\bar{\mathcal{G}}$ denotes the closure of $\mathcal{G}$; and define the following conditions:

[A1] There exists an integer $m, 1 \le m \le n$ such that $b_{mj}(x) > 0, b_{jm}(x) > 0$ in $\bar{\Omega}$ for all $j \ne m$, and the m-th component of $G(u)u$ is $\equiv 0$ in $\bar{\Omega}$.

[A2] For each $j \ne m$, the j-th component of $G(u)u$ is expressible as $k_j(u_1, \ldots, u_n)u_j$ with $k_j(u_1, \ldots, u_n) \in C^0$ for $u_i \ge 0$, $i = 1, \ldots, n$.

Theorem 4.1. *Under hypotheses* $[H1], [H1^*], [H2], [A1]$, *and* $[A2]$, *the component of* $\bar{\mathcal{G}}$ *containing the point* $(\lambda_0, 0)$ *is unbounded, and* $\bar{\mathcal{G}} \bigcap (R \times \partial P) = (\lambda_0, 0)$.

Proof. Theorem 2.1 and (2.5) imply by means of Theorem 29.2 in [2] that the component of $\bar{\mathcal{G}}$ containing the point $(\lambda_0, 0)$ is unbounded. Let $(\lambda_i, w_i) \in \mathcal{G}$, $i = 1, 2, ...$ be a sequence tending to a limit point $(\bar{\lambda}, \bar{w})$ in $R \times \partial P$, and $(\bar{\lambda}, \bar{w}) \ne (\lambda_0, 0)$. We now show $\bar{w} = \text{col}(\bar{w}_1, \ldots, \bar{w}_n)$ must satisfy $\bar{w}_i \equiv 0$, for each i, if $\bar{\lambda} > 0$. Consider the first case when there exists some $x_0 \in \Omega$, where $\bar{w}_m(x_0) = 0$. The equation

$$-\Delta \bar{w}_m = \bar{\lambda} \sum_{j=1}^{n} b_{mj} \bar{w}_j \geq 0$$

implies that $\bar{w}_m \equiv 0$ in $\bar{\Omega}$; and subsequently, the right hand side of this equation and [A1] imply that $\bar{w}_j \equiv 0$ for each $j \neq m$ too. Hence, $\bar{w} \equiv 0$ in this case.

Consider the second case when $\bar{w}_m(x) > 0$ for all $x \in \Omega$. For each $j \neq m$, consider the problem:

$$\begin{cases} -\Delta z(x) = \bar{\lambda} b_{jm} \bar{w}_m + \bar{\lambda} \sum_{k \neq m,j} b_{jk} \bar{w}_k + \bar{\lambda} b_{jj} z(x) \\ \qquad\qquad +\bar{\lambda} k_j(\bar{w}_1, .., \bar{w}_{j-1}, z(x), \bar{w}_{j+1}, .., \bar{w}_n) z \qquad \text{in } \Omega, \\ z = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

The function $z \equiv 0$ is a lower solution. From $[A1], [A2]$ and $\partial \bar{w}_m / \partial \nu < 0$ on $\partial\Omega$, we have

$$\begin{aligned} \lambda_1 \delta\phi(x) < \bar{\lambda} b_{jm}(x) \bar{w}_m + \bar{\lambda} \sum_{k \neq m,j} b_{jk} \bar{w}_k + \bar{\lambda} b_{jj} \delta\phi \\ + \bar{\lambda} k_j(\bar{w}_1, \ldots, \bar{w}_{j-1}, \delta\phi, \bar{w}_{j+1}, \ldots, \bar{w}_n) \delta\phi \qquad \text{in} \quad \Omega \end{aligned}$$

for $\delta > 0$ sufficiently small. Thus, $z = \delta\phi$ is a family of lower solution for problem (4.1) for such δ. Hence, the solution of (4.1) satisfies $\bar{w}_j(x) > \delta\phi(x)$ in Ω. This contradicts the fact that $\bar{w} \in \partial P$. We must thus have $\bar{w} \equiv 0$ in Ω in any case, for $\bar{\lambda} \geq 0$.

Next, define $z_i := w_i / \|w_i\|_E, i = 1, 2, \ldots$; they satisfy

$$z_i + \lambda_i \Delta^{-1}[B z_i] + \lambda_i \Delta^{-1}[G(w_i) w_i] / \|w_i\|_E = 0. \tag{4.2}$$

Since $\Delta^{-1} B$ is compact, there exists a subsequence (again denoted by $\{z_i\}$ for convenience) such that $\Delta^{-1}[B z_i]$ converges in E. Since $\Delta^{-1}[G(w_i) w_i] / \|w_i\|_E$ tends to zero in E, as $\|w_i\|_E \to \|\bar{w}\|_E = 0$, equation (4.2) implies that $\{z_i\}$ converges in E to a function z_0 say, and

$$z_0 = -\bar{\lambda} \Delta^{-1}[B z_0].$$

Moreover, we have $z_0 \in P$, since $w_i \in P$; and $z_0 \not\equiv 0$ since $\|z_i\|_E = 1$. Hence, we must have $\bar{\lambda} > 0$. The uniqueness part Theorem 2.1 implies that $\bar{\lambda} = \lambda_0$. Consequently, we must have $(\bar{\lambda}, \bar{w}) = (\lambda_0, 0)$. This completes the proof of Theorem 4.1.

Let λ_1 be the principal eigenvalue of the Laplace operator on Ω with zero Dirichlet boundary data, and $\phi > 0$ be a positive eigenfunction.

Theorem 4.2. *Assume all the hypotheses in Theorem* 4.1 *and* $b_{mm}(x) \equiv 0$ *in assumption* [*A*1]. *Suppose further that*

[*A*3] *There exists constant vector* $\vec{g} = \text{col}\ (g_1, ..., g_n)$, $g_i > 0$, *such that* $B(x)\vec{g} > \lambda_1 \vec{g}$ *for all* $x \in \Omega$.

[*A*4] *For each* $j \neq m$, *the function* k_j *described in* [*A*2] *has the property* $\limsup_{N\to\infty} N^{-1} \max\{k_j(u_1, .., u_{j-1}, N, u_{j+1}, ...u_n) : 0 \leq u_\tau \leq N, \tau \neq j\} < 0$. *Then* (1.1) *has a steady-state solution in* P *for* $\lambda = 1$.

Proof. Let $\hat{z} = \phi\vec{g}$, [*A*3] implies that there exists $\mu > 1$ such that $B\hat{z} \geq \mu\lambda_1\hat{z}$ in $\bar{\Omega}$. Hence, $(-\Delta)^{-1}(B\hat{z}) \geq \mu\lambda_1(-\Delta)^{-1}(\phi\vec{g}) = \mu\phi\vec{g} = \mu\hat{z}$. By Theorem 2.5 in [5], we have $1/\hat{\lambda}_0 \geq \mu > 1$, that is $\hat{\lambda}_0 < 1$.

The equation for u_m and $b_{mm} \equiv 0$ imply that if $\lambda \in [0, C]$ for some C, and all the components $u_j, j \neq m$ of a steady state solution of (1.1) have the property $|u_j| \leq M$, then u_m must satisfy $|u_m| \leq KM$ for some constant K independent of M. We now show that there must indeed exist some constant M, such that if $\lambda \in (0, C]$ and $(\lambda, u) \in \mathcal{G}$, then $0 \leq u_j \leq M$ for all $j \neq$ m. Otherwise, there exists u_r, $r \neq m$, and a point $x* \in \Omega$, where $u_r(x*) = \max_{j\neq m}\{\sup_{x\in\bar{\Omega}} u_j(x)\} > M$. Let $p = u_r(x*)/M > 1$, then we have the property: $0 \leq u_j(x) \leq pM$ for all $x \in \bar{\Omega}, j \neq m$. Consider the equation satisfied by u_r at the point $x* \in \Omega$.

$$\begin{aligned}
-\Delta u_r(x*) &= \sum_{j\neq m,r} \lambda b_{rj}u_j + \lambda b_{rm}u_m \\
&\quad + \lambda k_r(u_1, .., u_{r-1}(x*), pM, u_{r+1}(x*), .., u_n)\, u_r(x*) \\
&\leq \lambda[\sum_{j\neq m,r} b_{rj}pM + b_{rm}KpM + \max\{(pM)^{-1}k_r(u_1, .., u_{r-1}, pM, u_{r+1}, .., u_n) \\
&\quad : 0 \leq u_\tau \leq pM, \tau \neq r\}(pM)^2] < 0.
\end{aligned}$$

The last inequality is satisfied for M sufficiently large due to hypothesis [A4]. This contradicts the definition of $x*$. We thus assert that if $\lambda \in [0, C]$, then $|u_i|$ are bounded for each $i = 1, \ldots, n$, if $(\lambda, u) \in \mathcal{G}$. Finally, using gradient estimates by means of (2.6), we conclude that $\mathcal{G}$ cannot be unbounded if $\lambda \in [0, C]$, for some C. Thus, a solution of (2.6) must exist for $\lambda = 1$.

Example 4.1. We consider Example 3.2 with only a single modification. We change the entry b_{33} of B from -1 to 0. Clearly, condition [A1] is satisfied with $m = 3$. For [A2], we define

$$k_1(u_1, u_2) = \begin{cases} -u_1 - (u_2/u_1)\sin(u_1u_2) & \text{if} \quad u_1 \neq 0 \\ -u_2^2 & \text{if} \quad u_1 = 0 \end{cases}$$

$$k_2(u_1,u_2) = \begin{cases} -(u_1/u_2)\sin(u_1u_2) - u_2^2 & \text{if} \quad u_2^2 \neq 0 \\ -u_1^2 & \text{if} \quad u_2 = 0, \end{cases}$$

then k_1 and k_2 are continuous for $u_i \geq 0$. Condition [A3] is satisfied by choosing $\vec{g} = \text{col }(1,1,1)$. Using the formulas for k_1 and k_2, we can verify [A4] is satisfied. Thus, Theorem 4.2 can be applied, and we conclude that (1.1) has a steady-state solution for $\lambda = 1$ in this case. As for Example 3.2 in the last section, we also find the bifurcating positive steady-state is asymptotically stable for λ close to $\hat{\lambda}_0$ for this example. The stability of the steady-state when $\lambda = 1$ remains to be considered.

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