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ON STABILITY OF TRAVELING WAVE SOLUTIONS IN SYNAPTICALLY COUPLED NEURONAL NETWORKS

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(Submitted by: Jerry L. Bona)

Dedicated to Professor Yulin Zhou on the occasion of his eightieth birthday

Abstract. The author is concerned with the asymptotic stability of traveling wave solutions of integral differential equations arising from synaptically coupled neuronal networks. By using complex analytic functions, he proves that there is no nonzero spectrum of some linear operator $\mathcal{L}$ in the region Re $\lambda \geq 0$, and $\lambda = 0$ is a simple eigenvalue. By applying linearized stability criterion, he shows that the traveling wave solutions are asymptotically stable. Additionally, some explicit analytic functions are found for a scalar integral differential equation.

1. Introduction

Consider the following system of integral differential equations

$$\frac{\partial u}{\partial t} + u + w = \alpha K * [H(u-\theta)], \tag{1}$$

$$\frac{\partial w}{\partial t} = \varepsilon(u - \gamma w), \tag{2}$$

where $[f * g](x) = \int_{-\infty}^{\infty} f(x-y)g(y)dy$. The main purpose of this paper is to establish the asymptotic stability of a traveling pulse solution to this system. We will apply *linearized stability criterion* and *complex analytic functions* to achieve our goal. Let $\mathcal{L}$ be the operator obtained by linearizing the integral differential equations about their traveling wave solution, let $\sigma(\mathcal{L})$ be the spectrum of $\mathcal{L}$. The linearized stability criterion says that if

$$\max\{ \text{ Re } \lambda : \lambda \in \sigma(\mathcal{L}), \lambda \neq 0\} \leq -c_0,$$

for some constant $c_0 > 0$, and $\lambda = 0$ is a simple eigenvalue of $\mathcal{L}$, then the stability follows immediately, and vise versa.

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This model, describing spatially structured activity in synaptically coupled neuronal networks, was proposed by Ermentrout and McLeod [10] and then by Pinto and Ermentrout [25]. They have investigated traveling wave front and pulse solutions. They also found a clear relationship between the speed of traveling front solutions and various network features, such as synaptic parameters and axonal conduction velocity. In the presence of local inhibition, the existence of traveling pulse solutions was established as well. Spatially structured activity in synaptically coupled neuronal networks has an immediate consequence in terms of practical applications. Magnetic resonance, as well as other brain imaging techniques are employed both clinically and in research to visualize spatial temporal properties of activity in brain structures. Several disorders, such as cortical epilepsy and migraine are characterized by waves of activity spreading across the surface of the cortex. Thus, stable traveling wave solutions are biologically the most interesting and important solutions. There has been very little formal analysis attempting to understand the mathematical systems that describe spatially extended networks. Numerically, synaptically coupled spatial networks have been shown to exhibit stable traveling wave solutions.

In this system, K can be thought as a probability function, measuring the synaptic contributions from all other neurons to a particular one under consideration. It is an even, nonnegative, piecewise smooth function, for example $K(x)=\frac{1}{2}\cos x$ if $|x|\le\frac{\pi}{2}$, and $K(x)=0$ if $|x|>\frac{\pi}{2}$, such that

$$\int_{-\infty}^{\infty}K(x)dx=1,\ \int_{-\infty}^{\infty}|K'(x)|dx<\infty.$$

Furthermore, $K(x)\le C\exp(-\rho|x|)$, for all $x\in(-\infty,+\infty)$, and for some positive constants C and ρ. H is the Heaviside step function: $H(x)=0$ for all $x<0$ and $H(x)=1$ for all $x>0$. $0<\varepsilon\ll 1$ is sufficiently small. The positive parameters α and γ are in appropriate range, such that the system possesses a traveling wave solution. The parameter $0<\theta<\frac{1}{2}\alpha$ represents the threshold for excitation of certain neuronal networks.

There has been some very nice work in existence, uniqueness and asymptotic stability of traveling wave solutions of *scalar* integral differential equations. See, for example, Amari [2], Bates, Fife, Ren and Wang [3], Chen [6], Chen, Ermentrout and McLeod [8], Ermentrout and McLeod [10], Fife and Wang [16] and [33].

Definition 1. A traveling wave solution of (1)-(2) is a vector-valued function of the form $(\phi(z),\varphi(z))\equiv(\phi(x+vt),\varphi(x+vt))$, where $z\equiv x+vt$, for some constant v, the wave speed. Moreover,

$$\lim_{z\to\pm\infty}(\phi(z),\varphi(z)) = (0,0).$$

In another word, the traveling wave solutions solve the system

$$v\phi_z + \phi + \varphi = \alpha K * [H(\phi-\theta)], \tag{3}$$

$$v\varphi_z = \varepsilon(\phi - \gamma\varphi). \tag{4}$$

It is well known that there are precisely two wave speeds $v_1(\varepsilon)$ and $v_2(\varepsilon)$, for all sufficiently small $0 < \varepsilon \ll 1$, where $0 < v_1(\varepsilon) < v_2(\varepsilon)$. Moreover, there exist a "slow traveling wave" corresponding to $v_1(\varepsilon)$ and a "fast traveling wave" corresponding to $v_2(\varepsilon)$, see Pinto and Ermentrout [25]. The slow wave is unstable because there is a real positive eigenvalue to the operator obtained by linearizing (1)-(2) on the slow wave. On the other hand, it is not known whether the fast wave is stable or not. For simplicity, we write $v(\varepsilon) = v_2(\varepsilon)$ below. This wave speed is very close to the speed of the wave front.

Theorem 1. [25] *Suppose that* $0 < \varepsilon \ll 1$, $\alpha > 0$, $0 < 2\theta < \alpha$, $\gamma > 0$. *Then there exists a unique traveling pulse solution, with the speed* $v(\varepsilon) > 0$, *to the singularly perturbed system of integral differential equations* (1)–(2).

Remark 1.1. The existence of traveling pulse solutions for large $\varepsilon > 0$ is unknown. Thus, we are only concerned with stability of the pulse for $0 < \varepsilon \ll 1$. Motivated by the stability result of the fast traveling wave solution of the Fitzhugh-Nagumo equations: $u_t = u_{xx} + u(1-u)(u-a) - w, w_t = \varepsilon(u - \gamma w)$, which was proved by Jones [20], we study the asymptotic stability of the fast traveling wave solution of (1)-(2). However, the equations (1)-(2) are quite different from the Fitzhugh-Nagumo equations, so we cannot apply the method employed by Jones. We will use a different approach.

Definition 2. By asymptotic stability, we mean that if the initial data of (1)-(2) is L^∞-close to the traveling wave, i.e., $\|(u_0,w_0) - (\phi,\varphi)\|_{L^\infty} \le \delta$, for some small constant $\delta > 0$, then

$$\sup_{z\in(-\infty,+\infty)} |(U(z,t),W(z,t)) - (\phi(z+h),\varphi(z+h))| \le Ce^{-\rho t},$$

for some positive constants C, ρ and some $h \neq 0$, where $z \equiv x + vt$ and $(U(z,t),W(z,t)) \equiv (u(x,t),w(x,t))$ is the solution of (1)-(2) with

$$(u(x,0),w(x,0)) \equiv (u_0(x),w_0(x)).$$

Theorem 2. *The fast traveling wave solution of the singularly perturbed system* (1)–(2) *is asymptotically stable, as* $t \to +\infty$.

Remark 1.2. When $\varepsilon = 0$, the system (1)-(2) reduces to a scalar integral differential equation, where w is a parameter. For two different values of w, one can find "a traveling wave front" and "a traveling wave back". Both the front and the back are asymptotically stable, as $t \to +\infty$, see Theorems 2.2 and 2.4 in Section 2. Thus, $\lambda = 0$ is a simple eigenvalue and indeed it is the only eigenvalue of the corresponding linearized operators in some region Re $\lambda \geq -c$, where $c > 0$ is a constant. When $0 < \varepsilon \ll 1$, one would guess that at least one of these eigenvalues perturb to small, nonzero eigenvalues for $\mathcal{L}(\varepsilon)$, where the differential operator $\mathcal{L}(\varepsilon)$ is defined by

$$\mathcal{L}(\varepsilon)\psi = -v(\varepsilon)\frac{\partial \psi}{\partial z} - \begin{pmatrix} 1 & 1 \\ -\varepsilon & \varepsilon\gamma \end{pmatrix} \psi + \begin{pmatrix} \mathcal{N} \\ 0 \end{pmatrix}.$$

See (19) in section 3 for the definition of $\mathcal{N}$. The most possible result with the perturbation is that one eigenvalue is $\lambda = 0$, due to translation invariance, and the other one is real negative but small. Intuitively there is no other mechanisms to create a third eigenvalue. We will have to show this rigorously. The smallness of the parameter ε plays a very important role in the exploration of the eigenvalues of $\mathcal{L}(\varepsilon)$. Actually, when we prove that there is no nonzero eigenvalue in Re $\lambda \geq 0$ and when we show that $\lambda = 0$ is a simple eigenvalue, we always have to assume that ε is sufficiently small. In our stability proof, we will only show that there exists some positive constant ρ, such that all nonzero spectrum of $\mathcal{L}(\varepsilon)$ satisfy Re $\lambda \leq -\rho$, and $\lambda = 0$ is simple. This $-\rho$ should be a negative eigenvalue of $\mathcal{L}(\varepsilon)$. Furthermore, it should be of order ε, so that $-\rho = -c_0\varepsilon$, for some constant $c_0 > 0$.

There has been tremendous efforts in establishing stability of traveling wave solutions of reaction-diffusion equations, see [1], [15], [17], [18], [20], [21], [26], [27], [28], [29], [30] and [31]. In particular, during 1972 to 1975, John W. Evans [11]–[14] published a series of very important papers about nerve axon equations. He associated the normal spectrum of an operator obtained by linearizing a system about its traveling wave solutions to the zeroes of a complex analytic function in [14]. Later this function was referred to as the Evans function, denoted by $\mathcal{E}(\lambda)$.

Theorem of Evans. (See [14]) *The Evans function is a complex analytic function and it is real-valued if the eigenvalue parameter λ is real. The complex number λ is an eigenvalue of the operator if and only if $\mathcal{E}(\lambda) = 0$. Moreover, the algebraic multiplicity of an eigenvalue is exactly equal to the order of the zero of the Evans function.*

Evans also studied other good properties of the analytic functions. Nevertheless, for integral differential equations, the idea to construct the Evans functions is totally different from that for reaction-diffusion equations. We will use two steps to define the Evans function. First we construct *the intermediate Evans function* $D(\lambda)$ and then we construct *the Evans function* $\mathcal{E}(\lambda)$. Same results as in the Theorem of Evans also hold for this $\mathcal{E}(\lambda)$.

In the past thirty years, many authors have applied maximum principle (for example, the scalar bistable equation [15]: $u_t = u_{xx} + u(1-u)(u-a)$, where $0 < a < 1/2$ is a constant) or infinite-many conservation laws (for example, the Korteweg-de Vries equations [7], [9] and [24]: $u_t + u_{xxx} + u^p u_x = 0$, where $p = 1, 2$ is an integer) to prove the stability of traveling wave solutions. But for integral differential equations, because there is no available maximum principle or conservation laws, and Evans functions are hard to find, there are relatively much less work so far. We will only use basic ideas in ordinary differential equations to accomplish the stability. In particular, we use the method of variation of parameters and asymptotic analysis of solutions of differential equations to establish the asymptotic stability of the traveling waves. The asymptotic stability results in the present paper validate the numerical observations of stable traveling waves made by previous people. Similar problems, such as [8]:

$$\frac{\partial u}{\partial t} + u = \alpha(1-u)S(K * u^p - \theta),$$

where $S(x) = 1/[1 + \exp(-x/a)]$ and $a > 0$ is a constant, notice that as $a \to 0^+$, $S \to H$ pointwise; and

$$\frac{\partial u}{\partial t} + u = \alpha(1-u)H(K * u^p - \theta),$$

can also be solved using the method presented in this article.

This paper is organized as follows. In Section 2, we investigate the existence and the stability of a traveling wave front and a traveling wave back for a scalar integral differential equation. We also present some explicit Evans functions by choosing special kernel functions K. The explicit Evans functions for system (1)-(2) are extremely difficult to find. In Section 3, The asymptotic stability of the traveling pulse will be studied.

2. A scalar equation

In this section, we first study the spectrum of a linear operator $\mathcal{L}$ and then consider the asymptotic stability of the traveling wave front of the scalar integral differential equation

$$\frac{\partial u}{\partial t} + u = \alpha K * [H(u-\theta)]. \tag{5}$$

This equation is obtained formally by setting $\varepsilon = 0$ and $w = 0$ in (1)-(2).

Theorem 2.1. *Let α and θ be positive constants such that $2\theta < \alpha$. Then there exists a unique wave speed $v = v_0(\frac{\theta}{\alpha}) > 0$, such that*

$$\int_{-\infty}^{0} K(x)dx - \int_{-\infty}^{0} \exp(\frac{x}{v_0})K(x)dx = \frac{\theta}{\alpha}.$$

There exists a unique traveling wave front $\phi(z)$ to

$$v\phi_z + \phi = \alpha \int_{-\infty}^{\infty} K(z-y)H(\phi(y)-\theta)dy,$$

with the wave speed v_0, where $z = x + v_0 t$. The traveling wave front is given by

$$\phi(z) = \alpha \int_{-\infty}^{z} K(x)dx - \alpha \int_{-\infty}^{z} \exp(\frac{x-z}{v_0})K(x)dx,$$

such that $\phi(0) = \theta$, $\phi'(z) > 0$ on $(-\infty, +\infty)$ and

$$\lim_{z\to-\infty} \phi(z) = 0, \quad \lim_{z\to+\infty} \phi(z) = \alpha, \quad \lim_{z\to\pm\infty} \phi'(z) = 0.$$

Proof. Since the traveling wave front is translation invariant, we may fix $\phi(0) = \theta$ and require that $\phi'(z) > 0$ on $(-\infty, +\infty)$, so that $\phi(z) < \theta$ for $z < 0$, and $\phi(z) > \theta$ for $z > 0$. Therefore, the traveling wave equation becomes

$$v\phi_z + \phi = \alpha \int_{-\infty}^{z} K(x)dx.$$

Solving this ordinary differential equation, we get

$$\phi(z) = \frac{\alpha}{v} \int_{-\infty}^{z} \exp\Big[\frac{x-z}{v}\Big]\Big[\int_{-\infty}^{x} K(s)ds\Big]dx.$$

Then integration by parts completes the proof. The detail is straightforward and is omitted. □

In moving coordinate $z \equiv x + vt, U(z,t) \equiv u(z - vt, t)$, where $v > 0$ is the unique wave velocity, the equation (5) is equivalent to

$$\frac{\partial U}{\partial t} + v\frac{\partial U}{\partial z} + U = \alpha K * [H(U-\theta)]. \tag{6}$$

The traveling wave front is a stationary solution of this equation. Thus, the linearization of (6) on the wave front ϕ is

$$\frac{\partial U}{\partial t} + v\frac{\partial U}{\partial z} + U = \frac{\alpha}{\phi'(0)}K(z)U(0,t). \tag{7}$$

We define a linear differential operator $\mathcal{L} : BC^1(R,C) \to BC^0(R,C)$ by

$$\mathcal{L}\psi = -v\frac{\partial\psi}{\partial z} - \psi + \frac{\alpha}{\phi'(0)}K(z)\psi(0), \text{ for } \psi \in BC^1(R,C), \tag{8}$$

where $BC^1(R,C) = \{f$: both f and f' are complex-valued, bounded and continuous functions defined on $(-\infty,+\infty)\}$. The eigenvalue problem is $\mathcal{L}\psi = \lambda\psi$, namely

$$v\frac{\partial\psi}{\partial z} + (\lambda+1)\psi = \frac{\alpha}{\phi'(0)}K(z)\psi(0). \tag{9}$$

The essential spectrum of the operator $\mathcal{L}$ is $\lambda = -1 + ivx$, where $x \in (-\infty,+\infty)$ is a real number. This is a vertical line situated at Re $\lambda = -1$. Define the open, unbounded, simply connected region Ω by $\Omega = \{\lambda$: Re $\lambda > -1\}$. Obviously, $\lambda = 0 \in \Omega$.

Lemma 2.1. *The solution of the eigenvalue problem $\mathcal{L}\psi = \lambda\psi$ is given by*

$$\psi(\lambda,z) = c(\lambda)\psi_1(\lambda,z) + \frac{\alpha\psi(\lambda,0)}{v\phi'(0)}\Big[\int_{-\infty}^{z} K(s)\psi_2(\lambda,s)ds\Big]\psi_1(\lambda,z), \tag{10}$$

where $\psi_1(\lambda,z) = \exp[-\frac{\lambda+1}{v}z]$ and $\psi_2(\lambda,z) = \exp[+\frac{\lambda+1}{v}z]$, and $c(\lambda)$ is an appropriate complex constant.

Proof. Let us first consider *the intermediate operator* $\mathcal{L}_0 : BC^1(R,C) \to BC^0(R,C)$ defined by

$$\mathcal{L}_0\psi = -v\frac{\partial\psi}{\partial z} - \psi, \quad \text{for } \psi \in BC^1(R,C),$$

together with *the intermediate eigenvalue problem*

$$v\frac{\partial\psi}{\partial z} + (\lambda+1)\psi = 0.$$

The solution of this equation is given by $\psi_1(\lambda,z) = \exp\big[-\frac{\lambda+1}{v}z\big]$. The solution of the adjoint equation

$$v\frac{\partial\psi}{\partial z} = (\lambda+1)\psi,$$

is given by $\psi_2(\lambda,z) = \exp\big[+\frac{\lambda+1}{v}z\big]$.

The intermediate Evans function is defined to be the scalar product of the above functions ψ_1 and ψ_2, namely, $D(\lambda) = \psi_1(\lambda,z)\psi_2(\lambda,z) = 1 > 0$. This is independent of λ and z! This implies that there exists no eigenvalue of the operator $\mathcal{L}_0$ inside Ω. Let us use the method of variation of parameters to find the solution of $\mathcal{L}\psi = \lambda\psi$. Suppose that $\psi(\lambda, z) = c(\lambda, z)\psi_1(\lambda, z)$ is a solution. Then, without any difficulty, one obtains that

$$\frac{\partial c}{\partial z} = \frac{\alpha\psi(\lambda, 0)}{v\phi'(0)} K(z)\psi_2(\lambda, z).$$

Integrating in z from $-\infty$ yields

$$c(\lambda, z) = c(\lambda) + \frac{\alpha\psi(\lambda, 0)}{v\phi'(0)} \int_{-\infty}^{z} K(s)\psi_2(\lambda, s)ds.$$

Therefore the proof is completed. □

The Evans function is defined by

$$\mathcal{E}(\lambda) = 1 - \frac{\alpha}{v\phi'(0)} \int_{-\infty}^{0} K(z)\psi_2(\lambda, z)dz. \tag{11}$$

Lemma 2.2. *The Evans function is a complex analytic function and it is real-valued if the eigenvalue parameter λ is real. The complex number λ is an eigenvalue of the operator if and only if $\mathcal{E}(\lambda) = 0$.*

Proof. The first assertion is obviously true. Let us prove the second one. The complex number λ is an eigenvalue of $\mathcal{L}$ if and only if there is a bounded and uniformly continuous function ψ defined on $(-\infty, +\infty)$, such that $\mathcal{L}\psi = \lambda\psi$. By using L'Hospitals rule and the assumption on K, one easily asserts that the solution $\psi(\lambda, z)$ of the eigenvalue problem is bounded on the real line $(-\infty, +\infty)$ if and only if $c(\lambda) = 0$. Letting $z = 0$ in Lemma 2.1 gives

$$\psi(\lambda, 0)\Big[1 - \frac{\alpha\psi_1(\lambda, 0)}{v\phi'(0)} \int_{-\infty}^{0} K(z)\psi_2(\lambda, z)dz\Big] = c(\lambda)\psi_1(\lambda, 0).$$

Notice that $\psi_1(\lambda, 0) = 1 > 0$, for all $\lambda \in \Omega$. Hence, we obtain

$$c(\lambda) = \psi(\lambda, 0)\Big[1 - \frac{\alpha}{v\phi'(0)} \int_{-\infty}^{0} K(z)\psi_2(\lambda, z)dz\Big] = \psi(\lambda, 0)\mathcal{E}(\lambda).$$

Hence, λ is an eigenvalue of $\mathcal{L}$ if and only if $\mathcal{E}(\lambda) = 0$. □

Lemma 2.3. *The Evans function $\mathcal{E}(\lambda) \neq 0$, for all $\lambda \neq 0$ with Re $\lambda \geq 0$.*

Proof. Since $\lambda = 0$ is an eigenvalue, due to translation invariance of the traveling wave front, we get $c(0) = 0$ and $\mathcal{E}(0) = 0$. Hence,

$$\int_{-\infty}^{0} K(z)\psi_2(0, z)dz = \frac{v\phi'(0)}{\alpha}.$$

If Re $\lambda > 0$, then

$$\Big|\int_{-\infty}^{0} K(z)\psi_2(\lambda, z)dz\Big| < \int_{-\infty}^{0} K(z)\psi_2(0, z)dz.$$

So $\mathcal{E}(\lambda) \neq 0$. If Re $\lambda = 0$ but $\lambda \neq 0$, then applying the following lemma yields $\mathcal{E}(\lambda) \neq 0$. □

Lemma 2.4. *Let f be a positive, continuous and integrable function defined on $(-\infty, +\infty)$. Then for all open interval (a, b) and all nonzero real number ζ, where $-\infty < a < b < +\infty$, or $a = -\infty$, or $b = +\infty$, we have*

$$\Big| \int_a^b e^{ix\zeta} f(x) dx \Big| < \int_a^b f(x) dx. \tag{12}$$

Proof. Clearly, we have

$$\Big| \int_a^b e^{ix\zeta} f(x) dx \Big| \leq \int_a^b f(x) dx.$$

To show that for all nonzero real number ζ and for all interval (a, b), the above estimate holds, it suffices to show that there exists a smaller interval $(\alpha, \beta) \subset (a, b)$, such that

$$\Big| \int_\alpha^\beta e^{ix\zeta} f(x) dx \Big| < \int_\alpha^\beta f(x) dx.$$

Suppose that for all $(\alpha, \beta) \subset (a, b)$, there holds

$$\Big| \int_\alpha^\beta e^{ix\zeta} f(x) dx \Big| = \int_\alpha^\beta f(x) dx.$$

Then we have the identity

$$\Big| \int_\alpha^\beta \cos(x\zeta) f(x) dx \Big|^2 + \Big| \int_\alpha^\beta \sin(x\zeta) f(x) dx \Big|^2 = \Big| \int_\alpha^\beta f(x) dx \Big|^2.$$

We are treating α and β as independent variables. Differentiating this identity on β gives

$$\begin{aligned} &2 \int_\alpha^\beta \cos(x\zeta) f(x) dx \cos(\beta\zeta) f(\beta) + 2 \int_\alpha^\beta \sin(x\zeta) f(x) dx \sin(\beta\zeta) f(\beta) \\ &= 2 \int_\alpha^\beta f(x) dx f(\beta). \end{aligned}$$

Since $f > 0$ on the real line, we can cancel out $2f(\beta)$, so we have

$$\int_\alpha^\beta \cos(x\zeta) f(x) dx \cos(\beta\zeta) + \int_\alpha^\beta \sin(x\zeta) f(x) dx \sin(\beta\zeta) = \int_\alpha^\beta f(x) dx.$$

Now differentiating on α yields

$$\cos(\alpha\zeta)\cos(\beta\zeta) f(\alpha) + \sin(\alpha\zeta)\sin(\beta\zeta) f(\alpha) = f(\alpha).$$

Again canceling out $f(\alpha)$ we get

$$\cos(\alpha\zeta)\cos(\beta\zeta)+\sin(\alpha\zeta)\sin(\beta\zeta)=1.$$

This is equivalent to $\cos[(\alpha-\beta)\zeta]=1$ or $(\alpha-\beta)\zeta=2n\pi$, for some integer n. But α and β are arbitrary, so we get a contradiction. Thus, this implies that there exists at least a small interval, such that $|\int_\alpha^\beta e^{ix\zeta}f(x)dx|<\int_\alpha^\beta f(x)dx$. The proof of the lemma is now completed. □

Remark 2.1. Given a function $f\in L^1(-\infty,+\infty)$, define its Fourier transform by $\widehat{f}(\zeta)=\int_{-\infty}^{\infty}f(x)e^{ix\zeta}dx$. The following examples are special cases of this lemma. If $K(x)=\frac{1}{2}\exp(-|x|)$, then the Fourier transform $\widehat{K}(\zeta)=1/(1+|\zeta|^2)<1=\int_{-\infty}^{\infty}K(x)dx$, for all $\zeta\neq 0$. If $K(x)=\frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}|x|^2)$, then $\widehat{K}(\zeta)=\exp(-\frac{1}{2}|\zeta|^2)<1=\int_{-\infty}^{\infty}K(x)dx$, for all $\zeta\neq 0$. An application of Lemma 2.4 shows that if $\lambda=i\zeta$, where $0\neq\zeta\in(-\infty,+\infty)$, then

$$\Big|\int_{-\infty}^{0}K(z)\psi_2(\lambda,z)dz\Big|<\int_{-\infty}^{0}K(z)\psi_2(0,z)dz.$$

Therefore, $\mathcal{E}(\lambda)\neq 0$.

Lemma 2.5. *There holds the limit inside* Ω

$$\lim_{|\lambda|\to+\infty}\mathcal{E}(\lambda)=1. \tag{13}$$

There exists a constant $\kappa>0$*, such that*

$$\max\{\ Re\ \lambda:\lambda\in\sigma(\mathcal{L}),\lambda\neq 0\}\leq-\kappa. \tag{14}$$

Proof. Since $\psi_2(\lambda,z)=\exp[+\frac{\lambda+1}{v}z]$, it can be easily established that $\mathcal{E}(\lambda)\to 1$, as $|\lambda|\to\infty$, inside Ω. This means that there is a positive constant M, such that $\mathcal{E}(\lambda)\neq 0$, for all λ with $|\lambda|\geq M$. Then inside the circle $|\lambda|=M$, there exist at most finitely many eigenvalues, since $\mathcal{E}$ is complex analytic. Hence, if we take Lemma 2.3 into account, there is some constant $\kappa>0$, such that except for $\lambda=0$, there is no other eigenvalue of $\mathcal{L}$ inside the region $\{\lambda\colon \text{Re}\ \lambda>-\kappa\}$. □

Lemma 2.6. *The neutral eigenvalue* $\lambda=0$ *is simple.*

Proof. It suffices to show that there is no bounded solution on $(-\infty,+\infty)$ to the variation equation

$$v\frac{\partial\psi}{\partial z}+\psi+\phi_z=\frac{\alpha}{\phi'(0)}K(z)\psi(0). \tag{15}$$

By the method of variation of parameters again, the solution of this equation is found to be

$$\psi(z) = c\psi_1(0,z) + \frac{\alpha\psi(0)}{v\phi'(0)}\Big[\int_{-\infty}^{z} K(s)\psi_2(0,s)ds\Big]\psi_1(0,z) - \frac{1}{v}\Big[\int_{-\infty}^{z} \phi'(s)\psi_2(0,s)ds\Big]\psi_1(0,z).$$

To determine the constant c, setting $z = 0$ to reach the equation

$$\psi(0) = c\psi_1(0,0) + \frac{\alpha\psi(0)}{v\phi'(0)}\Big[\int_{-\infty}^{0} K(z)\psi_2(0,z)dz\Big]\psi_1(0,0) - \frac{1}{v}\Big[\int_{-\infty}^{0} \phi'(z)\psi_2(0,z)dz\Big]\psi_1(0,0).$$

Since $\mathcal{E}(0) = 0$, we obtain

$$c = \frac{1}{v}\int_{-\infty}^{0} \phi'(z)\psi_2(0,z)dz. \tag{16}$$

This implies that $c > 0$, equivalently, the solution $\psi(z)$ is not bounded as $z \to -\infty$. Hence, the neutral eigenvalue $\lambda = 0$ is simple. □

Remark 2.2. This result implies that the unstable manifold (of one fixed point of the traveling wave equation) and the stable manifold (of another fixed point) intersect transversally.

Theorem 2.2. *There exists no nonzero spectrum of* $\mathcal{L}$ *in the region Re* $\lambda \geq 0$. *The neutral eigenvalue* $\lambda = 0$ *is simple. The traveling wave front* ϕ *of the scalar integral differential equation* (6) *is exponentially stable.*

Proof. The first two statements follow from Lemmas 2.1 to 2.6. By the linearized stability criterion mentioned in the introduction, which was proved first by Evans [13], then by Bates and Jones in [4] and later by Zhang [36], we see the traveling wave front of the equation is exponentially stable. □

Theorem 2.3. *Let* α, θ *and* w *be positive constants, such that* $2\theta < \alpha$ *and* $w = \alpha - 2\theta$. *Then there exists a unique traveling wave solution to*

$$v\phi_z + \phi + w = \alpha\int_{-\infty}^{\infty} K(z-y)H(\phi(y)-\theta)dy,$$

with the same wave speed v_0 *as the front. The traveling wave solution is given by*

$$\phi(z) = \alpha\int_{z}^{\infty} K(x)dx - w + \alpha\int_{-\infty}^{z} \exp\Big(\frac{x-z}{v_0}\Big)K(x)dx,$$

such that $\phi(0) = \theta$, $\phi'(z) < 0$ *on* $(-\infty, +\infty)$ *and that*

$$\lim_{z\to-\infty}\phi(z) = \alpha - w,\ \ \lim_{z\to+\infty}\phi(z) = -w,\ \ \lim_{z\to\pm\infty}\phi'(z) = 0.$$

Proof. We require that $\phi(0) = \theta$ and $\phi'(z) < 0$ on $(-\infty, +\infty)$. Hence, $\phi(z) > \theta$ for $z < 0$ and $\phi(z) < \theta$ for $z > 0$. The traveling wave equation reduces to

$$v\phi_z + \phi + w = \alpha\int_z^{\infty} K(x)dx,$$

and the solution is

$$\phi(z) = \alpha\int_z^{\infty} K(x)dx - w + \alpha\int_{-\infty}^{z}\exp\Big(\frac{x-z}{v}\Big)K(x)dx.$$

Fixing $v = v_0$, the rest of the proof is basically the same as that of the front.

Theorem 2.4. *The traveling wave solution* ϕ_b *is asymptotically stable, as* $t \to +\infty$, *relative to* $u_t + u + w = \alpha K * [H(u-\theta)]$.

Proof. The proof is rather similar to that of Theorem 2.2 and is omitted. □

Concluding Remarks on the Evans Functions of the Eigenvalue Problem (9). For the eigenvalue problem $\mathcal{L}\psi = \lambda\psi$ corresponding to the scalar equation (6), we can find explicit Evans functions by choosing special kernels K. On the other hand, for systems of integral differential equations (1)-(2), the explicit Evans function is extremely difficult to find, even if the kernel K is very simple.

There are many nonlinear evolution equations which support traveling wave solutions. But there are very few equations which possesses explicit Evans functions. Only the Korteweg-de Vries equations [24]

$$\frac{\partial u}{\partial t} + \frac{\partial^3 u}{\partial x^3} + u^p\frac{\partial u}{\partial x} = 0,\ p = 1, 2,$$

has an “almost” explicit Evans function $\mathcal{E}(\lambda) = \{[\mu(\lambda)+\sqrt{c}]^2/[\mu(\lambda)-\sqrt{c}]^2\}^2$, where $c > 0$ is the wave speed and $\mu(\lambda) = (-\lambda)^{1/3} + |\lambda|^{-1/3}O(1)$ is the solution of a third order polynomial. We display several possibilities for which one can find the Evans functions (11) explicitly. One motivation to calculate the explicit Evans function is to find all the possible zeroes in the right half plane $\mathrm{Re}\,\lambda \geq -1$. Another motivation is to find critical eigenvalues of the operator $\mathcal{L}$. Critical eigenvalues are those nonzero eigenvalues closest to the origin. Since the traveling wave front for (6) is asymptotically stable, the critical eigenvalue is negative. Jones has demonstrated that the critical eigenvalue for the Fitzhugh-Nagumo systems is negative but very small. The rates (at which the solutions $U(z,t)$ of the initial value problems $U(z,0) = U_0(z)$ for the integral differential equations (6) converge to the traveling wave solutions ϕ, as $t \to +\infty$) depend on the corresponding critical eigenvalue. By definition of the Evans functions, we see that they depend on the function K. Thus, we need to choose particular K to compute the Evans functions. The functions K chosen below satisfy all the assumptions in this paper. Let us first consider the case $K > 0$ on $(-\infty, +\infty)$.

Example 1. Letting $K(x) = \frac{1}{2}\exp(-|x|)$, the Evans function (11) corresponding to the eigenvalue problem (9) is

$$\begin{aligned}\mathcal{E}(\lambda) &= 1 - \frac{\alpha}{2v\phi'(0)}\int_{-\infty}^{0} \exp\Big[z + \frac{\lambda+1}{v}z\Big]dz \\ &= 1 - \frac{\alpha}{2\phi'(0)(v+\lambda+1)} = 1 - \frac{v+1}{v+\lambda+1} = \frac{\lambda}{v+\lambda+1},\end{aligned}$$

where we have applied the fact that $\mathcal{E}(0) = 0$ which is equivalent to $\alpha = 2\phi'(0)(v+1)$. The explicit expression of the Evans function shows that $\lambda = 0$ is the only eigenvalue of the operator $\mathcal{L}$ inside the region Ω. It is fairly easy to see that

$$E'(0) = \frac{1}{1+v} > 0,$$

hence, $\lambda = 0$ is a simple eigenvalue.

Interesting readers can try to find the explicit Evans function for the function $K(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}x^2)$.

Now we consider the Evans functions $\mathcal{E}$ for which K has a compact support.

Example 2. Let $K(x) = \frac{1}{2}$ for $|x| \le 1$, and $K(x) = 0$ for $|x| > 1$. The Evans function

$$\begin{aligned}\mathcal{E}(\lambda) &= 1 - \frac{\alpha}{2v\phi'(0)}\int_{-1}^{0} \exp\Big[\frac{\lambda+1}{v}z\Big]dz \\ &= 1 - \frac{\alpha}{2\phi'(0)(\lambda+1)}\Big\{1 - \exp\Big[-\frac{\lambda+1}{v}\Big]\Big\} \\ &= 1 - \frac{1}{\lambda+1}\Big\{1 - \exp\Big[-\frac{\lambda+1}{v}\Big]\Big\}\Big/\Big\{1 - \exp\Big[-\frac{1}{v}\Big]\Big\},\end{aligned}$$

where $\frac{\alpha}{2\phi'(0)}\{1 - \exp[-\frac{1}{v}]\} = 1$. It is fairly easy to see that

$$E'(\lambda) = \frac{1}{(\lambda+1)^2\{1-\exp[-\frac{1}{v}]\}}\Big\{1 - \exp\Big[-\frac{\lambda+1}{v}\Big] - \frac{\lambda+1}{v}\exp\Big[-\frac{\lambda+1}{v}\Big]\Big\}.$$

The only eigenvalue of the operator $\mathcal{L}$ inside Ω is again $\lambda = 0$. A simple calculation demonstrates that $\lambda = 0$ is algebraically simple. Let us choose trigonometric functions as examples.

Example 3. Letting $K(x) = \frac{1}{2}\cos x$ for $|x| \leq \frac{\pi}{2}$ and $K(x) = 0$ for $|x| \geq \frac{\pi}{2}$; and $K(x) = \frac{1}{4}\sin|x|$ for $|x| \leq \pi$ and $K(x) = 0$ for $|x| \geq \pi$. Then by using the formulae

$$\int \exp(ax)\cos(bx)dx = \frac{1}{a^2+b^2}\exp(ax)[a\cos(bx) + b\sin(bx)],$$
$$\int \exp(ax)\sin(bx)dx = \frac{1}{a^2+b^2}\exp(ax)[a\sin(bx) - b\cos(bx)],$$

where both a and b are nonzero numbers, we find, respectively, that

$$\begin{aligned}\mathcal{E}(\lambda) &= 1 - \frac{\alpha v}{2\phi'(0)[(\lambda+1)^2+v^2]}\Big\{\frac{\lambda+1}{v} + \exp\Big[-\frac{\lambda+1}{2v}\pi\Big]\Big\} \\ &= 1 - \frac{1+v^2}{(\lambda+1)^2+v^2}\frac{\lambda+1+v\exp[-\frac{\lambda+1}{2v}\pi]}{1+v\exp[-\frac{\pi}{2v}]}, \\ \mathcal{E}(\lambda) &= 1 - \frac{\alpha v}{4\phi'(0)[(\lambda+1)^2+v^2]}\Big\{1 + \exp\Big[-\frac{\lambda+1}{v}\pi\Big]\Big\} \\ &= 1 - \frac{1+v^2}{(\lambda+1)^2+v^2}\frac{1+\exp[-\frac{\lambda+1}{v}\pi]}{1+\exp[-\frac{\pi}{v}]}.\end{aligned}$$

In each of these examples we have used the fact that $\mathcal{E}(0) = 0$ to simplify the expressions of $\mathcal{E}(\lambda)$. Finally, we consider a piecewise polynomial function K.

Example 4. Letting $K(x) = 1 + x$ for $-1 \leq x \leq 0$ and $K(x) = 1 - x$ for $0 \leq x \leq 1$, $K(x) = 0$ for all $|x| > 1$. Clearly if $a \neq 0$, then

$$\int (1+x)\exp(ax)dx = \frac{1}{a}(1+x)\exp(ax) - \frac{1}{a^2}\exp(ax).$$

Therefore,

$$\begin{aligned}\mathcal{E}(\lambda) &= 1 - \frac{\alpha v}{\phi'(0)(\lambda+1)^2}\Big\{\frac{\lambda+1}{v} - 1 + \exp\Big[-\frac{\lambda+1}{v}\Big]\Big\} \\ &= 1 - \frac{1}{(\lambda+1)^2}\frac{\lambda+1-v+v\exp[-\frac{\lambda+1}{v}]}{1-v+v\exp[-\frac{1}{v}]}.\end{aligned}$$

The Evans functions in Examples 3 and 4 are much more complicated, although still explicit. But one can demonstrate that $\lambda = 0$ is the only eigenvalue of $\mathcal{L}$ inside Ω and $\lambda = 0$ is simple.

These choice of K also enables us to find certain relationship between the wave speed v and the constant α. See also Pinto and Ermentrout [25].

3. The system (1)-(2)

In this section, we investigate the asymptotic stability of the traveling wave solution of system (1)-(2). As before, we need to rewrite this system in moving coordinate $z \equiv x + v(\varepsilon)t$, and setting $(U(z,t), W(z,t)) \equiv (u(z - v(\varepsilon)t, t), w(z - v(\varepsilon)t, t))$, so that we get

$$\begin{aligned} \frac{\partial U}{\partial t} + v(\varepsilon)\frac{\partial U}{\partial z} + U + W &= \alpha K * [H(U-\theta)], &(17)\\ \frac{\partial W}{\partial t} + v(\varepsilon)\frac{\partial W}{\partial t} &= \varepsilon(U - \gamma W). &(18)\end{aligned}$$

Recall that the traveling wave solutions $(\phi(\varepsilon, z), \varphi(\varepsilon, z))$ satisfy the conditions $\phi(\varepsilon, 0) = \phi(\varepsilon, z_0(\varepsilon)) = \theta$ and $\phi_z(\varepsilon, 0) > 0, \phi_z(\varepsilon, z_0(\varepsilon)) < 0$, where z_0 depends on ε and $z_0(\varepsilon) \to +\infty$, as $\varepsilon \to 0$. See [25]. The linearization of system (17)-(18) about the traveling wave solution is given by

$$\begin{aligned} \frac{\partial U}{\partial t} + v(\varepsilon)\frac{\partial U}{\partial z} + U + W &= \frac{\alpha}{\phi'(0)} K(z) U(0,t) - \frac{\alpha}{\phi'(z_0)} K(z - z_0) U(z_0, t),\\ \frac{\partial W}{\partial t} + v(\varepsilon)\frac{\partial W}{\partial z} &= \varepsilon(U - \gamma W).\end{aligned}$$

Let $\psi = \begin{pmatrix} \xi \\ \eta \end{pmatrix} \in BC^1(R, C^2) \equiv [BC^1(R, C)]^2$ and define

$$\mathcal{N}(\lambda, \varepsilon, z) = \frac{\alpha}{\phi'(0)} K(z)\xi(\lambda, \varepsilon, 0) - \frac{\alpha}{\phi'(z_0)} K(z - z_0)\xi(\lambda, \varepsilon, z_0). \quad (19)$$

The linear operator $\mathcal{L}(\varepsilon) : BC^1(R, C^2) \to BC^0(R, C^2)$ is defined by

$$\mathcal{L}(\varepsilon)\psi = \begin{pmatrix} -v(\varepsilon)\xi_z - \xi - \eta + \mathcal{N} \\ -v(\varepsilon)\eta_z + \varepsilon(\xi - \gamma\eta) \end{pmatrix} = -v(\varepsilon)\frac{\partial \psi}{\partial z} - \begin{pmatrix} 1 & 1 \\ -\varepsilon & \varepsilon\gamma \end{pmatrix}\psi + \begin{pmatrix} \mathcal{N} \\ 0 \end{pmatrix}.$$

The eigenvalue problem is $\mathcal{L}(\varepsilon)\psi = \lambda\psi$, explicitly,

$$v(\varepsilon)\frac{\partial \psi}{\partial z} + \Big[\lambda I + \begin{pmatrix} 1 & 1 \\ -\varepsilon & \varepsilon\gamma \end{pmatrix}\Big]\psi = \begin{pmatrix} \mathcal{N} \\ 0 \end{pmatrix}, \quad (20)$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the identity matrix. After some tedious computation, we find that the essential spectrum of the operator $\mathcal{L}(\varepsilon)$ contains the closed strip[1] bounded by the two vertical lines $\lambda = -\omega_\pm(\varepsilon) + iv(\varepsilon)x$ situated at $\mathrm{Re}\lambda = -\omega_\pm(\varepsilon)$, where

$$\omega_\pm(\varepsilon) = \frac{1 + \varepsilon\gamma \pm \sqrt{(1-\varepsilon\gamma)^2 - 4\varepsilon}}{2}.$$

[1]See D. Henry [19].

It is very easy to see that $1-2\varepsilon < \omega_+(\varepsilon) < 1$ and $\varepsilon(1+\gamma) < \omega_-(\varepsilon) < \varepsilon(2+\gamma)$. Define the open, unbounded, simply connected domain $\Omega(\varepsilon)$ by $\Omega(\varepsilon) = \{\lambda\colon \operatorname{Re}\lambda > -\omega_-(\varepsilon)\}$.

For the sake of convenience, define four vector-valued, complex analytic functions

$$\Psi_1(\lambda,\varepsilon,z) \equiv \exp\Big[-\frac{\lambda+\omega_+(\varepsilon)}{v(\varepsilon)}z\Big]\begin{pmatrix} 1 \\ \omega_+(\varepsilon)-1 \end{pmatrix}, \tag{21}$$

$$\Psi_2(\lambda,\varepsilon,z) \equiv \exp\Big[-\frac{\lambda+\omega_-(\varepsilon)}{v(\varepsilon)}z\Big]\begin{pmatrix} 1 \\ \omega_-(\varepsilon)-1 \end{pmatrix}, \tag{22}$$

$$\Psi_3(\lambda,\varepsilon,z) \equiv \exp\Big[+\frac{\lambda+\omega_+(\varepsilon)}{v(\varepsilon)}z\Big]\begin{pmatrix} \omega_-(\varepsilon)-1 \\ -1 \end{pmatrix}, \tag{23}$$

$$\Psi_4(\lambda,\varepsilon,z) \equiv \exp\Big[+\frac{\lambda+\omega_-(\varepsilon)}{v(\varepsilon)}z\Big]\begin{pmatrix} 1-\omega_+(\varepsilon) \\ 1 \end{pmatrix}, \tag{24}$$

and define *the intermediate Evans function*

$$D(\varepsilon) \equiv \omega_-(\varepsilon) - \omega_+(\varepsilon) = -\sqrt{(1-\varepsilon\gamma)^2 - 4\varepsilon} < 0.$$

Then

$$\Big(\Psi_1(\lambda,\varepsilon,z), \Psi_2(\lambda,\varepsilon,z)\Big)\Big(\Psi_3(\lambda,\varepsilon,z), \Psi_4(\lambda,\varepsilon,z)\Big)^T$$
$$= \Big(\Psi_3(\lambda,\varepsilon,z), \Psi_4(\lambda,\varepsilon,z)\Big)^T\Big(\Psi_1(\lambda,\varepsilon,z), \Psi_2(\lambda,\varepsilon,z)\Big) = D(\varepsilon)I.$$

Lemma 3.1. *One solution of the eigenvalue problem $\mathcal{L}(\varepsilon)\psi = \lambda\psi$ is*

$$\begin{aligned}\psi(\lambda,\varepsilon,z) &= \Big(\Psi_1(\lambda,\varepsilon,z), \Psi_2(\lambda,\varepsilon,z)\Big)\begin{pmatrix} \zeta_1(\lambda,\varepsilon) \\ \zeta_2(\lambda,\varepsilon) \end{pmatrix} \\ &+ \frac{1}{v(\varepsilon)D(\varepsilon)}\Big(\Psi_1(\lambda,\varepsilon,z), \Psi_2(\lambda,\varepsilon,z)\Big) \\ &\times \int_{-\infty}^{z} (\Psi_3(\lambda,\varepsilon,s), \Psi_4(\lambda,\varepsilon,s))^T \begin{pmatrix} \mathcal{N}(\lambda,\varepsilon,s) \\ 0 \end{pmatrix} ds,\end{aligned}$$

where ζ_1 and ζ_2 are complex functions of λ and ε.

Proof. First let us study *the intermediate operator* $\mathcal{L}_0(\varepsilon) : BC^1(R, C^2) \to BC^0(R, C^2)$, defined by

$$\mathcal{L}_0(\varepsilon)\psi = -v(\varepsilon)\frac{\partial\psi}{\partial z} - \begin{pmatrix} 1 & 1 \\ -\varepsilon & \varepsilon\gamma \end{pmatrix}\psi.$$

The solutions of *the intermediate eigenvalue problem* $\mathcal{L}_0(\varepsilon)\psi = \lambda\psi$ are given by $\varphi(\lambda,\varepsilon,z) = \zeta_1\Psi_1(\lambda,\varepsilon,z) + \zeta_2\Psi_2(\lambda,\varepsilon,z)$, with complex constants $\zeta_1(\lambda,\varepsilon)$ and $\zeta_2(\lambda,\varepsilon)$. There is no problem to see that $\Psi_1(\lambda,\varepsilon,z)$ and $\Psi_2(\lambda,\varepsilon,z)$ are analytic solutions in λ, smooth in ε and z, and they are linearly independent. There exists no eigenvalue of $\mathcal{L}_0(\varepsilon)$ inside $\Omega(\varepsilon)$ because

$$\exp\Big[\frac{2\lambda+1+\varepsilon\gamma}{v(\varepsilon)}z\Big]\det(\Psi_1(\lambda,\varepsilon,z),\Psi(\lambda,\varepsilon,z))=D(\varepsilon)\neq 0.$$

Suppose that

$$\psi(\lambda,\varepsilon,z)=\Big(\Psi_1(\lambda,\varepsilon,z),\Psi_2(\lambda,\varepsilon,z)\Big)\begin{pmatrix}\zeta_1(\lambda,\varepsilon,z)\\ \zeta_2(\lambda,\varepsilon,z)\end{pmatrix}$$

is a solution of the eigenvalue problem $\mathcal{L}(\varepsilon)\psi=\lambda\psi$. Then (20) yields

$$v(\varepsilon)\Big(\Psi_1(\lambda,\varepsilon,z),\Psi_2(\lambda,\varepsilon,z)\Big)\frac{\partial}{\partial z}\begin{pmatrix}\zeta_1(\lambda,\varepsilon,z)\\ \zeta_2(\lambda,\varepsilon,z)\end{pmatrix}=\begin{pmatrix}\mathcal{N}(\lambda,\varepsilon,z)\\ 0\end{pmatrix}.$$

By $(\Psi_3(\lambda,\varepsilon,z),\Psi_4(\lambda,\varepsilon,z))^T(\Psi_1(\lambda,\varepsilon,z),\Psi_2(\lambda,\varepsilon,z))=D(\varepsilon)I$, we have

$$\frac{\partial}{\partial z}\begin{pmatrix}\zeta_1(\lambda,\varepsilon,z)\\ \zeta_2(\lambda,\varepsilon,z)\end{pmatrix}=\frac{1}{v(\varepsilon)D(\varepsilon)}\Big(\Psi_3(\lambda,\varepsilon,z),\Psi_4(\lambda,\varepsilon,z)\Big)^T\begin{pmatrix}\mathcal{N}(\lambda,\varepsilon,z)\\ 0\end{pmatrix}.$$

Integrating in z from $-\infty$, we get

$$\begin{pmatrix}\zeta_1(\lambda,\varepsilon,z)\\ \zeta_2(\lambda,\varepsilon,z)\end{pmatrix}=\begin{pmatrix}\zeta_1(\lambda,\varepsilon)\\ \zeta_2(\lambda,\varepsilon)\end{pmatrix}$$

$$+\frac{1}{v(\varepsilon)D(\varepsilon)}\int_{-\infty}^{z}(\Psi_3(\lambda,\varepsilon,s),\Psi_4(\lambda,\varepsilon,s))^T\begin{pmatrix}\mathcal{N}(\lambda,\varepsilon,s)\\ 0\end{pmatrix}ds,$$

where $\mathcal{A}^T$ is the transposed matrix of $\mathcal{A}$. □

Let $\Psi_1=\begin{pmatrix}\xi_1\\ \eta_1\end{pmatrix}$, $\Psi_2=\begin{pmatrix}\xi_2\\ \eta_2\end{pmatrix}$ and $\psi=\begin{pmatrix}\xi\\ \eta\end{pmatrix}$. In the first component equation of the solution ψ, letting $z=0$ and $z=z_0$, we have

$$\xi(\lambda,\varepsilon,0)=(\xi_1(\lambda,\varepsilon,0),\xi_2(\lambda,\varepsilon,0))\begin{pmatrix}\zeta_1(\lambda,\varepsilon)\\ \zeta_2(\lambda,\varepsilon)\end{pmatrix}$$

$$+\frac{1}{v(\varepsilon)D(\varepsilon)}(\xi_1(\lambda,\varepsilon,0),\xi_2(\lambda,\varepsilon,0))\int_{-\infty}^{0}(\Psi_3(\lambda,\varepsilon,z),\Psi_4(\lambda,\varepsilon,z))^T\begin{pmatrix}\mathcal{N}(\lambda,\varepsilon,z)\\ 0\end{pmatrix}dz,$$

and

$$\xi(\lambda,\varepsilon,z_0)=(\xi_1(\lambda,\varepsilon,z_0),\xi_2(\lambda,\varepsilon,z_0))\begin{pmatrix}\zeta_1(\lambda,\varepsilon)\\ \zeta_2(\lambda,\varepsilon)\end{pmatrix}$$

$$+\frac{1}{v(\varepsilon)D(\varepsilon)}(\xi_1(\lambda,\varepsilon,z_0),\xi_2(\lambda,\varepsilon,z_0))\int_{-\infty}^{z_0}(\Psi_3(\lambda,\varepsilon,z),\Psi_4(\lambda,\varepsilon,z))^T\begin{pmatrix}\mathcal{N}(\lambda,\varepsilon,z)\\ 0\end{pmatrix}dz,$$

respectively. In a compact form, i.e., by using vectors and matrices, we have

$$B(\lambda,\varepsilon)=P(\lambda,\varepsilon)\begin{pmatrix}\zeta_1(\lambda,\varepsilon)\\ \zeta_2(\lambda,\varepsilon)\end{pmatrix}+Q(\lambda,\varepsilon)B(\lambda,\varepsilon),$$

or

$$P(\lambda,\varepsilon)\begin{pmatrix}\zeta_1(\lambda,\varepsilon)\\ \zeta_2(\lambda,\varepsilon)\end{pmatrix}=[I-Q(\lambda,\varepsilon)]B(\lambda,\varepsilon),$$

where

$$B(\lambda,\varepsilon)=\begin{pmatrix}\xi(\lambda,\varepsilon,0)\\ \xi(\lambda,\varepsilon,z_0)\end{pmatrix},\ P(\lambda,\varepsilon)=\begin{pmatrix}\xi_1(\lambda,\varepsilon,0) & \xi_2(\lambda,\varepsilon,0)\\ \xi_1(\lambda,\varepsilon,z_0) & \xi_2(\lambda,\varepsilon,z_0)\end{pmatrix}.$$

I is the identity matrix, $Q(\lambda,\varepsilon)$ is a 2×2 matrix and the two rows are given by

$$\frac{1}{v(\varepsilon)D(\varepsilon)}\Big(\xi_1(\lambda,\varepsilon,0),\xi_2(\lambda,\varepsilon,0)\Big)\int_{-\infty}^{0}\Big(\Psi_3(\lambda,\varepsilon,z),\Psi_4(\lambda,\varepsilon,z)\Big)^T G(z)dz,$$

$$\frac{1}{v(\varepsilon)D(\varepsilon)}\Big(\xi_1(\lambda,\varepsilon,z_0),\xi_2(\lambda,\varepsilon,z_0)\Big)\int_{-\infty}^{z_0}\Big(\Psi_3(\lambda,\varepsilon,z),\Psi_4(\lambda,\varepsilon,z)\Big)^T G(z)dz,$$

where $\begin{pmatrix}\mathcal{N}(\lambda,\varepsilon,z)\\ 0\end{pmatrix}=G(z)B(\lambda,\varepsilon)$ and

$$G(z)=\begin{pmatrix}\alpha K(z)/\phi'(0) & -\alpha K(z-z_0)/\phi'(z_0)\\ 0 & 0\end{pmatrix}.$$

Therefore, by Lemma 3.1, the solution ψ of the eigenvalue problem $\mathcal{L}(\varepsilon)\psi=\lambda\psi$ is bounded on $(-\infty,+\infty)$, if and only if $\begin{pmatrix}\zeta_1(\lambda,\varepsilon)\\ \zeta_2(\lambda,\varepsilon)\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$. Since $B(\lambda,\varepsilon)\neq 0$, ψ is bounded if and only if $\det[I-Q(\lambda,\varepsilon)]=0$. The Evans function is defined by

$$\mathcal{E}(\lambda,\varepsilon)=\det[I-Q(\lambda,\varepsilon)], \tag{25}$$

see also [36] for its definition. We have to find its zeros inside $\Omega(\varepsilon)$ to investigate the asymptotic stability of the pulse.

Lemma 3.2. *$\mathcal{E}(\lambda,\varepsilon)\neq 0$, for all $0<\varepsilon\ll 1$ and all $\lambda\neq 0$ with Re $\lambda\geq 0$.*

Proof. We find each integrand of the four entries in the first and the second rows of $Q(\lambda,\varepsilon)$ as follows:

$$(1,1)\quad \frac{\alpha}{v(\varepsilon)D(\varepsilon)\phi'(0)}K(z)\Big\{\xi_1(\lambda,\varepsilon,0)[\omega_-(\varepsilon)-1]\exp\Big[+\frac{\lambda+\omega_+(\varepsilon)}{v(\varepsilon)}z\Big]$$
$$+\xi_2(\lambda,\varepsilon,0)[1-\omega_+(\varepsilon)]\exp\Big[+\frac{\lambda+\omega_-(\varepsilon)}{v(\varepsilon)}z\Big]\Big\},$$

$$(1,2)\quad -\frac{\alpha}{v(\varepsilon)D(\varepsilon)\phi'(z_0)}K(z-z_0)\Big\{\xi_1(\lambda,\varepsilon,0)[\omega_-(\varepsilon)-1]\exp\Big[+\frac{\lambda+\omega_+(\varepsilon)}{v(\varepsilon)}z\Big]$$
$$+\xi_2(\lambda,\varepsilon,0)[1-\omega_+(\varepsilon)]\exp\Big[+\frac{\lambda+\omega_-(\varepsilon)}{v(\varepsilon)}z\Big]\Big\},$$

$$(2,1)\quad \frac{\alpha}{v(\varepsilon)D(\varepsilon)\phi'(0)}K(z)\Big\{\xi_1(\lambda,\varepsilon,z_0)[\omega_-(\varepsilon)-1]\exp\Big[+\frac{\lambda+\omega_+(\varepsilon)}{v(\varepsilon)}z\Big]$$
$$+\xi_2(\lambda,\varepsilon,z_0)[1-\omega_+(\varepsilon)]\exp\Big[+\frac{\lambda+\omega_-(\varepsilon)}{v(\varepsilon)}z\Big]\Big\},$$
$$(2,2)\quad -\frac{\alpha}{v(\varepsilon)D(\varepsilon)\phi'(z_0)}K(z-z_0)\Big\{\xi_1(\lambda,\varepsilon,z_0)[\omega_-(\varepsilon)-1]\exp\Big[+\frac{\lambda+\omega_+(\varepsilon)}{v(\varepsilon)}z\Big]$$
$$+\xi_2(\lambda,\varepsilon,z_0)[1-\omega_+(\varepsilon)]\exp\Big[+\frac{\lambda+\omega_-(\varepsilon)}{v(\varepsilon)}z\Big]\Big\},$$

respectively. There is an ε-dependent, negative number

$$z_1(\varepsilon)=\frac{v(\varepsilon)}{\omega_+(\varepsilon)-\omega_-(\varepsilon)}\ln\frac{1-\omega_+(\varepsilon)}{1-\omega_-(\varepsilon)}<0,$$

such that the integrand in the first row of $Q(\lambda,\varepsilon)$ is positive in the interval $(z_1(\varepsilon),0)$ and negative in the interval $(-\infty,z_1(\varepsilon))$. However, if the function K has a compact support, then we can assume without loss of generality that the integrand is nonnegative. If $K>0$ on $(-\infty,+\infty)$, by a change of variable, we have the estimate

$$\begin{aligned}\int_{-\infty}^{z_1(\varepsilon)} f(z)dz &= -\int_{z_1(\varepsilon)}^{0}\frac{1}{z}f\Big(z_1(\varepsilon)+\ln\frac{z}{z_1(\varepsilon)}\Big)dz, \text{ and}\\ \int_{-\infty}^{0} f(z)dz &= \int_{z_1(\varepsilon)}^{0} f(z)dz+\int_{-\infty}^{z_1(\varepsilon)} f(z)dz\\ &= \int_{z_1(\varepsilon)}^{0}\Big[f(z)-\frac{1}{z}f\Big(z_1(\varepsilon)+\ln\frac{z}{z_1(\varepsilon)}\Big)\Big]dz.\end{aligned}$$

We need to make sure that

$$f(z)-\frac{1}{z}f\Big(z_1(\varepsilon)+\ln\frac{z}{z_1(\varepsilon)}\Big)\geq 0, \qquad (26)$$

for all $z\in(z_1(\varepsilon),0)$. Choosing $\rho>0$ large enough in the estimate $K(x)\leq C\exp(-\rho|x|)$ so that K converges to zero sufficiently fast as $|z|\to\infty$ can guarantee this requirement. The integrand in the second row of $Q(\lambda,\varepsilon)$ can be dealt with in a similar way. The fact that the integrand is nonnegative yields some kind of monotonicity for the Evans function in λ, namely $|\mathcal{E}(\lambda)|>|\mathcal{E}(0)|$, for all $\lambda\neq 0$ with Re $\lambda\geq 0$. In fact, by using Lemma 2.4, one can easily show that each entry in the matrix $Q(\lambda,\varepsilon)$ is strictly "decreasing", namely, $|Q_{i,j}(\lambda,\varepsilon)|<|Q_{i,j}(0,\varepsilon)|$, for all $1\leq i,j\leq 2$ and all Re $\lambda\geq 0$ but $\lambda\neq 0$. Therefore, the Evans function

$$\mathcal{E}(\lambda,\varepsilon)=\det[Q(\lambda,\varepsilon)-I]\neq 0=\mathcal{E}(0,\varepsilon),$$

for all Re $\lambda \geq 0$ but $\lambda \neq 0$. □

Lemma 3.3. *There holds the asymptotic behavior for the Evans function:* $\mathcal{E}(\lambda, \varepsilon) \to 1$, *as* $|\lambda| \to \infty$, *inside* $\Omega(\varepsilon)$.

Proof. By Lebesgue's dominated convergence theorem, we know that

$$\lim_{|\lambda| \to +\infty} Q_{i,j}(\lambda, \varepsilon) = 0, \quad \lim_{|\lambda| \to +\infty} [I - Q(\lambda, \varepsilon)] = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

One can prove the desired result easily. □

Lemma 3.4. *There exists a positive but small constant* κ, *which may depend on* ε, *such that there is no eigenvalue of* $\mathcal{L}(\varepsilon)$ *in the region* $\{\lambda$: *Re* $\lambda > -\kappa, \lambda \neq 0\}$.

Proof. There exists a positive constant M, independent of $0 < \varepsilon \ll 1$, such that there is no eigenvalue of $\mathcal{L}(\varepsilon)$ outside the circle $|\lambda| = M$ but inside the domain $\Omega(\varepsilon)$. There are at most finitely many eigenvalues of $\mathcal{L}(\varepsilon)$ inside the compact region $\Omega(\varepsilon) \cap \{\lambda : |\lambda| \leq M\}$. It is easy to see that there is no eigenvalue of $\mathcal{L}(\varepsilon)$ in the right half plane Re $\lambda \geq 0$, except for $\lambda = 0$. Thus, all the eigenvalues except for $\lambda = 0$ necessarily have a negative real part. □

Now let us check if the neutral eigenvalue $\lambda = 0$ is algebraically simple.

Lemma 3.5. *The eigenvalue* $\lambda = 0$ *is simple.*

Proof. If there is no bounded solution on $(-\infty, +\infty)$ of the following variation equations, then the traveling wave solution is exponentially stable. Consider

$$v(\varepsilon)\frac{\partial \psi}{\partial z} + \begin{pmatrix} 1 & 1 \\ -\varepsilon & \varepsilon\gamma \end{pmatrix}\psi + \begin{pmatrix} \phi_z \\ \varphi_z \end{pmatrix} = \begin{pmatrix} \mathcal{N} \\ 0 \end{pmatrix}, \tag{27}$$

where

$$\mathcal{N}(0, \varepsilon, z) = \frac{\alpha}{\phi'(0)} K(z)\xi(0, \varepsilon, 0) - \frac{\alpha}{\phi'(z_0)} K(z - z_0)\xi(0, \varepsilon, z_0).$$

As before, applying the method of variation of parameters, one solves the system (27) to obtain the solution

$$\begin{aligned}
\psi(\varepsilon, z) &= \Big(\Psi_1(0, \varepsilon, z), \Psi_2(0, \varepsilon, z)\Big)\begin{pmatrix} \zeta_1(\varepsilon) \\ \zeta_2(\varepsilon) \end{pmatrix} \\
&+ \frac{1}{v(\varepsilon)D(\varepsilon)}\Big(\Psi_1(0, \varepsilon, z), \Psi_2(0, \varepsilon, z)\Big)\Big[\int_{-\infty}^{z} \Big(\Psi_3(0, \varepsilon, s), \Psi_4(0, \varepsilon, s)\Big)^T G(s)ds\Big]B(0, \varepsilon) \\
&- \frac{1}{v(\varepsilon)D(\varepsilon)}\Big(\Psi_1(0, \varepsilon, z), \Psi_2(0, \varepsilon, z)\Big)\int_{-\infty}^{z} \Big(\Psi_3(0, \varepsilon, s), \Psi_4(0, \varepsilon, s)\Big)^T \begin{pmatrix} \phi_z(\varepsilon, s) \\ \varphi_z(\varepsilon, s) \end{pmatrix} ds.
\end{aligned}$$

If the constant vector $\big(\zeta_1(\varepsilon), \zeta_2(\varepsilon)\big) \neq (0,0)$, then the solution $\psi(\varepsilon, z)$ is unbounded, as $z \to -\infty$, and the traveling wave solution would be asymptotically stable. To solve the vector $\big(\zeta_1(\varepsilon), \zeta_2(\varepsilon)\big)$ we let $z = 0$ and $z = z_0$, respectively, in the first component equation. Hence,

$$\begin{aligned}
\xi(\varepsilon, 0) &= \Big(\xi_1(0,\varepsilon,0), \xi_2(0,\varepsilon,0)\Big)\begin{pmatrix} \zeta_1(\varepsilon) \\ \zeta_2(\varepsilon) \end{pmatrix} \\
&+ \frac{1}{v(\varepsilon)D(\varepsilon)}\Big(\xi_1(0,\varepsilon,0), \xi_2(0,\varepsilon,0)\Big)\Big[\int_{-\infty}^{0}\Big(\Psi_3(0,\varepsilon,z), \Psi_4(0,\varepsilon,z)\Big)^T G(z)dz\Big]B(0,\varepsilon) \\
&- \frac{1}{v(\varepsilon)D(\varepsilon)}\Big(\xi_1(0,\varepsilon,0), \xi_2(0,\varepsilon,0)\Big)\int_{-\infty}^{0}\Big(\Psi_3(0,\varepsilon,z), \Psi_4(0,\varepsilon,z)\Big)^T \begin{pmatrix} \phi_z(\varepsilon,z) \\ \varphi_z(\varepsilon,z) \end{pmatrix} dz,
\end{aligned}$$

and

$$\begin{aligned}
\xi(\varepsilon, z_0) &= \Big(\xi_1(0,\varepsilon,z_0), \xi_2(0,\varepsilon,z_0)\Big)\begin{pmatrix} \zeta_1(\varepsilon) \\ \zeta_2(\varepsilon) \end{pmatrix} \\
&+ \frac{1}{v(\varepsilon)D(\varepsilon)}\Big(\xi_1(0,\varepsilon,z_0), \xi_2(0,\varepsilon,z_0)\Big)\Big[\int_{-\infty}^{z_0}\big(\Psi_3(0,\varepsilon,z), \Psi_4(0,\varepsilon,z)\big)^T G(z)dz\Big]B(0,\varepsilon) \\
&- \frac{1}{v(\varepsilon)D(\varepsilon)}\Big(\xi_1(0,\varepsilon,z_0), \xi_2(0,\varepsilon,z_0)\Big)\int_{-\infty}^{z_0}\Big(\Psi_3(0,\varepsilon,z), \Psi_4(0,\varepsilon,z)\Big)^T \begin{pmatrix} \phi_z(\varepsilon,z) \\ \varphi_z(\varepsilon,z) \end{pmatrix} dz.
\end{aligned}$$

Notice that $\xi_1(0,\varepsilon,0) = \xi_2(0,\varepsilon,0) = 1$, and

$$\xi_1(0,\varepsilon,z_0(\varepsilon)) = \exp\Big[-\frac{\omega_+(\varepsilon)z_0(\varepsilon)}{v(\varepsilon)}\Big],\ \xi_2(0,\varepsilon,z_0(\varepsilon)) = \exp\Big[-\frac{\omega_-(\varepsilon)z_0(\varepsilon)}{v(\varepsilon)}\Big].$$

We have the matrix equation

$$B(0,\varepsilon) = P(0,\varepsilon)\begin{pmatrix} \zeta_1(\varepsilon) \\ \zeta_2(\varepsilon) \end{pmatrix} + Q(0,\varepsilon)B(0,\varepsilon) - T(0,\varepsilon), \tag{28}$$

where

$$P(\lambda,\varepsilon) = \begin{pmatrix} \xi_1(\lambda,\varepsilon,0), \xi_2(\lambda,\varepsilon,0) \\ \xi_1(\lambda,\varepsilon,z_0), \xi_2(\lambda,\varepsilon,z_0) \end{pmatrix},\ T(0,\varepsilon) = \frac{1}{v(\varepsilon)D(\varepsilon)}\begin{pmatrix} R(0,\varepsilon) \\ S(0,\varepsilon) \end{pmatrix}.$$

Furthermore, $\det P(\lambda,\varepsilon) \neq 0$. As for $T(0,\varepsilon)$, we have

$$\begin{aligned}
R(0,\varepsilon) &= \big(\xi_1(0,\varepsilon,0), \xi_2(0,\varepsilon,0)\big)\int_{-\infty}^{0}\big(\Psi_3(0,\varepsilon,z), \Psi_4(0,\varepsilon,z)\big)^T \begin{pmatrix} \phi_z(\varepsilon,z) \\ \varphi_z(\varepsilon,z) \end{pmatrix} dz \\
&= O(1),
\end{aligned}$$

and

$$\begin{aligned}
S(0,\varepsilon) &= \big(\xi_1(0,\varepsilon,z_0), \xi_2(0,\varepsilon,z_0)\big)\int_{-\infty}^{z_0}\big(\Psi_3(0,\varepsilon,z), \Psi_4(0,\varepsilon,z)\big)^T \begin{pmatrix} \phi_z(\varepsilon,z) \\ \varphi_z(\varepsilon,z) \end{pmatrix} dz \\
&= O(1)\exp\Big(\frac{1}{\varepsilon}\Big).
\end{aligned}$$

Therefore, we get

$$P(0,\varepsilon)\begin{pmatrix} \zeta_1(\varepsilon) \\ \zeta_2(\varepsilon) \end{pmatrix} = [I - Q(0,\varepsilon)]B(0,\varepsilon) + T(0,\varepsilon). \tag{29}$$

Since $T(0,\varepsilon) \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and it is not perpendicular to the null space of the 2×2 matrix $[I - Q(0,\varepsilon)]^T$, and $\det[I - Q(0,\varepsilon)] = 0$, hence $(\zeta_1(\varepsilon), \zeta_2(\varepsilon)) \neq (0,0)$. This implies that there exists no bounded solution to the variation system (27). This establishes the desired simplicity result.

We conclude this section by stating the asymptotic stability result.

Theorem 3.1. *The traveling pulse solution of the singularly perturbed system* (17)-(18) *of integral differential equations is exponentially stable, as* $t \to +\infty$.

Proof. Follows from the linearized stability criterion and Lemmas 3.1 to 3.5.

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