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A MONOTONE ITERATION FOR AXISYMMETRIC VORTICES WITH SWIRL

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Abstract. We consider steady, inviscid axisymmetric vortex flows with swirl in a bounded channel, possibly with obstacles. Such flows can be obtained by solving the nonlinear equation

$$-\frac{\partial^2\psi}{\partial z^2} - r\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\psi}{\partial r}) = r^2 f(\psi) + h(\psi), \tag{0.1}$$

where f and h are given functions of the stream function ψ, with ψ prescribed on the boundary of the flow domain. We use an iterative procedure to prove the existence of solutions to this Dirichlet problem under certain conditions on f and h. Solutions are not unique, and relations between different families of solutions are obtained. These families include not only vortex rings with swirl, but also flows with tubular regions of swirling vorticity as occur in models of vortex breakdown.

1. Introduction

Swirling flows play an important role in engineering problems involving mixing and occur frequently in natural phenomena. These flows generally involve complicated interaction between different flow quantities and are not easy to characterize. Knowledge of relatively simple flow configurations in which the role of different components of vorticity is clear can be useful in understanding more general situations. We give results here which provide new solutions of the Euler equations in which a swirling vortex is in equilibrium with an irrotational flow past an obstacle.

Axisymmetric steady solutions of the Euler equations with swirl can be obtained by solving the nonlinear equation, known as the Bragg-Hawthorne [3] or Squire-Long [14], [11] equation,

$$L\psi =: -\frac{\partial^2\psi}{\partial z^2} - r\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\psi}{\partial r}) = -r^2\frac{dB}{d\psi} + C\frac{dC}{d\psi} \tag{1.1}$$

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for a stream function ψ which is related to the velocity components (u, v, w) in cylindrical coordinates (r, θ, z) by

$$u = -\frac{1}{r}\frac{\partial \psi}{\partial z}, \quad w = \frac{1}{r}\frac{\partial \psi}{\partial r}. \tag{1.2}$$

The swirl $C = rv$ and the Bernoulli function (or dynamic pressure)

$$B = (u^2 + v^2 + w^2)/2 + p$$

($p =$ pressure) are constant along streamlines, and in (1.1) are taken to be given functions of ψ. The cylindrical coordinate components of vorticity ω are given by

$$\omega_r = -\frac{1}{r}\frac{\partial C}{\partial z}, \quad \omega_\theta = \frac{1}{r}L\psi, \quad \omega_z = \frac{1}{r}\frac{\partial C}{\partial r}.$$

We consider flows in a bounded axisymmetric channel (which may contain axisymmetric obstacles) in two different contexts. First we consider vortex rings with swirl: flows with a compact toroidal region of nonzero vorticity and swirl inside the region, outside of which the flow is either irrotational or a nonswirling shear flow. Included is the limiting case when the inner radius of the vortex ring is 0. Such limiting cases are analogous to the explicit spherical vortices with swirl found by Hicks [10] and Moffatt [13] (generalizations of Hill's spherical vortex).

Second we consider flows with a tubular region of vorticity surrounding the axis and enclosing any obstacle, if present. Flows with this form of vorticity profile arise in connection with the problem of vortex breakdown; see Buntine-Saffman [4] and Wang-Rusak [17]. In the vortex breakdown problem model, conditions are assumed at the channel inlet which determine the functions B and C. The problem is then to determine the downstream character of the flow and a dramatic change, for example stagnation of the flow, is associated with vortex breakdown. In the present work we will consider a somewhat different problem with nonzero swirl at the inlet. The values of the stream function will be assigned at both the inlet and outlet; i.e., the normal component of the velocity is prescribed. The functions B and C will be prescribed as well, and we seek solutions of (1.1), the functions B and C being such that the vorticity and swirl are nonzero on at least a portion of the inlet and outlet.

Previous authors have generally considered only one or the other of these two types of vortex flow. Several authors have used variational methods in connection with the study of vortex flows with swirl. Fraenkel [8] and Tadie [15] generalize the variational approach pioneered by Fraenkel and

Berger [7] for vortex rings without swirl. Turkington [16] employs a different formulation of the problem in which ψ and $dB/d\psi$ are functions of the swirl and existence of solutions is obtained by maximizing a functional of the argument $\zeta = L\psi$. Moffatt [12] has applied a magnetic relaxation procedure to obtain rings with swirl with prescribed volume and volume flux profile functions of ψ. In both [16] and [12] the functions $B(\psi)$ and $C(\psi)$ are not prescribed, but determined *a posteriori*. Wang and Rusak [17] use variational methods to prove the existence of steady solutions in the context of the vortex breakdown problem.

Rather than a variational approach, we use a simple iteration procedure to prove the existence of solutions of (1.1). Rewriting that equation as

$$L\psi = r^2 f(\psi) + h(\psi), \tag{1.3}$$

where f and h are given profile functions, we solve iteratively the linear equation

$$L\psi_{n+1} = r^2 f(\psi_n) + h(\psi_n). \tag{1.4}$$

The basis for our approach is the following result (Theorems 5.1 and 6.2): if $\psi_1 \geq \psi_0$, where ψ_0 is the initialization, then the iterates converge monotonically to a solution of (1.3). The remainder of Sections 5 and 6 are a discussion of how this result implies the existence of various families of vortex flows with swirl.

One cannot expect solutions for arbitrary f and h; however, we show that in the vortex ring case, for arbitrary nonnegative, nondecreasing functions f and h there are solutions of

$$L\psi = r^2 \omega f(\psi) + \lambda h(\psi) \tag{1.5}$$

for ω and λ sufficiently large. We also show that initializing the iteration process with a known solution for smaller profile functions $F \leq f$ and $H \leq h$ always yields a solution of (1.3) (Theorems 5.2 and 6.5 below.) In particular for any vortex ring without swirl and arbitrary nonnegative, nondecreasing function h there is a corresponding vortex ring with swirl.

Our approach allows us to prove existence not only of vortex rings with swirl, but also the tubular type of vortex that occurs in the vortex breakdown problem. In [17] Wang and Rusak restrict attention to the Rankine and Burgers vortex models, models in which the profile functions f and h are related by

$$h(s) = csf(s). \tag{1.6}$$

We relax this condition by giving a weaker set of sufficient conditions, (2.4) and (2.5) below, for existence. In particular, under these conditions on f

and h there is at least one solution of (1.3): that obtained by initializing with a solution ψ_0 of $L\psi = 0$.

For axisymmetric flow in full space, Fraenkel [8] has proven uniqueness of the Hicks-Moffatt vortices. However for flow past an obstacle one cannot expect uniqueness for solutions of (1.3)—see, for example, [5] where multiple solutions were obtained numerically for nonswirling flow past a sphere. In connection with our procedure different initializations ψ_0 of the iteration process can lead to different solutions. In particular, in Section 6 we indicate why, in the presence of an obstacle, one would expect there to be more than one tubular solution for given f and h.

These results generalize recent work of the authors for axisymmetric vortices without swirl [6] which in turn traces back to Goldstik's work on two-dimensional vortex flows [9]. In the case of rings without swirl, the r^2 factor in front of $f(\psi)$ in (1.3) mitigates problems that might arise from the singularity at $r = 0$ in the operator L, and the analysis in [6] for rings without swirl was not substantially more difficult than the case of two-dimensional flow. When h is not zero this singularity presents more significant problems and a more detailed analysis is required, as carried out in Section 3 below.

2. Mathematical formulation

The meridional plane cross section of the flow domain is a bounded, simply connected region Ω of the half-plane $\mathbb{R}^2_+ = \{(z, r) : r \geq 0\}$. We assume that $\partial\Omega$ is piecewise C^1 and has no cusps. The stream-function ψ is required to satisfy the boundary condition $\psi = \chi$, where $\chi \leq 0$ is a prescribed nonconstant function such that $r^{-2}\chi$ is continuous on $\partial\Omega$. In particular $\chi = 0$ on the intersection of $\partial\Omega$ and the z-axis.

The previous paragraph gives the formal assumptions about Ω used in the sequel. Typically, however, Ω is channel-like, as in Figure 1, bounded by a channel wall Γ_1 on which $\chi = a < 0$, a constant, outlet and inlet segments V_1 and V_2 on each of which χ is strictly monotone, and possibly an obstacle Γ_2 (or multiple obstacles) together with segments of the z-axis on which $\chi = 0$. Near the inlet and outlet the flow is from right to left (by (1.2)). When Γ_1 is a horizontal line and V_1 and V_2 both extend to the z-axis, a typical choice of χ would be $\chi = -(U/2)r^2$, the boundary condition for flow with uniform horizontal component of velocity at both ends of the channel.

The profile function f is assumed to satisfy

$$f \text{ bounded, nonnegative, and nondecreasing.} \tag{2.1}$$

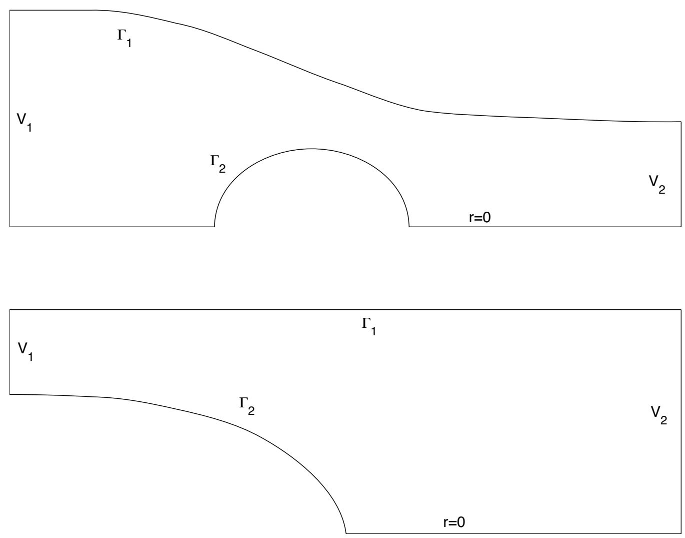

FIGURE 1. Meridional plane cross sections of two examples of an axisymmetric channel: Γ_1 the channel wall, V_1 and V_2 the outflow and inflow, and Γ_2 (if present) an axisymmetric obstacle in the channel.

Let $\alpha = \min\{\inf\ supp\ (df), \inf\ supp(h)\}$, where df is the derivative of f (a measure). Then $f(s) \equiv c$ for $s < \alpha$. If $c > 0$ there is a background nonswirling shear flow; if $c = 0$ then the flow is irrotational outside the set $\{\psi \geq \alpha\}$.

For vortex rings $\alpha > 0$ (see Figure 2); for tubular regions of vorticity as occur in the vortex breakdown problem model $\alpha < 0$; while $\alpha = 0$ gives analogues of the Hicks-Moffatt spherical vortices. For vortex rings with swirl typical choices for f and h are $f = I_\alpha$, the characteristic function of (α, ∞), and $h(s) = (s - \alpha)_+$.

We make the following assumptions concerning the function h:

$$h \text{ is bounded and nondecreasing for } s \geq 0, \tag{2.2}$$

$$h(s)/s \text{ bounded as } s \to 0^+. \tag{2.3}$$

If $\alpha < 0$, we make the following additional assumptions:

$$\frac{h(s)}{sf(s)} \text{ is nonnegative and nonincreasing for } \alpha < s < 0, \tag{2.4}$$

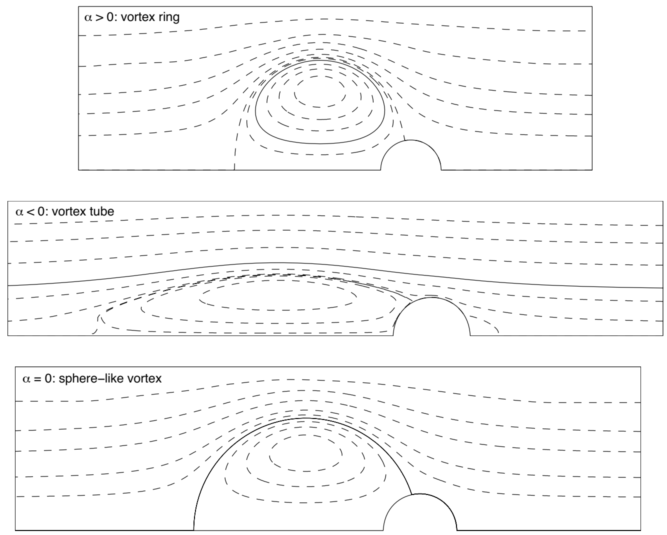

FIGURE 2. Types of vortices. For each case the boundary $\psi = \alpha$ of the vortex support is a solid curve, other level curves of ψ are dashed. ψ is constant along flowlines.

$$\frac{h(s)}{sf(s)} \leq \frac{1}{m} \text{ for } \alpha < s < 0, \tag{2.5}$$

where

$$m = -\inf_{\partial\Omega} r^{-2}\chi(z, r).$$

(By the hypotheses on χ, $m > 0$ is finite). Finally we assume that f and h are continuous from the left at any point of discontinuity. (Since f is monotone and h is the product of monotone functions, f and h have only jump discontinuities.)

Note that (2.3) and (2.5) imply $h(0) = 0$. This is necessary since the swirl $C = rv$, v is finite, and $\psi = 0$ when $r = 0$. Also the hypotheses (2.1)–(2.4) imply $h(s) \geq 0$ for $s \geq 0$ and $h(s) \leq 0$ for $s < 0$, so the function $C(s)$ can be defined to be either the positive or negative square root of the nonnegative function $2\int_0^s h(t)dt$ and the relation $h(s) = C(s)dC/ds$ will hold as required.

Note that of the four hypotheses (2.2)–(2.5), only (2.2) is needed for the vortex ring case, since $\alpha > 0$. Conditions (2.4) and (2.5) may seem rather obscure at this point. We note that in the usual models for vortex breakdown (the Rankine vortex [17], in which case $f(s) = H(s - \alpha)$, $\alpha < 0$; and the Burgers vortex [4], [17], in which case $f(s) = \frac{1}{2}k(e^{2ks} - e^{ks})/s$, with $\chi = -Ur^2/2$ in both cases), the inequality in (2.5) is an equality, so (2.4) is trivially satisfied. It will be shown in Section 6 that these hypotheses allow us to prove that, for appropriate ψ_0, the iterates ψ_n, given by (1.4), converge monotonically to a solution ψ of (1.3).

We will make extensive use of the following transformation, which has frequently been used in the study of axisymmetric flows (e.g. [1]): For $x \in \mathbb{R}^5$, let $x' = (x_2, x_3, x_4, x_5)$ and $r = |x'|$. If a function u of five variables is axisymmetric in the sense that $u(x_1, x') = u(x_1, y')$ when $|y'| = |x'|$, then define $\psi(x_1, r) = r^2 u(x_1, x')$. A straightforward calculation shows that

$$r^{-2}\, L\psi = -\Delta u$$

where Δ is the Laplace operator on $\mathbb{R}^5$. For any open subset D of $\mathbb{R}^2_+$ let

$$\hat{D} = \text{int cl}\{x \in \mathbb{R}^5 : (x_1, |x'|) \in D\}.$$

Define $\hat{\chi}$ on $\partial\hat{\Omega}$ by $\hat{\chi}(x) = |x'|^{-2}\chi(x_1, |x'|)$. Then $m = -(\inf_{\tilde{\Gamma}_1} \hat{\chi})$. A solution to (1.3) satisfying the boundary condition $\psi = \chi$ on $\partial\Omega$ can be found by solving the problem

$$\begin{aligned} -\Delta u &= f(r^2 u) + h(r^2 u)/r^2 \quad \text{on } \hat{\Omega}, \\ u &= \tilde{\chi} \qquad \text{on } \partial\hat{\Omega} \end{aligned} \tag{2.6}$$

for axisymmetric u.

Our approach to solving (2.6) is to iteratively solve the sequence of linear problems

$$\begin{aligned} -\Delta u_{n+1} &= f(r^2 u_n) + h(r^2 u_n)/r^2 \quad \text{on } \hat{\Omega}, \\ u_{n+1} &= \tilde{\chi} \qquad \text{on } \partial\hat{\Omega}, \end{aligned} \tag{2.7}$$

where $u_0 \in L^\infty(\hat{\Omega})$ is some given function. The solution to (2.7) can be expressed as $u_{n+1} = \tilde{u} + v_{n+1} + w_{n+1}$ where $\tilde{u}$ is the harmonic function satisfying $\tilde{u} = \tilde{\chi}$ on $\partial\hat{\Omega}$ and v_{n+1} and w_{n+1} are functions vanishing on $\partial\hat{\Omega}$ and satisfying respectively

$$-\Delta v_{n+1} = f(r^2 u_n), \qquad -\Delta w_{n+1} = h(r^2 u_n)/r^2 \tag{2.8}$$

on Ω.

In the next section we give integral representations of the solutions of (2.8), which are perhaps of interest in their own right. In Section 4 we give sufficient conditions to obtain monotone convergence of the sequence u_n to a solution u of (2.6). We apply these results to the study of vortex rings with swirl in Section 5 and tubular vortices with swirl in Section 6.

3. Integral representation of solutions of (2.8)

We would like to think of the iterants in (2.8) as deriving from level sets of the function obtained in the previous iterative step. In order to do this it is convenient to introduce representations of v and w in terms of certain set functions. Let g be the Green's function for $-\Delta$ on $\hat{\Omega}$. If D is a subset of $\hat{\Omega}$ define the functions $\Phi(D)$ and $\tilde{\Phi}(D)$ by

$$\Phi(D)(x) = \int_D g(x,y)dy$$

and

$$\tilde{\Phi}(D)(x) = \int_D g(x,y)|y'|^{-2}dy. \tag{3.1}$$

In $\mathbb{R}^5, g(x,y) \sim c|x-y|^{-3}$. Therefore, if $x' = 0$ and $dist(D, \{r=0\}) = 0$, the integral in (3.1) will generally be divergent. In what follows we consider $\tilde{\Phi}(D)$ as defined only when $dist(D, \{r=0\}) > 0$.

In this section the hypotheses on f and h can be reduced to those stated in the next theorem. The following notation will be used. $W^{2,p}(\hat{\Omega})$ refers to Sobolev space, with norms denoted $|| \quad ||_{2,p}$; $|| \quad ||_p$ denotes the norm in $L^p(\hat{\Omega})$. For subsets D of $\hat{\Omega}$, I_D denotes the characteristic function of D, and $|D|$ the Lebesgue measure of D.

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be functions of bounded variation that are continuous from the left. Assume there is a C such that $|h(s)| \leq C|s|$ on $\mathbb{R}$. Let $\hat{u}$ be the solution of $-\Delta\hat{u} = c$ on $\hat{\Omega}$, $\hat{u} = 0$ on $\partial\hat{\Omega}$, where $c = \lim_{s\to-\infty} f(s)$. Let $u \in L^\infty(\hat{\Omega})$ and $D(s) = \{x \in \hat{\Omega} : r^2u(x) > s, r = |x'|\}$. Then for all $p > 1$*

$$v = \hat{u} + \int_{-\infty}^{\infty} \Phi(D(s))df(s) \ \in W^{2,p}(\hat{\Omega}) \tag{3.2}$$

and solves the differential equation

$$-\Delta v = f(r^2u) \text{ on } \hat{\Omega}, \qquad v = 0 \text{ on } \partial\hat{\Omega};$$

$$w = \int_0^\infty \tilde{\Phi}(D(s))dh(s) \ - \int_{-\infty}^0 \tilde{\Phi}(\hat{\Omega} - D(s))dh(s) \in W^{2,p}(\hat{\Omega}) \tag{3.3}$$

and solves

$$-\Delta w = h(r^2u)/r^2 \text{ on } \hat{\Omega}, \qquad w = 0 \text{ on } \partial\hat{\Omega}.$$

Some comments are in order before turning to the proof. Equations such as $-\Delta v = f(r^2 u)$ are meant in the weak (or distribution) sense. The integral in (3.2) is a Lebesgue-Stieltjes integral. The integral may be considered pointwise as a real-valued integral, but it is more profitable to consider the integral as a vector-valued integral with values in the Sobolev space $W^{2,p}(\hat{\Omega})$. Since u and $\hat{\Omega}$ are bounded, there is a β such that $D(s)$ is empty for $|s| \geq \beta$, so $\tilde{\Phi}(D(s)) = 0$ for $s \geq \beta$ and

$$\int_{-\infty}^{\infty} \Phi(D(s))df(s) \ = \int_{-\infty}^{\beta} \Phi(D(s))df(s).$$

Because of the remark at the end of the first paragraph of this section, $\tilde{\Phi}(D(s))$ and $\tilde{\Phi}(\hat{\Omega} - D(s))$ are undefined at $s = 0$. Therefore the integrals in (3.3) must be interpreted as improper integrals at $s = 0$. Before proving the theorem we prove lemmas to show that the integrand in (3.2) is integrable and the improper integrals in (3.3) converge in $W^{2,p}(\hat{\Omega})$.

Lemma 3.2. *For any $p > 5/2$, there is a $C = C_p$ such that*

$$||\Phi(D)||_{2,p} \leq C|D|^{1/p}.$$

Proof. By Hölder's inequality $|\Phi(D)(x)| \leq ||g(x,\cdot)||_q ||I_D||_p$, for $p^{-1} + q^{-1} = 1$. Since $g(x,y) \sim c|x-y|^{-3}$, $||g(x,\cdot)||_q \leq C_q$ if $q < 5/3$. Since $-\Delta\Phi(D) = I_D$, it follows from the Calderon-Zygmund inequality that $||D^2\Phi(D)||_p \leq C||I_D||_p$, $p > 1$, where D^2 is the total second derivative. Since $||I_D||_p = |D|^{1/p}$, the lemma follows. □

Lemma 3.3. *Given $\gamma > 2/5$ and $\tau > 2$, there is a C, depending on γ, τ and $\hat{\Omega}$, but independent of $D \subset \hat{\Omega}$, such that*

$$||\tilde{\Phi}(D)||_\infty \leq C\delta^{-\gamma}, \tag{3.4}$$

$$||D^2\tilde{\Phi}(D)||_\tau \leq C\delta^{-2+4/\tau}, \tag{3.5}$$

where $\delta = dist(D, \{r = 0\})$ and D^2 is the total second derivative.

Proof. Define

$$R_D(y) = \begin{cases} 1/|y'|^2, & y \in D \\ 0, & y \notin D. \end{cases}$$

By Hölder's inequality $|\tilde{\Phi}(D)(x)| \leq ||g(x,\cdot)||_q ||R_D||_p$, for $p^{-1} + q^{-1} = 1$. Letting $r = |y'|$, the spherical coordinate in $\mathbb{R}^4$, one has

$$||R_D||_p^p \leq C \int_\delta^C \frac{r^3}{r^{2p}}\, dr \leq C\delta^{4-2p}, \quad p > 2. \tag{3.6}$$

Given $2/5 < \gamma < 2$, let $p = 4/(2-\gamma)$. Then $q < 5/3$, so $||g(x,\cdot)||_q \le C_q$ and inequality (3.4) follows. Since $-\Delta\tilde{\Phi}(D) = R_D$, the Calderon-Zygmund inequality implies that $||D^2\tilde{\Phi}(D)||_\tau \le C||R_D||_\tau$. Inequality (3.5) now follows from (3.6). □

With only slightly more work the first inequality can be improved to $||\tilde{\Phi}(D)||_\infty \le C\ln(1/\delta)$, but that is not needed here.

Lemma 3.4. *For every $p > 1$,*

a) the function $s \mapsto \Phi(D(s))$ is integrable from $[\alpha,\beta]$ to $W^{2,p}(\hat{\Omega})$ (with respect to the measure df);

b) for $\varepsilon > 0$, the functions $s \mapsto \tilde{\Phi}(D(s))$ from $[\varepsilon,\beta]$ to $W^{2,p}(\hat{\Omega})$ and $s \mapsto \tilde{\Phi}(\hat{\Omega}-D(s))$ from $[-\beta,-\varepsilon]$ to $W^{2,p}(\hat{\Omega})$ are integrable (with respect to the measure dh), and the improper integrals $\int_0^\beta \tilde{\Phi}(D(s))dh(s)$ and $\int_{-\beta}^0 \tilde{\Phi}(\hat{\Omega}-D(s))dh(s)$ converge in $W^{2,p}(\hat{\Omega})$.

Proof. We first show that the function in (a) is strongly measurable. (A Banach-space-valued function F is strongly measurable if there is a sequence of simple functions that converges strongly to F almost everywhere. By a theorem of Bochner, a strongly measurable F is integrable if and only if $||F||$ is integrable. See, for example, [18], pp. 130–133). Let w_n be a nondecreasing sequence of simple functions that converges to r^2u, $r = |x'|$, and let $D_n(s) = \{x \in \hat{\Omega} : w_n(x) > s\}$. For each n the function $s \mapsto \Phi(D_n(s))$ is simple. For each s, $\cup D_n(s) = D(s)$, so $\lim |D_n(s)| = |D(s)|$. It follows from Lemma 3.2 that $\Phi(D_n(s))$ converges strongly to $\Phi(D(s))$ for each s, proving the strong measurability of $\Phi(D(s))$. Lemma 3.2 also implies that $||\Phi(D(s))||_{2,p}$ is bounded as a function of s, proving (a). The strong measurability of the functions in (b) follows similarly.

For $s > 0$ let $\delta(s) = dist(D(s), \{r = 0\})$, and for $s < 0$ let $\delta(s) = dist(\hat{\Omega} - D(s), \{r = 0\})$. If $y \in D(s)$, $s > 0$, then $|y'|^2 > s/||u||_\infty$, and if $y \in \hat{\Omega} - D(s)$, $s < 0$, then $|y'|^2 \ge -s/||u||_\infty$. Since $\delta(s) = \inf\{|y'| : y \in D(s)\}$,

$$\delta(s) \ge |s|^{1/2}||u||_\infty^{-1/2}. \tag{3.7}$$

By the previous lemma there is a $\gamma < 2$ such that

$$||\tilde{\Phi}(D(s))||_{2,p} \le C\delta(s)^{-\gamma} \le Cs^{-\gamma/2}, \quad s > 0 \tag{3.8}$$

$$||\tilde{\Phi}(\hat{\Omega} - D(s))||_{2,p} \le C|s|^{-\gamma/2}, \quad s < 0. \tag{3.9}$$

Since every function of bounded variation is the difference of two nondecreasing functions, it suffices to assume h is nondecreasing. If h and g are nondecreasing functions, $0 \leq h \leq g$, and ϕ is a positive continuous nonincreasing function, then integration by parts,

$$\int_{\varepsilon}^{\beta} \phi\, dh - \int_{\varepsilon}^{\beta} \phi\, dg = (h-g)\phi|_{\varepsilon}^{\beta} + \int_{\varepsilon}^{\beta} (g-h)\, d\phi,$$

implies $\lim_{\varepsilon\to 0} \int_{\varepsilon}^{\beta} \phi\, dh \leq \lim_{\varepsilon\to 0} \int_{\varepsilon}^{\beta} \phi\, dg$ if $\lim_{\varepsilon\to 0} \phi(\varepsilon)g(\varepsilon) = 0$.

Applying this to $\phi(s) = s^{-\gamma/2}$ and $g(s) = Cs$, it follows from the hypothesis $h(s) \leq C|s|$ that $\int_0^{\beta} s^{-\gamma/2} dh$ converges. It follows from inequalities (3.8) and (3.9) that the improper integrals

$$\int_0^{\beta} ||\tilde{\Phi}(D(s))||_{2,p}\, dh(s) \quad \text{and} \int_{-\beta}^{0} ||\tilde{\Phi}(\hat{\Omega} - D(s))||_{2,p}\, dh(s)$$

converge. Hence the functions $s \mapsto \tilde{\Phi}(D(s))$ from $[\varepsilon, \beta]$ to $W^{2,p}(\hat{\Omega})$ and $s \mapsto \tilde{\Phi}(\hat{\Omega} - D(s))$ from $[-\beta, -\varepsilon]$ to $W^{2,p}(\hat{\Omega})$ are integrable by Bochner's theorem. Moreover the improper integrals in (3.3) converge by (3.8) and (3.9) and the completeness of $W^{2,p}(\hat{\Omega})$. □

Proof of Theorem 3.1. Since $\Delta : W^{2,p}(\hat{\Omega}) \longrightarrow L^p(\hat{\Omega})$ is a bounded operator, differentiation under the integral sign in (3.2) is valid:

$$\begin{aligned} -\Delta v(x) - c &= \int_{-\infty}^{\beta} -\Delta\Phi(D(s))(x) df(s) = \int_{-\infty}^{\beta} I_{D(s)}(x) df(s) \\ &= \int_{-\infty}^{\beta} I_{(-\infty, r^2 u(x))}(s) df(s) = f(r^2 u(x)) - c \end{aligned}$$

since f is continuous from the left. This proves (3.2). Similarly with $r = |x'|$,

$$-\Delta w(x) = \frac{1}{r^2} \int_0^{\infty} I_{D(s)}(x) dh(s) - \frac{1}{r^2} \int_{-\infty}^{0} I_{(\hat{\Omega} - D(s))}(x) dh(s). \qquad (3.10)$$

If $u(x) \geq 0$, the second integral in (3.10) vanishes and

$$\int_0^{\infty} I_{D(s)}(x) dh(s) = \int_0^{r^2 u(x)} dh(s) = h(r^2 u(x)).$$

If $u(x) \leq 0$ the first integral in (3.10) vanishes and

$$\int_{-\infty}^{0} I_{(\hat{\Omega} - D(s))}(x) dh(s) = \int_{r^2 u(x)}^{0} dh(s) = -h(r^2 u(x)).$$

Corollary 3.5. *Let $u \in L^\infty(\hat{\Omega})$ be a solution to (2.6). Then*

$$u = \tilde{u} + \omega v + \lambda w, \tag{3.11}$$

where v and w are given by (3.2) and (3.3) and $\tilde{u}$ is the harmonic function satisfying $\tilde{u} = \tilde{\chi}$ on $\partial\hat{\Omega}$. Also $u \in C^{1,\alpha}(\bar{\hat{\Omega}})$ for all $\alpha < 1$.

Proof. Let $U = \tilde{u} + \omega v + \lambda w$. By Theorem 3.1

$$-\Delta U = \omega f(r^2 u_n) + \lambda h(r^2 u_n)/r^2 = -\Delta u$$

on $\hat{\Omega}$, and $U = u$ on $\partial\hat{\Omega}$. Hence $U = \underline{u}$ on $\partial\hat{\Omega}$. Also by Theorem 3.1 $u \in W^{2,p}(\hat{\Omega})$ for all $p > 1$, so $u \in C^{1,\alpha}(\bar{\hat{\Omega}})$ for all $\alpha < 1$ by the Sobolev imbedding theorems. □

4. Monotone convergence

Let $u_0 \in L^\infty(\hat{\Omega})$ be given and define u_{n+1} iteratively to be solutions of the linear equation (2.7). We show that if the sequence u_n is monotone nondecreasing, it converges to a solution u of (2.6). Note that if u_0 is axisymmetric, then each u_n is axisymmetric and hence u is axisymmetric, and a solution to the Bragg-Hawthorne equation (1.3) is obtained by setting $\psi(x_1, r) = r^2 u(x_1, x')$, where $r = |x'|, x' = (x_2, \dots, x_5)$. Again f and h are assumed to satisfy the hypotheses in Theorem 3.1. As above, $\tilde{u}$ is the harmonic function satisfying $\tilde{u} = \tilde{\chi}$ on $\partial\hat{\Omega}$ and $\hat{u}$ is the solution of $-\Delta\hat{u} = c$ on $\hat{\Omega}$, $\hat{u} = 0$ on $\partial\hat{\Omega}$.

Lemma 4.1. *Given $u \in L^\infty(\hat{\Omega})$, let v and w be defined by (3.2) and (3.3). There is a C, independent of u, such that $||v||_\infty \le C$ and $||w||_\infty \le C||u||_\infty^{1/2}$.*

Proof. $||v||_\infty \le ||\hat{u}||_\infty + ||\Phi(\hat{\Omega})||_\infty \int_{-\infty}^{\infty} df(s) \le C$. Using Lemma 3.3 with $\gamma = 1$, (3.7) implies that

$$||w||_\infty \le C \int_{-\infty}^{\infty} \delta(s)^{-1} dh \le C||u||_\infty^{1/2} \int_{-\infty}^{\infty} |s|^{-1/2} dh.$$

$\int_0^1 |s|^{-1/2} dh < \infty$ by the argument following (3.9), and $\int_1^\infty |s|^{-1/2} dh < \infty$ since h is bounded. □

Theorem 4.2. *If for each $x \in \hat{\Omega}$ the sequence $u_n(x)$ is nondecreasing, then $u(x) = \lim u_n(x)$ exists and u solves (2.6).*

Proof. By the lemma, $||u_{n+1}||_\infty \le C||u_n||_\infty^{1/2}$. Therefore,

$$||u_{n+1}||_\infty \le C^p ||u_0||_\infty^{2^{-(n+1)}} \text{ where } p = 1 + 2^{-1} + \cdots + 2^{-n},$$

so $||u_n||_\infty \leq C^2 \max(1, ||u_0||_\infty)$. Hence the monotone sequence u_n converges pointwise. Let

$$D(s) = \{x \in \hat{\Omega} : r^2 u(x) > s\}, \quad D_n(s) = \{x \in \hat{\Omega} : r^2 u_n(x) > s\}. \tag{4.1}$$

Then for each s, $D(s) = \cup D_n(s)$; hence $\Phi(D_n(s))$ converges pointwise to $\Phi(D(s))$. Likewise, for $s > 0$, $\tilde{\Phi}(D_n(s))$ converges pointwise to $\tilde{\Phi}(D(s))$, and for $s < 0$, $\tilde{\Phi}(\hat{\Omega} - D_n(s))$ converges pointwise to $\tilde{\Phi}(\hat{\Omega} - D(s))$. Define v and w by (3.2) and (3.3) and define v_{n+1} and w_{n+1} by (3.2) and (3.3) respectively with $D(s)$ replaced by $D_n(s)$. By Theorem 3.1 and the uniqueness of solutions of the linear equation (2.7),

$$u_{n+1} = \tilde{u} + v_{n+1} + w_{n+1},$$

$\tilde{u}$ the harmonic function satisfying $\tilde{u} = \tilde{\chi}$ on $\partial\hat{\Omega}$. By the dominated convergence theorem v_{n+1} and w_{n+1} converge pointwise to v and w. Hence $u = \tilde{u} + v + w$. It follows from Theorem 3.1 that u solves (2.6). □

5. Existence of vortex rings with swirl

In this section $\alpha = \min\{\inf supp(df), \inf supp(h)\} \geq 0$. The functions f and h are assumed to satisfy (2.1), (2.2), and (2.3). u_0 is a nonnegative axisymmetric function in $L^\infty(\hat{\Omega})$, and u_n is defined iteratively by (2.7). Conditions will be given that are sufficient to guarantee that $u(x) = \lim u_n(x)$ exists and solves (2.6). Then $\psi(x_1, r) = r^2 u(x_1, x')$ is the stream function for a steady flow in Ω. Let $\underline{D}(s) = \{(x_1, r) \in \Omega : \psi(x_1, r) > s\}$. Then $cl(\underline{D}(\alpha))$ is the meridional plane cross section of the vortex support of the flow (excluding the background shear when $c > 0$).

Theorem 5.1. *If $u_1 \geq u_0$ on $\hat{\Omega}$, then u_n is nondecreasing, $u(x) = \lim u_n(x)$ exists and u solves* (2.6).

Proof. We make the inductive assumption $u_n \geq u_{n-1}$ on $\hat{\Omega}$.

$$\Delta(u_{n+1} - u_n) = \Delta(v_{n+1} - v_n) + \Delta(w_{n+1} - w_n),$$

where v_n and w_n are defined as in the preceding proof. $-\Delta(v_{n+1} - v_n) = f(r^2 u_n) - f(r^2 u_{n-1})$ and $-\Delta(w_{n+1} - w_n) = (h(r^2 u_n) - h(r^2 u_{n-1}))/r^2$. Since f and h are nondecreasing it follows that $-\Delta(u_{n+1} - u_n) \geq 0$ on $\hat{\Omega}$. Since $u_{n+1} = u_n$ on $\partial\hat{\Omega}$, it follows from the maximum principle that $u_{n+1} \geq u_n$ on $\hat{\Omega}$. Convergence follows from Theorem 4.2. □

Theorem 5.2. (Comparison theorem) *Let f, F, h, and H be nonnegative, nondecreasing functions such that $F \leq f$, $H \leq h$. Assume that df, dF, h and*

H are all supported in $[0,\infty)$ and that h satisfies (2.3) if $\alpha = 0$. Given u_0 define u_{n+1} iteratively to be solutions of (2.7), and setting $U_0 = u_0$, define U_{n+1} iteratively to be solutions of (2.7) with f, h, and u_n replaced by F, H, and U_n. If $U_1 \geq U_0$, then both sequences u_n and U_n are monotone and $u_n \geq U_n$ for all n. Hence $u = \lim u_n \geq U = \lim U_n$.

In particular if U is an axisymmetric solution of

$$\begin{aligned} -\Delta U &= F(r^2U) + H(r^2U)/r^2 \quad \textit{on } \hat{\Omega}, \\ U &= \tilde{\chi} \qquad \textit{on } \partial\hat{\Omega} \end{aligned} \tag{5.1}$$

and $u_0 = U$, then $u = \lim u_n$ exists and solves (2.6).

Proof. We prove $u_n \geq U_n$ by induction. Assuming $u_n \geq U_n$,

$$-\Delta(u_{n+1}-U_{n+1}) = [f(r^2u_n)-F(r^2U_n)]+[h(r^2u_n)-H(r^2U_n)]/r^2 \geq 0 \tag{5.2}$$

and $u_{n+1} = U_{n+1} = \tilde{\chi}$ on $\partial\hat{\Omega}$. The maximum principle implies $u_{n+1} \geq U_{n+1}$. In particular $u_1 \geq U_1 \geq U_0 = u_0$. Lemma 5.1 can now be applied to both sequences. □

It is frequently of interest to write equation (2.6) in the form

$$\begin{aligned} -\Delta u &= \omega f(r^2u) + \lambda h(r^2u)/r^2 \quad \text{on } \hat{\Omega}, \\ u &= \tilde{\chi} \qquad \text{on } \partial\hat{\Omega} \end{aligned} \tag{5.3}$$

where λ and ω are nonnegative constant parameters. As an immediate consequence of Theorem 5.2 we see that if $u^{(1)}$ is a solution of (5.3) for $\omega = \omega_1$, $\lambda = \lambda_1$, then for any $\omega_2 \geq \omega_1$, $\lambda_2 \geq \lambda_1$ there is a solution $u^{(2)}$ of (5.3) for $\omega = \omega_2$, $\lambda = \lambda_2$. Note that since $u^{(2)} \geq u^{(1)}$ the vortex support $cl(\underline{D}(\alpha))$ for the new flow contains the vortex support for the first flow. We also note that if the meridional plane cross section of the vortex support of the first flow is connected (i.e., the vortex support is topologically a ring for $\alpha > 0$), then the meridional plane cross section of the vortex support of the second flow is also connected. (See Proposition 4 of [6].)

In particular, given any vortex ring without swirl, i.e., any solution of (5.3) with $\lambda = 0$, $\omega > 0$, then for any nonnegative, nondecreasing h satisfying (2.3), $h(s) = 0$ for $s < \alpha$, there is a one-parameter family of vortex rings with swirl, parametrized by $\lambda > 0$, with the same ω and f as the given flow without swirl. The vortex supports increase with λ.

The following theorem shows that for arbitrary f, h, and u_0 there is a solution to (5.3) if ω and λ are sufficiently large. As above, $\tilde{u}$ is the harmonic function satisfying $\tilde{u} = \tilde{\chi}$ on $\partial\hat{\Omega}$ and $\hat{u}$ is the solution of $-\Delta\hat{u} = c$ on $\hat{\Omega}$, $\hat{u} = 0$ on $\partial\hat{\Omega}$.

Theorem 5.3. *If*

$$\sup_{D_0(\alpha)} (u_0 - \tilde{u}) \le \omega I_1 + \lambda I_2, \tag{5.4}$$

where

$$I_1 = \inf_{D_0(\alpha)} (\hat{u} + \int_\alpha^\beta \Phi(D_0(s))df(s)), \quad I_2 = \inf_{D_0(\alpha)} \int_\alpha^\beta \tilde{\Phi}(D_0(s))dh(s),$$

then $u(x) = \lim u_n(x)$ *exists and* u *solves* (5.3).

Proof. By Theorem 3.1

$$u_0 \le \tilde{u} + \omega(\hat{u} + \int_\alpha^\beta \Phi(D_0(s))df(s)) + \lambda \int_\alpha^\beta \tilde{\Phi}(D_0(s))dh(s) = u_1$$

for all $x \in D_0(\alpha)$. Therefore $D_0(s) \subset D_1(s)$ for all $s \ge \alpha$, and hence $\Phi(D_0(s)) \le \Phi(D_1(s))$ and $\tilde{\Phi}(D_0(s)) \le \tilde{\Phi}(D_1(s))$ on $\hat{\Omega}$. Since f and h are nondecreasing

$$u_2 = \tilde{u} + \omega(\hat{u} + \int_\alpha^\beta \Phi(D_1(s))df(s)) + \lambda \int_\alpha^\beta \tilde{\Phi}(D_1(s))dh(s) \ge u_1 \tag{5.5}$$

on $\hat{\Omega}$, and Lemma 5.1 applies (with u_0 replaced by u_1). □

By this theorem, given any f and h satisfying (2.1), (2.2), and (2.3), there is at least one two-parameter family of solutions of the Bragg-Hawthorne equation (1.3), parametrized by ω and λ, for ω and λ sufficiently large. The critical values are determined by u_0, $\tilde{\chi}$, and the geometry of Ω. We note that for fixed ω and λ the solution to (1.3) is not necessarily unique. For example in a numerical study of vortex rings without swirl in the region exterior to a sphere [5] we found five different one-parameter families of solutions, parametrized by ω, when $f(s) = I(s)$, the characteristic function of $(0, \infty)$. Different solutions for the same ω arise by taking different initializations u_0. The number of different solutions for fixed f, h, ω, and λ depends on the geometry of Ω.

6. Existence of tubular vortices with swirl

In this section we consider sufficient conditions for the existence of tubular regions of vorticity with swirl. For such flows $\alpha < 0$. The profile functions f and h are assumed to satisfy (2.1)–(2.5), and hence satisfy the hypotheses of Theorem 3.1. We also assume $c = \lim_{s \to -\infty} f(s) = 0$. For ease of reference we restate hypotheses (2.4) and (2.5) letting $q(s) = h(s)/(sf(s))$:

$$q(s) \text{ is nonnegative and nonincreasing for } \alpha < s < 0, \tag{2.4$'$}$$

$$q(s) \leq \frac{1}{m} \text{ for } \alpha < s < 0, \tag{2.5$'$}$$

where $-m = \inf_{\partial\hat{\Omega}} \hat{\chi} < 0$. Hypothesis (2.5′) implies $supp(h) \subseteq supp(f)$. Also $c = 0$ implies $\inf supp(df) = \inf supp(f)$, so $\alpha = \inf supp(f)$.

Lemma 6.1. *If f satisfies* (2.1) *and h satisfies* (2.2) *and* (2.5), *then $u \geq -m$ on $\hat{\Omega}$ implies $f(r^2u) + h(r^2u)/r^2 \geq 0$ on $\hat{\Omega}$.*

Proof. We consider the points where $u(x) \geq 0$ and where $u(x) < 0$ separately. If $u(x) \geq 0$ the desired inequality is trivial since $f(s)$ and $h(s)$ are both nonnegative when $s \geq 0$. By (2.5′) if $-m \leq u(x) < 0$, then $u(x)q(r^2u(x)) \geq -1$; hence,

$$f(r^2u(x)) + h(r^2u(x))/r^2 = f(r^2u(x))(1 + u(x)q(r^2u(x))) \geq 0. \qquad \square$$

Theorem 6.2. *Let f and h satisfy* (2.1)–(2.5) *with $\alpha < 0$, $c = 0$. Given $u_0 \geq -m$, define u_{n+1} iteratively by* (2.7). *If $u_1 \geq u_0$ on $\hat{\Omega}$, then $u(x) = \lim u_n(x)$ exists and u solves* (2.6).

Proof. We first show that $u_n \geq -m$ on $\hat{\Omega}$ implies $u_{n+1} \geq -m$ on $\hat{\Omega}$. By the previous lemma, if $u_n \geq -m$ on $\hat{\Omega}$, then

$$-\Delta u_{n+1} = f(r^2u_n) + h(r^2u_n)/r^2 \geq 0$$

on $\hat{\Omega}$. Since $u_{n+1} = \hat{\chi}$ on $\partial\hat{\Omega}$, $u_{n+1} \geq -m$ follows from the maximum principle. Therefore for all n, $u_n \geq -m$ on $\hat{\Omega}$. Moreover (2.5′) implies, as in the previous proof, that for all n

$$(1 + u_n(x)q(r^2u_n(x))) \geq 0 \text{ if } u_n(x) < 0. \tag{6.1}$$

Assume that $u_n \geq u_{n-1}$ on $\hat{\Omega}$.

$$-\Delta(u_{n+1} - u_n) = f(r^2u_n) - f(r^2u_{n-1}) + [h(r^2u_n) - h(r^2u_{n-1})]/r^2. \tag{6.2}$$

If $u_n(x) \geq 0$, then $h(r^2u_n(x)) \geq h(r^2u_{n-1}(x))$ (by (2.2) if $u_{n-1}(x) \geq 0$ and since $h(r^2u_{n-1}(x)) \leq 0$ if $u_{n-1}(x) < 0$) . So the right-hand side of (6.2) is positive if $u_n(x) \geq 0$. The right-hand side of (6.2) can also be written as

$$f(r^2u_n)(1 + u_nq(r^2u_n)) - f(r^2u_{n-1})(1 + u_{n-1}q(r^2u_{n-1})). \tag{6.3}$$

If $u_n(x) < 0$, then $u_n \geq u_{n-1}$ and (2.4′) imply

$$-u_n(x)q(r^2u_n(x)) \leq -u_{n-1}(x)q(r^2u_{n-1}(x)),$$

and hence, using (6.1),

$$(1 + u_nq(r^2u_n)) \geq (1 + u_{n-1}q(r^2u_{n-1})) \geq 0$$

at x. Since f is nonnegative and nondecreasing,

$$f(r^2u_n)(1+u_nq(r^2u_n)) \geq f(r^2u_{n-1})(1+u_{n-1}q(r^2u_{n-1})).$$

Therefore $-\Delta(u_{n+1}-u_n) \geq 0$ on $\hat{\Omega}$, and it follows from the maximum principle that $u_{n+1} \geq u_n$ on $\hat{\Omega}$. □

A natural choice for initialization is the potential flow satisfying the boundary conditions. The next theorem shows that this always leads to a solution.

Theorem 6.3. *Let f and h satisfy* (2.1)–(2.5) *with $\alpha < 0$, $c = 0$. Let $u_0 = \tilde{u}$, the axisymmetric harmonic function satisfying $\tilde{u} = \tilde{\chi}$ on $\partial\hat{\Omega}$. Define u_{n+1} iteratively by* (2.7). *Then $u(x) = \lim u_n(x)$ exists and u solves* (2.6).

Proof. By the maximum principle $u_0 \geq -m$ on $\hat{\Omega}$. By Lemma 6.1

$$-\Delta(u_1-u_0) = -\Delta u_1 = f(r^2u_0) + h(r^2u_0)/r^2 \geq 0 \text{ on } \hat{\Omega}.$$

So $u_1 - u_0 \geq 0$ on $\hat{\Omega}$, and the previous result applies. □

An immediate consequence is the following corollary concerning solutions of the parametrized equation (5.3) in the special case where, as in most studies of vortex breakdown (see e.g. [4], [17]), $q(s)$ is constant, $\alpha < s < 0$.

Corollary 6.4. *Let f satisfy* (2.1) *and h satisfy* (2.2) *and* (2.3) *with $\alpha < 0$, $c = 0$. Also assume $h(s) = sf(s)$ for $s < 0$. Let $u_0 = \tilde{u}$, the axisymmetric harmonic function satisfying $\tilde{u} = \tilde{\chi}$ on $\partial\hat{\Omega}$. Let ω and $\lambda \geq 0$ satisfy $m\lambda \leq \omega$, and define u_{n+1} iteratively by* (2.7) (*with f replaced by ωf and h by λh*). *Then $u(x) = \lim u_n(x)$ exists and u solves* (5.3).

Proof. (2.4′) is satisfied since $q \equiv 1$ for $s < 0$, and (2.5′) follows from $m\lambda \leq \omega$. □

A comparison theorem similar to Theorem 5.2 holds for $\alpha < 0$:

Theorem 6.5. *For $\alpha < 0$ the conclusions of Theorem* 5.2 *hold if* (f, h) *and* (F, H) *both satisfy* (2.1)–(2.5) *with $c = 0$, $F(s) \leq f(s)$ for all real s, $H(s) \leq h(s)$ for all $s \geq A = \inf supp(F)$, and if $u_0 \geq -m$ on $\hat{\Omega}$.*

Proof. Suppose $u_n \geq U_n$. As in the proof of Theorem 5.2 we wish to show that $-\Delta(u_{n+1} - U_{n+1}) \geq 0$ and then apply the maximum principle and Lemma 6.2. If $u_n(x) \geq 0$, then $-\Delta(u_{n+1} - U_{n+1})$ is positive at x by (5.2). Also, if $r^2U_n(x) < A$, then $-\Delta(u_{n+1} - U_{n+1}) = -\Delta u_{n+1} = f(r^2u_n) +$

$h(r^2u_n)/r^2 \geq 0$ by Lemma 6.1. Lemma 6.1 can be applied since for all n, $U_n \geq -m$ on $\hat{\Omega}$, as shown at the beginning of the proof of Lemma 6.2.

It remains to consider those x for which $u_n(x) < 0$ and $r^2U_n(x) \geq A$. In this case we write

$$-\Delta(u_{n+1} - U_{n+1}) = f(r^2u_n)(1 + u_nq(r^2u_n)) - F(r^2U_n)(1 + U_nQ(r^2U_n)), \tag{6.4}$$

where $Q(s) = H(s)/(sF(s))$ for $s < 0$. Note that the hypotheses imply $q(s) \leq Q(s)$ if $A \leq s < 0$. Hence, using the monotonicity of Q,

$$u_n(x)q(r^2u_n(x)) \geq U_n(x)Q(r^2U_n(x)).$$

Also $1 + U_n(x)Q(r^2U_n(x)) \geq 0$ since $U_n \geq -m$ on $\hat{\Omega}$. Therefore the right-hand side of (6.4) is nonnegative. □

If $u^{(1)}$ and $u^{(2)}$ are solutions of (5.3) obtained using Corollary 6.4 for the same f, h, and λ but with different values of ω, $\omega_2 > \omega_1 \geq m\lambda$, then it follows immediately from Theorem 6.5 that $u^{(2)} \geq u^{(1)}$. On the other hand if $u^{(1)}$ and $u^{(2)}$ are solutions of (5.3) obtained using Corollary 6.4 for the same f, h, and ω but with different values of λ, $\lambda_2 > \lambda_1$, the comparison theorem doesn't apply, and indeed the maximum principle apparently does not imply either $u^{(1)} \geq u^{(2)}$ or $u^{(1)} \leq u^{(2)}$. ($-\Delta(u^{(1)} - u^{(2)})$ assumes both positive and negative values on $\hat{\Omega}$.) However, it is an immediate consequence of Theorem 6.5 that in the special case where $h(s) \equiv 0$ for $s \geq 0$, then $\lambda_2 > \lambda_1$ implies $u^{(2)} \leq u^{(1)}$. (Note that the opposite ordering holds when $\alpha \geq 0$.) Thus, in this special case, for the one-parameter family of solutions parametrized by λ, $0 \leq \lambda \leq \omega/m$, the vortex supports are ordered, with the nonswirling flow having the largest support.

We now use Theorem 6.5 to show that, in the presence of an obstacle, there are tubular swirling vortices other than those obtained by initializing with potential flow. To be specific we restrict attention to Rankine vortices; i.e., $f = \omega I_\alpha$, I_α the characteristic function of (α, ∞), $\alpha < 0$, and $h(s) = \lambda sf(s)$. In a numerical study of nonswirling flow past a sphere in unbounded space, [5], in addition to perturbations of potential flow, other families of tubular solutions were found as shown in Figure 3.

These families were found by perturbing from the $\alpha = 0$ solutions shown in dashed lines in Figure 3. Assuming similar nontrivial $\alpha = 0$ solutions exist for flow past an obstacle in a bounded channel, let U be such a solution, $F = \omega I_0$, $H \equiv 0$. Then $U \geq -m$ on $\hat{\Omega}$, since $-\Delta U = F(U) \geq 0$ and $U = \tilde{\chi}$ on $\partial\hat{\Omega}$. If $m\lambda \leq \omega$, then the hypotheses of Theorem 6.5 are satisfied, so there is a Rankine vortex with stream function $u \geq U$. The vortex support

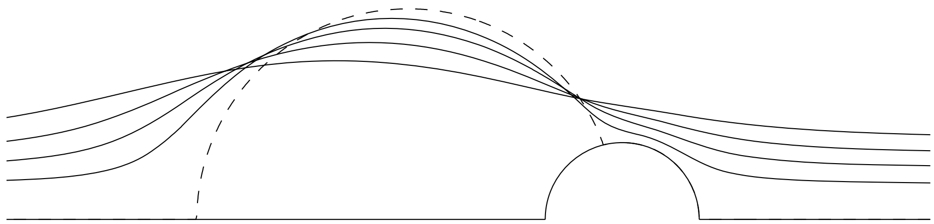

FIGURE 3. A family of nonswirling tubular vortices for flow past a sphere. The vortex strength is fixed at $\omega = 1$. The parameter α varies from -0.1 to -0.4. These flows can be considered as perturbations of a spherical-like vortex (the dashed curve) standing behind the obstacle.

$\{u > \alpha\}$ will contain $\{U > 0\}$. For α close to 0, $\alpha < 0$, the vortex support of the solution obtained by perturbing from potential flow will not contain $\{U > 0\}$. So for $\alpha < 0$ sufficiently close to 0, the solutions obtained by initializing with a nontrivial $\alpha = 0$ solution U will differ from the solutions obtained by initializing with potential flow. However, there is no reason to believe these solutions will remain distinct as $-\alpha$ becomes large, and in fact in [5] the two families eventually come together.

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