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ON THE DOMAINS OF ELLIPTIC OPERATORS IN L^1

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Abstract. We prove optimal embedding estimates for the domains of second-order elliptic operators in L^1 spaces. Our procedure relies on general semigroup theory and interpolation arguments, and on estimates for $\nabla T(t)f$ in L^1, in L^∞, and possibly in fractional Sobolev spaces, for $f \in L^1$. It is applied to a number of examples, including some degenerate hypoelliptic operators, and operators with unbounded coefficients.

1. Introduction

It is well known that the domains of the realizations of elliptic operators in L^1 spaces are not as nice as in the case of L^p spaces with $p \in (1, \infty)$. For instance, it is easy to see that the domain of the realization of the Laplacian in $L^1(\mathbb{R}^N)$ is not contained in $W^{2,1}(\mathbb{R}^N)$ if $N > 1$. However, it is continuously embedded in $W^{1+\beta,1}(\mathbb{R}^N)$ for each $\beta \in (0, 1)$. Therefore, if u and Δu are in $L^1(\mathbb{R}^N)$, then any first-order derivative $D_i u$, $i = 1, \dots, N$, belongs to $W^{\beta,1}(\mathbb{R}^N)$ for each $\beta \in (0, 1)$, and by Sobolev embedding it belongs also to $L^p(\mathbb{R}^N)$ for each $p < N/(N-1)$, with

$$\|D_i u\|_{L^p} \le C(p)\big(\|u\|_{L^1} + \|\Delta u\|_{L^1}\big). \tag{1.1}$$

Easy counterexamples show that we cannot take $p = \frac{N}{N-1}$. They show also that, in general, $D_i u$ does not belong to any Lorentz space $L^{N/(N-1),q}_{\text{loc}}(\mathbb{R}^N)$ with $q < \infty$; see Example 6.2. On the other hand, it is possible to prove that each derivative $D_i u$ belongs to the Besov space $B^{1,1}_\infty(\mathbb{R}^N)$, and

$$\|D_i u\|_{B^{1,1}_\infty(\mathbb{R}^N)} \le C\big(\|u\|_{L^1} + \|\Delta u\|_{L^1}\big). \tag{1.2}$$

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See e.g. [5, Corollary 4.3.16]. Since $B^{1,1}_\infty(\mathbb{R}^N)$ is continuously embedded in the Lorentz space $L^{N/(N-1),\infty}(\mathbb{R}^N)$, it follows that each derivative $D_i u$ belongs to $L^{N/(N-1),\infty}(\mathbb{R}^N)$, and

$$\|D_i u\|_{L^{N/(N-1),\infty}(\mathbb{R}^N)} \le C\big(\|u\|_{L^1} + \|\Delta u\|_{L^1}\big). \tag{1.3}$$

For definitions and properties of Lorentz and Besov spaces see Section 2. In this paper we show that similar embeddings hold for a large class of second-order elliptic operators,

$$A = \sum_{i,j=1}^N a_{ij} D_{ij} + \sum_{i=1}^N b_i D_i + c. \tag{1.4}$$

We assume that the realization of A, endowed with a suitable domain $D(A)$, is the infinitesimal generator of a semigroup $(T(t))_{t\ge 0}$ in $L^1(\Omega)$, Ω being an open set in $\mathbb{R}^N$ with empty or smooth boundary. Our approach is based on interpolation arguments, on estimates for the semigroup $(T(t))_{t\ge 0}$, and on the well-known representation formula for the resolvent

$$R(\lambda, A) = \int_0^\infty e^{-\lambda t} T(t)\,dt,$$

for λ large enough. The needed estimates are

$$\begin{aligned} &(i) \quad \|\nabla T(t) f\|_{L^1} \le C e^{\omega t} t^{-\gamma_1} \|f\|_{L^1},\\ &(ii) \quad \|\nabla T(t) f\|_{L^\infty} \le C e^{\omega t} t^{-\gamma_2} \|f\|_{L^1}, \quad t>0,\ f \in L^1(\Omega), \end{aligned} \tag{1.5}$$

for Lorentz regularity, with $0 \le \gamma_1 < 1 < \gamma_2$ and $\omega \in \mathbb{R}$, and

$$\|T(t) f\|_{W^{\beta,1}} \le C e^{\omega t} t^{-\gamma} \|f\|_{L^1}, \quad t>0, \tag{1.6}$$

for Besov regularity, with $\beta > 0$, $\gamma > 1$, and $\omega \in \mathbb{R}$. Then we show that the map

$$\nabla : D(A) \to L^{\frac{\gamma_2-\gamma_1}{\gamma_2-1},\infty}(\Omega, \mathbb{R}^N)$$

is bounded (respectively, $D(A)$ is continuously embeded in $B^{\beta/\gamma,1}_\infty(\Omega,\mathbb{R}^N)$), by a procedure which was used up to now to find optimal Schauder estimates. See e.g. [17, Theorem 2.1] and [20]. In the case of the Laplace operator in $\mathbb{R}^N$, the heat semigroup $(T(t))_{t\ge0}$ is easily seen to satisfy both (1.5) with $\gamma_1 = 1/2$ and $\gamma_2 = (N+1)/2$, and (1.6) with $\beta = 3$ and $\gamma = 3/2$, for each $f \in L^1(\mathbb{R}^N)$. Hence we get (1.2) and (1.3).

Embedding results in weaker norms are much easier. Estimates (1.5), assuming for simplicity that $\omega = 0$, imply by interpolation

$$\|\nabla T(t) f\|_{L^p} \le C(N) t^{-\frac{N(1-1/p)+1}{2}} \|f\|_{L^1},$$

for every $p \in (1, \infty)$. Let $u \in D(A)$ and, for $\lambda > 0$, set $f = \lambda u - Au$. Then $u = \int_0^\infty e^{-\lambda t} T(t) f \, dt$, and taking $p < N/(N-1)$ we get

$$\|\nabla u\|_{L^p} \le C(N,p)\lambda^{\gamma-1}\|f\|_{L^1} \le C(N,p)(\lambda^\gamma \|u\|_{L^1} + \lambda^{\gamma-1}\|Au\|_{L^1}),$$

where $\gamma = \frac{N(1-1/p)+1}{2} < 1$. Taking the minimum for $\lambda > 0$ one obtains the inequality

$$\|\nabla u\|_{L^p} \le C(N,p)\|u\|_1^{1-\gamma}\|Au\|_1^\gamma,$$

which is well known for elliptic operators with bounded, smooth coefficients, see e.g. [27, Theorem 5.8], [4, Theorem 8], and [2, Proposition 9.2]. Taking $p = N/(N-1)$ we get $\gamma = 1$ and this direct approach fails. To treat the limiting case we need the more refined argument used in the proofs of Theorems 3.1 and 4.1.

Estimates of the type (1.5) or (1.6) are satisfied by a number of elliptic operators, including elliptic operators in bounded domains with Dirichlet or Neumann boundary conditions, a class of elliptic operators with unbounded Lipschitz-continuous coefficients in $\mathbb{R}^N$, and degenerate elliptic operators such as Kolmogorov-type operators, the Heisenberg Laplacian in $\mathbb{R}^3$, and more generally sub-Laplacians in nilpotent stratified Lie groups. For all these operators the application of Theorems 3.1 and 4.1 gives optimal embedding results. See Sections 5 and 6. Some of these embeddings were already known, and ours is just an alternative approach. This is the case of elliptic operators with bounded and Hölder-continuous coefficients in the whole of $\mathbb{R}^N$, and in regular, bounded, open sets in $\mathbb{R}^N$ with Dirichlet boundary condition. See [14, Theorem 1.7, Theorem 3.3], where a class of general $2m$-order elliptic systems is considered. On the other hand, the study of elliptic operators with unbounded coefficients is much more recent and the general theory is still under development. Here we consider operators of the type

$$A = \Delta + \sum_{i=1}^N F_i D_i,$$

with globally Lipschitz-continuous coefficients F_i. It has been recently proved that the domain of the realization of A in $L^p(\mathbb{R}^N)$ is contained in $W^{2,p}(\mathbb{R}^N)$ for $1 < p < \infty$ ([24]), and that the domain of the realization of A in $C^\alpha(\mathbb{R}^N)$ is contained in $C^{2+\alpha}(\mathbb{R}^N)$ for $0 < \alpha < 1$ ([21]). We prove here that the semigroup $T(t)$ generated by the realization of A in $L^1(\mathbb{R}^N)$ satisfies (1.5) with $\gamma_1 = 1/2$ and $\gamma_2 = (N+1)/2$, as in the case of bounded regular coefficients. As a first step, ultracontractivity of $T(t)$ is proved adapting to our situation a classical argument based on Nash's inequality. This gives $\|T(t)\|_{\mathcal{L}(L^1,L^\infty)} \le Ce^{\omega t}t^{-(N+1)/2}$. Then the estimate $\|\nabla T(t)\|_{\mathcal{L}(L^\infty)} \le Ce^{\omega t}t^{-1/2}$, which is known for smooth F, is extended to our situation by a perturbation argument. Using the semigroup law, we arrive at (1.5)(ii). For (1.5)(i), we use the fact that the commutator $T(t)\nabla - \nabla T(t)$ may be extended to a bounded operator from $L^\infty(\mathbb{R}^N, \mathbb{R}^N)$ to itself, and then we use a duality argument and again the estimate $\|\nabla T(t)\|_{\mathcal{L}(L^\infty)} \le Ce^{\omega t}t^{-1/2}$ to conclude. Theorem 4.1 about Lorentz regularity implies then that for each $u \in D(A)$, $D_i u$ belongs to $L^{N/(N-1),\infty}(\mathbb{R}^N)$ for $i = 1, \dots, N$, and

$$\|D_i u\|_{L^{N/(N-1),\infty}(\mathbb{R}^N)} \le C\big(\|u\|_{L^1} + \|Au\|_{L^1}\big).$$

2. Preliminaries, Besov and Lorentz spaces

In the next sections we shall consider the usual real interpolation spaces $(X, D)_{\theta,q}$ for couples of Banach spaces (X, D) with $D \subset X$. We refer to the book [28] for an extensive treatment of these spaces. We recall that if X, Y, and D are Banach spaces such that $D \subset Y \subset X$ with continuous embeddings, and $0 \le \theta \le 1$, Y is said to belong to the class $J_\theta(X, D)$ if there is $C > 0$ such that

$$\|f\|_Y \le C\|f\|_X^{1-\theta}\|f\|_D^\theta, \ \ \forall f \in D.$$

The reiteration theorem (see e.g. [28, Section 1.10.2]) implies that if Y_1 belongs to $J_{\theta_1}(X, D)$ and Y_2 belongs to $J_{\theta_2}(X, D)$, then

$$(Y_1, Y_2)_{\theta,q} \supset (X, D)_{(1-\theta)\theta_1+\theta\theta_2,q}$$

for all $\theta \in (0, 1)$ and $q \in [1, +\infty]$, and the embedding is continuous.

Let Ω be an open set in $\mathbb{R}^N$. By $L^p(\Omega)$, $1 \le p \le \infty$, we mean $L^p(\Omega, dx)$ where dx is the usual Lebesgue measure. The norm of a function f in $L^p(\Omega)$ is denoted by $\|f\|_p$.

Now we recall briefly definitions and main properties of the Besov and Lorentz spaces $B^{s,1}_\infty(\Omega)$ and $L^{p,q}(\Omega)$. We refer the reader to [29, 28, 13] for a detailed treatment of these spaces. Let $s = r + \alpha$ with $r \in \mathbb{N}$ and $0 < \alpha \le 1$. A function $f \in W^{r,1}(\mathbb{R}^N)$ belongs to $B^{s,1}_\infty(\mathbb{R}^N)$ if

$$\|f\|_{B^{s,1}_\infty} = \|f\|_{W^{r,1}(\mathbb{R}^N)} + \sum_{|\beta|=r} \sup_{h\in\mathbb{R}^N} |h|^{-\alpha}\|D^\beta u(\cdot+h) - 2D^\beta u + D^\beta u(\cdot-h)\|_1 < \infty.$$

$B^{s,1}_\infty(\mathbb{R}^N)$ is a Banach space with the above norm. If $\Omega \ne \mathbb{R}^N$, the space $B^{s,1}_\infty(\Omega)$ consists of the restrictions to Ω of the functions in $B^{s,1}_\infty(\mathbb{R}^N)$. The norm of f is the infimum of $\|\widetilde{f}\|_{B^{s,1}_\infty(\mathbb{R}^N)}$ over all the extensions $\widetilde{f}$ of f to the whole of $\mathbb{R}^N$.

Besov spaces arise naturally as real interpolation spaces between Sobolev spaces. We recall the definition of Sobolev spaces of noninteger order. Let

$s = r + \alpha$ with $r \in \mathbb{N}$ and $0 < \alpha < 1$. A function $f \in W^{r,1}(\mathbb{R}^N)$ belongs to $W^{s,1}(\mathbb{R}^N)$ if

$$\|f\|_{W^{s,1}(\mathbb{R}^N)} = \|f\|_{W^{r,1}(\mathbb{R}^N)} + \sum_{|\beta|=r} \int_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|D^\beta f(x+h) - D^\beta f(x)|}{|h|^{N+\alpha}}\, dx\, dh < \infty.$$

If $\Omega \neq \mathbb{R}^N$, the space $W^{s,1}(\Omega)$ consists of the restrictions to Ω of the functions in $W^{s,1}(\mathbb{R}^N)$. The norm of f is the infimum of $\|\widetilde{f}\|_{W^{s,1}(\mathbb{R}^N)}$ over all the extensions $\widetilde{f}$ of f to the whole of $\mathbb{R}^N$. If Ω is the whole of $\mathbb{R}^N$, or a half space, or a bounded, open set with C^ρ boundary, then for $0 < \theta < 1$, $1 \le q \le \infty$, we have $(L^1(\Omega), W^{s,1}(\Omega))_{\theta,\infty} = B^{\theta s,1}_{\infty}(\Omega)$, for $0 < s < \rho$, with equivalence of the respective norms; see e.g. [29, 13]. This is in fact the only property of Besov spaces that we will need in the sequel.

Let now Ω be any measurable subset of $\mathbb{R}^N$. Let $f : \Omega \mapsto \mathbb{C}$ be a measurable function and define for $\sigma \ge 0$

$$m(\sigma, f) = \big|\{x \in \Omega : |f(x)| > \sigma\}\big|,$$

where $|E|$ denotes the Lebesgue measure of E. Next, we introduce the decreasing rearrangement of f, that is, $f^*(t) = \inf\{\sigma : m(\sigma, f) \le t\}$. The function f^* is decreasing on $(0,\infty)$, it satisfies $f^*(0) = \sup \text{ess}|f|$, and moreover

$$\big|\{t > 0 : f^*(t) > \sigma\}\big| = \big|\{x \in \Omega : |f(x)| > \sigma\}\big|.$$

The Lorentz spaces $L^{p,q}(\Omega)$ ($1 \le p \le \infty$, $1 \le q \le \infty$) are defined by

$$L^{p,q}(\Omega) = \Big\{ f \in L^1(\Omega) + L^\infty(\Omega) : \|f\|_{p,q} := \Big(\int_0^\infty (t^{1/p} f^*(t))^q \frac{dt}{t} \Big)^{1/q} < \infty \Big\}$$

for $q < \infty$, and

$$L^{p,\infty}(\Omega) = \Big\{ f \in L^1(\Omega) + L^\infty(\Omega) : \|f\|_{p,\infty} := \sup_{t>0} t^{1/p} f^*(t) < \infty \Big\}.$$

We remark that $L^{p,q_1}(\Omega) \hookrightarrow L^{p,q_2}(\Omega)$ if $q_1 < q_2$ and that $L^{p,p}(\Omega) = L^p(\Omega)$. We shall consider the Lorentz spaces mainly with $q = \infty$, using the following characterization in terms of the distribution function $m(f, \sigma)$; see [28, Lemma 1.18.6].

Proposition 2.1. *The following equality holds:*

$$L^{p,\infty}(\Omega) = \Big\{ f : \Omega \to \mathbb{C} : \sup_{\sigma > 0} \sigma (m(f,\sigma))^{1/p} < \infty \Big\}.$$

Moreover, $\|f\|_{p,\infty} := \sup_{\sigma>0} \sigma (m(f,\sigma))^{1/p}$, *for every* $f \in L^{p,\infty}(\Omega)$.

Also the Lorentz spaces arise naturally as real interpolation spaces between Lebesgue spaces. Indeed, for $0 < \theta < 1$ and $1 \le q \le \infty$ we have

$$(L^1(\Omega), L^\infty(\Omega))_{\theta,q} = L^{1/(1-\theta),q}(\Omega),$$

with equivalence of the respective norms. See [28, Section 1.18.6].

The following lemma allows us to treat the vector-valued integrals appearing in the proofs of Theorems 3.1 and 4.1. The space $V^{1,\infty}(\Omega)$ is defined by

$$V^{1,\infty}(\Omega) = \Big\{ f \in L^1(\Omega) : \ \nabla f \in L^\infty(\Omega, \mathbb{R}^N) \Big\}, \tag{2.1}$$

and it is a Banach space when endowed with the norm $\|f\|_{V^{1,\infty}(\Omega)} = \|f\|_1 + \| \, |\nabla f| \, \|_\infty$.

Lemma 2.2. *Let Ω be an open set in $\mathbb{R}^N$. Then the following properties hold.*

(i) *Closed balls in $W^{1,1}(\Omega) \cap V^{1,\infty}(\Omega)$ are closed in $L^1(\Omega)$.*

(ii) *If s is not an integer and $\Omega = \mathbb{R}^N$, or Ω is a half space, or Ω is bounded with C^ρ boundary, $\rho > s$, then closed balls in $W^{s,1}(\Omega)$ are closed in $L^1(\Omega)$.*

Proof. (i) Let $(f_n) \subset W^{1,1}(\Omega) \cap V^{1,\infty}(\Omega)$ converge to f in $L^1(\Omega)$, and be such that $\|D_i f_n\|_1 \le C_1$, $\|D_i f_n\|_\infty \le C_2$ for $i = 1, \dots, N$, with C_1 and C_2 independent of n. Then $\|D_i f_n\|_2 \le C = \max\{C_1, C_2\}$ for each i, and hence (∇f_n) converges to ∇f weakly in $L^2(\Omega, \mathbb{R}^N)$. It follows that for every $\phi \in C_0^\infty(\Omega)$ and for each $i = 1, \dots, N$

$$\Big| \int_\Omega D_i f \phi \Big| = \lim_{n \to \infty} \Big| \int_\Omega D_i f_n \phi \Big| \le \begin{cases} C_1 \|\phi\|_\infty \\ C_2 \|\phi\|_1. \end{cases}$$

Hence, $\|D_i f\|_1 \le C_1$ and $\|D_i f\|_\infty \le C_2$.

(ii) Let $(f_n) \subset W^{s,1}(\Omega)$ and $\|f_n\|_{W^{s,1}(\Omega)} \le C$ be convergent to $f \in L^1(\Omega)$. Write $s = r + \alpha$ with $r \in \mathbb{N}$ and $0 < \alpha < 1$. By interpolation, (f_n) is a Cauchy sequence in $W^{r,1}(\Omega)$, and hence $(f_n) \to f$ in $W^{r,1}(\Omega)$. Fatou's lemma yields easily $\|f\|_{W^{s,1}(\Omega)} \le C$. □

Observe that (ii) of Lemma 2.2 fails if s is an integer. For example, if $s = 1$, the function f in the proof above is just of bounded variation and it does not belong necessarily to $W^{1,1}(\Omega)$.

3. BESOV REGULARITY

Throughout the section we shall consider the following assumptions ($C > 0$, $\omega \in \mathbb{R}$).

(H1) $(T(t))_{t\geq 0}$ is a strongly continuous semigroup in $L^1(\Omega)$ with generator $A : D(A) \mapsto L^1(\Omega)$, and

$$\|T(t)f\|_1 \leq Ce^{\omega t}\|f\|_1, \quad f \in L^1(\Omega).$$

(H2) There are $\gamma > 1$ and $\beta > 0$ such that for $t > 0$ and $f \in L^1(\Omega)$, $T(t)f$ belongs to $W^{\beta,1}(\Omega)$ and satisfies the estimate

$$\|T(t)f\|_{W^{\beta,1}} \leq Ce^{\omega t}t^{-\gamma}\|f\|_1.$$

Moreover, either $\Omega = \mathbb{R}^N$, or Ω is a half space, or Ω is bounded with C^ρ boundary, $\rho > \beta$.

We shall adapt to the present situation the procedure of [20, Theorem 2.5], to get an optimal embedding result for $D(A)$. To be more precise, we get an optimal embedding result for $(L^1(\Omega), D(A^2))_{1/2,\infty}$, in which $D(A)$ is continuously embedded (see [28, Theorem 1.13.2(b)]. Since we work in specific Sobolev spaces, we do not need the measurability assumption made in [20].

Theorem 3.1. *Let (H1) and (H2) hold. Then* $(L^1(\Omega), D(A^2))_{1/2,\infty} \subset B^{\beta/\gamma,1}_\infty(\Omega)$ *with continuous embedding, and therefore* $D(A) \subset B^{\beta/\gamma,1}_\infty(\Omega)$ *with continuous embedding.*

Proof. As a first step, we prove that the statement holds for $\beta \notin \mathbb{N}$. Since $D(A^2) = D((A - \omega I)^2)$ with equivalence of the norms, possibly replacing A by $A - \omega I$ we may assume also that $\omega = 0$. We shall show that

$$(L^1(\Omega), D(A^m))_{1/m,\infty} \subset (L^1(\Omega), W^{\beta,1}(\Omega))_{1/\gamma,\infty},$$

with continuous embedding, for m large enough. Indeed, it is well known (see e.g. [28, Theorem 1.13.5]) that $(L^1(\Omega), D(A^2))_{1/2,\infty} = (L^1(\Omega), D(A^m))_{1/m,\infty}$ for every integer $m \geq 2$. Fix any integer $m > \gamma$. Let $u \in D(A^m)$, $\lambda > 0$ and set $(\lambda I - A)^m u = f$. Then $u = (R(\lambda, A))^m f$, so that

$$u = \frac{(-1)^{m-1}}{(m-1)!}\,\frac{d^{m-1}}{d\lambda^{m-1}}R(\lambda, A)f = \frac{1}{(m-1)!}\int_0^\infty t^{m-1}e^{-\lambda t}T(t)f\,dt, \tag{3.1}$$

the integral being understood in $L^1(\Omega)$. However, u belongs to $W^{\beta,1}(\Omega)$, with

$$\|u\|_{W^{\beta,1}(\Omega)} \leq \frac{C}{(m-1)!}\int_0^\infty t^{m-1-\gamma}e^{-\lambda t}dt\,\|f\|_1. \tag{3.2}$$

Indeed,

$$u = L^1 - \lim_{a\to 0,\, b\to\infty} \frac{1}{(m-1)!}\int_a^b t^{m-1}e^{-\lambda t}T(t)f\,dt,$$

and for fixed $0 < a < b < \infty$,

$$\int_a^b t^{m-1-\gamma} e^{-\lambda t} T(t) f\, dt = L^1 - \lim_{n\to\infty} I_n,$$

where

$$I_n = \frac{b-a}{n} \sum_{k=1}^{n} (a + k(b-a)/n)^{m-1} e^{-\lambda(a+k(b-a)/n)} T(a + k(b-a)/n) f. \quad (3.3)$$

Due to assumption $(H2)$, $I_n \in W^{\beta,1}(\Omega)$ and

$$\|I_n\|_{W^{\beta,1}(\Omega)} \le \frac{C(b-a)}{n} \sum_{k=1}^{n} (a + k(b-a)/n)^{m-1-\gamma} e^{-\lambda(a+k(b-a)/n)} \|f\|_1.$$

The right-hand side goes to $C\int_a^b t^{m-1-\gamma} e^{-\lambda t} dt \|f\|_{L^1}$ as n goes to $+\infty$. By Lemma 2.2(ii) closed balls in $W^{\beta,1}(\Omega)$ are closed in $L^1(\Omega)$, and then $\int_a^b t^{m-1} e^{-\lambda t} T(t) f\, dt \in W^{\beta,1}(\Omega)$ and

$$\Big\| \int_a^b t^{m-1} e^{-\lambda t} T(t) f\, dt \Big\|_{W^{\beta,1}(\Omega)} \le C \int_a^b t^{m-1-\gamma} e^{-\lambda t} dt \|f\|_1.$$

Letting $a \to 0$ and $b \to \infty$ and using again Lemma 2.2(ii) it follows that u belongs to $W^{\beta,1}(\Omega)$ and that it satisfies (3.2). The proof now goes on as in [20]; we write it down for the reader's convenience. From (3.2) we get

$$\begin{aligned}\|u\|_{W^{\beta,1}(\Omega)} &\le \frac{C}{(m-1)!} \int_0^\infty e^{-\lambda t} t^{m-\gamma-1} dt \|f\|_1 = \frac{C\Gamma(m-\gamma)}{(m-1)!} \lambda^{\gamma-m} \|f\|_1 \\ &= \frac{C\Gamma(m-\gamma)}{(m-1)!} \lambda^{\gamma-m} \Big\| \sum_{r=0}^{m} \binom{m}{r} \lambda^{m-r} (-1)^r A^r u \Big\|_1 \le C_1 \sum_{r=0}^{m} \lambda^{\gamma-r} \|A^r u\|_1,\end{aligned}$$

$\lambda > 0$. We recall now the interpolation inequalities

$$\|A^r u\|_1 \le C_2 \|u\|_{D(A^m)}^{r/m} \|u\|_1^{1-r/m}, \;\; r = 1, \dots, m, \;\; u \in D(A^m)$$

(see e.g. [28, Section 1.14.3]). Using such inequalities and then $ab \le (a^2 + b^2)/2$ we get

$$\|u\|_{W^{\beta,1}(\Omega)} \le C_3 \lambda^\gamma (\lambda^{-m} \|u\|_{D(A^m)} + \|u\|_{L^1}), \;\; \lambda > 0,$$

so that, taking the minimum for $\lambda > 0$,

$$\|u\|_{W^{\beta,1}(\Omega)} \le C_4 \|u\|_1^{1-\gamma/m} \|u\|_{D(A^m)}^{\gamma/m}.$$

This means that $W^{\beta,1}(\Omega) \in J_{\gamma/m}(L^1, D(A^m))$, and therefore, by the reiteration theorem, $(L^1(\Omega), D(A^m))_{1/m,\infty} \subset (L^1(\Omega), W^{\beta,1}(\Omega))_{1/\gamma,\infty} = B_\infty^{\beta/\gamma,1}(\Omega)$, with continuous embedding. So, the statement is proved if $\beta \notin \mathbb{N}$. If $\beta \in \mathbb{N}$

fix $\beta' \in (\beta - 1, \beta)$ such that $\gamma' = \gamma\beta'/\beta > 1$. Using assumptions $(H1)$ and $(H2)$ we get by interpolation

$$\|T(t)f\|_{W^{\beta',1}} \le Ce^{\omega t}t^{-\gamma'}\|f\|_1, \;\; t > 0, \; f \in L^1.$$

Since β' is not integer, the first part of the proof yields

$$(L^1(\Omega), D(A^2))_{1/2,\infty} \subset B_\infty^{\beta'/\gamma',1}(\Omega) = B_\infty^{\beta/\gamma,1}(\Omega). \qquad \square$$

In the following corollary we consider an important special case of Theorem 3.1.

Corollary 3.2. *Let either $\Omega = \mathbb{R}^N$, or Ω be a half space, or Ω be a bounded, open set with C^ρ boundary, $\rho > 2$. Let $(H1)$ and $(H2)$ hold, with $2 < \beta < \rho$ and $\gamma = \beta/2$. Then $D(A) \subset B_\infty^{2,1}(\Omega)$, with continuous embedding. Therefore, the gradient of each function $u \in D(A)$ belongs to $B_\infty^{1,1}(\Omega, \mathbb{R}^N)$, which is continuously embedded in the space $L^{N/(N-1),\infty}(\Omega, \mathbb{R}^N)$.*

In the case of the Laplacian and of other elliptic operators with smooth coefficients in $\mathbb{R}^N$, $(H2)$ is satisfied with $\beta = 3$ and $\gamma = 3/2$, and we may apply Corollary 3.2. Therefore, we get

$$D(A) \subset B_\infty^{2,1}(\mathbb{R}^N), \tag{3.4}$$

with continuous embedding. So, the gradient of each function $u \in D(A)$ belongs to $B_\infty^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$, which is continuously embedded in the space

$$L^{N/(N-1),\infty}(\mathbb{R}^N, \mathbb{R}^N).$$

In other situations, the semigroup $T(t)$ does not map L^1 into $W^{\beta,1}$ for $\beta > 2$, and the embedding (3.4) is not known. However, we are still able to get optimal Lorentz regularity provided a somewhat weaker condition than $(H2)$ holds. See the next section.

4. Lorentz regularity

Throughout the section we shall consider assumption $(H1)$ together with the following one ($C > 0$, $\omega \in \mathbb{R}$).

(H3) There are constants $\gamma_1 \in (0, 1)$ and $\gamma_2 > 1$ such that for $t > 0$ and $f \in L^1(\Omega)$ the function $T(t)f$ belongs to $W^{1,1}(\Omega) \cap V^{1,\infty}(\Omega)$ and satisfies

$$\|\nabla T(t)f\|_1 \le Ce^{\omega t}t^{-\gamma_1}\|f\|_1, \qquad \|\nabla T(t)f\|_\infty \le Ce^{\omega t}t^{-\gamma_2}\|f\|_1.$$

By a method similar to the one of Theorem 3.1 we get the following result.

Theorem 4.1. *Assume that $(T(t))_{t\geq 0}$ satisfies hypotheses (H1) and (H3). Then the map $\nabla : (L^1, D(A^2))_{1/2,\infty} \to L^{\frac{\gamma_2-\gamma_1}{\gamma_2-1},\infty}(\Omega, \mathbb{R}^N)$ is bounded. In particular, the map $\nabla : D(A) \to L^{\frac{\gamma_2-\gamma_1}{\gamma_2-1},\infty}(\Omega, \mathbb{R}^N)$ is bounded.*

Proof. We shall show that $W^{1,1}(\Omega)$ and $V^{1,\infty}(\Omega)$ belong to $J_{\gamma_1/m}(L^1(\Omega), D(A^m))$ and to $J_{\gamma_2/m}(L^1(\Omega), D(A^m))$, respectively, for any integer $m > \gamma_2$. By the reiteration theorem it will follow that $(L^1(\Omega), D(A^m))_{1/m,\infty} \subset (W^{1,1}(\Omega), V^{1,\infty}(\Omega))_{\frac{1-\gamma_1}{\gamma_2-\gamma_1},\infty}$, with continuous embedding. The latter space consists of functions in $W^{1,1}(\Omega) + V^{1,\infty}(\Omega)$ whose first-order derivatives belong to $(L^1(\Omega), L^\infty(\Omega))_{\frac{1-\gamma_1}{\gamma_2-\gamma_1},\infty} = L^{\frac{\gamma_2-\gamma_1}{\gamma_2-1},\infty}(\Omega)$. See [28, Section 1.18.6]. This yields the statement because of the fact that $(L^1(\Omega), D(A^m))_{1/m,\infty} = (L^1(\Omega), D(A^2))_{1/2,\infty}$, with equivalence of the respective norms ([28, Theorem 1.12.5]). We may assume that $\omega = 0$ without loss of generality. Let $u \in D(A^m)$ with $m > \gamma_2$, let $\lambda > 0$, and set $(\lambda I - A)^m u = f$. Then (3.1) holds, the integral being well defined in $L^1(\Omega)$. We show now that $u \in W^{1,1}(\Omega) \cap V^{1,\infty}(\Omega)$, and

$$\begin{aligned} \|D_i u\|_1 &\leq \frac{C}{(m-1)!}\int_0^\infty t^{m-1-\gamma_1} e^{-\lambda t}\, dt \|f\|_1, \\ \|D_i u\|_\infty &\leq \frac{C}{(m-1)!}\int_0^\infty t^{m-1-\gamma_2} e^{-\lambda t}\, dt \|f\|_1. \end{aligned} \tag{4.1}$$

As in the proof of Theorem 3.1, we remark that for $0 < a < b < \infty$, the sums I_n defined in (3.3) belong to $W^{1,1}(\Omega) \cap V^{1,\infty}(\Omega)$, and for each $i = 1, \ldots, N$ we have

$$\|D_i I_n\|_1 \leq \frac{C(b-a)}{n}\sum_{k=1}^n (a + k(b-a)/n)^{m-1-\gamma_1} e^{-\lambda(a+k(b-a)/n)}\|f\|_1,$$

$$\|D_i I_n\|_\infty \leq \frac{C(b-a)}{n}\sum_{k=1}^n (a + k(b-a)/n)^{m-1-\gamma_2} e^{-\lambda(a+k(b-a)/n)}\|f\|_1.$$

The right-hand sides go to $C\int_a^b t^{m-1-\gamma_i} e^{-\lambda t} dt \|f\|_{L^1}$, $i = 1, 2$, as n goes to $+\infty$. It follows that the function $\int_a^b t^{m-1} e^{-\lambda t} T(t) f\, dt$ belongs to $W^{1,1}(\Omega) \cap V^{1,\infty}(\Omega)$, and the proof of Lemma 2.2(i) gives

$$\|D_i \int_a^b t^{m-1} e^{-\lambda t} T(t) f\, dt\|_1 \leq C\int_a^b t^{m-1-\gamma_1} e^{-\lambda t} dt \|f\|_1,$$

$$\|D_i \int_a^b t^{m-1} e^{-\lambda t} T(t) f\, dt\|_\infty \leq C\int_a^b t^{m-1-\gamma_2} e^{-\lambda t} dt \|f\|_1,$$

for $i = 1, \dots, N$. Letting $a \to 0$ and $b \to \infty$ we find that u belongs to $W^{1,1}(\Omega) \cap V^{1,\infty}(\Omega)$ and that it satisfies (4.1). Once (4.1) is established, arguing as in the proof of Theorem 3.1 we obtain that $W^{1,1}(\Omega)$ belongs to $J_{\gamma_1/m}(L^1, D(A^m))$, and $V^{1,\infty}(\Omega)$ belongs to $J_{\gamma_2/m}(L^1, D(A^m))$, just what we needed. □

Let us write down explicitly a particular but important case of Theorem 4.1.

Corollary 4.2. *Assume that the hypotheses in Theorem* 4.1 *are fulfilled with* $\gamma_1 = \frac{1}{2}$ *and* $\gamma_2 = \frac{N+1}{2}$. *Then the map* $\nabla : (L^1(\Omega), D(A^2))_{\frac{1}{2},\infty} \to L^{\frac{N}{N-1},\infty}(\Omega, \mathbb{R}^N)$ *is bounded. In particular, the map* $\nabla : D(A) \to L^{\frac{N}{N-1},\infty}(\Omega, \mathbb{R}^N)$ *is bounded.*

A less sharp version of Theorem 4.1 has a simpler proof, which we write below.

Theorem 4.3. *Assume that* $(T(t))_{t\geq 0}$ *satisfies hypotheses* $(H1)$ *and* $(H3)$. *Then the map* $\nabla : D(A) \to L^{\frac{\gamma_2-\gamma_1}{\gamma_2-1},\infty}(\Omega, \mathbb{R}^N)$ *is bounded.*

Proof. Let $\lambda > \omega$ and let $u \in D(A)$. Setting $\lambda u - Au = f$, u is the Laplace transform of $T(t)f$; that is,

$$u = \int_0^\infty e^{-\lambda t} T(t) f \, dt.$$

Let us fix $\xi > 0$ and split $u = a(\xi) + b(\xi)$, where

$$a(\xi) = \int_0^\xi e^{-\lambda t} T(t) f \, dt, \qquad b(\xi) = \int_\xi^\infty e^{-\lambda t} T(t) f \, dt.$$

The procedure of Theorem 4.1 gives $a(\xi) \in W^{1,1}(\Omega)$, $a(\xi) \in V^{1,\infty}(\Omega)$, and

$$\|\nabla a(\xi)\|_1 \leq C_1 \xi^{1-\gamma_1} \|f\|_1, \qquad \|\nabla b(\xi)\|_\infty \leq C_2 \xi^{1-\gamma_2} \|f\|_1.$$

Since $\{|\nabla u| > \sigma\} \subset \{|\nabla a(\xi)| > \sigma/2\} \cup \{|\nabla b(\xi)| > \sigma/2\}$ for every $\sigma > 0$, choosing ξ such that $\xi^{1-\gamma_2} = \sigma/2C_2$ we get $\{|\nabla b(\xi)| > \sigma/2\} = \emptyset$, and therefore

$$m(|\nabla u|, \sigma) \leq |\{|\nabla a(\xi)| > \sigma/2\}| \leq C_3 \sigma^{-\frac{\gamma_2-\gamma_1}{\gamma_2-1}} \|f\|_1,$$

with C_3 independent of f. Applying Proposition 2.1 concludes the proof. □

Remark 4.4. Assume that the hypotheses in Theorem 4.1 are fulfilled with $\gamma_1 = 1/2$ and $\omega = 0$. The first inequality in (4.1) with $m = 1$ yields

$$\|\nabla u\|_1 \leq K \lambda^{-1/2} \|f\|_1 \leq K\big(\lambda^{1/2} \|u\|_1 + \lambda^{-1/2} \|Au\|_1\big)$$

for each $u \in D(A)$. Minimizing over $\lambda > 0$, we get

$$\|\nabla u\|_1 \le C\|u\|_1^{1/2}\|Au\|_1^{1/2};$$

that is, $W^{1,1}(\Omega)$ belongs to the class $J_{1/2}(L^1(\Omega), D(A))$.

5. Operators with Lipschitz-continuous coefficients

In this section we apply the results of Section 4 to operators in $\mathbb{R}^N$ of the form $A = \Delta + F \cdot \nabla$, assuming that the drift vector field F is globally Lipschitz continuous. An important example is the Ornstein-Uhlenbeck operator, which may be reduced to this one by an obvious change of coordinates:

$$A = \sum_{i,j=1}^{n} q_{ij} D_{ij} + \sum_{i,j=1}^{n} b_{ij} x_j D_i,$$

where $Q = (q_{ij})$ is a real, symmetric, positive-definite matrix, and $B = (b_{ij})$ is a nonzero, real matrix. The Ornstein-Uhlenbeck semigroup in $L^1(\mathbb{R}^N)$ has the explicit representation

$$(T(t)f)(x) = \frac{1}{(4\pi)^{n/2}(\det Q_t)^{1/2}} \int_{\mathbb{R}^N} e^{-\langle Q_t^{-1}y,y\rangle/4} f(e^{tB}x - y)\, dy,$$

where

$$Q_t = \int_0^t e^{sB} Q e^{sB^*}\, ds. \tag{5.1}$$

Having an explicit representation formula, the hypotheses of Corollary 3.2 can be checked immediately. Therefore, $u \in B^{2,1}_\infty(\mathbb{R}^N)$, and hence $\nabla u \in L^{N/(N-1),\infty}(\mathbb{R}^N, \mathbb{R}^N)$, for every $u \in D(A)$. We shall prove that the last result still holds for general F. This will be done in some steps; the first one consists in showing that a suitable realization of A in $L^1(\mathbb{R}^N)$ generates a strongly continuous semigroup. For technical reasons we need to consider the realizations of the operator A in every $L^p(\mathbb{R}^N)$; hence, we set for $1 < p < \infty$

$$D_p = \Big\{ u \in W^{2,p}(\mathbb{R}^N) : F \cdot \nabla u \in L^p(\mathbb{R}^N) \Big\}, \tag{5.2}$$

and

$$D_\infty = \Big\{ u \in C_0(\mathbb{R}^N) \cap W^{2,p}_{\rm loc}(\mathbb{R}^N) \text{ for every } p < \infty : Au \in C_0(\mathbb{R}^N) \Big\}. \tag{5.3}$$

Here $C_0(\mathbb{R}^N)$ is the space of the continuous functions on $\mathbb{R}^N$ vanishing as $|x| \to +\infty$, endowed with the sup norm. Observe that $C_0^\infty(\mathbb{R}^N)$ is dense in D_p for $1 < p < \infty$. We set

$$k = \sup_{x \in \mathbb{R}^N} \big(-\operatorname{div} F(x) \big). \tag{5.4}$$

Proposition 5.1. *For $1 < p < \infty$ the operator (A, D_p) generates a strongly continuous positive semigroup $(T_p(t))_{t\geq 0}$ in $L^p(\mathbb{R}^N)$ satisfying*

$$\|T_p(t)f\|_p \leq e^{kt/p}\|f\|_p, \quad f \in L^p(\mathbb{R}^N).$$

The operator (A, D_∞) generates a strongly continuous semigroup of positive contractions in $C_0(\mathbb{R}^N)$.

Proof. The proof is in [24] for $1 < p < \infty$. The statement about $C_0(\mathbb{R}^N)$ follows from [25, Corollary 3.18], since F has at most linear growth. □

Note that for $q \neq p$, $T_p(t)f = T_q(t)f$ for each $f \in L^p(\mathbb{R}^N) \cap L^q(\mathbb{R}^N)$. Letting $p \to 1$ it follows that $T_p(t)$ may be extended to a bounded operator $T(t)$ in $L^1(\mathbb{R}^N)$, and

$$\|T(t)f\|_1 \leq e^{kt}\|f\|_1, \quad f \in L^1(\mathbb{R}^N). \tag{5.5}$$

Actually more is true, as the next proposition shows.

Proposition 5.2. *$(T(t))_{t\geq 0}$ is strongly continuous in $L^1(\mathbb{R}^N)$, and its infinitesimal generator is the L^1 closure of $A : C_0^\infty(\mathbb{R}^N) \mapsto L^1(\mathbb{R}^N)$.*

Proof. Let $B = A + \operatorname{div} F$ and $u \in C_0^\infty(\mathbb{R}^N)$ be real valued. Since

$$\int_{\mathbb{R}^N} Au \operatorname{sign} u \, dx \leq 0, \tag{5.6}$$

the operator $B : C_0^\infty(\mathbb{R}^N) \mapsto L^1(\mathbb{R}^N)$ is dissipative. Next we show that $(\lambda I - B)(C_0^\infty(\mathbb{R}^N))$ is dense in $L^1(\mathbb{R}^N)$ for $\lambda > 0$. Let $f \in L^\infty(\mathbb{R}^N)$ be such that

$$\int_{\mathbb{R}^N} (\lambda u - Bu) f \, dx = 0$$

for every $u \in C_0^\infty(\mathbb{R}^N)$. By local elliptic regularity (see [1, Theorem 6.2]), f belongs to $W^{1,2}_{\text{loc}}(\mathbb{R}^N)$ and it is a weak solution of the equation $\lambda f - B^* f = 0$, where $B^* = \Delta - F \cdot \nabla$ is the formal adjoint of B. Since the coefficients of B^* are locally bounded, the classical method of difference quotients yields $f \in W^{2,2}_{\text{loc}}(\mathbb{R}^N)$; hence, $\lambda f - B^* f = 0$ almost everywhere in $\mathbb{R}^N$ and, finally, $f \in W^{2,p}_{\text{loc}}(\mathbb{R}^N)$ for all $p < \infty$, by [12, Lemma 9.16]. Taking $\lambda > 0$ and observing that the coefficients of B^* have at most linear growth, we may apply the maximum principle in $\mathbb{R}^N$ (see e.g. [19, Proposition 2.2]) to conclude that $f = 0$. Since $A = B - \operatorname{div} F$, the Lumer-Phillips theorem implies that the closure of $A : C_0^\infty(\mathbb{R}^N) \mapsto L^1(\mathbb{R}^N)$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ satisfying $\|S(t)f\|_1 \leq e^{kt}\|f\|_1$. On the other hand, since the generator of $(S(t))_{t\geq 0}$ coincides with the generator of $(T(t))_{t\geq 0}$ on $C_0^\infty(\mathbb{R}^N)$ which is a core both in L^1 and in L^p, it follows that the Laplace transform of

$S(t)f$ and $T(t)f$ coincide for $f \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$. Therefore $S(t)f = T(t)$ for $f \in L^1(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$, and the proof is complete. □

In some arguments below we need the adjoint semigroup $(T(t)^*)_{t\geq 0}$ of $(T(t))_{t\geq 0}$. Let us consider the formal adjoint A^* of A defined by

$$A^* = \Delta - F \cdot \nabla - \operatorname{div} F. \tag{5.7}$$

Since $\operatorname{div} F$ is bounded, it follows from Proposition 5.1 that $A^* : D_p \mapsto L^p(\mathbb{R}^N)$ generates a strongly continuous semigroup in $L^p(\mathbb{R}^N)$ for $1 < p < \infty$.

Lemma 5.3. *Let $1 < p < \infty$. Then the adjoint semigroup $(T(t)^*)_{t\geq 0}$ in $L^{p'}(\mathbb{R}^N)$ is generated by $(A^*, D_{p'})$.*

Proof. The adjoint semigroup in $L^{p'}(\mathbb{R}^N)$ is generated by the dual operator of (A, D_p); let it be $C : D(C) \mapsto L^{p'}(\mathbb{R}^N)$. An elementary integration by parts yields $C\phi = A^*\phi$ for each $\phi \in C_0^\infty(\mathbb{R}^N)$. Since $C_0^\infty(\mathbb{R}^N)$ is a core for $(A^*, D_{p'})$, by Proposition 5.1 we obtain that $(C, D(C))$ is an extension of $(A^*, D_{p'})$; hence, they do coincide because both of them are generators in $L^{p'}(\mathbb{R}^N)$. □

Next we prove an ultracontractivity property of $(T(t))_{t\geq 0}$, showing that $T(t)$ maps $L^1(\mathbb{R}^N)$ into $L^\infty(\mathbb{R}^N)$ with a behavior similar to that of the heat semigroup near $t = 0$. We adapt a classical argument based on the so-called Nash inequality,

$$\|g\|_2^{2+4/N} \leq C\|g\|_{W^{1,2}}^2 \|g\|_1^{4/N}, \quad g \in W^{1,2}(\mathbb{R}^N) \cap L^1(\mathbb{R}^N). \tag{5.8}$$

See e.g. [7, Theorem 2.4.6].

Proposition 5.4. *There exist constants M and ω such that for every $f \in L^1(\mathbb{R}^N)$, $t > 0$*

$$\|T(t)f\|_\infty \leq \frac{Me^{\omega t}}{t^{N/2}} \|f\|_1. \tag{5.9}$$

Moreover, for every $t > 0$, $T(t)$ maps $L^1(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$.

Proof. Let $\lambda \geq 1 + k/2$, where k is given by (5.4). For each $u \in C_0^\infty(\mathbb{R}^N)$ we have

$$\int_{\mathbb{R}^N} u(\lambda u - Au)dx = \int_{\mathbb{R}^N} \Big(|\nabla u|^2 + (\lambda + \frac{1}{2}\operatorname{div} F)|u|^2\Big)dx \geq \int_{\mathbb{R}^N} \Big(|\nabla u|^2 + |u|^2\Big)dx. \tag{5.10}$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in D_2, (5.10) is true for each $u \in D_2$.

Let $f \in C_0^\infty(\mathbb{R}^N)$, $f \not\equiv 0$, and set $v(t) = \|e^{-\lambda t}T(t)f\|_2^2$, $t \geq 0$. Since $T(t)f$ goes to f as $t \to 0$, then $v(t) > 0$ at least for t in a right neighborhood I of

0. Moreover v is differentiable, and

$$v'(t) = 2e^{-2\lambda t}\int_{\mathbb{R}^N} T(t)f((A-\lambda)T(t)f)dx, \quad t \geq 0.$$

Since $T(t)f \in D_2 \cap L^1$ for each t, we may apply estimate (5.10) and then estimate (5.8), getting for $t \in I$

$$v'(t) \leq -2\|e^{-\lambda t}T(t)f\|^2_{W^{1,2}(\mathbb{R}^N)} \leq -C\|e^{-\lambda t}T(t)f\|_2^{2+4/N}\|e^{-\lambda t}T(t)f\|_1^{-4/N}.$$

Therefore, for each $t \in I$,

$$\frac{d}{dt}(v(t)^{-2/N}) \geq \frac{2C}{N}\|e^{-\lambda t}T(t)f\|_1^{-4/N} \geq C_1 e^{4(\lambda-k)t/N}\|f\|_1^{-4/N}.$$

(We used (5.5) to get the last inequality.) Integrating between 0 and t we get

$$v(t)^{-2/N} \geq C_2\|f\|_1^{-4/N}\int_0^t e^{4(\lambda-k)s/N}\,ds \geq C_2 t\|f\|_1^{-4/N}, \quad t \in I,$$

provided $\lambda \geq k$. This gives

$$\|T(t)f\|_2 \leq C_3 e^{\lambda t}t^{-N/4}\|f\|_1, \quad t > 0. \tag{5.11}$$

Thanks to Lemma 5.3, the same argument may be applied to $T(t)^*$, and it yields for $g \in C_0^\infty(\mathbb{R}^N)$ and $\lambda \geq \max\{1+k/2, 0\}$

$$\|T(t)^*g\|_2 \leq C_4 e^{\lambda t}t^{-N/4}\|g\|_1, \quad t > 0. \tag{5.12}$$

Now we use a standard duality argument to get a bound for $\|T(t)f\|_\infty$. We have

$$\|T(t)f\|_\infty = \sup_{g\in C_0^\infty(\mathbb{R}^N),\, \|g\|_1=1}\int_{\mathbb{R}^N} T(t)f\,g\,dx = \sup_{g\in C_0^\infty(\mathbb{R}^N),\, \|g\|_1=1}\int_{\mathbb{R}^N} f\,T(t)^*g\,dx,$$

and for each $g \in C_0^\infty(\mathbb{R}^N)$ with $\|g\|_1 = 1$, inequality (5.12) yields

$$\int_{\mathbb{R}^N} f\,T(t)^*g\,dx \leq \|f\|_2\|T(t)^*g\|_2 \leq \|f\|_2\,C_4 e^{\lambda t}t^{-N/4}, \quad t > 0;$$

hence,

$$\|T(t)f\|_\infty \leq C_4 e^{\lambda t}t^{-N/4}\|f\|_2, \quad t > 0. \tag{5.13}$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in $L^1(\mathbb{R}^N)$, inequalities (5.11) and (5.13) hold for every $f \in L^1(\mathbb{R}^N)$, and we get (5.9) using the semigroup law. Finally, let us show that $T(t)$ maps $L^1(\mathbb{R}^N)$ into $C_0(\mathbb{R}^N)$. If $f \in C_0^\infty(\mathbb{R}^N)$, then $T(t)f \in C_0(\mathbb{R}^N)$, since $(T(t))_{t\geq 0}$ preserves $C_0(\mathbb{R}^N)$ (see Proposition 5.1). The general case where $f \in L^1(\mathbb{R}^N)$ follows by approximation, using estimate (5.9). $\square$

Now we turn to gradient estimates. We recall that inequalities like

$$\|\nabla T(t)f\|_\infty \le e^{\omega t} t^{-1/2} \|f\|_\infty, \tag{5.14}$$

with $f \in C_b(\mathbb{R}^N)$, are valid under a dissipativity condition on the drift F, that is, when the quadratic form defined by the Jacobian matrix of F is bounded from above. Clearly this condition is satisfied in our case. We refer to [3] and to the references therein for a discussion of inequalities of the type (5.14). The paper [3] deals with a general class of second-order elliptic operators with coefficients in $C^{1+\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$. For this reason we assume for the moment that $F \in C^{1+\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$; this extra assumption will be removed later.

Corollary 5.5. *Assume that $F \in C^{1+\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\alpha > 0$. Then the semigroup $(T(t))_{t\ge 0}$ satisfies*

$$\|\nabla T(t)f\|_\infty \le C e^{\omega t} t^{-(N+1)/2} \|f\|_1, \quad f \in L^1(\mathbb{R}^N),$$

for suitable $C > 0$ and $\omega \in \mathbb{R}$.

Proof. Since $\|\nabla F\|_\infty < \infty$, the assumptions of [3] are satisfied, and (5.14) holds for each $f \in C_b(\mathbb{R}^N)$. The statement then follows using Proposition 5.4 and the semigroup law. □

Now we estimate the operator norm of $\nabla T(t)$ from $L^1(\mathbb{R}^N)$ to $(L^1(\mathbb{R}^N))^N$. It will be deduced from the analogous result for $p = \infty$ via the following lemma.

Lemma 5.6. *Assume that $F \in C^{1+\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\alpha > 0$. Then the operator $\nabla T(t) - T(t)\nabla$ may be extended to a bounded operator $\Gamma(t)$ from $(C_0(\mathbb{R}^N))^N$ into itself, satisfying $\|\Gamma(t)f\|_\infty \le C e^{\omega t}\sqrt{t}\|f\|_\infty$ for suitable $C > 0$ and $\omega \in \mathbb{R}$.*

Proof. Let $f \in C_0^\infty(\mathbb{R}^N)$. The function

$$u(t,x) = T(t)f(x)$$

is in $C^{1+\alpha/2,3+\alpha}_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^N)$. Moreover, ∇u is bounded in $[0,\infty)\times\mathbb{R}^N$ and it satisfies $\nabla u(t,x) \to \nabla f(x)$ as $t \to 0$; see [3]. Set

$$v(t,\cdot) = \nabla T(t)f - T(t)\nabla f, \quad t > 0.$$

Then $v \in C^{1+\alpha/2,2+\alpha}_{\mathrm{loc}}((0,\infty)\times\mathbb{R}^N) \cap L^\infty((0,\infty)\times\mathbb{R}^N)$ and v is continuous up to $t = 0$. A straightforward computation shows that v satisfies

$$\begin{cases} D_t v - Av = F_x \nabla u, & t > 0,\ x \in \mathbb{R}^N, \\ v(0,x) = 0, & x \in \mathbb{R}^N, \end{cases}$$

where F_x denotes the Jacobian matrix of F. Since F_x is bounded, using estimates (5.14) we get

$$|D_t v(t,x) - Av(t,x)| \le C_1 e^{\omega t} t^{-1/2} \|f\|_\infty, \ \ t>0, \ x \in \mathbb{R}^N.$$

For $i = 1, \dots, N$ let $w_i = v_i - 2C_1 e^{\omega t}\sqrt{t}\|f\|_\infty$. Then $w_i \in C^{1,2}((0,T]\times\mathbb{R}^N)\cap C_b([0,T]\times\mathbb{R}^N)$ for every $T<\infty$ and it satisfies

$$\begin{cases} D_t w_i - Aw_i \le 0, & t>0, \ x \in \mathbb{R}^N, \\ w_i(0,x) = 0, & x \in \mathbb{R}^N. \end{cases}$$

The maximum principle (in the form stated e.g. in [15, Exercise 8.1.22] for operators with unbounded coefficients) yields $w_i \le 0$. The same argument applied to v eventually gives

$$\|\nabla T(t) f - T(t)\nabla f\|_\infty = \|v(t,\cdot)\|_\infty \le C_2 e^{\omega t}\sqrt{t}\|f\|_\infty,$$

for $f \in C_0^\infty(\mathbb{R}^N)$, hence for $f \in C_0(\mathbb{R}^N)$, by density. Finally we show that $\Gamma(t) f \in C_0(\mathbb{R}^N)$. If $f \in C_0^\infty(\mathbb{R}^N)$, then $T(t)f \in D_p \subset W^{2,p}(\mathbb{R}^N)$ for each $p>1$, and taking $p>N$, we get $\nabla T(t) f \in C_0(\mathbb{R}^N)$ by Sobolev embedding. Since $T(t)\nabla f$ also belongs to $C_0(\mathbb{R}^N)$ we deduce that $\Gamma(t) f \in C_0(\mathbb{R}^N)$. The case $f \in C_0(\mathbb{R}^N)$ easily follows by approximation, using the continuity of $\Gamma(t)$ with respect to the sup norm. □

Corollary 5.7. *Assume that $F \in C^{1+\alpha}_{\rm loc}(\mathbb{R}^N)$ for some $\alpha>0$, and that F is Lipschitz continuous. Then $T(t)f \in W^{1,1}(\mathbb{R}^N)$ for every $f \in L^1(\mathbb{R}^N)$, $t>0$ and*

$$\|\nabla T(t) f\|_1 \le C e^{\omega t} t^{-1/2}\|f\|_1, \ \ t>0, \tag{5.15}$$

for suitable $C>0$ and $\omega \in \mathbb{R}$.

Proof. Let $f, g \in C_0^\infty(\mathbb{R}^N)$. Then $T(t)f$, $T(t)g \in D_2$, and therefore

$$-\langle \nabla T(t) f, g\rangle = \langle f, T(t)^*\nabla g\rangle = \langle f, \nabla T(t)^* g\rangle + \langle f, (T(t)^*\nabla - \nabla T(t)^*)g\rangle.$$

Using estimate (5.14) and Lemma 5.6 we obtain

$$\big|\langle \nabla T(t) f, g\rangle\big| \le C e^{\omega t}(t^{-1/2} + t^{1/2})\|f\|_1\|g\|_\infty,$$

and the statement follows for $f \in C_0^\infty(\mathbb{R}^N)$. By density, (5.15) holds for every $f \in L^1(\mathbb{R}^N)$. □

Now we are ready to remove the extra regularity assumption on F and to prove gradient regularity for functions belonging to the domain $D(A)$ of A in $L^1(\mathbb{R}^N)$.

Theorem 5.8. *Assume that F is globally Lipschitz continuous in $\mathbb{R}^N$. Then the map $\nabla : D(A) \to L^{N/(N-1),\infty}(\mathbb{R}^N, \mathbb{R}^N)$ is bounded.*

Proof. As a first step, assume that F is in $C^{1+\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $\alpha > 0$. Corollaries 5.5 and 5.7 show that $(T(t))_{t\geq 0}$ satisfies hypotheses $(H1)$ and $(H3)$ of Section 4, and Theorem 4.1 yields the statement. The general case can be handled by perturbation. Let $0 \leq \rho \in C_0^\infty(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N} \rho = 1$, and define $F_0 = F \star \rho$. Set moreover $A_0 = \Delta + \langle F_0, \nabla\rangle$, and let D_0 be the domain of A_0 in $L^1(\mathbb{R}^N)$. Since F_0 is smooth and Lipschitz continuous, for each $u \in C_0^\infty(\mathbb{R}^N)$ we have

$$\|\nabla u\|_{N/(N-1),\infty} \leq C\big(\|u\|_1 + \|A_0 u\|_1\big). \tag{5.16}$$

Since F is globally Lipschitz continuous, $F - F_0$ is bounded, and therefore

$$\|Au - A_0 u\|_1 = \|\langle F - F_0, \nabla u\rangle\|_1 \leq C\|\nabla u\|_1. \tag{5.17}$$

Moreover, using Remark 4.4, for each $\varepsilon > 0$ we have

$$\|\nabla u\|_1 \leq \varepsilon\|A_0 u\|_1 + \frac{C}{\varepsilon}\|u\|_1 \leq \varepsilon\|Au\|_1 + C\varepsilon\|\nabla u\|_1 + \frac{C}{\varepsilon}\|u\|_1,$$

and, taking ε small enough,

$$\|\nabla u\|_1 \leq C\big(\|u\|_1 + \|Au\|_1\big). \tag{5.18}$$

Using (5.16), (5.17), and (5.18) we get

$$\|\nabla u\|_{N/(N-1),\infty} \leq C(\|u\|_1 + \|A_0 u\|_1) \leq C(\|u\|_1 + \|\nabla u\|_1 + \|Au\|_1) \leq C\|u\|_{D(A)}$$

for $u \in C_0^\infty(\mathbb{R}^N)$, and by density for $u \in D(A)$. □

6. Remarks and other examples

In this section we give further examples to show how Theorems 3.1 and 4.1 and their corollaries may be applied to a wide class of second-order elliptic operators in $L^1(\Omega)$.

Example 6.1. Assume that the operator A in (1.4) is uniformly elliptic in $\mathbb{R}^N$ with bounded and Hölder-continuous coefficients. It is well known that the realization of A in $L^p(\mathbb{R}^N)$ generates an analytic semigroup $T_p(t)$, $1 < p < \infty$, which can be expressed through a kernel $G(t, \cdot, \cdot)$ independent of p; i.e.,

$$(T_p(t)f)(x) = \int_{\mathbb{R}^N} G(t, x, y) f(y)\, dy. \tag{6.1}$$

Since

$$|G(t,x,y)| \leq Ce^{\omega t} t^{-N/2} e^{-b|x-y|^2/t}, \quad |\nabla G(t,x,y)| \leq Ce^{\omega t} t^{-(N+1)/2} e^{-b|x-y|^2/t} \tag{6.2}$$

for suitable $C, b > 0$ and $\omega \in \mathbb{R}$ (see [9, Chapter 1, Section 6]), it is easy to see that $T_p(t)$ may be extended to a strongly continuous semigroup $T(t)$ in

$L^1(\mathbb{R}^N)$ which satisfies $(H1)$ and $(H3)$ with $\gamma_1 = 1/2$ and $\gamma_2 = (N+1)/2$. Denoting by $D(A)$ the domain of the generator of $T(t)$, Corollary 4.2 then shows that $\nabla u \in L^{N/(N-1),\infty}$ for every $u \in D(A)$.

Next we consider the case of a uniformly elliptic operator A in a bounded domain Ω with smooth boundary. We assume again that the coefficients a_{ij}, b_i, and c are Hölder continuous. The realization of A in $L^p(\Omega)$ with Dirichlet boundary condition, with domain $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, generates an analytic semigroup $T_p(t)$ represented as in (6.1). Using [16, Theorem 16.3] we see that the kernel G satisfies (6.2); hence, $(H1)$ and $(H3)$ hold for the extension $T(t)$ to $L^1(\Omega)$ with $\gamma_1 = 1/2$ and $\gamma_2 = (N+1)/2$, and Corollary 4.2 implies that $\nabla u \in L^{N/(N-1),\infty}$ for every $u \in D(A)$.

Let now $\nu \in C^1(\partial\Omega, \mathbb{R}^N)$ be an outward-pointing, nowhere-tangent vector field, and let $\beta \in C^1(\partial\Omega)$. We define the oblique derivative boundary operator

$$Bu = \frac{\partial u}{\partial \nu} + \beta u.$$

For $1 < p < \infty$, the realization of A in $L^p(\Omega)$ with domain $\{u \in W^{2,p}(\Omega) : Bu = 0 \text{ on } \partial\Omega\}$ generates an analytic semigroup $T_p(t)$ in $L^p(\Omega)$, for which (6.1) and (6.2) hold, if the coefficients a_{ij} and b_i belong to $C^1(\overline{\Omega})$. See [27, Theorem 5.7]. Therefore $T_p(t)$ is extended to a strongly continuous semigroup in $L^1(\Omega)$ to which Corollary 4.2 may be applied, yielding again $\nabla u \in L^{N/(N-1),\infty}$ for every $u \in D(A)$.

We remark that in the case of elliptic operators in the whole of $\mathbb{R}^N$ or in Ω with Dirichlet boundary condition, even the inclusion $D(A) \subset B^{2,1}_\infty$ can be proved. See [14, Theorem 1.7 and Theorem 3.3]. For the oblique boundary condition our embedding result seems to be new. For operators in divergence form with C^1 coefficients, $Au = \sum_{i,j=1}^N D_i(a_{ij} D_j u)$, we mention also the paper [8], where estimates (1.5)(i) were proved in the case of Dirichlet boundary condition.

Example 6.2. Let us prove that Theorem 4.3 and Corollary 4.2 are sharp by showing that, in general, $\nabla u \notin L^{N/(N-1),q}_{\text{loc}}(\Omega)$ for $u \in D(A)$ and $q < \infty$. This implies also that $\nabla u \notin L^{N/(N-1)}(\Omega)$.

Let $N \geq 3$ and let $A = \Delta$ be the realization of the Laplace operator in $L^1(\mathbb{R}^N)$. Suppose for the sake of contradiction that $\nabla u \in L^{N/(N-1),q}$ for every $u \in D(A)$ and some $q < \infty$. From the closed-graph theorem there is a positive constant C such that $\|\nabla u\|_{N/(N-1),q} \leq C(\|u\|_1 + \|\Delta u\|_1)$, for every $u \in C_0^\infty(\mathbb{R}^N)$. Let $u(x) = \frac{|x|^{2-N}}{2-N}$ be the fundamental solution of the Laplace operator, and let η_ε be a smooth cutoff function such that

$\eta_\varepsilon(x)=1$ if $2\varepsilon\le|x|\le 1$, $\eta_\varepsilon(x)=0$ if $|x|\le\varepsilon$ or $|x|\ge 2$, $\|\,|\nabla\eta_\varepsilon|\,\|_\infty\le c/\varepsilon$, and $\|\Delta\eta_\varepsilon\|_\infty\le c/\varepsilon^2$. Define $u_\varepsilon=\eta_\varepsilon u\in C_0^\infty(\mathbb{R}^N)$. Clearly $\|u_\varepsilon\|_1\le K$, with K independent of ε. Since $\Delta u_\varepsilon=u\Delta\eta_\varepsilon+2\nabla u\cdot\nabla\eta_\varepsilon$, we get easily $\|\Delta u_\varepsilon\|_1\le K$. A somewhat longer but still elementary computation yields $\|\,|\nabla u_\varepsilon|\,\|_{N/(N-1)}\ge C|\log\varepsilon|^{1-1/N}$ and shows that $D(\Delta)$ is not embedded in $W^{1,N/(N-1)}(\mathbb{R}^N)$.

To proceed further and obtain the full counterexample for Lorentz spaces we must estimate the decreasing rearrangement of $f_\varepsilon:=|\nabla u_\varepsilon|$. Observe that $f_\varepsilon(x)=|x|^{1-N}$ for $2\varepsilon\le|x|\le 1$. Let $1\le\delta\le(2\varepsilon)^{1-N}$. Then

$$\{|f_\varepsilon|\ge\delta\}\supset\{2\varepsilon\le|x|\le\delta^{-1/(N-1)}\},$$

and hence $\big|\{|f_\varepsilon|\ge\delta\}\big|\ge c_N\big(\delta^{-N/(N-1)}-(2\varepsilon)^N\big)$. Now we estimate $f_\varepsilon^*(t)=\inf\{\delta:|\{|f_\varepsilon|\ge\delta\}|\le t\}$. If $1\le\delta\le(2\varepsilon)^{1-N}$ and $|\{|f_\varepsilon|\ge\delta\}|\le t\}$, then the above inequality implies that $\delta\ge\big(c_Nt+(2\varepsilon)^N\big)^{-N/(N-1)}$. This last inequality obviously holds if $\delta\ge(2\varepsilon)^{1-N}$. Finally, if $\delta\le 1$, then

$$|\{|f_\varepsilon|\ge\delta\}\big|\ge\big|\{2\varepsilon\le|x|\le 1\}\big|=c_N(1-(2\varepsilon)^N)\ge c_N/2$$

for small ε. It follows that, for small ε and $t\le c_N/2$, $f_\varepsilon^*(t)\ge\big(c_Nt+(2\varepsilon)^N\big)^{-N/(N-1)}$, and then for $q<+\infty$

$$\|\,|\nabla u_\varepsilon|\,\|^q_{N/(N-1),q}=\int_0^\infty t^{q\frac{N-1}{N}-1}f_\varepsilon^*(t)^q\,dt\ge\int_0^{c_N/2}\frac{t^{q\frac{N-1}{N}-1}}{(c_Nt+(2\varepsilon)^N)^{q\frac{N-1}{N}}}\,dt$$

goes to $+\infty$ as $\varepsilon\to 0$.

Example 6.3. Let us consider again the Ornstein-Uhlenbeck operator A as in Section 5, replacing the positivity assumption on Q by $Q\ge 0$ plus a hypoellipticity assumption, $\det Q_t>0$, $t>0$ (Q_t is defined in (5.1)), which is equivalent to Rank $[Q^{1/2},BQ^{1/2},\dots,B^{N-1}Q^{1/2}]=N$. Then $\mathbb{R}^N$ may be decomposed as the direct sum of subspaces naturally associated to the operator A, as follows. Let $k\in\{0,\dots,N-1\}$ be the smallest integer such that Rank $[Q^{1/2},BQ^{1/2},\dots,B^kQ^{1/2}]=N$. Set $V_0=$ Range $Q^{1/2}$ and $V_h=$ Range $Q^{1/2}$ +Range $BQ^{1/2}+\cdots+$ Range $B^hQ^{1/2}$. We have $V_h\subset V_{h+1}$ for every h, and $V_k=\mathbb{R}^N$. Define the orthogonal projections P_h as

$$\begin{cases} P_0=\text{projection on } V_0,\\ P_{h+1}=\text{projection on } (V_h)^\perp \text{ in } V_{h+1},\quad h=1,\dots,k-1.\end{cases}$$

Then $\mathbb{R}^N=\bigoplus_{h=0}^k P_h(\mathbb{R}^N)$. By possibly changing coordinates in $\mathbb{R}^N$ we will consider an orthogonal basis $\{e_1,\dots,e_N\}$ consisting of generators of the subspaces $P_h(\mathbb{R}^N)$. For every $h = 0, \ldots, k$ we define I_h as the set of indices i such that the vectors e_i with $i \in I_h$ span $P_h(\mathbb{R}^N)$. The number

$$d = \sum_{h=0}^{k} (h+1)\dim(P_h(\mathbb{R}^N))$$

is of relevance in our analysis. Indeed, it can be proved that

$$\det Q_t^{-1} \le Ce^{\omega t} t^{-d}, \ \ t > 0,$$

with $C > 0$ and $\omega \in \mathbb{R}$. Note that $d \ge N$, and $d = N$ if and only if $Det\, Q \neq 0$. Moreover,

$$\|P_h e^{tB^*} Q_t^{-1/2}\| \le \frac{Ce^{\omega t}}{t^h}, \ \ t > 0, \ h = 0, \ldots, k.$$

See e.g. [26] and [18, Lemma 3.1]. These estimates, used in the representation formula for $\nabla T(t)f$,

$$\nabla T(t)f = -\frac{t^{-1/2} \det Q_t^{-1/2}}{2(4\pi)^{N/2}} \int_{\mathbb{R}^N} e^{tB^*} Q_t^{-1/2} (e^{tB}x - z) e^{-|e^{tB}x - z|^2/4} f(z)dz,$$

$f \in L^1(\mathbb{R}^N)$, give, for $h = 0, \ldots, k$,

$$\|D_i T(t) f\|_\infty \le \frac{Ce^{\omega t}}{t^{h+(d+1)/2}} \|f\|_1, \ \ i \in I_h.$$

On the other hand,

$$\|D_i T(t) f\|_1 \le \frac{Ce^{\omega t}}{t^{(h+1)/2}} \|f\|_1, \ \ i \in I_h.$$

We may apply Theorem 4.1 to the derivatives D_i for $i \in I_0$, and we get, for every $u \in D(A)$,

$$D_i u \in L^{d/(d-1),\infty}(\mathbb{R}^N), \ \ i \in I_0.$$

Example 6.4. The most popular subelliptic operator is perhaps the Heisenberg Laplacian in $\mathbb{R}^3$,

$$Af(x, y, z) = f_{xx} + f_{yy} + 4y f_{xz} - 4x f_{yz} + 4(x^2 + y^2) f_{zz},$$

which can be seen as the sum of squares of vector fields, $A = D_x^2 + D_y^2$, where

$$D_x = \frac{\partial}{\partial x} + 2y \frac{\partial}{\partial z}, \ \ D_y = \frac{\partial}{\partial y} - 2x \frac{\partial}{\partial z}.$$

It is well known that the realization of A in $L^1(\mathbb{R}^3)$ generates a strongly continuous semigroup $T(t)$; see e.g. [10]. A nice representation formula for $T(t)$ is in [11]. Estimates for $D_x T(t) f$ and $D_y T(t) f$ may be found either using such a representation formula, or by homogeneity arguments. Indeed, the operators D_x and D_y are homogeneous with degree 1 with respect to the family of dilations δ_r defined by $\delta_r(x,y,z) = (rx, ry, r^2 z)$, $r > 0$. It follows that A is homogeneous with degree 2, and this implies easily that

$$T(t)f = (T(r^2 t)(f \circ \delta_{1/r})) \circ \delta_r, \;\; f \in L^1(\mathbb{R}^3),\ r > 0,\ t > 0.$$

Therefore,

$$D_x T(t)f = r D_x (T(r^2 t)(f \circ \delta_{1/r})) \circ \delta_r, \;\; D_y T(t)f = r D_y (T(r^2 t)(f \circ \delta_{1/r})) \circ \delta_r.$$

Note that for every $g \in L^1$ we have $\|g \circ \delta_r\|_{L^1} = r^{-4}\|g\|_{L^1}$. Choosing $r = 1/\sqrt{t}$ we get, for $t > 0$,

$$\|D_x T(t)f\|_1 + \|D_y T(t)f\|_1 = t^{-1/2}(\|D_x T(1)\|_{\mathcal{L}(L^1)} + \|D_y T(1)\|_{\mathcal{L}(L^1)})\|f\|_1,$$

and

$$\begin{aligned} &\|D_x T(t)f\|_\infty + \|D_y T(t)f\|_\infty \\ &= t^{-5/2}(\|D_x T(1)\|_{\mathcal{L}(L^1, L^\infty)} + \|D_y T(1)\|_{\mathcal{L}(L^1,L^\infty)})\|f\|_1. \end{aligned}$$

The fact that $D_x T(1)$ and $D_y T(1)$ are bounded operators from L^1 to itself and to L^∞ is a consequence of hypoellipticity and homogeneity; see [10, Corollaries 3.5 and 3.6]. Therefore we may apply Theorem 4.1, with the usual derivatives replaced by D_x and D_y, and $\gamma_1 = 1/2$, $\gamma_2 = 5/2$. We get, for every $u \in D(A)$, $D_x u, D_y u \in L^{4/3,\infty}(\mathbb{R}^3)$.

Let us consider now a more general situation. We refer to [30] for notation, definitions, and references. Let G be a simply connected, nilpotent Lie group. By L^1 we mean now $L^1(G, dx)$, where dx is the right-invariant Haar measure of G. Let $X = \{X_1, \ldots, X_k\}$ be a Hörmander system of vector fields on G, and let d and D be the local dimension of (G, X) and its dimension at infinity, respectively. We have always $d \le D$, and they do coincide if and only if G is stratified and X spans the first slice of the stratification. For instance, in the case of the Heisenberg group we have $X = \{D_x, D_y\}$, where D_x and D_y are the above vector fields, and $d = D = 4$. We denote by A the associated sub-Laplacian,

$$A = \sum_{i=1}^{k} X_i^2,$$

and we define the gradient by $\nabla f = (X_1 f, \ldots, X_k f)$. The realization of A in L^1, with a suitable domain, generates a semigroup $T(t)$ which is usually called a "heat semigroup." It may be represented as

$$T(t)f(x) = \int_G f(y) h(t, y^{-1}x)\,dy,$$

where the "heat kernel" h satisfies ([30, Theorem IV.4.2])

$$|X_i h(t,x)| \le Ct^{-1/2}V(\sqrt{t})^{-1}\exp(-c\rho^2(x)/t), \;\; t>0, \; x\in G,$$

$\rho(x)$ is the distance associated to X, $V(t)$ is the volume (Haar measure) of the ball $B(x,t)$ for every $x \in G$, and C and c are positive constants. Therefore,

$$\|\nabla T(t)f\|_{L^1(G)} \le Ct^{-1/2}V(\sqrt{t})^{-1}\|f\|_{L^1(G)},$$
$$\|\nabla T(t)f\|_{L^\infty(G)} \le Ct^{-1/2}V(\sqrt{t})^{-1}\|f\|_{L^1(G)},$$

for every $f \in L^1$ and $t > 0$. Lemma 2.2(i) still holds in $L^1(G)$, with the derivatives D_i replaced by the vector fields X_i. Since $V(t) \ge Ct^d$ for t small, say $0 < t \le 1$, and $V(t) \ge Ct^D$ for t large ([30, Proposition IV.5.6]), we may apply Theorem 4.1 with $\gamma_1 = 1/2$ and $\gamma_2 = (1+d)/2$, and we get, for every $u \in D(A)$, and $i = 1,\dots,k$,

$$X_i u \in L^{d/(d-1),\infty}(G), \; \|X_i u\|_{L^{d/(d-1),\infty}(G)} \le C(\|u\|_{L^1(G)} + \|Au\|_{L^1(G)}).$$

Example 6.5. Suppose that $(T(t))_{t\ge 0}$ is a semigroup in $C_b(\mathbb{R}^N)$, the space of the bounded, continuous functions on $\mathbb{R}^N$, that has an invariant measure μ, i.e., a finite, positive Borel measure on $\mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N} T(t)f\,d\mu = \int_{\mathbb{R}^N} f\,d\mu$$

for every $t \ge 0$ and $f \in C_b(\mathbb{R}^N)$. It is well known that $(T(t))_{t\ge 0}$ has a natural extension to a strongly continuous semigroup in $L^1(\mathbb{R}^N,\mu)$; let $A : D(A) \mapsto L^1(\mathbb{R}^N,\mu)$ be its generator. If we replace $L^1(\mathbb{R}^N)$ by $L^1(\mathbb{R}^N,\mu)$, assumptions $(H1)$, $(H2)$, and $(H3)$ are not satisfied in general, and the conclusions of Theorems 3.1 and 4.1 do not hold in general. Let us consider, for instance, the Ornstein-Uhlenbeck operator considered at the beginning of Section 5, with $Q = B = I$, that is, $A = \Delta - x\cdot\nabla$. In this case the invariant measure μ is the Gaussian measure $d\mu(x) = e^{-|x|^2/2}\,dx$ and even the inclusion of $D(A)$ in $W^{1,1}(\mathbb{R}^N,\mu)$ fails. In fact it can be proved that the inclusion of $W^{1,1}(\mathbb{R}^N,\mu)$ into $L^1(\mathbb{R}^N,\mu)$ is compact, whereas the operator A has not compact resolvent in $L^1(\mu)$; see [23].

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References

[1] S. Agmon, "Lectures on Elliptic Boundary Problems," Van Nostrand, 1965.

[2] H. Amann, *Dual semigroups and second order linear elliptic boundary value problems,* Isr. J. Math., 45 (1983), 225–254.

[3] M. Bertoldi and S. Fornaro, *Gradient inequalities in parabolic problems with unbounded coefficients,* preprint.

[4] H. Brezis and W. Strauss, *Semi-linear second-order elliptic equations in* L^1*,* J. Math. Soc. Japan, 25 (1973), 565–590.

[5] P.L. Butzer and H. Berens, "Semi-Groups of Operators and Approximation," Springer-Verlag, 1967.

[6] G. Da Prato and A. Lunardi, *On the Ornstein-Uhlenbeck operator in spaces of continuous functions,* J. Funct. Anal., 131 (1995), 94–114.

[7] E.B. Davies, "Heat Kernels and Spectral Theory," Cambridge Univ. Press, 1989.

[8] G. Di Blasio, *Analytic semigroups generated by elliptic operators in* L^1 *and parabolic equations,* Osaka J. Math., 28 (1991), 367–384.

[9] A. Friedman, "Partial Differential Equations of Parabolic Type," Prentice-Hall, 1964.

[10] G.B. Folland, *Subelliptic estimates and function spaces on nilpotent Lie groups,* Ark. Mat., 13 (1975), 161–207.

[11] B. Gaveau, *Principe de moindre action, propagation de la chaleur et estimèes sous elliptiques sur certains groupes nilpotents,* Acta Math., 139 (1977), 95–153.

[12] D. Gilbarg and N. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer-Verlag, 1977.

[13] D. Guidetti, *On interpolation with boundary conditions,* Math. Z., 207 (1991), 439–460.

[14] D. Guidetti, *On elliptic systems in* L^1*,* Osaka J. Math., 30 (1993), 397–429.

[15] N.V. Krylov, "Lectures on Elliptic and Parabolic Equations in Hölder Spaces," GSM 12, Amer. Math. Soc., 1996.

[16] O.A. Ladyzenskaja, V.A. Solonnikov, and N.N. Ural'ceva, "Linear and Quasi-linear Equations of Parabolic Type," Amer. Math. Soc., 1968.

[17] A. Lunardi, *An interpolation method to characterize domains of generators of semigroups,* Semigroup Forum, 53 (1996), 155–169.

[18] A. Lunardi, *Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients,* Ann. Sc. Norm. Sup. Pisa, 24 (1997), 133–164.

[19] A. Lunardi, *Schauder theorems for elliptic and parabolic problems with unbounded coefficients in* $\mathbb{R}^N$*,* Studia Math., 128 (1998), 171–198.

[20] A. Lunardi, *Regularity for a class of sums of noncommuting operators,* in "Topics in Nonlinear Analysis," The Herbert Amann Anniversary volume, J. Escher and G. Simonett, eds., Birkhäuser, 1999, 517–533.

[21] A. Lunardi and V. Vespri, *Optimal* L^∞ *and Schauder estimates for elliptic and parabolic operators with unbounded coefficients,* proceedings of the conference "Reaction-Diffusion Systems," Trieste, 1995, G. Caristi and E. Mitidieri, eds., Lect. Notes in Pure and Applied Math., 194, M. Dekker, 1998, 217–239.

[22] G. Metafune, J. Prüss, A. Rhandi, and R. Schnaubelt, L^p *regularity for elliptic operators with unbounded coefficients,* preprint.

[23] G. Metafune, D. Pallara, and E. Priola, *Spectrum of Ornstein-Uhlenbeck operators in* L^p *spaces with respect to invariant measures,* J. Funct. Anal., 196 (2002), 40–60.

[24] G. Metafune, D. Pallara, and V. Vespri, *L^p-estimates for a class of elliptic operators with unbounded coefficients in $\mathbb{R}^N$*, preprint.

[25] G. Metafune, D. Pallara, and M. Wacker, *Feller semigroups in $\mathbb{R}^N$*, Semigroup Forum, 65 (2002), 159–205.

[26] T. Seidman, *How violent are fast controls?* Math. Control Signals Systems, 1 (1988), 89–95.

[27] H. Tanabe, "Functional Analytic Methods for Partial Differential Equations," Marcel Dekker, 1997.

[28] H. Triebel, "Interpolation Theory, Function Spaces, Differential Operators," North-Holland, 1978.

[29] H. Triebel, "Theory of Function Spaces," Birkhäuser, 1983.

[30] N.Th. Varopoulos, L. Saloff-Coste, and T. Coulhon, "Analysis and Geometry on Groups," Cambridge Univ. Press, 1992.