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# STABILITY ANALYSIS OF POSITIVE SOLUTIONS TO CLASSES OF REACTION-DIFFUSION SYSTEMS

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**Abstract.** We analyze the stability of positive solutions to systems of the form

$$\begin{cases} -\Delta u_i = f_i(u_1, u_2, \ldots, u_m) & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Omega$ is a bounded region in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$, and $f_i : [0,\infty)^m \to \mathbb{R}$ are $C^1$ functions for $i = 1, \ldots, m$. In particular, we establish conditions for stability/instability when the system is cooperative and strictly coupled ($\frac{\partial f_i}{\partial u_j} \geq 0,\ i \neq j,\ \sum_{j=1, j\neq i}^m (\frac{\partial f_i}{\partial u_j})^2 > 0$). When $m = 2$, we extend this analysis for strictly coupled competitive systems ($\frac{\partial f_i}{\partial u_j} < 0,\ i \neq j$). We apply our results to various examples, each one of different characteristics, and further analyze systems with unequal diffusion coefficients.

## 1. Introduction

We consider positive solutions $u := (u_1, \ldots, u_m)$ ($u_i \geq 0$ and $\sum_{i=1}^m u_i^2 > 0$) to systems of the form

$$\begin{cases} -\Delta u_i = f_i(u_1, u_2, \ldots, u_m) & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega \end{cases} \tag{1.1}$$

for $i = 1, \ldots, m$, where $\Omega$ is a bounded region in $\mathbb{R}^n$ $(n \geq 1)$ with smooth boundary $\partial\Omega$. For $i = 1, \ldots, m$, $f_i : [0,\infty)^m \to \mathbb{R}$ are $C^1$ functions satisfying

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either

$$\frac{\partial f_i}{\partial u_j} \geq 0,\ i \neq j,\ j = 1, \ldots, m \text{ and } \sum_{j=1, j\neq i}^{m} (\frac{\partial f_i}{\partial u_j})^2 > 0 \tag{1.2}$$

(strictly coupled cooperative system) or

$$\frac{\partial f_i}{\partial u_j} < 0,\ i \neq j,\ j = 1, 2 \text{ and } m = 2 \tag{1.3}$$

(strictly coupled competitive system).

One can think of a solution $u := (u_1, \ldots, u_m)$ of (1.1) as an equilibrium solution to the parabolic problem

$$\begin{cases} (v_i)_t - \Delta v_i = f_i(v_1, \ldots, v_m) & \text{in } (0, \infty) \times \Omega \\ v_i(t, x) = 0 & \text{on } (0, \infty) \times \partial\Omega \\ v_i(0, x) = k_i(x) & \text{in } \Omega \end{cases} \tag{1.4}$$

for $i = 1, \ldots, m$. Hence, denoting by $k(x) := (k_1(x), \ldots, k_m(x))$ and the corresponding solution of (1.4) by $v(t, x) := (v_1(t, x), \ldots, v_m(t, x))$, we say a positive solution $u$ of (1.1) is stable (in the maximum norm) if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $\|k - u\|_\infty < \delta$ implies $\|v(t, \cdot) - u\|_\infty < \epsilon$ for all $t > 0$. We say $u$ is unstable if it is not stable. We define the maximum norm of $u$ by $\|u\|_\infty := \|u_1\|_\infty + \cdots + \|u_m\|_\infty$.

Equilibrium solutions of (1.4) are of importance since, under certain conditions, they play a role in the possible limiting behavior of the model (1.4) as time increases. In general, the problems of existence and location of equilibrium solutions are difficult. This difficulty makes the stability properties of possible equilibrium solutions a very important problem.

In this paper, we provide sufficient conditions on $f(z) := (f_1(z), \ldots, f_m(z))$, where $z := (z_1, \ldots, z_m)$, $z_i \geq 0$ for $i = 1, \ldots, m$, that determine the stability properties of any possible positive equilibrium solution of (1.4). However, if one knows the range of a given solution, then these sufficient conditions need to be satisfied only on that range to yield its stability properties.

The scalar case ($m = 1$) has been extensively studied; the reader is referred to [1], [2], [3], [10], and [14] for a rich history. However, very little is known in the systems case (see [12]). The purpose of this paper is to provide useful contributions to the systems case ($m > 1$). In particular, for cooperative systems and when $m = 2$ for competitive systems, we establish the following results:

**Theorem 1.1.** (Cooperative systems) *Let* (1.2) *hold, and assume that for* $i = 1, \dots, m$ *there exist constants* $a_{ij} \geq 0$ *such that* $\sum_{j=1}^{m} a_{ij}^2 > 0$ *and*

$$\sum_{j=1}^{m} \left\{ a_{ij} f_j(z) - A_j(z) \frac{\partial f_j}{\partial z_i}(z) \right\} < 0 \tag{1.5}$$

*for all* $z := (z_1, \dots, z_m)$, $z_k \geq 0$, $k = 1, \dots, m$, *where* $A_j(z) := \sum_{k=1}^{m} a_{jk} z_k$. *Then any positive solution of* (1.1) *is unstable.*

**Theorem 1.2.** (Cooperative systems) *Let* (1.2) *hold, and assume that for* $i = 1, \dots, m$ *there exist constants* $a_{ij} \geq 0$ *such that* $\sum_{j=1}^{m} a_{ij}^2 > 0$ *and*

$$\sum_{j=1}^{m} \left\{ a_{ij} f_j(z) - A_j(z) \frac{\partial f_j}{\partial z_i}(z) \right\} > 0 \tag{1.6}$$

*for all* $z := (z_1, \dots, z_m)$, $z_k \geq 0$, $k = 1, \dots, m$, *and* $A_j(z) := \sum_{k=1}^{m} a_{jk} z_k$. *Then any positive solution of* (1.1) *is stable.*

**Theorem 1.3.** (Competitive systems for the case $m = 2$) *Let* (1.3) *hold, and assume that for* $i = 1, 2$ *there exist constants* $a_{ij} \geq 0$ *such that* $\sum_{j=1}^{2} a_{ij}^2 > 0$ *and*

$$\sum_{j=1}^{2} \left\{ a_{ij} f_j(z) - A_j(z) (-1)^{(i+j)} \frac{\partial f_j}{\partial z_i}(z) \right\} < 0 \tag{1.7}$$

*for all* $z := (z_1, z_2)$, $z_k \geq 0$, $k = 1, 2$, *and* $A_j(z) := \sum_{k=1}^{2} a_{jk} z_k$. *Then any positive solution of* (1.1) *is unstable.*

**Theorem 1.4.** (Competitive systems for the case $m = 2$) *Let* (1.3) *hold, and assume that for* $i = 1, 2$ *there exist constants* $a_{ij} \geq 0$ *such that* $\sum_{j=1}^{2} a_{ij}^2 > 0$ *and*

$$\sum_{j=1}^{2} \left\{ a_{ij} f_j(z) - A_j(z) (-1)^{(i+j)} \frac{\partial f_j}{\partial z_i}(z) \right\} > 0 \tag{1.8}$$

*for all* $z := (z_1, z_2)$, $z_k \geq 0$, $k = 1, 2$, *and* $A_j(z) := \sum_{k=1}^{2} a_{jk} z_k$. *Then any positive solution of* (1.1) *is stable.*

For cooperative systems, we prove Theorem 1.1 and Theorem 1.2 by determining the sign of the principal eigenvalue, $\lambda_1$, of the linearized equation of (1.1) about a positive solution $\bar{u} := (\bar{u}_1, \dots, \bar{u}_m)$, namely,

$$-\Delta \phi_i - \sum_{j=1}^{m} \frac{\partial f_i}{\partial u_j}(\bar{u}) \phi_j = \lambda_1 \phi_i \quad \text{in} \ \ \Omega \tag{1.9}$$

$$\phi_i = 0 \qquad \text{on} \ \ \partial\Omega \tag{1.10}$$

for $i = 1, \ldots, m$.

It follows from Proposition 3.1 of [13] that there is a unique eigenvalue $\lambda_1$ with strictly positive eigenfunction $\phi = (\phi_1, \ldots, \phi_m)$ of (1.9)–(1.10) (i.e., $\phi_i > 0$ in $\Omega$ for every $i = 1, \ldots, m$). Once the sign of $\lambda_1$ is established (which we do by carefully analyzing (1.1) and (1.9)–(1.10)), we construct suitable sub- and supersolutions to (1.4) with initial data $k$ near $\bar{u}$, and deduce our stability results via comparison results established for (1.4) in [12] (Theorem 3.1 on page 393 and Corollary 3.2 on page 397).

In the case of cooperative systems ($\frac{\partial f_i}{\partial v_j} \geq 0$; $i \neq j$, $i, j = 1, \ldots, m$), a function $\tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_m)$ in $[C(\overline{(0,\infty) \times \Omega})]^m \cap [C^{1,2}((0,\infty) \times \Omega)]^m$ is called a supersolution of (1.4) if for each $i = 1, \ldots, m$,

$$
\begin{aligned}
(\tilde{v}_i)_t - \Delta \tilde{v}_i \geq f_i(\tilde{v}_1, \ldots, \tilde{v}_m) &\text{ in } \quad (0,\infty) \times \Omega \\
\tilde{v}_i(t,x) \geq 0 &\text{ on } \quad (0,\infty) \times \partial\Omega \\
\tilde{v}_i(0,x) \geq k_i(x) &\text{ in } \quad \Omega,
\end{aligned}
$$

and $\hat{v} = (\hat{v}_1, \ldots, \hat{v}_m)$ in $[C(\overline{(0,\infty) \times \Omega})]^m \cap [C^{1,2}((0,\infty) \times \Omega)]^m$ is called a subsolution if it satisfies the reverse inequalities.

In the case of competitive systems with $m = 2$, we establish Theorem 1.3 and Theorem 1.4 by reducing (1.1) to a cooperative system via the transformation $w_1 = \bar{u}_1$ and $w_2 = -\bar{u}_2$, and then proceeding as in the case of cooperative systems.

We prove our theorems in Section 2. In Section 3, we provide a wide variety of examples satisfying our hypotheses. In particular, while discussing examples of positone type ($f_i(0, \ldots, 0) > 0$ for $i = 1, \ldots, m$) and semipositone type ($f_i(0, \ldots, 0) < 0$ for $i = 1, \ldots, m$), we also consider examples of the mixed type. In Section 4, we discuss systems involving unequal diffusion coefficients.

## 2. Proof of main results

**Proof of Theorem 1.1.** Let $\bar{u} := (\bar{u}_1, \ldots, \bar{u}_m)$ be a positive solution to (1.1). For each $i = 1, \ldots, m$, multiplying (1.9) by $A_i(\bar{u})$ and integrating over $\Omega$, we have

$$
\int_\Omega (-\Delta\phi_i) A_i(\bar{u})\, dx - \int_\Omega \sum_{j=1}^m \frac{\partial f_i}{\partial u_j}(\bar{u}) \phi_j A_i(\bar{u})\, dx = \int_\Omega \lambda_1 \phi_i A_i(\bar{u})\, dx. \quad (2.1)
$$

But by the divergence theorem,

$$
\int_\Omega (-\Delta\phi_i) A_i(\bar{u})\, dx = \int_\Omega \Big( \sum_{k=1}^m a_{ik} f_k(\bar{u}) \Big) \phi_i\, dx,
$$

and thus (2.1) becomes

$$\int_\Omega \sum_{k=1}^m a_{ik} f_k(\bar u)\phi_i\,dx - \int_\Omega \sum_{j=1}^m \frac{\partial f_i}{\partial u_j}(\bar u)\phi_j A_i(\bar u)\,dx = \int_\Omega \lambda_1 \phi_i A_i(\bar u)\,dx. \quad (2.2)$$

Now adding (2.2) over $i = 1, \ldots, m$, we obtain

$$\begin{aligned}\sum_{i=1}^m \Big(\int_\Omega \sum_{j=1}^m a_{ij} f_j(\bar u)\phi_i\,dx\Big) \quad &- \quad \sum_{i=1}^m \Big(\int_\Omega \sum_{j=1}^m \frac{\partial f_i}{\partial u_j}(\bar u)\phi_j A_i(\bar u)\,dx\Big) \\ &= \quad \sum_{i=1}^m \Big(\int_\Omega \lambda_1 \phi_i A_i(\bar u)\,dx\Big).\end{aligned}$$

Rearranging the terms, we have

$$\begin{aligned}\sum_{i=1}^m \Big(\int_\Omega \sum_{j=1}^m a_{ij} f_j(\bar u)\phi_i\,dx\Big) \quad &- \quad \sum_{j=1}^m \Big(\int_\Omega \sum_{i=1}^m \frac{\partial f_j}{\partial u_i}(\bar u)\phi_i A_j(\bar u)\,dx\Big) \\ &= \quad \lambda_1 \sum_{i=1}^m \Big(\int_\Omega \phi_i A_i(\bar u)\,dx\Big),\end{aligned}$$

and hence

$$\int_\Omega \sum_{i=1}^m \phi_i \Big(\sum_{j=1}^m \Big\{a_{ij} f_j(\bar u) - \frac{\partial f_j}{\partial u_i}(\bar u) A_j(\bar u)\Big\}\Big)dx = \lambda_1 \int_\Omega \Big(\sum_{i=1}^m \phi_i A_i(\bar u)\Big)\,dx. \quad (2.3)$$

By hypothesis, for each $i = 1, \ldots, m$, $A_i(\bar u) > 0$ and $\phi_i > 0$ in $\Omega$, and so the integrand on the right-hand side is positive. Thus, if (1.5) holds, then $\lambda_1 < 0$.

Now, by using the fact that $\lambda_1 < 0$, we show that $\bar u$ is unstable. Consider

$$(\hat v_1, \ldots, \hat v_m) = (\bar u_1 + \rho(t)\phi_1, \ldots, \bar u_m + \rho(t)\phi_m),$$

where $\rho(t) = \rho_0(1 - be^{-\delta t})$ for $0 < b < 1$ and $\delta > 0$. We note that

$$\rho'(t) = -\rho_0 b(-\delta)e^{-\delta t} \le \epsilon\rho(t) \quad \text{if} \quad \delta \le \epsilon[1/b - 1].$$

Now, for each $i = 1, \ldots, m$,

$$\begin{aligned}(\hat v_i)_t - \Delta \hat v_i \quad &- \quad f_i(\hat v_1, \ldots, \hat v_m) = \rho'(t)\phi_i + f_i(\bar u_1, \ldots, \bar u_m) \\ &+ \quad \rho(t)\Big\{\lambda_1\phi_i + \sum_{j=1}^m \frac{\partial f_i}{\partial u_j}(\bar u)\phi_j\Big\} - f_i(\hat v_1, \ldots, \hat v_m) \\ \le [\epsilon\rho(t)\phi_i \quad &+ \quad \lambda_1\rho(t)\phi_i] + f_i(\bar u_1, \ldots, \bar u_m) + \sum_{j=1}^m \frac{\partial f_i}{\partial u_j}(\bar u)\rho(t)\phi_j\end{aligned}$$

$$-f_i(\bar{u}_1 + \rho(t)\phi_1, \ldots, \bar{u}_m + \rho(t)\phi_m). \tag{2.4}$$

But $\lambda_1 < 0$. Hence, for $\rho_0$ sufficiently small and choosing $\epsilon < |\lambda_1|$ from (2.4) we obtain

$$(\hat{v}_i)_t - \Delta \hat{v}_i \leq f_i(\hat{v}_1, \ldots, \hat{v}_m).$$

Also clearly $\hat{v}_i(t, x) = 0$ for $x \in \partial\Omega$ and $t > 0$ for $i = 1, \ldots, m$. Now consider (1.4) with $k_i(x)$ such that

$$k_i(x) \geq \bar{u}_i + \rho_0(1 - b)\|\phi_i\|_\infty \quad \text{for } i = 1, \ldots, m. \tag{2.5}$$

Then $k_i(x) \geq \hat{v}_i(0, x)$ for $i = 1, \ldots, m$. Thus $(\hat{v}_1, \ldots, \hat{v}_m)$ is a subsolution of (1.4).

Now, if $(v_1, \ldots, v_m)$ is unbounded as $t \to \infty$, then we are done (since we can take $1 - b$ sufficiently small). Suppose for each $i = 1, \ldots, m$, $\|v_i\|_\infty$ are bounded. Since $(\hat{v}_1, \ldots, \hat{v}_m)$ is a subsolution to (1.4), for any $b < 1$, we have

$$\liminf(v_1 - \bar{u}_1, \ldots, v_m - \bar{u}_m) \geq (\rho_0\phi_1, \ldots, \rho_0\phi_m) \quad \text{as } t \to \infty. \tag{2.6}$$

However, by (2.5) we can make $\|k - \bar{u}\|_\infty$ arbitrarily small by taking $(1 - b)$ sufficiently small. So (2.6) implies that $\bar{u}$ cannot be stable.

**Proof of Theorem 1.2.** Let $\bar{u}$ be a positive solution to (1.1). Then proceeding as in the proof of Theorem 1.1, from (2.3) we infer that $\lambda_1 > 0$ if (1.6) holds. We now wish to prove that $\bar{u}$ is stable; in other words, we need to show the following:

Given $\epsilon_0 > 0$, $\exists\, \delta_0 > 0$ such that if $\|k - \bar{u}\|_\infty < \delta_0$, then

$$\|v(t, \cdot) - \bar{u}\|_\infty < \epsilon_0 \quad \text{for all} \quad t \geq 0.$$

Let $(\tilde{\phi}_1, \ldots, \tilde{\phi}_m)$ be an eigenfunction corresponding to the first eigenvalue $\tilde{\lambda}_1(\tilde{\Omega})$ of the linearized equation of (1.1) in $\tilde{\Omega} \supset \Omega$. Let us choose $\tilde{\Omega}$ close enough to $\Omega$ such that the first eigenvalue $\tilde{\lambda}_1(\tilde{\Omega})$ corresponding to the linearized equation of (1.1) in $\tilde{\Omega}$ is positive, and let $(\tilde{\phi}_1, \ldots, \tilde{\phi}_m)$ be a corresponding eigenfunction such that $\min\limits_{\overline{\Omega}} \tilde{\phi}_i(x) > 0$ for $i = 1, \ldots, m$. We also extend $(\bar{u}_1, \ldots, \bar{u}_m)$ to $\tilde{\Omega}$ such that $\bar{u}_1(x) = \cdots = \bar{u}_m(x) = 0$ in $\tilde{\Omega} \backslash \Omega$. Now, let

$$(\tilde{v}_1, \ldots, \tilde{v}_m) := (\bar{u}_1 + \rho e^{-\epsilon t}\tilde{\phi}_1, \ldots, \bar{u}_m + \rho e^{-\epsilon t}\tilde{\phi}_m),$$

where $\rho > 0$ and $\epsilon > 0$ are to be chosen later. Then for each $i = 1, \ldots, m$ and $x$ in $\Omega$,

$$\begin{aligned} & (\tilde{v}_i)_t - \Delta \tilde{v}_i - f_i(\tilde{v}_1, \ldots, \tilde{v}_m) \\ = \quad & -\epsilon\rho e^{-\epsilon t}\tilde{\phi}_i + f_i(\bar{u}_1, \ldots, \bar{u}_m) + \rho e^{-\epsilon t}\Big[\tilde{\lambda}_1(\tilde{\Omega})\tilde{\phi}_i + \sum_{j=1}^{m} \frac{\partial f_i}{\partial u_j}(\bar{u})\tilde{\phi}_j\Big] \\ & -f_i(\tilde{v}_1, \ldots, \tilde{v}_m) \end{aligned}$$

$$= \quad [\tilde{\lambda}_1(\tilde{\Omega}) - \epsilon]e^{-\epsilon t}\rho\tilde{\phi}_i + f_i(\bar{u}_1, \ldots, \bar{u}_m) + \sum_{j=1}^{m} \frac{\partial f_i}{\partial u_j}(\bar{u})\rho e^{-\epsilon t}\tilde{\phi}_j$$

$$-f_i(\bar{u}_1 + \rho e^{-\epsilon t}\tilde{\phi}_1, \ldots, \bar{u}_m + \rho e^{-\epsilon t}\tilde{\phi}_m). \tag{2.7}$$

But $\tilde{\lambda}_1(\tilde{\Omega}) > 0$. Hence, for $\rho$ sufficiently small and choosing $\epsilon < \tilde{\lambda}_1(\tilde{\Omega})$, from (2.7) we obtain

$$(\hat{v}_i)_t - \Delta\hat{v}_i \geq f_i(\tilde{v}_1, \ldots, \tilde{v}_m).$$

Also, clearly $\tilde{v}_i > 0$ on $\bar{\Omega}$ due to the choice of $\tilde{\phi}_i$ for $i = 1, \ldots, m$. Now consider (1.4) with $k$ satisfying

$$\|k - \bar{u}\|_\infty \leq \rho \min\{\min_{\overline{\Omega}} \tilde{\phi}_1, \ldots, \min_{\overline{\Omega}} \tilde{\phi}_m\}.$$

Then for each $i = 1, \ldots, m$

$$k_i(x) \leq \bar{u}_m + \rho \min_{\overline{\Omega}} \tilde{\phi}_i \leq \tilde{v}_i(0, x) \text{ in } \quad \Omega.$$

This implies that $(\tilde{v}_1, \ldots, \tilde{v}_m) = (\bar{u}_1 + e^{-\epsilon t}\rho\tilde{\phi}_1, \ldots, \bar{u}_m + e^{-\epsilon t}\rho\tilde{\phi}_m)$ is a supersolution to (1.4).

Similarly we can show that $(\hat{v}_1, \ldots, \hat{v}_m) = (\bar{u}_1 - e^{-\epsilon t}\rho\tilde{\phi}_1, \ldots, \bar{u}_m - e^{-\epsilon t}\rho\tilde{\phi}_m)$ is a subsolution to (1.4).

Then, by Theorem 3.1 of [12], for each $i = 1, \ldots, m$, we have

$$\bar{u}_i - e^{-\epsilon t}\rho\tilde{\phi}_i \leq v_i(t, x) \leq \bar{u}_i + e^{-\epsilon t}\rho\tilde{\phi}_i \quad \forall t \geq 0,$$

which implies

$$\|v(t, \cdot) - \bar{u}\|_\infty \leq \rho \max\{\max_{\overline{\Omega}} \tilde{\phi}_1, \ldots, \max_{\overline{\Omega}} \tilde{\phi}_m\} \quad \forall t \geq 0.$$

Hence, given $\epsilon_0 > 0$, let $\rho > 0$ be such that

$$\rho \max\{\max_{\overline{\Omega}} \tilde{\phi}_1, \ldots, \max_{\overline{\Omega}} \tilde{\phi}_m\} < \epsilon_0,$$

and choose $\delta_0 > 0$ such that

$$\delta_0 < \rho \min\{\min_{\overline{\Omega}} \tilde{\phi}_1, \ldots, \min_{\overline{\Omega}} \tilde{\phi}_m\}.$$

This proves that $\bar{u}$ is stable, and hence Theorem 1.2 is complete.

**Proof of Theorem 1.3.** Let $\bar{u} := (\bar{u}_1, \bar{u}_2)$ be a positive solution to (1.1). First we transform this competitive system to a cooperative system. If we let $w_1 = u_1$ and $w_2 = -u_2$, we get

$$\left.\begin{aligned} -\Delta w_1 &= f_1(u_1, u_2) = f_1(w_1, -w_2) = g_1(w_1, w_2) \text{ (say) in } \quad \Omega \\ -\Delta w_2 &= -f_2(u_1, u_2) = -f_2(w_1, -w_2) = g_2(w_1, w_2) \text{ (say) in } \quad \Omega. \end{aligned}\right\} \tag{2.8}$$

It is easy to verify that (2.8) is cooperative with respect to new functions $g_1$ and $g_2$. Let $\bar{w}_1 = \bar{u}_1$ and $\bar{w}_2 = -\bar{u}_2$. Linearizing (2.8) about $\bar{w} := (\bar{w}_1, \bar{w}_2)$, we obtain

$$\begin{aligned} -\Delta\phi_1 - \frac{\partial g_1}{\partial w_1}(\bar{w})\phi_1 - \frac{\partial g_1}{\partial w_2}(\bar{w})\phi_2 &= \lambda_1\phi_1 \text{ in } \Omega && (2.9)\\ -\Delta\phi_2 - \frac{\partial g_2}{\partial w_1}(\bar{w})\phi_1 - \frac{\partial g_2}{\partial w_2}(\bar{w})\phi_2 &= \lambda_1\phi_2 \text{ in } \Omega && (2.10)\\ \phi_1 = 0 &= \phi_2 \text{ on } \partial\Omega. && (2.11) \end{aligned}$$

Now, we multiply (2.9) by $A_1 := a_{11}\bar{w}_1 - a_{12}\bar{w}_2$, (2.10) by $A_2 := a_{21}\bar{w}_1 - a_{22}\bar{w}_2$, integrate by parts over $\Omega$, and add the two resulting expressions to get

$$\begin{aligned} &\int_\Omega \phi_1 \left\{ a_{11}g_1(\bar{w}) - a_{12}g_2(\bar{w}) - A_1\frac{\partial g_1}{\partial w_1}(\bar{w}) - A_2\frac{\partial g_2}{\partial w_1}(\bar{w}) \right\} \\ &+ \int_\Omega \phi_2 \left\{ a_{21}g_1(\bar{w}) - a_{22}g_2(\bar{w}) - A_1\frac{\partial g_1}{\partial w_2}(\bar{w}) - A_2\frac{\partial g_2}{\partial w_2}(\bar{w}) \right\} \\ &= \lambda_1 \int_\Omega [\phi_1 A_1 + \phi_2 A_2]\,. \end{aligned}$$

By hypothesis, the integrand on the right-hand side is positive, and thus $\lambda_1 < 0$ if

$$a_{11}g_1(\bar{w}) - a_{12}g_2(\bar{w}) - A_1\frac{\partial g_1}{\partial w_1}(\bar{w}) - A_2\frac{\partial g_2}{\partial w_1}(\bar{w}) < 0$$

and

$$a_{21}g_1(\bar{w}) - a_{22}g_2(\bar{w}) - A_1\frac{\partial g_1}{\partial w_2}(\bar{w}) - A_2\frac{\partial g_2}{\partial w_2}(\bar{w}) < 0,$$

that is, if (1.7) holds. Now, one can follow the proof of Theorem 1.1 to show that every positive solution is unstable.

**Proof of Theorem 1.4.** Following the first part of Theorem 1.3, we see that $\lambda_1 > 0$ if (1.8) holds. Then proceeding as in Theorem 1.2 we can conclude that every positive solution is stable.

## 3. Examples

In this section we first discuss systems with the same diffusion coefficients, namely, examples of $f$ and $g$ for systems of the form

$$\left.\begin{aligned} -\Delta u &= \lambda f(u,v) \text{ in } \Omega \\ -\Delta v &= \lambda g(u,v) \text{ in } \Omega \\ u &= 0 = v \text{ on } \partial\Omega \end{aligned}\right\} \qquad (3.1)$$

where $\lambda > 0$ is a parameter.

We begin with cooperative systems for which positive solutions are unstable for every $\lambda > 0$. We note that the hypotheses of Theorem 1.1 are satisfied if one of the following sets of conditions hold:

$$\left.\begin{aligned} f - u\tfrac{\partial f}{\partial u} - v\tfrac{\partial g}{\partial u} < 0 \\ g - u\tfrac{\partial f}{\partial v} - v\tfrac{\partial g}{\partial v} < 0 \end{aligned}\right\} \tag{3.2}$$

$$\left.\begin{aligned} f - v\tfrac{\partial f}{\partial v} - u\tfrac{\partial g}{\partial v} < 0 \\ g - v\tfrac{\partial f}{\partial u} - u\tfrac{\partial g}{\partial u} < 0 \end{aligned}\right\} \tag{3.3}$$

$$\left.\begin{aligned} (f+g) - (u+v)\tfrac{\partial f}{\partial u} - (u+v)\tfrac{\partial g}{\partial u} < 0 \\ (f+g) - (u+v)\tfrac{\partial f}{\partial v} - (u+v)\tfrac{\partial g}{\partial v} < 0 \end{aligned}\right\} \tag{3.4}$$

Here (3.2) follows from (1.5) with $a_{11} = 1 = a_{22}$ and $a_{21} = 0 = a_{12}$, (3.3) follows from (1.5) with $a_{11} = 0 = a_{22}$ and $a_{21} = 1 = a_{12}$, and (3.4) follows from (1.5) with $a_{11} = a_{22} = a_{21} = a_{12} = 1$.

We now discuss our first set of examples (1–4) satisfying one of the conditions (3.2), (3.3), or (3.4), and hence for such $f$ and $g$, all positive solutions of (3.1) **are unstable** for all $\lambda > 0$.

**Example 1.** Consider the semipositone system:

$$\left.\begin{aligned} f(u,v) = u^p + uv^\alpha - \epsilon_1 \\ g(u,v) = v^q + vu^\beta - \epsilon_2 \end{aligned}\right\}; p,q > 1,\ \alpha,\beta > 0,\ \epsilon_i > 0,\ i = 1,2.$$

Then

$$f - u\frac{\partial f}{\partial u} - v\frac{\partial g}{\partial u} = (1-p)u^p - \epsilon_1 - \beta v^2 u^{\beta-1} < 0$$

and

$$g - u\frac{\partial f}{\partial v} - v\frac{\partial g}{\partial v} = (1-q)v^q - \epsilon_2 - \alpha u^2 u^{\alpha-1} < 0.$$

Hence, (3.2) is satisfied.

**Example 2.** Consider the semipositone system:

$$\left.\begin{aligned} f(u,v) = v^p - \epsilon_1 \\ g(u,v) = u^q - \epsilon_2, \end{aligned}\right\}; p,\ q > 1,\ \epsilon_i > 0,\ i = 1,2.$$

Then

$$f - v\frac{\partial f}{\partial v} - u\frac{\partial g}{\partial v} = (1-p)v^p - \epsilon_1 < 0$$

and

$$g - v\frac{\partial f}{\partial u} - u\frac{\partial g}{\partial u} = (1-q)u^q - \epsilon_2 < 0.$$

Hence, (3.3) is satisfied.

**Example 3.** Consider the semipositone system:

$$\left.\begin{aligned} f(u,v) &= u^p + v^q - \epsilon_1 \\ g(u,v) &= u^{p-1}v - \epsilon_2, \end{aligned}\right\} ;\ p,q > 1,\ \epsilon_i > 0,\ i = 1,2.$$

Then

$$f - v\frac{\partial f}{\partial v} - u\frac{\partial g}{\partial v} = (1-q)v^q - \epsilon_1 < 0$$

and

$$g - u\frac{\partial g}{\partial u} - v\frac{\partial f}{\partial u} = -2(p-1)vu^{p-1} - \epsilon_2 < 0.$$

Hence, (3.4) is satisfied.

**Example 4.** Consider the mixed system:

$$\begin{aligned} f(u,v) &= u^2v + 1 \\ g(u,v) &= u^3 - 2. \end{aligned}$$

Note here that since $f(0,0) > 0$, (3.2) or (3.3) cannot hold. However,

$$(f+g) - (u+v)\frac{\partial f}{\partial u} - (u+v)\frac{\partial g}{\partial u} = -2u^3 - 2u^2v - 2u^2 - 2uv - 1 < 0$$

and

$$(f+g) - (u+v)\frac{\partial f}{\partial v} - (u+v)\frac{\partial g}{\partial v} = -1 < 0.$$

Hence, (3.4) is satisfied.

We next consider a cooperative system of the form (3.1) for which positive solutions are stable for every $\lambda > 0$. We note that the hypotheses of Theorem 1.2 are satisfied if one of the following sets of conditions hold:

$$\left.\begin{aligned} f - u\tfrac{\partial f}{\partial u} - v\tfrac{\partial g}{\partial u} > 0 \\ g - u\tfrac{\partial f}{\partial v} - v\tfrac{\partial g}{\partial v} > 0 \end{aligned}\right\} \tag{3.5}$$

$$\left.\begin{aligned} f - v\tfrac{\partial f}{\partial v} - u\tfrac{\partial g}{\partial v} > 0 \\ g - v\tfrac{\partial f}{\partial u} - u\tfrac{\partial g}{\partial u} > 0 \end{aligned}\right\} \tag{3.6}$$

$$\left.\begin{aligned} (f+g) - (u+v)\tfrac{\partial f}{\partial u} - (u+v)\tfrac{\partial g}{\partial u} > 0 \\ (f+g) - (u+v)\tfrac{\partial f}{\partial v} - (u+v)\tfrac{\partial g}{\partial v} > 0 \end{aligned}\right\} \tag{3.7}$$

Here (3.5) follows from (1.6) with $a_{11} = 1 = a_{22}$ and $a_{21} = 0 = a_{12}$, (3.6) follows from (1.6) with $a_{11} = 0 = a_{22}$ and $a_{21} = 1 = a_{12}$, and (3.7) follows from (1.6) with $a_{11} = a_{22} = a_{21} = a_{12} = 1$.

We now discuss our second set of examples (5–7) satisfying one of the conditions (3.5), (3.6), or (3.7), and hence for such $f$ and $g$ all positive solutions of (3.1) **are stable** for every $\lambda > 0$.

**Example 5.** Consider the positone system: Let

$$\left.\begin{aligned} f(u,v) &= (u+1)^p + (v+1)^q + 1 \\ g(u,v) &= u(v+1)^{q-1} + 1 \end{aligned}\right\}; \ 0 \le p < 1, \ q < 1.$$

Then

$$\begin{aligned} &f - u\frac{\partial f}{\partial u} - v\frac{\partial g}{\partial u} = (u+1)^p + (v+1)^q + 1 - up(u+1)^{p-1} - v(v+1)^{q-1} \\ &> (u+1)^p + (v+1)^q + 1 - p(u+1)(u+1)^{p-1} - (v+1)(v+1)^{q-1} \\ &= (1-p)(u+1)^p + 1 > 0 \end{aligned}$$

and

$$\begin{aligned} &g - u\frac{\partial f}{\partial v} - v\frac{\partial g}{\partial v} = u(v+1)^{q-1} + 1 - uq(v+1)^{q-1} - uv(q-1)(v+1)^{q-2} \\ &> (v+1)^{q-1}[u - uq - u(q-1)] + 1 = 2u(v+1)^{q-1} + 1 > 0. \end{aligned}$$

Hence, (3.5) is satisfied.

**Example 6.** Consider the positone system:

$$\left.\begin{aligned} f(u,v) &= (v+\epsilon_1)^p \\ g(u,v) &= (u+\epsilon_2)^q \end{aligned}\right\}; \ 0 \le p,q < 1, \ \epsilon_i \ge 0, \ \ i = 1,2.$$

Then

$$\begin{aligned} &f - v\frac{\partial f}{\partial v} - u\frac{\partial g}{\partial v} = (v+\epsilon_1)^q - vq(v+\epsilon_1)^{q-1} \\ &\ge (v+\epsilon_1)^q - q(v+\epsilon_1)(v+\epsilon_1)^{q-1} = (1-q)(v+\epsilon_1)^q > 0. \end{aligned}$$

Similarly it can be easily shown that

$$g - v\frac{\partial f}{\partial u} - u\frac{\partial g}{\partial u} > 0.$$

Hence, (3.6) is satisfied.

**Example 7.** Consider the mixed system:

$$\left.\begin{aligned} f(u,v) &= (u+v+1)^{1/2} - u + 4 \\ g(u,v) &= (u+v+1)^{1/2} - v - 2 \end{aligned}\right\}$$

Note here that since $g(0,0) < 0$, (3.5) or (3.6) cannot hold. However,

$$(f+g) - (u+v)\frac{\partial f}{\partial u} - (u+v)\frac{\partial g}{\partial u}$$

$$= 2(u+v+1)^{1/2} - (u+v) + 2 - (u+v)\{\frac{1}{2}(u+v+1)^{-1/2} - 1\}$$
$$- \frac{1}{2}(u+v)(u+v+1)^{-1/2}$$
$$= 2(u+v+1)^{1/2} + 2 - (u+v)(u+v+1)^{-1/2}$$
$$> 2(u+v+1)^{1/2} + 2 - (u+v+1)(u+v+1)^{-1/2} > 0.$$

Similarly one can easily show that

$$(f+g) - (u+v)\frac{\partial f}{\partial v} - (u+v)\frac{\partial g}{\partial v} > 0.$$

Hence, (3.7) is satisfied.

We now turn our attention to competitive systems for which positive solutions are unstable for every $\lambda > 0$. We note that the hypotheses of Theorem 1.3 are satisfied if one of the following sets of conditions hold:

$$\left.\begin{array}{l} f - u\frac{\partial f}{\partial u} + v\frac{\partial g}{\partial u} < 0 \\ g + u\frac{\partial f}{\partial v} - v\frac{\partial g}{\partial v} < 0 \end{array}\right\} \tag{3.8}$$

$$\left.\begin{array}{l} f + v\frac{\partial f}{\partial v} - u\frac{\partial g}{\partial v} < 0 \\ g - v\frac{\partial f}{\partial u} + u\frac{\partial g}{\partial u} < 0 \end{array}\right\} \tag{3.9}$$

$$\left.\begin{array}{l} (f+g) - (u+v)\frac{\partial f}{\partial u} + (u+v)\frac{\partial g}{\partial u} < 0 \\ (f+g) + (u+v)\frac{\partial f}{\partial v} - (u+v)\frac{\partial g}{\partial v} < 0 \end{array}\right\} \tag{3.10}$$

Here (3.8) follows from (1.7) with $a_{11} = 1 = a_{22}$ and $a_{21} = 0 = a_{12}$, (3.9) follows from (1.7) with $a_{11} = 0 = a_{22}$ and $a_{21} = 1 = a_{12}$, and (3.10) follows from (1.7) with $a_{11} = a_{22} = a_{21} = a_{12} = 1$.

We now discuss examples (8–10) satisfying one of the conditions (3.8), (3.9), or (3.10), and hence for such $f$ and $g$ all positive solutions of (3.1) **are unstable** for every $\lambda > 0$.

**Example 8.** Consider the semipositone system:

$$\left.\begin{array}{l} f(u,v) = u^p - uv^\alpha - \epsilon_1 \\ g(u,v) = v^q - vu^\beta - \epsilon_2 \end{array}\right\} ; p,q > 1,\ \alpha,\beta > 0,\ \epsilon_i > 0,\ i = 1,2.$$

Then

$$f - u\frac{\partial f}{\partial u} + v\frac{\partial g}{\partial u} = (1-p)u^p - \epsilon_1 - \beta v^2 u^{\beta-1} < 0$$

and

$$g - u\frac{\partial f}{\partial v} - v\frac{\partial g}{\partial v} = (1-q)v^q - \epsilon_2 - \alpha u^2 u^{\alpha-1} < 0.$$

Hence, (3.8) is satisfied.

**Example 9.** Consider the semipositone system:

$$\left.\begin{array}{l} f(u,v) = u - u^2 - uv - \epsilon_1 \\ g(u,v) = v - v^2 - uv - \epsilon_2 \end{array}\right\}; \ \epsilon_i > 0,\ i = 1, 2.$$

Then $f + v\frac{\partial f}{\partial v} - u\frac{\partial g}{\partial v} = -\epsilon_1 < 0$ and $g - v\frac{\partial f}{\partial u} + u\frac{\partial g}{\partial u} = -\epsilon_2 < 0$. Hence, (3.9) is satisfied.

**Example 10.** Consider the mixed system:

$$\left.\begin{array}{l} f(u,v) = u - uv + \epsilon_1 \\ g(u,v) = v - uv - \epsilon_2 \end{array}\right\}; \ 0 < \epsilon_1 < \epsilon_2.$$

Note here that since $f(0,0) > 0$, (3.8) or (3.9) cannot hold. However,

$$f + g - (u+v)\frac{\partial f}{\partial u} + (u+v)\frac{\partial g}{\partial u} = -2uv + (\epsilon_1 - \epsilon_2) < 0$$

and

$$f + g + (u+v)\frac{\partial f}{\partial v} - (u+v)\frac{\partial g}{\partial v} = -2uv + (\epsilon_1 - \epsilon_2) < 0.$$

Hence, (3.10) is satisfied.

Next we discuss competitive systems of the form (3.1) for which positive solutions are stable for every $\lambda > 0$. We note that the hypotheses of Theorem (1.4) are satisfied if one of the following conditions hold:

$$\left.\begin{array}{l} f - u\frac{\partial f}{\partial u} + v\frac{\partial g}{\partial u} > 0 \\ g + u\frac{\partial f}{\partial v} - v\frac{\partial g}{\partial v} > 0 \end{array}\right\} \qquad (3.11)$$

$$\left.\begin{array}{l} f + v\frac{\partial f}{\partial v} - u\frac{\partial g}{\partial v} > 0 \\ g - v\frac{\partial f}{\partial u} + u\frac{\partial g}{\partial u} > 0 \end{array}\right\} \qquad (3.12)$$

$$\left.\begin{array}{l} (f+g) - (u+v)\frac{\partial f}{\partial u} + (u+v)\frac{\partial g}{\partial u} > 0 \\ (f+g) + (u+v)\frac{\partial f}{\partial v} - (u+v)\frac{\partial g}{\partial v} > 0 \end{array}\right\} \qquad (3.13)$$

Here (3.11) follows from (1.8) with $a_{11} = 1 = a_{22}$ and $a_{21} = 0 = a_{12}$, (3.12) follows from (1.8) with $a_{11} = 0 = a_{22}$ and $a_{12} = 1 = a_{21}$, and (3.13) follows from (1.8) with $a_{11} = a_{12} = a_{21} = a_{22} = 1$.

We now discuss our fourth set of examples (11)–(13) satisfying one of the conditions (3.11), (3.12), or (3.13), and hence for such $f$ and $g$ all positive solutions of (3.1) **are stable** for every $\lambda > 0$.

**Example 11.** Consider the positone system:

$$\left.\begin{array}{l} f(u,v) = u^p + \frac{1}{(u+v+1)^\alpha} + \epsilon_1 \\ g(u,v) = v^q + \frac{1}{(u+v+1)^\alpha} + \epsilon_2 \end{array}\right\}; \ \epsilon_i > 0,\ i = 1, 2,\ 0 < p,\ q,\ \alpha < 1.$$

Then

$$\begin{aligned}
&f - u\frac{\partial f}{\partial u} + v\frac{\partial g}{\partial u} \\
&u^p + \frac{1}{(u+v+1)^\alpha} + \epsilon_1 - u\Big[pu^{p-1} - \frac{\alpha}{(u+v+1)^\alpha}\Big] - \frac{\alpha v}{(u+v+1)^{\alpha+1}} \\
&\geq (1-p)u^p + \epsilon_1 + \frac{1}{(u+v+1)^\alpha} - \frac{\alpha(u+1)}{(u+v+1)^{\alpha+1}} - \frac{\alpha v}{(u+v+1)^{\alpha+1}} \\
&= (1-p)u^p + \epsilon_1 + \frac{1-\alpha}{(u+v+1)^\alpha} > 0.
\end{aligned}$$

Similarly we can show that

$$g + u\frac{\partial f}{\partial v} - v\frac{\partial g}{\partial v} > 0.$$

Hence, (3.11) is satisfied.

**Example 12.** Consider the positone system:

$$\left.\begin{aligned} f(u,v) &= u - u^2 - uv + \epsilon_1 \\ g(u,v) &= v - v^2 - uv + \epsilon_2 \end{aligned}\right\}; \ \epsilon_i > 0,\ i = 1,2.$$

Then $f + v\frac{\partial f}{\partial v} - u\frac{\partial g}{\partial v} = \epsilon_1 > 0$ and $g - v\frac{\partial f}{\partial u} + u\frac{\partial g}{\partial u} = \epsilon_2 > 0$. Hence, (3.12) is satisfied.

**Example 13.** Consider the mixed system:

$$\left.\begin{aligned} f(u,v) &= u + \tfrac{1}{(u+v+1)^\alpha} + \epsilon_1 \\ g(u,v) &= v + \tfrac{1}{(u+v+1)^\alpha} - \epsilon_2 \end{aligned}\right\}; \ \epsilon_1 \geq \epsilon_2 > 0,\ \alpha > 0.$$

Note here that since $g(0,0) < 0$, (3.11) or (3.12) cannot hold. However,

$$\begin{aligned}
&f + g - (u+v)\frac{\partial f}{\partial u} + (u+v)\frac{\partial g}{\partial u} \\
&= u + v + \frac{2}{(u+v+1)^\alpha} + \epsilon_1 - \epsilon_2 - (u+v)\Big[1 - \frac{\alpha}{(u+v+1)^{\alpha+1}}\Big] \\
&+ (u+v)\Big[\frac{-\alpha}{(u+v+1)^{\alpha+1}}\Big] \\
&= \frac{2}{(u+v+1)^\alpha} + \epsilon_1 - \epsilon_2 > 0,
\end{aligned}$$

and similarly one can show that

$$f + g + (u+v)\frac{\partial f}{\partial v} - (u+v)\frac{\partial g}{\partial v} > 0.$$

Hence, (3.13) is satisfied.

**Remark.** Readers are referred to [4], [5], [6], [7], [8], [12], and references therein for existence results for systems.

## 4. Systems with unequal diffusion coefficients

In this section we discuss systems with unequal diffusion coefficients. Namely, we consider systems of the form

$$\left.\begin{array}{c} -\Delta u = \lambda f(u,v) = \tilde{f}(u,v) \ \text{ in } \quad \Omega \\ -\Delta v = \mu g(u,v) = \tilde{g}(u,v) \ \text{ in } \quad \Omega \\ u = 0 = v \ \text{ on } \quad \partial\Omega \end{array}\right\} \tag{4.1}$$

where $\lambda$ and $\mu$ are positive parameters. For brevity, we will discuss only cooperative systems for which positive solutions are unstable for all $\lambda > 0$ and all $\mu > 0$. We recall that the hypotheses of Theorem 1.1 are satisfied if $\tilde{f}$ and $\tilde{g}$ satisfy

$$\left.\begin{array}{l} a_{11}\tilde{f} + a_{12}\tilde{g} - (a_{11}u + a_{12}v)\frac{\partial \tilde{f}}{\partial u} - (a_{21}u + a_{22}v)\frac{\partial \tilde{g}}{\partial u} < 0 \\ a_{21}\tilde{f} + a_{22}\tilde{g} - (a_{11}u + a_{12}v)\frac{\partial \tilde{f}}{\partial v} - (a_{21}u + a_{22}v)\frac{\partial \tilde{g}}{\partial v} < 0 \end{array}\right\}. \tag{4.2}$$

But taking $a_{11} = 1/\lambda$, $a_{12} = 0 = a_{21}$ and $a_{22} = 1/\mu$ it follows that the hypotheses of Theorem 1.1 will be satisfied if $f$ and $g$ satisfy (3.2). Hence, if $f$ and $g$ are as in Example 1, then positive solutions of (4.1) **are unstable** for all $\lambda > 0$ and all $\mu > 0$.

Further, if we let $a_{11} = 0 = a_{22}$ and $a_{12} = 1 = a_{21}$, then (4.2) is satisfied if

$$\left.\begin{array}{l} \lambda f - \lambda v\frac{\partial f}{\partial v} - \mu u\frac{\partial g}{\partial v} < 0 \\ \mu g - \lambda v\frac{\partial f}{\partial u} - \mu u\frac{\partial g}{\partial u} < 0 \end{array}\right\}. \tag{4.3}$$

It is easy to see that $f$ and $g$ in Example 2 satisfy (4.3). Hence, with $f$ and $g$ as in Example 2, positive solutions of (4.1) **are unstable** for all $\lambda > 0$ and $\mu > 0$.

We next consider cooperative systems of the form (4.1) for which all positive solutions are stable for all $\lambda > 0$ and $\mu > 0$. We recall that the hypotheses of Theorem 1.2 are satisfied if $\tilde{f}$ and $\tilde{g}$ satisfy

$$\left.\begin{array}{l} a_{11}\tilde{f} + a_{12}\tilde{g} - (a_{11}u + a_{12}v)\frac{\partial \tilde{f}}{\partial u} - (a_{21}u + a_{22}v)\frac{\partial \tilde{g}}{\partial u} > 0 \\ a_{21}\tilde{f} + a_{22}\tilde{g} - (a_{11}u + a_{12}v)\frac{\partial \tilde{f}}{\partial v} - (a_{21}u + a_{22}v)\frac{\partial \tilde{g}}{\partial v} > 0 \end{array}\right\}. \tag{4.4}$$

But taking $a_{11} = 1/\lambda$, $a_{12} = 0 = a_{21}$, and $a_{22} = 1/\mu$ we see that the hypotheses of Theorem 1.2 are satisfied if $f$ and $g$ satisfy (3.5). Hence, if $f$

and $g$ are as in Example 5, then all positive solutions of (4.1) **are stable** for all $\lambda > 0$ and all $\mu > 0$.

Finally, taking $a_{11} = 0 = a_{22}$, and $a_{12} = 1 = a_{21}$ we notice that (4.4) is satisfied if

$$\left.\begin{aligned} \lambda f - \lambda v \tfrac{\partial f}{\partial v} - \mu u \tfrac{\partial g}{\partial v} > 0 \\ \mu g - \lambda v \tfrac{\partial f}{\partial u} - \mu u \tfrac{\partial g}{\partial u} > 0 \end{aligned}\right\}. \tag{4.5}$$

And it is easy to see that $f$ and $g$ in Example 6 satisfy (4.5). Hence, with $f$ and $g$ as in Example 6, positive solutions of (4.1) **are stable** for all $\lambda > 0$ and all $\mu > 0$.

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