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IDENTIFICATION PROBLEMS FOR INTEGRO-DIFFERENTIAL DELAY EQUATIONS

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Abstract. This paper is devoted to recovering a scalar time-dependent function in a source term in an integro-differential parabolic equation with delay. For such a problem existence and uniqueness results as well as continuous dependence upon the data are proved.

1. Introduction

The main aim of this paper consists of recovering the unknown right-hand side $f : [0, T] \to \mathbf{R}$ in the following delay functional differential equation in a Banach space E:

$$u'(t) = Au(t) + A_1 u(t-r) + \int_{-r}^{0} a(s) A_2 u(t+s)\, ds + f(t) z, \tag{1.1}$$

for almost every $t \in (0, T)$. Here, $A : D(A) \subset E \to E$ is the infinitesimal generator of an analytic semigroup in E, and A_1 and A_2 are given linear and closed operators defined on $D(A)$. For example, A is the realization, in some function space, of an elliptic operator of order $2m$ with suitable boundary conditions, and A_1 and A_2 are differential operators of order less than or equal to $2m$. Furthermore, $z \in E$ and $a : (-r, 0) \to \mathbf{R}$ are given.

To study (1.1) one has to know the history of u in the interval $[-r, 0]$. Here we are interested in an L^p setting, and hence we consider the following

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initial conditions:

$$u(s) = \varphi_1(s), \quad \text{for a.e. } s \in (-r, 0), \qquad u(0) = \varphi_0, \tag{1.2}$$

where $\varphi_0 \in E$ and $\varphi_1 : (-r, 0) \to E$ are given.

Finally, to recover the scalar-valued function f we prescribe the additional information

$$\Phi[u(t)] = g(t), \qquad t \in [0, T], \tag{1.3}$$

$g : [0, T] \to \mathbf{R}$ and Φ being, respectively, a given function and a linear, continuous functional defined on the whole of E.

We notice that, to the authors' knowledge, identification problems related to delay integro-differential equations in the *present form* are scarcely studied. This paper wants to be a contribution in this field. However, in the case of *no* delay, i.e., when $A_1 = 0$ and $a = 0$, identification problems of several kinds are widely studied. See e.g. the monographs [14] and [7].

Moreover, we recall that in the case of no *discrete* delay, i.e., when $A_1 = 0$, identification problems for various kinds of integro-differential equations with memory are studied in [9], [10], and [8].

In this paper we study problem (1.1)–(1.3) in an L^p setting, $1 \le p < \infty$. We establish existence and regularity results, as well as continuous dependence upon the data, under various assumptions (depending on the values of p). See Theorems 4 and 5 of Section 4. Finally, in Theorems 7 and 8 of Section 5, we give an application of the abstract results to delay partial differential problems.

2. Preliminaries

Applying the linear functional Φ to both sides of (1.1) and assuming that the function g in (1.3) is differentiable, we obtain the following equation for f:

$$g'(t) = \Phi[Au(t) + A_1u(t-r)] + \int_{-r}^{0} a(s)\Phi[A_2u(t+s)]\, ds + f(t)\Phi[z],$$

for almost every $t \in (0, T)$. Assume now that

$$\chi^{-1} := \Phi[z] \neq 0. \tag{2.1}$$

Then from (2.1) we easily derive that the function f can be expressed in terms of u by

$$f(t) = \chi g'(t) - \chi\Phi\Big[Au(t) + A_1u(t-r) + \int_{-r}^{0} a(s)A_2u(t+s)\, ds \Big], \tag{2.2}$$

for almost every $t \in (0,T)$. Consequently, our identification problem is equivalent to the delay problem

$$\begin{cases} u'(t) = Au(t) + A_1 u(t-r) + \displaystyle\int_{-r}^{0} a(s) A_2 u(t+s)\, ds \\ \qquad\qquad +\chi g'(t) z - \chi[\Phi A u(t)] z - \chi[\Phi A_1 u(t-r)] z \\ \qquad\qquad -\chi \displaystyle\int_{-r}^{0} a(s) \Phi[A_2 u(t+s) z]\, ds, \qquad \text{for a.e. } t \in (0,T), \\ u(s) = \varphi_1(s), \quad \text{for a.e. } s \in (-r,0), \qquad u(0) = \varphi_0. \end{cases} \tag{2.3}$$

As far as the continuous dependence of the solution on the data is concerned, assume now that (u_j, f_j), $j = 1,2$, are two solutions to the identification problem (1.1)–(1.3) corresponding to the data $(z_j, \varphi_{0,j}, \varphi_{1,j}, g_j)$, respectively. It is immediate to check that the pair $(u,f) = (u_2 - u_1, f_2 - f_1)$ satisfies the following problem:

$$\begin{cases} u'(t) = Au(t) + A_1 u(t-r) + \displaystyle\int_{-r}^{0} a(s) A_2 u(t+s)\, ds \\ \qquad\qquad +f_1(t) z + f(t) z_2, \quad \text{for a.e. } t \in (0,T), \\ u(s) = \varphi_1(s), \quad \text{for a.e. } s \ \in (-r,0), \qquad u(0) = \varphi_0, \\ \Phi[u(t)] = g(t), \qquad t \in [0,T], \end{cases} \tag{2.4}$$

where

$$(z, \varphi_0, \varphi_1, g) = (z_2 - z_1, \varphi_{0,2} - \varphi_{0,1}, \varphi_{1,2} - \varphi_{1,1}, g_2 - g_1). \tag{2.5}$$

Assuming

$$\chi_2^{-1} := \Phi[z_2] \neq 0, \tag{2.6}$$

we easily deduce for f the representation

$$\begin{aligned} f(t) \ &= \ \chi_2 \{ g'(t) - f_1(t) \Phi[z] \} - \chi_2 \Phi[Au(t) + A_1 u(t-r)] \\ &\quad -\chi_2 \int_{-r}^{0} a(s) \Phi[A_2 u(t+s)]\, ds, \qquad \text{a.e. } t \in (0,T). \end{aligned} \tag{2.7}$$

Consequently, our identification problem (2.4)–(2.5) is equivalent to the following *direct* problem:

$$\begin{cases} u'(t) = Au(t) + A_1 u(t-r) + \displaystyle\int_{-r}^{0} a(s) A_2 u(t+s)\, ds + f_1(t) z \\ \qquad +\chi_2 \{g'(t) - f_1(t)\Phi[z]\} z_2 - \chi_2[\Phi A u(t)] z_2 - \chi_2[\Phi A_1 u(t-r)] z_2 \\ \qquad -\chi_2 \displaystyle\int_{-r}^{0} a(s)\Phi[A_2 u(t+s)] z_2\, ds, \quad \text{for a.e. } t \in (0,T), \\ u(s) = \varphi_1(s), \quad \text{for a.e. } s \in (-r,0), \qquad u(0) = \varphi_0, \end{cases}$$

where

$$\begin{aligned} f_1(t) \quad = \quad & \chi_1 g_1'(t) - \chi_1 \Phi[A u_1(t) + A_1 u_1(t-r)] \\ & -\chi_1 \int_{-r}^{0} a(s) \Phi[A_2 u_1(t+s)]\, ds, \qquad \text{a.e. } t \in (0,T), \end{aligned} \tag{2.8}$$

is assumed to be *known,* χ_1 being defined by

$$\chi_1^{-1} := \Phi[z_1] \neq 0. \tag{2.9}$$

We conclude this section by observing that the previous problems can be unified introducing the following formulation:

$$\begin{cases} u'(t) = Au(t) + A_1 u(t-r) + \displaystyle\int_{-r}^{0} a(s) A_2 u(t+s)\, ds + \rho_1(t)\zeta_1 \\ \qquad +\rho_2(t)\zeta_2 - \chi[\Phi A u(t)]\zeta - \chi[\Phi A_1 u(t-r)]\zeta_3 \\ \qquad -\chi \displaystyle\int_{-r}^{0} a(s)\Phi[A_2 u(t+s)]\zeta_4\, ds, \qquad \text{for a.e. } t \in (0,T), \\ u(s) = \varphi_1(s), \qquad \text{for a.e. } s \in (-r,0), \qquad u(0) = \varphi_0, \end{cases} \tag{2.10}$$

where $\chi \neq 0$, $\rho_1, \rho_2 : [0,T] \to \mathbf{R}$, and $\zeta, \zeta_1, \dots, \zeta_4 \in E$ are given.

In the case of the continuous dependence we stress that actually ρ_1 and ρ_2 *do* depend on one of the *unknowns* u_1 or u_2, e.g., on u_1. However, in this occurrence we suppose having at our disposal an existence, uniqueness, and continuous-dependence theorem for problem (2.3) or, at least, having some *a priori information* on such functions—such as what spaces they belong to and what bounds they satisfy in terms of the data. This allows us to estimate f_1 in terms of the data, only.

3. The problem without delay

We begin to introduce some definitions and notation which are used in the sequel.

Given a Banach space X we will be concerned with the following spaces of X-valued Bochner-measurable functions defined on a bounded interval $[a,b] \subset \mathbf{R}$:

- $L^p(a,b;X)$, $1 \le p < \infty$, is the space of all u such that $\|u(\cdot)\|_X^p$ is integrable in (a,b);
- $C(a,b;X)$ is the space of all continuous functions on $[a,b]$;
- $W^{\alpha,p}(a,b;X)$, $0 < \alpha < 1$, is the Sobolev space of all $u \in L^p(a,b;X)$ for which

$$N_{\alpha,p}(u) := \Big(\int_a^b dt \int_a^t \|u(t) - u(s)\|_X^p \, (t-s)^{-1-p\alpha} ds \Big)^{1/p} < +\infty \ ; \qquad (3.1)$$

- $W_*^{\alpha,p}(a,b;X)$, $0 < \alpha < 1$, is the space of all $u \in W^{\alpha,p}(a,b;X)$ for which

$$H_{\alpha,p}(u) := \Big(\int_a^b (t-a)^{-\alpha p} \|u(t)\|_X^p \, dt \Big)^{1/p} < +\infty \ ; \qquad (3.2)$$

- $W^{1,p}(a,b;X)$ is the Sobolev space of all functions $u \in L^p(a,b;X)$ having distributional derivatives $u' \in L^p(a,b;X)$.

In what follows, we will denote by $\|\cdot\|_{L^p(X)}$, $\|\cdot\|_{C(X)}$, $\|\cdot\|_{W^{\alpha,p}(X)}$, and $\|\cdot\|_{W_*^{\alpha,p}(X)}$ the usual norms in the previous spaces. The following equivalence and continuous embedding properties are needed in the sequel (for a direct proof see e.g. [4, Appendix, Lemmas 6, 7, and 8]):

$$W^{\alpha,p}(a,b;X) = W_*^{\alpha,p}(a,b;X), \quad 0 < \alpha < 1/p; \qquad (3.3)$$

$$W^{\alpha,p}(a,b;X) \hookrightarrow C(a,b;X), \quad \alpha > 1/p; \qquad (3.4)$$

$$W_0^{\alpha,p}(a,b;X) := \{u \in W^{\alpha,p}(a,b;X),\ u(a) = 0\} \hookrightarrow W_*^{\alpha,p}(a,b;X), \quad \alpha > 1/p. \qquad (3.5)$$

We now establish some results concerning identification problems for parabolic differential equations of *specific form*. We will consider an L^p setting, $1 \le p < +\infty$, which is more convenient for our purposes.

Following the notation of Section 2 we consider the problem

$$\begin{cases} u'(t) = Au(t) - \chi\Phi[Au(t)]\zeta + \rho_1(t)\zeta_1 + \rho_2(t)\zeta_2, & \text{for a.e. } t \in (0,T) \\ u(0) = \varphi_0. \end{cases} \qquad (3.6)$$

We study (3.6) under the assumption that $A : D(A) \subset E \to E$ is the infinitesimal generator of a bounded analytic semigroup $\{S(t)\}_{t\geq 0}$ in E. It is further assumed, without loss of generality for our purposes, that $0 \in \rho(A)$. Hence $D(A)$ is a Banach space under the norm $|x|_{D(A)} := \|Ax\|$, where, from now on, $\|\cdot\|$ denotes the norm in E.

Concerning ρ_1 and ρ_2 we assume that

$$\rho_1, \rho_2 \in L^p((0,T)) \quad \text{for some } p \in [1, +\infty). \tag{3.7}$$

We are interested in solutions of (3.6) in the sense of L^p. By this we mean a function $u \in L^p(0,T;D(A)) \cap W^{1,p}(0,T;E)$ satisfying (3.6). We recall that the study of the existence of such a solution requires the introduction of the spaces $E_{\alpha,p}$, $0 < \alpha < 1$, defined (see Butzer and Berens [3, Chapter 3.5]) by

$$E_{\alpha,p} := \Big\{ x \in E : |x|_{\alpha,p} := \Big(\int_0^{+\infty} \|t^{1-\alpha} AS(t)x\|^p t^{-1}\, dt \Big)^{1/p} < +\infty \Big\}. \tag{3.8}$$

It easily follows from (3.8) that a necessary condition for the existence of L^p solutions is $\varphi_0 \in E_{1-1/p,p}$. We also recall the inclusions

$$D(A) \hookrightarrow E_{\alpha',p} \hookrightarrow E_{\alpha,p} \hookrightarrow E, \qquad 0 < \alpha < \alpha' < 1. \tag{3.9}$$

For convenience we denote by $E_{1+\alpha,p}$, $0 < \alpha < 1$ the space defined by

$$E_{1+\alpha,p} := \big\{ x \in D(A) : Ax \in E_{\alpha,p} \big\}. \tag{3.10}$$

Together with (3.6) we consider the following integral equation:

$$\begin{aligned} u(t) &= S(t)\varphi_0 - \chi \int_0^t S(t-s)\Phi[Au(s)]\zeta\, ds + \int_0^t S(t-s)h(s)\, ds \\ &\quad + \int_0^t S(t-s)[\,\rho_1(s)\zeta_1 + \rho_2(s)\zeta_2\,]ds, \end{aligned} \tag{3.11}$$

where $h \in L^p(0,T;E)$ is a given function. It is readily seen that if $h = 0$, then (3.11) is the integrated version of (3.6). Again a function $u \in L^p(0,T;D(A))$ satisfying (3.11) will be called a solution, in the sense of L^p.

To solve (3.11) it is convenient to collect some preliminary results. Set

$$U(t) := S(t)x; \qquad V(t) := \int_0^t S(t-s)h(s)\, ds.$$

The following lemmas concern the properties of the functions U and V. For the sake of simplicity we do not treat the singular case $\theta = 1/p$ and refer to [4] for a more complete study.

We have

Lemma 1. *Let $x \in E_{\theta+1-1/p,p}$, for some $p \in [1,+\infty)$ and $\theta \in (0,1)\backslash\{1/p\}$. Then $U' = AU \in W^{\theta,p}(0,T;E)$, $U \in C(0,T;E_{\theta+1-1/p,p})$, and there exists a positive constant $c = c(\theta,p)$ such that*

$$\|AU\|_{W^{\theta,p}(E)} \leq c|x|_{\theta+1-1/p,p}.$$

Proof. The results are proved in [4, Section 4]. □

Lemma 2. *Let $h \in W_*^{\theta,p}(0,T;E)$ for some $p \in [1,+\infty)$ and $\theta \in (0,1)\backslash\{1/p\}$. Then $V', AV \in W^{\theta,p}(0,T;E)$ and $V' = AV + h$. Moreover, there exists a positive constant $c' = c'(\theta,p,T)$, which is bounded for bounded T, such that*

$$\|AV\|_{W_*^{\theta,p}(E)} \leq c'\|h\|_{W_*^{\theta,p}(E)}.$$

Furthermore, we have $V \in C(0,T;E_{\theta+1-1/p,p})$ if $\theta < 1/p$, whereas $V' \in C(0,T;E_{\theta-1/p,p})$ if $\theta > 1/p$.

Proof. The results are proved in [4, Section 7]. □

Lemma 3. *Let $h \in L^p(0,T;E_{\theta,p})$, for some $p \in [1,+\infty)$ and $\theta \in (0,1)\backslash\{1/p\}$. Then $V' \in L^p(0,T;E_{\theta,p})$, $AV \in W^{\theta,p}(0,T;E)$, $V \in C(0,T;E_{\theta+1-1/p,p})$, and $V' = AV + h$. Moreover, there exists a positive constant $c'' = c''(\theta,p,T)$, which is bounded for bounded T, such that*

$$\|AV\|_{W^{\theta,p}(E)} \leq c''\|h\|_{L^p(E_{\theta,p})}.$$

Proof. The results are proved in [4, Section 6]. □

We now study some basic properties of the following function, for any given $\gamma \in L^p((0,T))$:

$$\Gamma(t) := \int_0^t S(t-s)\gamma(s)z\,ds. \tag{3.12}$$

Lemma 4. *Let $z \in E_{\theta,p}$, for some $p \in [1,+\infty)$ and $\theta \in (0,1)$. Then there exists a positive constant $c_0 = c_0(\theta,p)$ satisfying (cf. (3.2))*

$$H_{\theta,p}(A\Gamma) \leq c_0\,\|\gamma\|_{L^p}\,|z|_{T,\theta,p},$$

where

$$|z|_{T,\theta,p} := \Big(\int_0^T t^{p-1-\theta p}\|AS(t)z\|^p\,dt\Big)^{1/p}. \tag{3.13}$$

Proof. From (3.2) and (3.12) we have

$$H^p_{\theta,p}(A\Gamma) \leq \int_0^T t^{-\theta p}\Big(\int_0^t (t-s)^{\frac{1}{p}-1+\theta}(t-s)^{1-\frac{1}{p}-\theta}|\gamma(s)|\|AS(t-s)z\|\,ds\Big)^p dt.$$

Therefore, if $p=1$, we have

$$H^p_{\theta,p}(A\Gamma) \leq \int_0^T |\gamma(s)|\,ds \int_s^T t^{-\theta}\|AS(t-s)z\|\,dt.$$

If $p>1$ we use Hölder's inequality and obtain

$$\begin{aligned} H^p_{\theta,p}(A\Gamma) &\leq (p-1)^{p-1}(p\theta)^{1-p}\int_0^T dt \int_0^t (t-s)^{p-1-\theta p}\|AS(t-s)z\|^p\,|\gamma(s)|^p\,ds \\ &= (p-1)^{p-1}(p\theta)^{1-p}\int_0^T |\gamma(s)|^p ds \int_s^T (t-s)^{p-1-\theta p}\|AS(t-s)z\|^p\,dt, \end{aligned}$$

and the result follows. □

Lemma 5. *Let $z \in E_{\theta,p}$, for some $p \in [1,+\infty)$ and $\theta \in (0,1)$. Then there exists a positive constant $c_1 = c_1(\theta,p)$ such that*

$$\|A\Gamma\|_{W^{\theta,p}_*(E)} \leq c_1\,\|\gamma\|_{L^p}\,|z|_{T,\theta,p}.$$

Proof. Owing to Lemma 4 it suffices to estimate $N_{\theta,p}(A\Gamma)$. From (3.1) we have

$$\begin{aligned} N_{\theta,p}(A\Gamma) &= \int_0^T dt\int_0^t (t-s)^{-1-\theta p}\|A\Gamma(t)-A\Gamma(s)\|^p\,ds \\ &\leq 2^{p-1}\int_0^T dt\int_0^t (t-s)^{-1-\theta p}\Big\|\int_0^s A[S(t-\sigma)-S(s-\sigma)]\gamma(\sigma)z\,d\sigma\Big\|^p ds \\ &+ 2^{p-1}\int_0^T dt\int_0^t (t-s)^{-1-\theta p}\Big\|\int_s^t AS(t-\sigma)\gamma(\sigma)z\,d\sigma\Big\|^p ds =: 2^{p-1}I_1 + 2^{p-1}I_2. \end{aligned}$$

First we estimate I_1. Interchanging the order of integration, we obtain

$$\begin{aligned} I_1 &= \int_0^T dt\int_0^t (t-s)^{-1-\theta p}\,ds\Big\|\int_0^s d\sigma\int_0^{t-s} A^2S(\tau+s-\sigma)\gamma(\sigma)z\,d\tau\Big\|^p \\ &= \int_0^T ds\int_s^T (t-s)^{-1-\theta p}\,dt\Big\|\int_0^{t-s} d\tau\int_0^s A^2S(\tau+s-\sigma)\gamma(\sigma)z\,d\sigma\Big\|^p \\ &= \int_0^T ds\int_0^{T-s} t^{-1-\theta p}\,dt\Big\|\int_0^t d\tau\int_0^s A^2S(\tau+s-\sigma)\gamma(\sigma)z\,d\sigma\Big\|^p. \end{aligned}$$

Therefore, using Hardy's inequality (see e.g. [3, page 199]), we get

$$\begin{aligned} I_1 &\le \theta^{-p}\int_0^T ds\int_0^T t^{p-1-\theta p}\Big(\int_0^s \|A^2S(t+s-\sigma)\gamma(\sigma)z\| d\sigma\Big)^p dt \\ &= \theta^{-p}\int_0^T ds\int_0^T t^{p-1-\theta p}\Big(\int_0^s \|A^2S(t+\sigma)\gamma(s-\sigma)z\|\, d\sigma\Big)^p dt\,. \end{aligned}$$

As A generates a bounded analytic semigroup $\{S(t)\}_{t\ge 0}$, there exists a positive constant M such that

$$\|A^2S(t+\sigma)z\| \le M(t+\sigma)^{-1}\|AS(t+\sigma)z\| \le M^2(t+\sigma)^{-1}\|AS(\sigma)z\|$$

so that

$$\begin{aligned} I_1 &\le M^{2p}\theta^{-p}\,2^{p-1}\int_0^T ds\int_0^T t^{-1-\theta p}\Big(\int_0^t \|AS(\sigma)\gamma(s-\sigma)z\| d\sigma\Big)^p dt \\ &\quad + M^{2p}\theta^{-p}\,2^{p-1}\int_0^T ds\int_0^T t^{p-1-\theta p}\Big|\int_t^s \sigma^{-1}\|AS(\sigma)\gamma(s-\sigma)z\| d\sigma\Big|^p dt\,. \end{aligned}$$

Therefore, using Hardy's inequalities ([3, page 199]),

$$I_1 \le M^{2p}\,2^{p-1}\theta^{-p}[\theta^{-p}+(1-\theta)^{-p}]\int_0^T |\gamma(s)|^p\, ds\int_0^T t^{p-1-\theta p}\|AS(t)z\|^p\, dt. \tag{3.14}$$

We now estimate I_2. We have

$$\begin{aligned} I_2 &\le \int_0^T dt\int_0^t (t-s)^{-1-\theta p}\Big(\int_0^{t-s} \|AS(\sigma)\gamma(t-\sigma)z\|\, d\sigma\Big)^p ds \\ &= \int_0^T dt\int_0^t s^{-1-\theta p}\Big(\int_0^s \|AS(\sigma)\gamma(t-\sigma)z\|\, d\sigma\Big)^p ds \\ &\le \theta^{-p}\int_0^T dt\int_0^t |\gamma(t-s)|^p s^{p-1-\theta p}\|AS(s)z\|^p\, ds, \end{aligned} \tag{3.15}$$

where we have again used Hardy's inequality.

Combining estimates (3.14) and (3.15) for I_1 and I_2 and Lemma 4, the conclusion follows. □

Theorem 1. *Let p be given by* (3.7), *and, for some $\theta\in(0,1/p)$, let*

(i) $\rho_1,\rho_2\in L^p(0,T)$, $h\in W^{\theta,p}(0,T;E)$, $\zeta,\zeta_1,\zeta_2\in E_{\theta,p}$, *and* $\varphi_0\in E_{\theta+1-1/p,p}$. *Then there exists a unique solution u of* (3.11), *and we have*

$$u\in W^{\theta,p}(0,T;D(A))\cap C(0,T;E_{\theta+1-1/p}).$$

Moreover, there exists a positive constant $C_1 = C_1(T, \Phi, \zeta, p)$*, which is bounded for bounded* T*, satisfying*

(ii) $\|Au\|_{W^{\theta,p}(E)}, \|u'\|_{L^p(E)} \le C_1(|\varphi_0|_{\theta+1-1/p,p} + \|h\|_{W_*^{\theta,p}(E)} + \|\rho_1\|_{L^p} |\zeta_1|_{\theta,p} + \|\rho_2\|_{L^p} |\zeta_2|_{\theta,p})$.

Proof. To solve (3.11) we proceed as follows. For any fixed $w \in L^p(0,T;D(A))$ we set

$$(Lw)(t) := -\chi \int_0^t S(t-s)\Phi[Aw(s)]\zeta \, ds \tag{3.16}$$

and consider the function $u = u(w)$ defined by

$$u(w)(t) = S(t)\varphi_0 + (Lw)(t) + \int_0^t S(t-s)h(s)\, ds \tag{3.17}$$

$$+ \sum_{j=1}^{2} \int_0^t S(t-s)\rho_j(s)\zeta_j \, ds =: u_0(t) + u_1(t) + u_2(t) + u_3(t)\,.$$

Let now $\varphi_0 \in E_{\theta+1-1/p}$. From Lemma 1, we obtain

$$\|Au_0\|_{W^{\theta,p}(E)} \le c\,|\varphi_0|_{\theta+1-1/p,p}.$$

Furthermore, let $\zeta, \zeta_1, \zeta_2 \in E_{\theta,p}$. From Lemma 5 we obtain

$$\begin{aligned} \|u_3\|_{W^{\theta,p}(D(A))} &\le c_1\,[\,\|\rho_1\|_{L^p}\,|\zeta_1|_{T,\theta,p} + \|\rho_2\|_{L^p}\,|\zeta_2|_{T,\theta,p}\,], \\ \|u_1\|_{W^{\theta,p}(D(A))} &\le c_1\,|\chi|\,\|\Phi\|\,|\zeta|_{T,\theta,p}\,\|w\|_{L^p(D(A))}. \end{aligned}$$

Finally, let $h \in W^{\theta,p}(0,T;E)$. From (3.3) we have $h \in W_*^{\theta,p}(0,T;E)$ so that from Lemma 2 we obtain

$$\|Au_2\|_{W^{\theta,p}(E)} \le c'\,\|h\|_{W_*^{\theta,p}(E)}.$$

Hence, the application $w \to u(w)$ maps $L^p(0,T;D(A))$ into $W^{\theta,p}(0,T;D(A))$. Furthermore, for any given $w_1, w_2 \in L^p(0,T;D(A))$, we have

$$\|u(w_1) - u(w_2)\|_{W^{\theta,p}(D(A))} \le c_1\,|\chi|\;\|\Phi\|\;|\zeta|_{T,\theta,p}\,\|w_1 - w_2\|_{L^p(D(A))}.$$

Now the assumption $\zeta \in E_{\theta,p}$ implies that the function $t \to t^{p-1-\theta p}\|AS(t)\zeta\|^p$ belongs to $L^1((0,+\infty))$. Therefore it is possible to choose $T_0 > 0$ satisfying, for some $R < 1$, the inequality

$$c_1\,|\chi|\;\|\Phi\|\;|\zeta|_{T_0,\theta,p} \le R < 1\,. \tag{3.18}$$

Hence, if $T \le T_0$ the map $w \to u(w)$ is a contraction mapping. Consequently, (3.17) admits a unique $u \in L^p(0,T;D(A))$ satisfying, for almost every $t \in$

$(0,T)$,

$$u(t) = S(t)\varphi_0 + (Lu)(t) + \int_0^t S(t-s)h(s)\,ds + \sum_{j=1}^{2}\int_0^t S(t-s)\rho_j(s)\zeta_j ds.$$

Whence, we deduce also $u \in W^{\theta,p}(0,T;D(A))$ and the estimate

$$\|u\|_{W^{\theta,p}(D(A))} \le (1-R)^{-1}\left[c\,|\varphi_0|_{\theta+1-1/p,p} + c'\,\|h\|_{W_*^{\theta,p}(E)} + c_1(\|\rho_1\|_{L^p}|\zeta_1|_{\theta,p} + \|\rho_2\|_{L^p}|\zeta_2|_{\theta,p})\right],$$

where $R < 1$ satisfies (3.18). Therefore $\|Au\|$ satisfies estimate (ii).

It remains to prove $u \in C(0,T;E_{\theta+1-1/p,p}) \cap W^{1,p}(0,T;E)$. To do this we prove that the functions u_i defined in (3.17), $i = 0,1,2,3$, satisfy

$$u_i \in C(0,T;E_{\theta+1-1/p,p}) \cap W^{1,p}(0,T,E). \tag{3.19}$$

Lemmas 1 and 2 imply that u_0 and u_2 satisfy (3.19). Furthermore, it is easy to see that, if $\Phi[Au] \in L^p((0,T))$ and $\zeta \in E_{\theta,p}$, then $\Phi[Au]\zeta \in L^p(0,T;E_{\theta,p})$. Therefore from Lemma 3 we deduce that u_1 also satisfies (3.19). In the same way we find that u_3 satisfies (3.19). Hence, from (3.17), $u \in C(0,T;E_{\theta+1-1/p,p}) \cap W^{1,p}(0,T;E)$.

Finally, from equation (3.11) and the fact that $\|Au\|$ satisfies estimate (ii) it can be easily deduced that also $\|u'\|$ satisfies (ii).

Therefore, the theorem is proved if $T \in (0,T_0]$, with T_0 satisfying (3.18). Since (3.18) does not depend on φ_0 we can iterate this procedure and prove, step by step, the results for all $T > 0$. We use here the following property: if $f_1 \in W^{\theta,p}(a_1,a_2;E)$ and $f_2 \in W^{\theta,p}(a_2,a_3;E)$, $\theta \in (0,1/p)$, then

$$f(t) = \begin{cases} f_1(t), & t \in (a_1,a_2), \\ f_2(t), & t \in (a_2,a_3), \end{cases}$$

belongs to $W^{\theta,p}(a_1,a_3;E)$. □

To study the case $\theta > 1/p$ we will assume that

$$\rho_1, \rho_2 \in C([0,T]). \tag{3.20}$$

Then we have

Theorem 2. *Let* (3.20) *be satisfied and assume that, for some* $p \in (1,+\infty)$ *and* $\theta \in (1/p,1)$ *the following property is satisfied:*

(i) $h \in W^{\theta,p}(0,T;E)$, $\zeta,\zeta_1,\zeta_2 \in E_{\theta,p}$, $\varphi_0 \in D(A)$, *and* $A\varphi_0 + h(0) \in E_{\theta-1/p,p}$.

Then there exists a unique solution u of (3.11), *and we have*

$$u \in C^1(0,T;E_{\theta-1/p}) \cap W^{\theta,p}(0,T;D(A)).$$

Moreover, there exists a positive constant $C_1 = C_1(T,\Phi,\zeta,p)$, which is bounded for bounded T, such that

(ii) $\|Au\|_{W^{\theta,p}(E)}, \|u'\|_{L^p(E)} \le C_1(\|A\varphi_0 + h(0)\|_{\theta-1/p,p} + \|h_*\|_{W_*^{\theta,p}(E)} + \|\rho_1\|_{L^p} \, |\zeta_1|_{\theta,p} + \|\rho_2\|_{L^p}|\zeta_2|_{\theta,p})$,

where $h_*(t) := h(t) - h(0)$.

Proof. We use the notation of Theorem 1. From (3.4) we have $h \in C(0,T;E)$ so that we can define

$$h_*(t) = h(t) - h(0).$$

Then from (3.5) we have $h_* \in W_*^{\theta,p}(0,T;E)$, and we can rewrite (3.17) in the following equivalent way:

$$\begin{aligned} u(w)(t) &= S(t)[\varphi_0 + A^{-1}h(0)] - A^{-1}h(0) + (Lw)(t) \\ &\quad + \int_0^t S(t-s)h_*(s)\,ds + \sum_{j=1}^2 \int_0^t S(t-s)\rho_j(s)\zeta_j\,ds. \end{aligned} \tag{3.21}$$

As $A^{-1}h(0) \in D(A)$ we can repeat the arguments used in the proof of Theorem 1 and obtain that the map $w \to u(w)$ from $L^p(0,T;D(A))$ into $W^{\theta,p}(0,T;D(A))$ is a contraction mapping if T satisfies (3.18). Consequently there exists a unique $u \in L^p(0,T;D(A))$ satisfying

$$\begin{aligned} u(t) &= S(t)[\varphi_0 + A^{-1}h(0)] - A^{-1}h(0) + (Lu)(t) \\ &\quad + \int_0^t S(t-s)h_*(s)\,ds + \sum_{j=1}^2 \int_0^t S(t-s)\rho_j(s)\zeta_j\,ds \\ &=: v_0(t) + v_1(t) + v_2(t) + v_3(t) + v_4(t). \end{aligned} \tag{3.22}$$

Furthermore, by arguments similar to those used in the proof of Theorem 1, we deduce $u \in W^{\theta,p}(0,T;D(A))$ and $u \in W^{1,p}(0,T;E)$.

It remains to prove $u' \in C(0,T;E_{\theta-1/p,p})$. For this purpose we show that such a property is enjoyed by any function v_i, $i = 0,\dots,4$, in (3.22). From Lemmas 1 and 2 we have $v_0' = Av_0 \in C(0,T;E_{\theta-1/p,p})$ and $v_3' \in C(0,T;E_{\theta-1/p,p})$, respectively.

We now study v_2 and v_4. Using the assumption $\zeta \in E_{\theta,p}$, the property $u \in W^{\theta,p}(0,T;D(A))$, and the embedding (3.3), we find $\Phi[Au]\zeta \in C(0,T;E_{\theta,p})$. Hence, from Lemma 3 and the inclusion (3.9), we obtain $v_2' = Av_2 - \Phi[Au]\zeta \in$

$C(0,T;E_{\theta-1/p,p})$. In the same way, using assumptions (3.20) and $\zeta_1,\zeta_2 \in E_{\theta,p}$ we obtain $v_4' \in C(0,T;E_{\theta-1/p,p})$. Furthermore, as $v_1' = 0$, we deduce $u' \in C(0,T;E_{\theta-1/p,p})$.

Finally, from equation (3.11) we easily deduced that $\|u'\|$ satisfies (ii). Therefore the theorem is proved if T satisfies (3.18).

Now if $u' \in C(0,T;E_{\theta-1/p,p})$, then $Au+h = u' + \chi\Phi[Au]\zeta - \rho_1\zeta_1 - \rho_2\zeta_2 \in C(0,T;E_{\theta-1/p,p})$, so that $Au(T)+h(T) \in E_{\theta-1/p,p}$. Therefore we can iterate this procedure over $[T,2T]$ with $A\varphi_0 + h(0)$ being replaced by $Au(T)+h(T)$ and prove the result for all $T>0$. We use here the following property: if $f_1 \in W^{\theta,p}(a_1,a_2;E)$, $f_2 \in W^{\theta,p}(a_2,a_3;E)$, $\theta \in (1/p,1)$, and $f_1(a_2) = f_2(a_2)$, then

$$f(t) = \begin{cases} f_1(t), & t \in (a_1,a_2), \\ f_2(t), & t \in (a_2,a_3), \end{cases}$$

belongs to $W^{\theta,p}(a_1,a_3;E)$. □

Finally, concerning problem (3.6), we have the following result.

Theorem 3. *Assume that*

(i) $\zeta,\zeta_1,\zeta_2 \in E_{\theta,p}$, $\varphi_0 \in E_{\theta+1-1/p,p}$, *for some* $\theta \in (0,1)\backslash\{1/p\}$ *and* $p \in [1,+\infty)$.

Then if $\theta < 1/p$ *and* $\rho_1,\rho_2 \in L^p((0,T))$, *there exists a unique solution* u *of* (3.6) *and we have* $u \in W^{\theta,p}(0,T;D(A)) \cap C(0,T;E_{\theta+1-1/p,p})$. *Moreover, there exists a positive constant* $C_1 = C_1(T,\Phi,\zeta,p)$, *which is bounded for bounded* T, *such that*

(ii) $\|Au\|_{W^{\theta,p}(E)}, \|u'\|_{L^p(E)} \le C_1(\|\varphi_0\|_{\theta+1-1/p,p} + \|\rho_1\|_{L^p}\|\zeta_1\|_{\theta,p}$ $+\|\rho_2\|_{L^p}\|\zeta_2\|_{\theta,p})$.

If $\theta > 1/p$ *and* $\rho_1,\rho_2 \in C((0,T))$, *then there exists a unique solution* u *of* (3.6). *Moreover,* $u \in C^1(0,T;E_{\theta-1/p,p}) \cap W^{\theta,p}(0,T;D(A))$, *and satisfies properties* (ii).

Proof. The results follow from Theorems 1 and 2 with $h=0$. □

4. Solving the identification problem

We can now study the delay problem (2.10). We assume that A satisfies the hypotheses of Section 3 and that $A_1, A_2 \in \mathcal{L}(D(A),E)$. Hence, there exist two positive constants k_1,k_2 satisfying, for each $x \in D(A)$,

$$\|A_1x\| \le k_1\|Ax\|, \qquad \|A_2x\| \le k_2\|Ax\|. \tag{4.1}$$

Furthermore, we assume

$$a \in L^p((-r,0)) \quad \text{for some } p \in [1,+\infty), \tag{4.2}$$

and we set

$$\|a\|_\tau := \Big(\int_{-\tau}^{0} |a(s)|^p \, ds \Big)^{1/p}, \qquad \tau \in (0, r]. \tag{4.3}$$

As in Section 3, we are interested in solutions u to problem (2.10) in the sense of L^p; i.e., $u \in L^p(-r, T; D(A)) \cap W^{1,p}(0, T; E)$.

Take now $\varphi_1 \in L^p(-r, 0; D(A))$ and fix $T \in (0, r]$. We denote by h_0, ρ_3, and ρ_4 the following functions defined, for $t \in [0, T]$, by

$$h_0(t) = A_1\varphi_1(t-r) + \int_{-r}^{-t} a(s)A_2\varphi_1(t+s)\, ds =: h_1(t) + h_2(t) \tag{4.4}$$

$$\rho_3(t) = -\chi\Phi[A_1\varphi_1(t-r)], \quad \rho_4(t) := -\chi \int_{-r}^{-t} a(s)\Phi[A_2\varphi_1(t+s)]\, ds. \tag{4.5}$$

Furthermore, for $w \in L^p(0, T; D(A))$ and $t \in [0, T]$, we set

$$\begin{aligned}(L_0 w)(t) &:= \int_{-t}^{0} a(s)A_2 w(t+s)\, ds - \chi \int_{-t}^{0} a(s)\Phi[A_2 w(t+s)]\zeta_4\, ds \\ &:= (L_1 w)(t) + (L_2 w)(t). \end{aligned} \tag{4.6}$$

Then, if $T \in (0, r]$, problem (2.10) can be rewritten in the form

$$\begin{cases} u'(t) = Au(t) - \chi\Phi[Au(t)]\zeta + h_0(t) + (L_0 u)(t) + \sum_{j=1}^{4} \rho_j(t)\zeta_j, \\ u(0) = \varphi_0. \end{cases} \tag{4.7}$$

We begin by studying (4.7). For this purpose we consider its integrated version

$$\begin{aligned} u(t) = S(t)\varphi_0 &- \chi \int_0^t S(t-s)\Phi[Au(s)]\zeta\, ds + \int_0^t S(t-s)h_0(s)\, ds \\ &+ \int_0^t S(t-s)(L_0 u)(s)\, ds + \sum_{j=1}^{4} \int_0^t \rho_j(s)S(t-s)\zeta_j\, ds. \end{aligned} \tag{4.8}$$

To state our results it is convenient to introduce some further notation.

For any given $\theta \in (0,1)$, let α be given by

$$\alpha := \max(0, \theta - 1 + 1/p). \tag{4.9}$$

Then $\alpha \le \theta$ and $\alpha < 1/p$, so that for $\psi \in W^{\theta,p}(a, b; E)$ we have $H_{\alpha,p}(\psi) < +\infty$, and we can set

$$N_{\alpha,\theta,p}(\psi) := H_{\alpha,p}(\psi) + N_{\theta,p}(\psi). \tag{4.10}$$

In what follows we denote by $W^{\alpha,\theta,p}(0,T;D(A))$ the space $W^{\theta,p}(0,T;D(A))$ endowed with the equivalent norm $N_{\alpha,\theta,p}$:

$$W^{\alpha,\theta,p}(0,T;D(A)) := \{u \in L^p(0,T;D(A)) : N_{\alpha,\theta,p}(Au) < +\infty\}. \quad (4.11)$$

We begin by establishing some preliminary results.

Lemma 6. *Let $w \in W^{\theta,p}(0,T;D(A))$ for some $\theta \in (0,1)$. Then $L_1 w, L_2 w \in W_*^{\theta,p}(0,T;E)$, and there exists a positive constant $c_2 = c_2(k_2,T,\theta,p)$, which is bounded for bounded T, such that*

(i) $\|L_1 w\|_{W_*^{\theta,p}(E)} \le c_2 \,\|a\|_T \, N_{\alpha,\theta,p}(Aw)$;

(ii) $\|L_2 w\|_{W_*^{\theta,p}(E)} \le c_2 \,|\chi|\, \|a\|_T \, N_{\alpha,\theta,p}(Aw) \ \|\Phi\| \ \|\zeta_4\|$.

Proof. From (3.2) we have (here, when $p > 1$, we use Hölder's inequality)

$$H^p_{\theta,p}(L_1 w) \le \int_0^T t^{p-1-\theta p}\Big(\int_{-t}^0 \|a(s)A_2 w(t+s)\|^p\, ds\Big)\, dt.$$

Hence, if $p - 1 - \theta p > 0$, we get

$$H^p_{\theta,p}(L_1 w) \le T^{p-1-\theta p}\, k_2^p \ \|a\|_T^p \int_0^T \|Aw(t)\|^p\, dt = T^{p-1-\theta p}\, k_2^p \ \|a\|_T^p H^p_{\alpha,p}(Aw)$$

whereas, if $p - 1 - \theta p < 0$, we obtain

$$H^p_{\theta,p}(L_1 w) \le \ k_2^p \ \|a\|_T^p \int_0^T t^{p-1-\theta p}\|Aw(t)\|^p\, dt = k_2^p \ \|a\|_T^p H^p_{\alpha,p}(Aw).$$

Furthermore, from (3.1) we have

$$\begin{aligned} N^p_{\theta,p}(L_1 w) \le{}& 2^{p-1}\int_0^T dt \int_0^t (t-s)^{-1-\theta p} \\ & \times \Big\| \int_{-s}^0 a(\sigma)(A_2 w(t+\sigma) - A_2 w(s+\sigma))\, d\sigma \Big\|^p\, ds \\ & + 2^{p-1}\int_0^T dt \int_0^t (t-s)^{-1-\theta p}\Big\| \int_{-t}^{-s} a(\sigma) A_2 w(t+\sigma) d\sigma\Big\|^p ds \\ & =: 2^{p-1} I_1 + 2^{p-1} I_2. \end{aligned}$$

Hence (using again Hölder's inequality when $p > 1$), we find

$$\begin{aligned} I_1 &\le \int_0^T dt \int_0^t (t-s)^{-1-\theta p} s^{p-1}\, ds \int_{-s}^0 |a(\sigma)|^p \|A_2(w(t+\sigma) - w(s+\sigma))\|^p\, d\sigma \\ &= \int_{-T}^0 |a(\sigma)|^p\, d\sigma \int_{-\sigma}^T dt \int_{-\sigma}^t (t-s)^{-1-\theta p} s^{p-1} \|A_2(w(t+\sigma) - w(s+\sigma))\|^p ds \end{aligned}$$

$$\leq T^{p-1} k_2^p \, \|a\|_T^p N_{\theta,p}^p(Aw).$$

To estimate I_2 we note that

$$I_2 \leq \int_0^T dt \int_0^t (t-s)^{-1-\theta p} \Big(\int_0^{t-s} |a(\sigma - t)| \, \|A_2 w(\sigma)\| \, d\sigma \Big)^p ds$$
$$= \int_0^T dt \int_0^t s^{-1-\theta p} \Big(\int_0^s |a(\sigma - t)| \, \|A_2 w(\sigma)\| \, d\sigma \Big)^p ds \, .$$

Finally, using Hardy's inequality, we get

$$I_2 \leq \theta^{-p} \, k_2^p \int_0^T dt \int_0^t s^{p-1-\theta p} |a(s-t)|^p \|Aw(s)\|^p \, ds$$
$$= \theta^{-p} \, k_2^p \int_0^T s^{p-1-\theta p} \|Aw(s)\|^p \, ds \int_s^T |a(s-t)|^p \, dt \, ,$$

which completes the proof of (i).

Assertion (ii) can be proved in a similar way. □

Lemma 7. *Let $\varphi_1 \in W^{\theta,p}(-r,0;D(A))$ for some $\theta \in (0,1)$. Then $h_1, h_2 \in W^{\theta,p}(0,T;E)$ and satisfy the estimates*

(i) $\|h_1\|_{W^{\theta,p}(E)} \leq k_1 \|\varphi_1\|_{W^{\theta,p}(D(A))}$;
(ii) $\|h_2\|_{W^{\theta,p}(E)} \leq c_2 \, \|a\|_r \, N_{\alpha,\theta,p} \, (A\varphi_1)$.

Proof. Assertion (i) can be easily proved by a direct inspection. Assertion (ii) can be proved by arguments similar to those of Lemma 6. □

Lemma 8. *Let $\varphi_1 \in W^{\theta,p}(-r,0,D(A))$ for some $\theta \in (0,1)$. Then $\rho_3, \rho_4 \in L^p(0,T;E)$, and there exists a positive constant $c_3 = c_3(\chi, \Phi, k_1, k_2)$ such that*

(i) $\|\rho_3\|_{L^p(E)} \leq c_3 \, \|A\varphi_1\|_{L^p(E)}$;
(ii) $\|\rho_4\|_{L^p(E)} \leq c_3 \, \|a\|_r \, \|A\varphi_1\|_{L^p(E)}$.

If $\theta > 1/p$ we have in addition

(iii) $\rho_3, \rho_4 \in C(0,T;E)$.

Proof. Assertions (i) and (ii) can be easily proved by a direct inspection. Assertion (iii) follows from (3.4). □

We now are able to prove the following result.

Theorem 4. *Let p be given by* (4.2), *and assume that, for some $\theta \in (0,1/p)$,*

(i) $\rho_1, \rho_2 \in L^p((0,T))$, $\zeta, \zeta_1, \zeta_2, \zeta_3, \zeta_4 \in E_{\theta,p}$;
(ii) $\varphi_1 \in W^{\theta,p}(-r,0;D(A))$, $\varphi_0 \in E_{\theta+1-1/p,p}$.

Then there exists a unique solution $u \in W^{\theta,p}(-r,T;D(A))$ *of* (2.10), *and we have* $u \in C(0,T;E_{\theta+1-1/p,p})$. *Moreover, there exists a positive constant* $C_2 = C_2(T,\Phi,\zeta,p,k_1,k_2,a)$ *which is bounded for bounded* T, *satisfying*

(iii) $\|Au\|_{W^{\theta,p}(E)}, \|u'\|_{L^p(E)} \le C_2[|\varphi_0|_{\theta+1-1/p,p} + \|\varphi_1\|_{W^{\theta,p}(D(A))}(1+|\zeta_3|_{\theta,p} +|\zeta_4|_{\theta,p}) + |\zeta_1|_{\theta,p}\|\rho_1\|_{L^p} + |\zeta_2|_{\theta,p}\|\rho_2\|_{L^p}]$.

Proof. We begin by proving the theorem in the case when $T \in (0,r]$. We consider the integrated version (4.8) and use a fixed-point argument. For any given $w \in W^{\theta,p}(0,T;D(A))$ we consider the integral equation

$$u(t) = S(t)\varphi_0 - \chi \int_0^t S(t-s)\Phi[Au(s)]\zeta\, ds + \int_0^t S(t-s)h_0(s)\, ds + \int_0^t S(t-s)(L_0 w)(s)\, ds + \sum_{j=1}^4 \int_0^t \rho_j(s)S(t-s)\zeta_j\, ds.$$

Now property (3.3) and Lemma 7 imply that $h_0 \in W_*^{\theta,p}(0,T;E)$ and that there exists a positive constant k such that

$$\|h_0\|_{W_*^{\theta,p}(E)} \le k\,(k_1 + c_2\|a\|_r)\,\|\varphi_1\|_{W^{\theta,p}(D(A))}. \tag{4.12}$$

Moreover, from Lemma 6 we deduce $L_0 \in W_*^{\theta,p}(0,T;E)$ and

$$\|L_0 w\|_{W_*^{\theta,p}(E)} \le c_2\|a\|_T\,(1 + |\chi|\,\|\Phi\|\;\|\zeta_4\|\,)N_{\alpha,\theta,p}(Aw). \tag{4.13}$$

Therefore, using Theorem 1, with h being replaced by $h_0 + L_0 w$, we can ensure that there exists a unique $u = u(w) \in W^{\theta,p}(0,T;D(A))$ satisfying

$$\begin{aligned} u(w)(t) &= S(t)\varphi_0 - \chi \int_0^t S(t-s)\Phi[Au(w)(s)]\zeta\, ds \\ &\quad + \int_0^t S(t-s)[h_0(s) + (L_0 w)(s)\,]\, ds \\ &\quad + \sum_{j=1}^4 \int_0^t \rho_j(s)S(t-s)\zeta_j\, ds =: u_0(t) + u_1(t) + u_2(t) + u_3(t). \end{aligned} \tag{4.14}$$

Furthermore, from Theorem 1 (ii), Lemma 8, and estimates (4.12) and (4.13) we obtain

$$\begin{aligned} N_{\theta,p}(Au(w)) &\le C_1\Big[|\varphi_0|_{\theta+1-1/p,p} + (k_1' + c_3)\|\varphi_1\|_{W^{\theta,p}(D(A))} \\ &\times (1+|\zeta_3|_{\theta,p}+\|a\|_r|\zeta_4|_{\theta,p}) + c_2'\|a\|_T N_{\alpha,\theta,p}(Aw) + \|\rho_1\|_{L^p}|\zeta_1|_{\theta,p} + \|\rho_2\|_{L^p}|\zeta_2|_{\theta,p}\Big], \end{aligned}$$

where

$$k_1' = k\,(k_1 + c_2\|a\|_r), \qquad c_2' = c_2[1 + \|\Phi\|\ |\chi|\ \|\zeta_4\|\,].$$

We now estimate $H_{\alpha,p}(Au_i)$, where the u_i's, $i = 0, \ldots, 3$, are given by (4.14). From (3.2) and (3.8), since $\alpha \le \theta$, it follows easily that

$$H_{\alpha,p}(Au_0) \le T^{\theta-\alpha} H_{\theta,p}(Au_0) = T^{\theta-\alpha}|\varphi_0|_{\theta+1-1/p,p}.$$

From Lemmas 4 and 8 and the inequalities $0 \le \alpha \le \theta$ we further obtain

$$\begin{aligned} H_{\alpha,p}(Au_1) &\le T^{\theta-\alpha} H_{\theta,p}(Au_1) \le T^{\theta-\alpha} c_0\,|\chi|\ \|\Phi\|\ \|Au(w)\|_{L^p(E)} |\zeta|_{T,\theta,p} \\ &\le T^{\theta} c_0\,|\chi|\ \|\Phi\|\ H_{\alpha,p}(Au(w))|\zeta|_{T,\theta,p} \end{aligned}$$

and

$$\begin{aligned} H_{\alpha,p}(Au_3) \le c_0[&\|\rho_1\|_{L^p(E)}|\zeta_1|_{T,\theta,p} + \|\rho_2\|_{L^p(E)}|\zeta_2|_{T,\theta,p} + c_3\|A\varphi_1\|_{L^p(E)} \\ &\times (|\zeta_3|_{\theta,p} + \|a\|_r|\zeta_4|_{\theta,p})]. \end{aligned}$$

Finally, from Lemma 2 and (4.12) and (4.13) we derive

$$H_{\alpha,p}(Au_2) \le c'\|h_0 + L_0 w\|_{W_*^{\alpha,p}} \le c' k_1'\,\|\varphi_1\|_{W^{\theta,p}(D(A))} + c' c_2' \|a\|_T N_{\alpha,\theta,p}(Aw).$$

Now the assumption $\zeta \in E_{\theta,p}$ implies that there exists $0 < T_0 \le r$ satisfying, for some $R' < 1$,

$$c_0|\chi|\ \|\Phi\|\ |\zeta|_{T_0,\theta,p} \le R' < 1. \tag{4.15}$$

Therefore, if $T \le T_0$, we obtain

$$\begin{aligned} &H_{\alpha,p}(Au(w)) \\ &\le (1-R')^{-1}[|\varphi_0|_{\theta+1-1/p} + c_0\,(\,\|\rho_1\|_{L^p(E)}|\zeta_1|_{T,\theta,p} + \|\rho_2\|_{L^p(E)}|\zeta_2|_{T,\theta,p}) \\ &+ c_3'\|\varphi_1\|_{W^{\theta,p}(D(A))}\,(\,1 + |\zeta_3|_{\theta,p} + \|a\|_r\,|\zeta_4|_{\theta,p}) + c' c_2' \|a\|_T\,N_{\alpha,\theta,p}(Aw)\,], \end{aligned}$$

where $c_3' = c_0 c_3 + c' k_1'$. Denote now by $u(w_1)$ and $u(w_2)$ the solutions of (4.14) for given $w_1, w_2 \in W^{\alpha,\theta,p}(0,T;D(A))$. Since

$$\begin{aligned} u(w_1)(t) - u(w_2)(t) = -\chi \int_0^t S(t-s)\Phi[Au(w_1)(s) - Au(w_2)(s)]\zeta\,ds \\ + \int_0^t S(t-s)L_0(w_1-w_2)(s)\,ds, \end{aligned}$$

then, combining the preceding estimates, we obtain

$$N_{\alpha,\theta,p}(Au(w_1) - Au(w_2)) \le\ c_2'\|a\|_T \Big[\,C_1 + c'(1-R')^{-1}\Big] N_{\alpha,\theta,p}(Aw_1 - Aw_2).$$

Now the assumption $a \in L^p((-r,0))$ implies that there exists $0 < T_0 \le r$ satisfying, in addition to (4.15), also the inequality

$$c_2' \|a\|_{T_0} \Big[C_1 + c'(1-R')^{-1} \Big] \le R'' \tag{4.16}$$

for some $R'' < 1$. Therefore, if $T \in (0, T_0]$, the map $w \to u(w)$ is a contraction mapping. Consequently, there exists a unique $u \in W^{\alpha,\theta,p}(0,T;D(A))$ satisfying (4.8). Moreover, Au satisfies estimate (iii).

Furthermore, using Theorem 1, with h being replaced by $h_0 + L_1 u + L_2 u$, we have also $u \in W^{1,p}(0,T;E) \cap C(0,T;E_{\theta+1-1/p,p})$ so that u is a solution to problem (4.7), too. Finally, from (4.7), and the fact that Au satisfies estimate (iii), we easily deduce that u' also satisfies (iii). Hence the theorem is proved if $T \in (0,T_0]$ with $T_0 \in (0,r]$ satisfying (4.15) and (4.16). As (4.15) and (4.16) do not depend on the data φ_1 and φ_0, we can iterate this procedure and prove, step by step, our results for all $T > 0$. □

The case $\theta > 1/p$ is dealt with in the following theorem.

Theorem 5. *Let* (4.2) *hold with* $p > 1$ *and assume that, for some* $\theta \in (1/p, 1)$,

(i) $\rho_1, \rho_2 \in C((0,T))$, $\zeta, \zeta_1, \zeta_2, \zeta_3, \zeta_4 \in E_{\theta,p}$,

(ii) $\varphi_1 \in W^{\theta,p}(-r,0;D(A))$, $\varphi_0 = \varphi_1(0)$,

$$A\varphi_0 + A_1\varphi_1(-r) + \int_{-r}^0 A_2\varphi_1(s)ds \in E_{\theta-1/p,p}.$$

Then there exists a unique solution u *of* (2.10) *such that* $u \in W^{\theta,p}(-r,T; D(A))$ *and we have* $u' \in C(0,T;E_{\theta-1/p,p})$. *Moreover, there exists a positive constant* $C_2 = C_2(T,\Phi,\zeta,p,k_1,k_2,a)$, *which is bounded for bounded* T, *satisfying*

(iii)

$$\|Au\|_{W^{\theta,p}(E)}, \|u'\|_{L^p(E)} \le C_2 \Big[\Big| A\varphi_0 + A_1\varphi_1(-r) + \int_{-r}^0 A_2\varphi_1(s)ds \Big|_{\theta-1/p,p}$$
$$+ \|\varphi_1\|_{W^{\theta,p}(D(A))}(1 + |\zeta_3|_{\theta,p} + |\zeta_4|_{\theta,p}) + |\zeta_1|_{\theta,p}\|\rho_1\|_{L^p} + |\zeta_2|_{\theta,p}\|\rho_2\|_{L^p} \Big].$$

Proof. We proceed as in the proof of Theorem 4. First we prove the theorem for $T < r$ and consider equation (4.14), with a fixed $w \in W^{\theta,p}(0,T;D(A))$.

Assumption (ii) and Lemmas 6 and 7 imply that the function $h := h_0 + L_0(w)$ belongs to $W^{\theta,p}(0,T;E)$ and $A\varphi_0 + h(0) = A\varphi_0 + h_1(0) + h_2(0) \in E_{\theta-1/p,p}$. Furthermore, from Lemma 8 the functions ρ_3 and ρ_4 defined by (4.5) belong to $C(0,T;E)$.

Hence, from Theorem 2 and a fixed-point argument similar to the one used in the proof of Theorem 4 we can prove that there exists a unique u satisfying all the assertions for $T \in (0, T_0]$, with $T_0 \in (0, r]$ satisfying (4.15) and (4.16). To be able to iterate this procedure we have to prove that $Au(T) + h(T) \in E_{\theta-1/p,p}$.

Using the property $u \in W^{\theta,p}(0, T; D(A))$ and (3.4) we find $\Phi[Au] \in C([0, T])$, so that from the assumption $\zeta \in E_{\theta,p}$ we have

$$\Phi[Au]\zeta \in C(0, T; E_{\theta,p}).$$

Furthermore, from $\rho_i \in C([0, T])$ and $\zeta_i \in E_{\theta,p}$ we find $\rho_i\zeta_i \in C(0, T; E_{\theta,p})$, $i = 1, \ldots, 4$. Since $u' \in C(0, T; E_{\theta-1/p,p})$, we deduce

$$Au + h = u' + \chi\Phi[Au]\zeta - \rho_1\zeta_1 - \rho_2\zeta_2 - \rho_3\zeta_3 - \rho_4\zeta_4 \in C(0, T; E_{\theta-1/p,p})$$

so that $Au(T) + h(T) \in E_{\theta-1/p,p}$. Therefore we can iterate the above procedure over $[T, 2T]$ and prove the results for all $T > 0$. □

We conclude this section by showing that in the case where $\Phi[z] = 0$, the delay problem (2.3) *may* admit infinitely many solutions. For this purpose assume that z is also a common eigenvector of operators A, A_1, and A_2:

$$Az = \lambda_0 z, \qquad A_j z = \lambda_j z, \quad j = 1, 2,$$

where $\lambda_0, \lambda_1, \lambda_2 \in \mathbf{R}$. For given $\psi_0 \in \mathbf{R}$ and $\psi_1 \in L^p((-r, 0))$ we look for solutions $(\psi z, f)$, ψ being a *scalar function*, to problem (1.1)–(1.3) with $g = 0$ and

$$\varphi_1(s) = \psi_1(s)z, \quad s \in (-r, 0), \qquad \varphi_0 = \psi_0 z.$$

Then the pair (ψ, f) is easily seen to be a solution to the *scalar* delay problem

$$\psi'(t) = \lambda_0\psi(t) + \lambda_1\psi(t - r) + \lambda_2 \int_{-r}^{0} a(s)\psi(t + s)\, ds + f(t), \tag{4.17}$$

for a.e. $t \in (0, T)$,

$$\psi(s) = \psi_1(s), \quad \text{for a.e. } s \in (-r, 0), \qquad \psi(0) = \psi_0. \tag{4.18}$$

From known results concerning scalar delay differential equations we easily deduce the following theorem.

Theorem 6. *Let $a \in L^p((-r, 0))$ and $\psi_1 \in L^p((-r, 0))$, for some $p \in [1, +\infty)$, and let $\psi_0 \in \mathbf{R}$. Then for any $f \in L^p((0, T))$ there exists a unique solution $\psi \in W^{1,p}((0, T))$ to problem* (4.17)–(4.18).

5. An application

We can now apply the abstract results of the previous sections to recover the source term $f : [0,T] \to \mathbf{R}$ in the following parabolic identification problem related to a bounded domain Ω in $\mathbf{R}^n$ of class C^2:

$$D_t u(t,x) = \mathcal{A}_0 u(t,x) + \mathcal{A}_1 u(t-r,x) + \int_{-r}^{0} a(s)\mathcal{A}_2 u(t+s,x)\,\mathrm{d}s + f(t)z(x),$$

$$\text{for a.e. } (t,x) \in [0,T]\times\Omega, \tag{5.1}$$

$$Bu(t,x) = 0,\ \text{for a.e. } (t,x) \in [0,T]\times\partial\Omega, \tag{5.2}$$

$$u(s,x) = \varphi_1(s,x) \text{ for a.e. } (s,x) \in (-r,0)\times\Omega,\ u(0,x) = \varphi_0(x), \tag{5.3}$$

$$\text{for a.e. } x \in \Omega,$$

$$\Phi[u(t,\cdot)] = g(t), \qquad \text{for a.e. } t \in [0,T]. \tag{5.4}$$

Here, $\mathcal{A}_i$, $i = 0,1,2$, denote the following second-order linear differential operators:

$$\mathcal{A}_i = \sum_{j,k=1}^{n} a_{j,k}^{(i)}(x) D_{x_j} D_{x_k} + \sum_{j=1}^{n} a_j^{(i)}(x) D_{x_j} + a_0^{(i)}(x). \tag{5.5}$$

We will assume that the coefficients of $\mathcal{A}_i$ are continuous on $\overline{\Omega}$ and that the coefficients of $\mathcal{A}_0$ enjoy the following property:

$$\sum_{j,k=1}^{n} a_{j,k}^{(0)}(x)\xi_j\xi_k \geq \alpha|\xi|^2, \qquad \forall\,(x,\xi) \in \Omega\times\mathbf{R}^n, \tag{5.6}$$

α being a positive constant.

Moreover, the boundary operator B is defined by either of the following equations:

$$\text{(D)} \quad Bu(x) = u(x), \qquad x \in \partial\Omega, \tag{5.7}$$

$$\text{(O)} \quad Bu(x) = \sum_{j=1}^{n} b_j(x) D_{x_j} u(x) + b_0(x)u(x), \qquad x \in \partial\Omega, \tag{5.8}$$

where $b_i \in C^1(\overline{\Omega})$ and

$$\Big|\sum_{j=1}^{n} b_j(x) n_j(x)\Big| \geq \nu_0 > 0, \qquad x \in \partial\Omega. \tag{5.9}$$

Here D and O stand, respectively, for the Dirichlet and the oblique boundary operators, $n(x)$ denoting the outward-normal vector at $x \in \partial\Omega$. Now, for

$q \in [1, +\infty)$ and $0 < s \le 2$, we introduce the spaces $E_q := L^q(\Omega)$ and

$$W_B^{s,q}(\Omega) := \begin{cases} W^{s,q}(\Omega), & s \in (0, q^{-1} + \operatorname{ord} B), \\ \{u \in W^{s,q}(\Omega) : \int_\Omega |Bu|^p d^{-1}\, dx < +\infty\}, & s = q^{-1} + \operatorname{ord} B, \\ \{u \in W^{s,q}(\Omega) : Bu = 0 \text{ on } \partial\Omega\} & s \in (q^{-1} + \operatorname{ord} B, 2], \end{cases} \tag{5.10}$$

where $\operatorname{ord} B$ and d denote, respectively, the order of the operator B and the distance from x to $\partial\Omega$.

Next we denote by $A_{(q)}$ the *realization* in E_q of $\mathcal{A}_0$, with the boundary conditions $Bu_{|\partial\Omega} = 0$. Then it is known that $A_{(q)}$ is the infinitesimal generator of an analytic semigroup on E_q. See Agmon [1] for the case $q > 1$ and Amann [2], Pazy [13], and Tanabe [15] for the case $q = 1$. See also the monograph by Lunardi [11, Chapter 3] and Lunardi and Metafune [12]. If $q > 1$ we have

$$D(A_{(q)}) = W_B^{2,q}(\Omega), \qquad A_{(q)}u = \mathcal{A}_0 u. \tag{5.11}$$

Consequently, according to Grisvard [6, Theorem 7.5], we deduce the following characterization for the related interpolation spaces $E_{\theta,q}$ defined by (3.8):

$$E_{\theta,q} = W_B^{2\theta,p}(\Omega).$$

If $q = 1$ the study of the realization of $\mathcal{A}_0$, with homogeneous Dirichlet boundary conditions, is less simple. Assuming that the coefficients of $\mathcal{A}_0$ satisfy $a_{i,j}^{(0)} \in C^1(\overline{\Omega})$, $i, j = 1, \dots, n$, we have (see [13, Theorem 3.10])

$$D(A_{(1)}) = \{u \in W_0^{1,1}(\Omega) : \mathcal{A}_0 u \in L^1(\Omega)\}, \qquad A_{(1)}u = \mathcal{A}_0 u, \tag{5.12}$$

where $\mathcal{A}_0 u$ is understood in the sense of distributions.

Concerning the related interpolation spaces we have (see [5, Theorem 3.2])

$$E_{\theta,1} = \begin{cases} W^{2\theta,1}(\Omega), & \text{if } 0 < \theta < 1/2, \\ \{u \in B^{1,1}(\Omega) : \int_\Omega |Bu| d^{-1}\, dx < +\infty\}, & \text{if } \theta = 1/2, \\ W^{2\theta,1}(\Omega) \cap W_0^{1,1}(\Omega), & \text{if } 1/2 < \theta < 1, \end{cases} \tag{5.13}$$

where $B^{1,1}(\Omega)$ denotes the Besov space of indices $(1, 1)$.

Finally, Φ stands for *any* linear functional in L^q.

We can now state in the present case our global-in-time existence and uniqueness Theorems 4 and 5. For brevity of exposition we do not treat the case $p = 1$. The corresponding results can be stated explicitly by using Theorems 4 and 5 and the characterizations (5.12) and (5.13).

Theorem 7. *Let $g \in W^{1,p}((0,T))$ and $a \in L^p((-r,0))$ for some $p \in (1,+\infty)$, and assume that, for some $\theta \in (0,1/p)$,*

$$z \in W_B^{2\theta,p}(\Omega), \quad \chi^{-1} := \Phi[z] \neq 0;$$
$$\varphi_1 \in W^{\theta,p}(-r,0;W_B^{2,p}(\Omega)), \quad \varphi_0 \in W_B^{2(\theta+1-1/p),p}(\Omega).$$

Then there exists a unique pair

$$(u,f) \in [W^{1,p}(0,T;L^p(\Omega)) \cap W^{\theta,p}(-r,T;W_B^{2,p}(\Omega))] \times L^p((0,T)),$$

solving problem (5.1)–(5.4), and we have $u \in C(0,T;W_B^{2(\theta+1-1/p),p}(\Omega))$. Moreover, there exists a positive constant $C_4 = C_4(T,\Phi,z,p,a)$, which is bounded for bounded T, satisfying

$$\begin{aligned}&\|u\|_{W^{1,p}(0,T;L^p(\Omega))} + \|u\|_{W^{\theta,p}(0,T;W_B^{2,p}(\Omega))} + \|f\|_{L^p}\\ &\le C_4\,[\,|\varphi_0|_{W_B^{2(\theta+1-1/p),p}(\Omega)} + \|\varphi_1\|_{W^{\theta,p}(-r,0;W_B^{2,p}(\Omega))} + |\chi|\ \|g'\|_{L^p}\,|z|_{W_B^{2\theta,p}(\Omega)}].\end{aligned}$$

Proof. The assertions follow from Theorem 4 and (2.2). □

Theorem 8. *Let $g \in C^1([0,T])$ and $a \in L^p((-r,0))$ for some $p \in (1,+\infty)$, and assume that, for some $\theta \in (1/p,1)$,*

$$z \in W_B^{2\theta,p}(\Omega), \;\; \chi^{-1} := \Phi[z] \neq 0;$$
$$\varphi_1 \in W^{\theta,p}(-r,0;W_B^{2,p}(\Omega)), \;\; \varphi_1(0) = \varphi_0;$$
$$\mathcal{A}_0\varphi_0 + \mathcal{A}_1\varphi_1(-r) + \textstyle\int_{-r}^0 a(s)\mathcal{A}_2\varphi_1(s)\,ds \in W_B^{2(\theta-1/p),p}(\Omega).$$

Then there exists a unique pair

$$(u,f) \in [W^{1,p}(0,T;L^p(\Omega)) \cap W^{\theta,p}(-r,T;W_B^{2,p}(\Omega))] \times L^p((0,T)),$$

solving problem (5.1)–(5.4), and we have $u' \in C(0,T;W_B^{2(\theta-1/p),p}(\Omega))$. Moreover, there exists a positive constant $C_4 = C_4(T,\Phi,z,p,a)$, which is bounded for bounded T, satisfying

$$\begin{aligned}&\|u\|_{W^{1,p}(0,T;L^p(\Omega))} + \|u\|_{W^{\theta,p}(0,T;W_B^{2,p}(\Omega))} + \|f\|_{L^p}\\ &\le C_4\Big[\Big|\mathcal{A}_0\varphi_0 + \mathcal{A}_1\varphi_1(-r) + \int_{-r}^0 a(s)\mathcal{A}_2\varphi_1(s)\,ds\Big|_{W_B^{2(\theta-1/p),p}(\Omega)}\\ &+ \|\varphi_1\|_{W^{\theta,p}(-r,0;W_B^{2,p}(\Omega))} + |\chi|\ \|g'\|_{L^p}\,|z|_{W_B^{2\theta,p}(\Omega)}\Big].\end{aligned}$$

Proof. The assertions follow from Theorem 5 and (2.2). □

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