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DAMPED WAVE EQUATION WITH SUPER CRITICAL NONLINEARITIES

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Abstract. We study global existence in time of small solutions to the Cauchy problem for the nonlinear damped wave equation

$$\begin{cases} \partial_t^2 u + \partial_t u - \Delta u = \mathcal{N}(u), \ x \in \mathbf{R}^n, \ t > 0, \\ u(0,x) = \varepsilon u_0(x), \partial_t u(0,x) = \varepsilon u_1(x), x \in \mathbf{R}^n, \end{cases} \tag{0.1}$$

where $\varepsilon > 0$. The nonlinearity $\mathcal{N}(u) \in \mathbf{C}^k(\mathbf{R})$ satisfies the estimate

$$\left| \frac{d^j}{du^j} \mathcal{N}(u) \right| \leq C |u|^{\rho - j}, \ 0 \leq j \leq k \leq \rho.$$

The power $\rho > 1 + \frac{2}{n}$ is considered as super critical for large time. We assume that the initial data

$$u_0 \in \mathbf{H}^{\alpha,0} \cap \mathbf{H}^{0,\delta}, \ u_1 \in \mathbf{H}^{\alpha-1,0} \cap \mathbf{H}^{0,\delta},$$

where $\delta > \frac{n}{2}$, $[\alpha] \leq \rho, \alpha \geq \frac{n}{2} - \frac{1}{\rho-1}$ for $n \geq 2$, and $\alpha \geq \frac{1}{2} - \frac{1}{2(\rho-1)}$ for $n = 1$. Weighted Sobolev spaces are

$$\mathbf{H}^{l,m} = \left\{ \phi \in \mathbf{L}^2; \| \langle x \rangle^m \langle i\partial_x \rangle^l \phi(x) \|_{\mathbf{L}^2} < \infty \right\},$$

where $\langle x \rangle = \sqrt{1+x^2}$. Then we prove that there exists a small $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ there exists a unique global solution

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$u \in \mathbf{C}\left([0,\infty);\mathbf{H}^{\alpha,0}\cap\mathbf{H}^{0,\delta}\right)$ for the Cauchy problem (0.1) and solutions satisfy the time decay property

$$\|u(t)\|_{\mathbf{L}^p} \le Ct^{-\frac{n}{2}(1-\frac{1}{p})}$$

for all $t > 0$, where $2 \le p \le \frac{2n}{n-2\alpha}$ if $\alpha < \frac{n}{2}$, $2 \le p < \infty$ if $\alpha = \frac{n}{2}$, and $2 \le p \le \infty$ if $\alpha > \frac{n}{2}$.

1. Introduction

We study the global existence in time of small solutions to the Cauchy problem for the nonlinear damped wave equation

$$\begin{cases} \mathcal{L}u = \mathcal{N}(u),\ x \in \mathbf{R}^n,\ t > 0 \\ u(0,x) = \varepsilon u_0(x),\ \partial_t u(0,x) = \varepsilon u_1(x),\ x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

where $\mathcal{L} = \partial_t^2 + \partial_t - \Delta$, $\varepsilon > 0$. The nonlinearity $\mathcal{N}(u) \in \mathbf{C}^k(\mathbf{R})$ satisfies the estimate

$$\left|\frac{d^j}{du^j}\mathcal{N}(u)\right| \le C|u|^{\rho-j},\ \ 0 \le j \le k \le \rho.$$

The power $\rho > 1 + \frac{2}{n}$ is considered as super critical for large time since it is known that solutions of (1.1) blow up, when the data are positive, $\mathcal{N}(u) = |u|^\rho$, $\varepsilon > 0$, and $1 < \rho \le 1 + \frac{2}{n}$; see [4], [8], and [12]. From the previous works [3] and [5] we know that under the condition $\langle\xi\rangle^\delta \widehat{u}_0(\xi)$, $\langle\xi\rangle^{\delta-1}\widehat{u}_1(\xi) \in \mathbf{L}^2$ with $\delta > \frac{n}{2}$, the Fourier transform of a solution to the linearized problem corresponding to (1.1) decays exponentially in time and behaves like a solution of the linear wave equation in the high-frequency part $|\xi| \ge \frac{1}{2}$. In the low-frequency part $|\xi| \le \frac{1}{2}$ the solution resembles that of the linear heat equation. Therefore some regularity assumptions on the data are required to get $\mathbf{L}^\infty$ time decay estimates of solutions in the high-frequency part for the case of higher space dimensions. This fact is an obstacle for considering the problem with fractional-order nonlinear terms in function spaces including $\mathbf{L}^\infty$ space, except the low space dimensions $n = 1, 2, 3$. In the one-dimensional case $n = 1$, the global existence in time of small solutions can be obtained by the method of paper [5]. When the initial data are in the Sobolev space $\partial^\alpha u_0 \in \mathbf{L}^1 \cap \mathbf{L}^\infty$, $|\alpha| = 0, 1$, $u_1 \in \mathbf{L}^1 \cap \mathbf{L}^\infty$, and $n = 3$, problem (1.1) was considered in [7]. By making use of the fundamental solution of the linear problem there were proved the global existence of small solutions and large time decay estimates $\|u\|_{\mathbf{L}^q} \le Ct^{-\frac{n}{2}(1-\frac{1}{q})}$, $1 \le q \le \infty$ for $n = 3$. Later these requirements on the initial data were relaxed in [9] as follows: $u_0 \in \mathbf{L}^1$, $\partial^\alpha u_0 \in \mathbf{L}^2$, $|\alpha| = 0, 1$, and $u_1 \in \mathbf{L}^1 \cap \mathbf{L}^2$, under the additional assumptions on ρ and q such that $\rho \le 5$ and $q \le 6$ for the space dimension $n = 3$ and

$q < \infty$ for the two-dimensional case $n = 2$. For the case of higher dimensions $n = 4, 5$, global existence and $\mathbf{L}^q$ time decay estimates for $\rho \le q \le \frac{\rho}{\rho-1}$ were obtained via Fourier analysis in paper [6], when the power of the nonlinearity ρ is such that $1 + \frac{2}{n} < \rho \le \frac{n+2}{n-2}$ and the initial data are small enough and satisfy $u_0, \partial^\alpha u_0 \in \mathbf{L}^1 \cap \mathbf{L}^{\frac{\rho}{\rho-1}}$, $\partial^\beta u_0 \in \mathbf{L}^2$, $u_1 \in \mathbf{L}^1 \cap \mathbf{L}^{\frac{\rho}{\rho-1}}$, $\partial^\alpha u_1 \in \mathbf{L}^2$, $|\alpha| \le 1$, and $|\beta| \le 2$. Applying energy type estimates obtained in papers [5] and [3] it was proved in [2] that solutions of the nonlinear damped wave equation (1.1) in the super critical cases $1 + \frac{4}{n} < \rho \le \frac{n}{n-2}$, if $n = 3$, and $1 + \frac{4}{n} < \rho < \infty$, if $n = 1, 2$, with arbitrary initial data $u_0 \in \mathbf{H}^1 \cap \mathbf{L}^1$, $u_1 \in \mathbf{L}^2 \cap \mathbf{L}^1$ (i.e., without a smallness assumption on the initial data) have the same large time asymptotics as that for the linear heat equation $\mathcal{L} = \partial_t - \Delta$; that is,

$$\left\| u(t) - MG_0(t) \right\|_{\mathbf{L}^p} = o(t^{-\frac{n}{2}(1-\frac{1}{p})})$$

as $t \to \infty$, where $2 \le p < \frac{2n}{n-2}$ for $n = 3$, $2 \le p < \infty$ for $n = 2$, and $2 \le p \le \infty$ for $n = 1$; here $G_0(t, x) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ is the heat kernel and M is a constant. For higher space dimensions, in paper [11], the global existence and energy decay estimates of solutions to the Cauchy problem for the damped wave equation (1.1) with sufficiently small initial data having a compact support was proved, in the case $\rho > 1 + \frac{2}{n}$.

Our purpose in the present paper is to remove the support condition on the data and to find the large time asymptotics of solutions in the case of higher space dimensions.

Define the weighted Sobolev space by

$$\mathbf{H}^{l,m} = \left\{ \phi \in \mathbf{L}^2 : \ \left\| \langle x \rangle^m \langle i\partial_x \rangle^l \phi(x) \right\|_{\mathbf{L}^2} < \infty \right\},$$

where $\langle x \rangle = \sqrt{1 + x^2}$. By $[\alpha]$ we denote the integer part of α.

Our main result is the following.

Theorem 1.1. *Let $\rho > 1 + \frac{2}{n}$. Suppose that the initial data*

$$u_0 \in \mathbf{H}^{\alpha,0} \cap \mathbf{H}^{0,\delta}, \ u_1 \in \mathbf{H}^{\alpha-1,0} \cap \mathbf{H}^{0,\delta},$$

where $\delta > \frac{n}{2}$, $[\alpha] \le \rho$; $\alpha \ge \frac{n}{2} - \frac{1}{\rho-1}$ for $n \ge 2$ and $\alpha \in [\frac{1}{2} - \frac{1}{2(\rho-1)}, 1)$ for $n = 1$. Then there exists a sufficiently small $\varepsilon > 0$ such that the Cauchy problem (1.1) has a unique global solution $u \in \mathbf{C}\left([0,\infty); \mathbf{H}^{\alpha,0} \cap \mathbf{H}^{0,\delta}\right)$. Moreover, the following asymptotics are valid:

$$\left\| u(t) - AG_0(t) \right\|_{\mathbf{L}^p} \le Ct^{-\frac{n}{2}\left(1-\frac{1}{p}\right) - \min\left(1, \frac{\delta}{2} - \frac{n}{4}, \frac{n}{2}(\rho-1)-1\right)} \tag{1.2}$$

for all $t > 0$*, where*

$$A = \varepsilon\theta_1 + \varepsilon\theta_0 + \int_0^\infty \int_{\mathbf{R}^n} \mathcal{N}\left(u\left(t,x\right)\right) dxdt,$$

$\theta_j = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} u_j\left(x\right) dx$, $2 \le p \le \frac{2n}{n-2\alpha}$ *if* $\alpha < \frac{n}{2}$, $2 \le p < \infty$ *if* $\alpha = \frac{n}{2}$, *and* $2 \le p \le \infty$ *if* $\alpha > \frac{n}{2}$; $G_0\left(t,x\right) = \left(4\pi t\right)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$ *is the heat kernel.*

We organize our paper as follows. In Section 2 we prove some preliminary estimates. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminary lemmas

We consider the linear Cauchy problem

$$\begin{cases} \mathcal{L}u = f(t,x),\ x \in \mathbf{R}^n,\ t > 0, \\ u\left(0,x\right) = \varepsilon u_0\left(x\right),\ \partial_t u\left(0,x\right) = \varepsilon u_1\left(x\right),\ x \in \mathbf{R}^n, \end{cases} \tag{2.1}$$

where $\mathcal{L} = \partial_t^2 + \partial_t - \Delta$ and $\varepsilon > 0$. The solution of (2.1) can be written by the Duhamel formula

$$u\left(t\right) = \left(\partial_t + \frac{1}{2}\right)\mathcal{G}\left(t\right)\varepsilon u_0 + \mathcal{G}\left(t\right)\varepsilon u_1 + \int_0^t \mathcal{G}\left(t-\tau\right) f(\tau) d\tau, \tag{2.2}$$

where $\mathcal{G}\left(t\right) = e^{-\frac{t}{2}}\mathcal{F}^{-1}L\left(t,\xi\right)\mathcal{F}$ and $L\left(t,\xi\right) = \frac{\sin(t\sqrt{|\xi|^2-\frac{1}{4}})}{\sqrt{|\xi|^2-\frac{1}{4}}}$.

Note that the symbol $L\left(t,\xi\right)$ is smooth and bounded: $L\left(t,\xi\right) \in \mathbf{C}^\infty\left(\mathbf{R}^n\right)$; moreover, the symbol $L\left(t,\xi\right)$ decays like $\frac{1}{|\xi|}$ for $|\xi| \to \infty$. This implies a gain of one derivative concerning the initial datum u_1 and a forcing term f. We first prove some preliminary estimates for the Green's operator $\mathcal{G}\left(t\right)$.

Lemma 2.1. *The estimates*

$$\left\||\nabla|^\alpha \mathcal{G}\left(t\right)\psi\right\|_{\mathbf{L}^2} \le C\left\langle t\right\rangle^{-\frac{\alpha-\beta}{2}-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \left\| |\nabla|^\beta \psi\right\|_{\mathbf{L}^q} + Ce^{-\frac{t}{4}}\left\| |\nabla|^\alpha \left\langle\Delta\right\rangle^{-\frac{1}{2}}\psi\right\|_{\mathbf{L}^2}$$

and

$$\begin{aligned} \left\| |\cdot|^\delta \mathcal{G}\left(t\right)\psi\right\|_{\mathbf{L}^2} &\le C\left\langle t\right\rangle^{\frac{\delta}{2}-\frac{n}{4}}\left\|\psi\right\|_{\mathbf{L}^1} + C\left\langle t\right\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}\left\| |\cdot|^\delta \psi\right\|_{\mathbf{L}^q} \\ &\quad + Ce^{-\frac{t}{4}}\left\| \left\langle\Delta\right\rangle^{-\frac{1}{2}}\left\langle\cdot\right\rangle^\delta \psi\right\|_{\mathbf{L}^2} \end{aligned}$$

are true for all $t > 0$*, where* $1 \le q \le 2$*,* $\delta > \frac{n}{2}$*, and* $\alpha \ge \beta \ge 0$*, provided that the right-hand sides are finite.*

Proof. Taking into account the estimates

$$e^{-\frac{t}{2}}\left\||\xi|^{\alpha}L(t,\xi)\right\|_{\mathbf{L}^{r}(|\xi|\le 1)}\le C\left\langle t\right\rangle^{-\frac{\alpha}{2}-\frac{n}{2r}}$$

and

$$\left\|\left\langle\xi\right\rangle L\left(t,\xi\right)\right\|_{\mathbf{L}^{\infty}(|\xi|\ge 1)}\le C$$

for all $t>0$, we get

$$\begin{aligned}&\left\||\nabla|^{\alpha}\mathcal{G}(t)\psi\right\|_{\mathbf{L}^{2}}\le Ce^{-\frac{t}{2}}\left\||\xi|^{\alpha-\beta}L(t,\xi)\right\|_{\mathbf{L}^{\frac{2q}{2-q}}(|\xi|\le 1)}\left\||\xi|^{\beta}\widehat{\psi}(\xi)\right\|_{\mathbf{L}^{\frac{q}{q-1}}(|\xi|\le 1)}\\&+Ce^{-\frac{t}{2}}\left\|\left\langle\xi\right\rangle L\left(t,\xi\right)\right\|_{\mathbf{L}^{\infty}(|\xi|\ge 1)}\left\||\xi|^{\alpha}\left\langle\xi\right\rangle^{-1}\widehat{\psi}\left(\xi\right)\right\|_{\mathbf{L}^{2}(|\xi|\ge 1)}\\&\le C\left\langle t\right\rangle^{-\frac{\alpha-\beta}{2}-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}\left\||\nabla|^{\beta}\psi\right\|_{\mathbf{L}^{q}}+Ce^{-\frac{t}{4}}\left\||\nabla|^{\alpha}\left\langle\Delta\right\rangle^{-\frac{1}{2}}\psi\right\|_{\mathbf{L}^{2}},\end{aligned}$$

whence the first estimate of the lemma follows.

To prove the second estimate we write

$$\begin{aligned}&\left\||x|^{\delta}\mathcal{G}\left(t\right)\psi\right\|_{\mathbf{L}^{2}}=C\left\||\nabla|^{\delta}e^{-\frac{t}{2}}L\left(t,\xi\right)\chi_{1}\left(\xi\right)\widehat{\psi}\left(\xi\right)\right\|_{\mathbf{L}^{2}}\\&+C\left\||\nabla|^{\delta}\left\langle\xi\right\rangle e^{-\frac{t}{2}}L\left(t,\xi\right)\chi_{2}\left(\xi\right)\left\langle\xi\right\rangle^{-1}\widehat{\psi}\left(\xi\right)\right\|_{\mathbf{L}^{2}},\end{aligned}\tag{2.3}$$

where we denote $\chi_{1}\left(\xi\right)\in\mathbf{C}^{\infty}\left(\mathbf{R}^{n}\right)$ such that $\chi_{1}\left(\xi\right)=1$ for $|\xi|\le 1$ and $\chi_{1}\left(\xi\right)=0$ for $|\xi|\ge 2$; also we write $\chi_{2}\left(\xi\right)=1-\chi_{1}\left(\xi\right)$. Note that there exists a smooth and rapidly decaying kernel

$$K\left(t,x\right)=e^{-\frac{t}{2}}\mathcal{F}^{-1}\left(L\left(t,\xi\right)\chi_{1}\left(\xi\right)\right),$$

so that by the Young inequality we have

$$\begin{aligned}&\left\||\cdot|^{\delta}\mathcal{F}^{-1}\left(e^{-\frac{t}{2}}L\left(t,\xi\right)\chi_{1}\left(\xi\right)\right)\widehat{\psi}\left(\xi\right)\right\|_{\mathbf{L}^{2}}\\=&\left\||x|^{\delta}\int_{\mathbf{R}^{n}}K\left(t,x-y\right)\psi\left(y\right)dy\right\|_{\mathbf{L}^{2}}\\\le&C\left\|\int_{\mathbf{R}^{n}}\left(|x-y|^{\delta}\left|K\left(t,x-y\right)\right|+\left|K\left(t,x-y\right)\right||y|^{\delta}\right)\left|\psi\left(y\right)\right|dy\right\|_{\mathbf{L}^{2}}\\\le&C\left\||\cdot|^{\delta}K\left(t\right)\right\|_{\mathbf{L}^{2}}\left\|\psi\right\|_{\mathbf{L}^{1}}+C\left\|K\left(t\right)\right\|_{\mathbf{L}^{\frac{2q}{3q-2}}}\left\||\cdot|^{\delta}\psi\right\|_{\mathbf{L}^{q}}.\end{aligned}\tag{2.4}$$

By the estimate

$$\left|\left(-\Delta\right)^{k}\left(e^{-\frac{t}{2}}L\left(t,\xi\right)\chi_{1}\left(\xi\right)\right)\right|\le C\left\langle t\right\rangle^{k}e^{-Ct|\xi|^{2}}\chi_{1}\left(\xi\right)$$

for all $t>0$, $|\xi|\le 2$, and $k\ge 0$, we have

$$\left\||\cdot|^{2k}K\left(t\right)\right\|_{\mathbf{L}^{2}}\le C\left\|\left(-\Delta\right)^{k}\left(e^{-\frac{t}{2}}L\left(t,\xi\right)\chi_{1}\left(\xi\right)\right)\right\|_{\mathbf{L}^{2}}\le C\left\langle t\right\rangle^{k-\frac{n}{4}};$$

hence,

$$\left\| \left| \cdot \right|^{\delta} K\left(t\right) \right\|_{\mathbf{L}^2} \leq C \left\langle t \right\rangle^{\frac{\delta}{2}-\frac{n}{4}}.$$

By virtue of the Cauchy-Schwartz inequality we obtain

$$\begin{aligned} \left\|\phi\right\|_{\mathbf{L}^1} &\leq \int_{\mathbf{R}^n} \left(\rho^2+x^2\right)^{-\frac{\delta}{2}} \left(\rho^2+x^2\right)^{\frac{\delta}{2}} \left|\phi\left(x\right)\right| dx \\ &\leq \left(\int_{\mathbf{R}^n} \left(\rho^2+x^2\right)^{\delta} \left|\phi\left(x\right)\right|^2 dx\right)^{\frac{1}{2}} \left(\int_{\mathbf{R}^n} \left(\rho^2+x^2\right)^{-\delta} dx\right)^{\frac{1}{2}} \\ &\leq C\rho^{\frac{n}{2}-\delta} \left(\int_{\mathbf{R}^n} \left(\rho^2+x^2\right)^{\delta} \left|\phi\left(x\right)\right|^2 dx\right)^{\frac{1}{2}} \\ &\leq C\rho^{\frac{n}{2}} \left\|\phi\right\|_{\mathbf{L}^2} + C\rho^{\frac{n}{2}-\delta} \left\| \left|\cdot\right|^{\delta}\phi\right\|_{\mathbf{L}^2} \leq C \left\| \left|\cdot\right|^{\delta}\phi\right\|_{\mathbf{L}^2}^{\frac{n}{2\delta}} \left\|\phi\right\|_{\mathbf{L}^2}^{1-\frac{n}{2\delta}}, \end{aligned}$$

where we have taken $\rho = \| |\cdot|^{\delta}\phi\|_{\mathbf{L}^2}^{\frac{1}{\delta}} \|\phi\|_{\mathbf{L}^2}^{-\frac{1}{\delta}} > 0$ (provided that $\delta > \frac{n}{2}$). Then applying the Hölder inequality we get

$$\left\|\phi\right\|_{\mathbf{L}^p} \leq \left\|\phi\right\|_{\mathbf{L}^1}^{\frac{2}{p}-1} \left\|\phi\right\|_{\mathbf{L}^2}^{2-\frac{2}{p}} \leq C \left\| \left|\cdot\right|^{\delta}\phi\right\|_{\mathbf{L}^2}^{\frac{n}{\delta}\left(\frac{1}{p}-\frac{1}{2}\right)} \left\|\phi\right\|_{\mathbf{L}^2}^{1-\frac{n}{\delta}\left(\frac{1}{p}-\frac{1}{2}\right)} \tag{2.5}$$

for $p \in [1,2]$. Hence we obtain

$$\left\|K\left(t\right)\right\|_{\mathbf{L}^{\frac{2q}{3q-2}}} \leq \left\| \left|\cdot\right|^{\delta} K\left(t\right) \right\|_{\mathbf{L}^2}^{\frac{n}{\delta}\left(1-\frac{1}{q}\right)} \left\|K\left(t\right)\right\|_{\mathbf{L}^2}^{1-\frac{n}{\delta}\left(1-\frac{1}{q}\right)} \leq C \left\langle t\right\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}.$$

Therefore, (2.4) yields

$$\begin{aligned} &\left\| \left|\cdot\right|^{\delta} \mathcal{F}^{-1} \left(e^{-\frac{t}{2}} L\left(t,\xi\right) \chi_1\left(\xi\right)\right) \widehat{\psi}\left(\xi\right) \right\|_{\mathbf{L}^2} \\ \leq\ & C\left\langle t\right\rangle^{\frac{\delta}{2}-\frac{n}{4}} \left\|\psi\right\|_{\mathbf{L}^1} + C\left\langle t\right\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)} \left\| \left|\cdot\right|^{\delta}\psi\right\|_{\mathbf{L}^q}. \end{aligned} \tag{2.6}$$

We have

$$\left| \left(-\Delta\right)^k \left(\left\langle \xi\right\rangle e^{-\frac{t}{2}} L\left(t,\xi\right) \chi_2\left(\xi\right)\right) \right| \leq Ce^{-\frac{t}{4}}$$

for all $t > 0$, $\xi \in \mathbf{R}^n$, and $k \geq 0$.

Denote $m = [\delta]$ and $\omega = \delta - m$. For the fractional derivative ∂_j^{ω} (see [10]) we have

$$\begin{aligned} \partial_j^{\omega}\left(\phi\left(x\right)\psi\left(x\right)\right) &= C\int_{\mathbf{R}^1} \left(\phi\left(x\right)\psi\left(x\right) - \phi\left(x+y\right)\psi\left(x+y\right)\right) \frac{dy_j}{|y_j|^{1+\omega}} \\ &= \phi\left(x\right)\partial_j^{\omega}\psi\left(x\right) + C\int_{\mathbf{R}^1} \left(\phi\left(x\right) - \phi\left(x+y\right)\right)\psi\left(x+y\right) \frac{dy_j}{|y_j|^{1+\omega}}, \end{aligned}$$

where $y \equiv (0, \dots, y_j, \dots, 0)$. Then we have the Leibniz rule

$$\|(-\Delta)^{\frac{\delta}{2}}\phi\psi\|_{\mathbf{L}^2} \le C\sum_{j=1}^{n}\left\|\partial_j^{\omega}\partial_j^{m}(\phi\psi)\right\|_{\mathbf{L}^2} \le C\sum_{j=1}^{n}\sum_{k=1}^{m}\left\|\partial_j^{\omega}\left((\partial_j^{m-k}\phi)\partial_j^{k}\psi\right)\right\|_{\mathbf{L}^2}$$

$$\le C\sum_{j=1}^{n}\sum_{k=1}^{m}\left\|\left(\partial_j^{m-k}\phi\right)\partial_j^{k+\omega}\psi\right\|_{\mathbf{L}^2}$$

$$+ C\sum_{j=1}^{n}\sum_{k=1}^{m}\left\|\int_{\mathbf{R}^1}\left(\partial_j^{m-k}\phi(\cdot) - \partial_j^{m-k}\phi(\cdot + y)\right)\partial_j^{k}\psi(\cdot + y)\frac{dy_j}{|y_j|^{1+\omega}}\right\|_{\mathbf{L}^2}$$

$$\le C\sum_{j=1}^{n}\sum_{k=1}^{m}\left(\left\|\partial_j^{m-k}\phi\right\|_{\mathbf{L}^\infty}\left\|\partial_j^{k+\omega}\psi\right\|_{\mathbf{L}^2} + \left\|\partial_j^{m-k+1}\phi\right\|_{\mathbf{L}^\infty}\left\|\partial_j^{k}\psi\right\|_{\mathbf{L}^2}\right)$$

$$\le C\left\|\langle\Delta\rangle^{\frac{[\delta]+1}{2}}\phi\right\|_{\mathbf{L}^\infty}\left\|\langle\Delta\rangle^{\frac{\delta}{2}}\psi\right\|_{\mathbf{L}^2}.$$

Therefore we get

$$\begin{aligned}&\left\|(-\Delta)^{\frac{\delta}{2}}\left(\langle\xi\rangle e^{-\frac{t}{2}}L(t,\xi)\chi_2(\xi)\right)\left(\langle\xi\rangle^{-1}\widehat{\psi}(\xi)\right)\right\|_{\mathbf{L}^2}\\ \le\ & Ce^{-\frac{t}{4}}\left\|\langle\xi\rangle^{-1}\langle\Delta\rangle^{\frac{\delta}{2}}\widehat{\psi}(\xi)\right\|_{\mathbf{L}^2} \le Ce^{-\frac{t}{4}}\left\|\langle\Delta\rangle^{-\frac{1}{2}}\langle\cdot\rangle^{\delta}\psi\right\|_{\mathbf{L}^2}.\end{aligned} \tag{2.7}$$

Substitution of (2.6) and (2.7) into (2.3) yields the third estimate of the lemma. Lemma 2.1 is proved. □

In the same manner we estimate the operator

$$\widetilde{\mathcal{G}}(t) = \left(\partial_t + \frac{1}{2}\right)\mathcal{G}(t) = \mathcal{F}^{-1}\widetilde{L}(t,\xi)\mathcal{F},$$

$$\widetilde{L}(t,\xi) = e^{-\frac{t}{2}}\cos\left(t\sqrt{|\xi|^2 - \frac{1}{4}}\right)$$

to obtain the following result.

Lemma 2.2. *The estimates*

$$\left\||\nabla|^{\alpha}\widetilde{\mathcal{G}}(t)\psi\right\|_{\mathbf{L}^2} \le C\langle t\rangle^{-\frac{\alpha}{2}-\frac{n}{4}}\|\psi\|_{\mathbf{L}^1} + Ce^{-\frac{t}{4}}\left\||\nabla|^{\alpha}\psi\right\|_{\mathbf{L}^2}$$

and

$$\left\||\cdot|^{\delta}\widetilde{\mathcal{G}}(t)\psi\right\|_{\mathbf{L}^2} \le C\langle t\rangle^{\frac{\delta}{2}-\frac{n}{4}}\|\psi\|_{\mathbf{L}^1} + C\left\|\langle\cdot\rangle^{\delta}\psi\right\|_{\mathbf{L}^2}$$

are true for all $t > 0$, where $\delta > \frac{n}{2}$ and $\alpha \ge 0$, provided that the right-hand sides are finite.

We now define two norms

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0}\left(\langle t\rangle^{\frac{n}{4}}\|\phi(t)\|_{\mathbf{L}^2} + \langle t\rangle^{\frac{n}{4}+\frac{\alpha}{2}}\left\||\nabla|^{\alpha}\phi(t)\right\|_{\mathbf{L}^2} + \langle t\rangle^{\frac{n}{4}-\frac{\delta}{2}}\left\||\cdot|^{\delta}\phi(t)\right\|_{\mathbf{L}^2}\right),$$

$$\|\psi\|_{\mathbf{Y}} = \sup_{t>0}\Big(\langle t\rangle^{\eta}\left\||\nabla|^{[\alpha]}\psi(t)\right\|_{\mathbf{L}^{\widetilde{q}}} + \sup_{1\le r\le\widetilde{q}}\langle t\rangle^{\frac{n}{2}\left(\rho-\frac{1}{r}\right)}\|\psi(t)\|_{\mathbf{L}^r}$$
$$+ \langle t\rangle^{(\rho-1)\left(\frac{\mu}{2}+\frac{n}{4}\right)+\frac{n}{4}-\frac{\delta}{2}}\left\|\langle\cdot\rangle^{\delta}\psi(t)\right\|_{\mathbf{L}^q}\Big),$$

where $\delta > \frac{n}{2}$, $\alpha \ge \mu$, and $[\alpha] \le \rho$; $\mu = \frac{n}{2} - \frac{1}{\rho-1}$, $\widetilde{q} = \frac{2n}{n+2+2[\alpha]-2\alpha}$, $q = \frac{2n}{n+2}$, and $\eta = \frac{\alpha}{2} + \frac{n}{2}\rho - \frac{n}{4} - \frac{1}{2}$ for $n \ge 2$; and $\mu = \frac{1}{2} - \frac{1}{2(\rho-1)}$, $\widetilde{q} = 2$, $q = 1$, and $\eta = \frac{\rho}{2} - \frac{1}{4}$ for $n = 1$.

The following lemma states the interpolation inequalities.

Lemma 2.3. *The estimates*

$$\sup_{t>0}\left(\sup_{0\le\beta\le\alpha}\langle t\rangle^{\frac{n}{4}+\frac{\beta}{2}}\left\||\nabla|^{\beta}\phi(t)\right\|_{\mathbf{L}^2} + \sup_{0\le\sigma\le\delta}\langle t\rangle^{\frac{n}{4}-\frac{\sigma}{2}}\left\||\cdot|^{\sigma}\phi(t)\right\|_{\mathbf{L}^2}\right) \le C\|\phi\|_{\mathbf{X}}$$

and

$$\sup_{t>0}\sup_{1\le r\le\frac{2n}{n-2\alpha}}\langle t\rangle^{\frac{n}{2}\left(1-\frac{1}{r}\right)}\|\phi(t)\|_{\mathbf{L}^r} \le C\|\phi\|_{\mathbf{X}}$$

are valid, provided that the right-hand sides are bounded.

Proof. By the Hölder inequality we have

$$\left\||\nabla|^{\beta}\phi\right\|_{\mathbf{L}^2} = \left\||\xi|^{\beta}\widehat{\phi}\right\|_{\mathbf{L}^2} \le \|\widehat{\phi}\|_{\mathbf{L}^2}^{1-\frac{\beta}{\alpha}}\left\||\xi|^{\alpha}\widehat{\phi}\right\|_{\mathbf{L}^2}^{\frac{\beta}{\alpha}} = \|\phi\|_{\mathbf{L}^2}^{1-\frac{\beta}{\alpha}}\left\||\nabla|^{\alpha}\phi\right\|_{\mathbf{L}^2}^{\frac{\beta}{\alpha}};$$

therefore,

$$\langle t\rangle^{\frac{n}{4}+\frac{\beta}{2}}\left\||\nabla|^{\beta}\phi\right\|_{\mathbf{L}^2} \le \left(\langle t\rangle^{\frac{n}{4}}\|\phi(t)\|_{\mathbf{L}^2}\right)^{1-\frac{\beta}{\alpha}}\left(\langle t\rangle^{\frac{n}{4}+\frac{\alpha}{2}}\left\||\nabla|^{\alpha}\phi(t)\right\|_{\mathbf{L}^2}\right)^{\frac{\beta}{\alpha}} \le \|\phi\|_{\mathbf{X}}.$$

In the same manner we obtain

$$\langle t\rangle^{\frac{n}{4}-\frac{\sigma}{2}}\left\||\cdot|^{\sigma}\phi(t)\right\|_{\mathbf{L}^2} \le \left(\langle t\rangle^{\frac{n}{4}}\|\phi(t)\|_{\mathbf{L}^2}\right)^{1-\frac{\sigma}{\delta}}\left(\langle t\rangle^{\frac{n}{4}-\frac{\delta}{2}}\left\||\cdot|^{\delta}\phi(t)\right\|_{\mathbf{L}^2}\right)^{\frac{\sigma}{\delta}} \le \|\phi\|_{\mathbf{X}}.$$

Thus the first estimate of the lemma is true.

If $2 \le r \le \frac{2n}{n-2\alpha}$ we apply the Sobolev imbedding theorem with $\beta = \frac{n}{2} - \frac{n}{r} \in [0,\alpha]$,

$$\|\phi\|_{\mathbf{L}^r} \le C\left\||\nabla|^{\beta}\phi\right\|_{\mathbf{L}^2};$$

therefore,

$$\langle t\rangle^{\frac{n}{2}\left(1-\frac{1}{r}\right)}\|\phi(t)\|_{\mathbf{L}^r} \le C\langle t\rangle^{\frac{n}{2}\left(1-\frac{1}{r}\right)}\left\||\nabla|^{\beta}\phi(t)\right\|_{\mathbf{L}^2} \le C\|u\|_{\mathbf{X}}.$$

And for the case $1 \leq r \leq 2$ applying estimate (2.5) we have

$$\begin{aligned} & \langle t\rangle^{\frac{n}{2}\left(1-\frac{1}{r}\right)} \left\|\phi(t)\right\|_{\mathbf{L}^r} \\ \leq \ & C\left(\langle t\rangle^{\frac{n}{4}-\frac{\delta}{2}} \left\| |\cdot|^{\delta}\phi\right\|_{\mathbf{L}^2}\right)^{\frac{n}{\delta}\left(\frac{1}{r}-\frac{1}{2}\right)} \left(\langle t\rangle^{\frac{n}{4}} \left\|\phi\right\|_{\mathbf{L}^2}\right)^{1-\frac{n}{\delta}\left(\frac{1}{r}-\frac{1}{2}\right)} \leq C\left\|u\right\|_{\mathbf{X}}. \end{aligned}$$

The second estimate of the lemma is true. Lemma 2.3 is proved. □

In the next lemma we estimate the integral in the Duhamel formula (2.2).

Lemma 2.4. *Suppose that $\delta > \frac{n}{2}$ and $\rho > 1 + \frac{2}{n}$. Then the estimate*

$$\left\| \int_0^t \mathcal{G}(t-\tau)\,\psi(\tau)\,d\tau \right\|_{\mathbf{X}} \leq C\left\|\psi\right\|_{\mathbf{Y}}$$

is true.

Proof. By the Sobolev embedding theorem we have

$$\left\| |\nabla|^{\alpha} \langle\Delta\rangle^{-\frac{1}{2}} \psi\right\|_{\mathbf{L}^2} \leq C\left\| |\nabla|^{[\alpha]}\psi\right\|_{\mathbf{L}^{\widetilde{q}}},$$

where $\widetilde{q} = \frac{2n}{n+2+2[\alpha]-2\alpha}$ for $n \geq 2$ and $\widetilde{q} = 2$, $[\alpha] = 0$ for $n = 1$. Applying the first estimate of Lemma 2.1 with $\beta = 0$ and $q = 1$ in the case $0 < \tau < \frac{t}{2}$, and $\beta = [\alpha]$ and $q = \widetilde{q}$ in the case $\frac{t}{2} < \tau < t$, we obtain, taking $\zeta = \frac{1}{2}$ for $n \geq 2$ and $\zeta = \frac{\alpha}{2}$ for $n = 1$,

$$\begin{aligned} & \left\| |\nabla|^{\alpha} \int_0^t \mathcal{G}(t-\tau)\,\psi(\tau)\,d\tau\right\|_{\mathbf{L}^2} \\ \leq \ & C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{\alpha}{2}-\frac{n}{4}} \left(\left\|\psi(\tau)\right\|_{\mathbf{L}^1} + \left\| |\nabla|^{[\alpha]}\psi(\tau)\right\|_{\mathbf{L}^{\widetilde{q}}}\right) d\tau \\ & + C\int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-\frac{\alpha-[\alpha]}{2}-\frac{n}{2}\left(\frac{1}{\widetilde{q}}-\frac{1}{2}\right)} \left\| |\nabla|^{[\alpha]}\psi(\tau)\right\|_{\mathbf{L}^{\widetilde{q}}} d\tau \\ \leq \ & C\left\|\psi\right\|_{\mathbf{Y}} \int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{\alpha}{2}-\frac{n}{4}} \langle\tau\rangle^{-\frac{n}{2}(\rho-1)}\,d\tau \\ & + C\left\|\psi\right\|_{\mathbf{Y}} \int_{\frac{t}{2}}^t \langle t-\tau\rangle^{-\zeta} \langle\tau\rangle^{-\eta}\,d\tau \leq C\langle t\rangle^{-\frac{\alpha}{2}-\frac{n}{4}} \left\|\psi\right\|_{\mathbf{Y}} \end{aligned}$$

for all $t > 0$, since $\rho > 1 + \frac{2}{n}$. Similarly, we find

$$\left\| \int_0^t \mathcal{G}(t-\tau)\,\psi(\tau)\,d\tau\right\|_{\mathbf{L}^2} \leq C\int_0^{\frac{t}{2}} \langle t-\tau\rangle^{-\frac{n}{4}} \left(\left\|\psi(\tau)\right\|_{\mathbf{L}^1} + \left\|\psi(\tau)\right\|_{\mathbf{L}^{\widetilde{q}}}\right) d\tau$$

$$+ C\int_{\frac{t}{2}}^{t}\langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\widetilde{q}}-\frac{1}{2}\right)}\left\|\psi(\tau)\right\|_{\mathbf{L}^{\widetilde{q}}}d\tau \le C\left\|\psi\right\|_{\mathbf{Y}}\int_{0}^{\frac{t}{2}}\langle t-\tau\rangle^{-\frac{n}{4}}\langle\tau\rangle^{-\frac{n}{2}(\rho-1)}d\tau$$

$$+C\left\|\psi\right\|_{\mathbf{Y}}\int_{\frac{t}{2}}^{t}\langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{\widetilde{q}}-\frac{1}{2}\right)}\langle\tau\rangle^{-\frac{n}{2}\left(\rho-\frac{1}{\widetilde{q}}\right)}d\tau \le C\langle t\rangle^{-\frac{n}{4}}\left\|\psi\right\|_{\mathbf{Y}}.$$

Finally, by the second estimate of Lemma 2.1, using the Sobolev embedding theorem

$$\left\|\langle\Delta\rangle^{-\frac{1}{2}}\langle\cdot\rangle^{\delta}\psi\right\|_{\mathbf{L}^2} \le C\left\|\langle\cdot\rangle^{\delta}\psi\right\|_{\mathbf{L}^q}$$

with $q=\max(1,\frac{2n}{n+2})$, we have

$$\begin{aligned}
&\left\||\cdot|^{\delta}\int_0^t \mathcal{G}(t-\tau)\psi(\tau)d\tau\right\|_{\mathbf{L}^2} \le C\int_0^t\langle t-\tau\rangle^{\frac{\delta}{2}-\frac{n}{4}}\left\|\psi(\tau)\right\|_{\mathbf{L}^1}d\tau\\
&+C\int_0^t\langle t-\tau\rangle^{-\frac{n}{2}\left(\frac{1}{q}-\frac{1}{2}\right)}\left\|\langle\cdot\rangle^{\delta}\psi(\tau)\right\|_{\mathbf{L}^q}d\tau\\
\le\ & C\left\|\psi\right\|_{\mathbf{Y}}\int_0^t\langle t-\tau\rangle^{\frac{\delta}{2}-\frac{n}{4}}\langle\tau\rangle^{-\frac{n}{2}(\rho-1)}d\tau\\
&+C\left\|\psi\right\|_{\mathbf{Y}}\int_0^t\langle t-\tau\rangle^{-\frac{\min(2,n)}{4}}\langle\tau\rangle^{-(\rho-1)\left(\frac{\mu}{2}+\frac{n}{4}\right)+\frac{\delta}{2}-\frac{n}{4}}d\tau\\
\le\ & C\langle t\rangle^{\frac{\delta}{2}-\frac{n}{4}}\left\|\psi\right\|_{\mathbf{Y}}.
\end{aligned}$$

Lemma 2.4 is proved. □

Lemma 2.5. *The estimate*

$$\left\|\mathcal{N}(u)\right\|_{\mathbf{Y}} \le C\left\|u\right\|_{\mathbf{X}}^{\rho}$$

is true, provided that the right-hand side is bounded.

Proof. As above we choose $\delta>\frac{n}{2}$, $\mu=\frac{n}{2}-\frac{1}{\rho-1}$, $\widetilde{q}=\frac{2n}{n+2+2[\alpha]-2\alpha}$ and $q=\frac{2n}{n+2}$ for $n\ge 2$, and $\mu=\frac{1}{2}-\frac{1}{2(\rho-1)}>0$, $\widetilde{q}=2$ and $q=1$ for $n=1$.

By the Sobolev imbedding theorem we have with $r=n(\rho-1)$ for $n\ge 2$ and $r=2(\rho-1)$ for $n=1$

$$\left\|u\right\|_{\mathbf{L}^r} \le C\left\||\nabla|^{\mu}u\right\|_{\mathbf{L}^2},$$

whence by the Hölder inequality we have taking $q=\max(1,\frac{2n}{2+n})$

$$\begin{aligned}
\left\|\langle\cdot\rangle^{\delta}\mathcal{N}(u)\right\|_{\mathbf{L}^q} &\le C\left\|\langle\cdot\rangle^{\delta}|u|^{\rho}\right\|_{\mathbf{L}^q} \le C\left\|\langle\cdot\rangle^{\delta}u\right\|_{\mathbf{L}^2}\left\|u\right\|_{\mathbf{L}^r}^{\rho-1}\\
&\le C\left\|\langle\cdot\rangle^{\delta}u\right\|_{\mathbf{L}^2}\left\||\nabla|^{\mu}u\right\|_{\mathbf{L}^2}^{\rho-1}. \qquad (2.8)
\end{aligned}$$

Using (2.8) we get

$$\langle t\rangle^{(\rho-1)\left(\frac{\mu}{2}+\frac{n}{4}\right)+\frac{n}{4}-\frac{\delta}{2}}\left\|\langle\cdot\rangle^{\delta}\mathcal{N}(u)\right\|_{\mathbf{L}^{q}} \\ \leq C\langle t\rangle^{\frac{n}{4}-\frac{\delta}{2}}\left\|\langle\cdot\rangle^{\delta}u\right\|_{\mathbf{L}^{2}}\left(\langle t\rangle^{\frac{\mu}{2}+\frac{n}{4}}\left\||\nabla|^{\mu}u\right\|_{\mathbf{L}^{2}}\right)^{\rho-1}\leq C\|u\|_{\mathbf{X}}^{\rho}. \tag{2.9}$$

We now consider the estimates of the norm $\left\||\nabla|^{[\alpha]}\mathcal{N}(u)\right\|_{\mathbf{L}^{\widetilde{q}}}$, where $\alpha\geq\mu$; $\widetilde{q}=\frac{2n}{n+2+2[\alpha]-2\alpha}$ for $n\geq 2$ and $\widetilde{q}=2$ for $n=1$.

First let us consider the case $[\alpha]=0$; then via Lemma 2.3 we have by the Sobolev imbedding theorem with $\beta=\frac{n}{2}-\frac{n}{\rho\widetilde{q}}$

$$\langle t\rangle^{\eta}\left\||\nabla|^{[\alpha]}\mathcal{N}(u)\right\|_{\mathbf{L}^{\widetilde{q}}}=\langle t\rangle^{\eta}\|u\|_{\mathbf{L}^{\rho\widetilde{q}}}^{\rho} \\ \leq C\left(\langle t\rangle^{\frac{\beta}{2}+\frac{n}{4}}\left\||\nabla|^{\beta}u\right\|_{\mathbf{L}^{2}}\right)^{\rho}\leq C\|u\|_{\mathbf{X}}^{\rho}. \tag{2.10}$$

Now we consider the case $[\alpha]\geq 1$, $n\geq 2$. By the Leibniz rule and by the Hölder inequality with $1\leq q_j\leq\infty$ such that $\sum_{j=0}^{[\alpha]}\frac{1}{q_j}=\frac{1}{\widetilde{q}}$, we get (provided that $[\alpha]\leq\rho$)

$$\left\||\nabla|^{[\alpha]}\mathcal{N}(u)\right\|_{\mathbf{L}^{\widetilde{q}}}\leq C\left\|u^{\rho-[\alpha]}\right\|_{\mathbf{L}^{q_0}}\sum_{k_j\geq 0,\ k_1+\cdots+k_{[\alpha]}=[\alpha]}\prod_{j=1}^{[\alpha]}\left\||\nabla|^{k_j}u\right\|_{\mathbf{L}^{q_j}}.$$

We now choose $\frac{1}{q_j}=\frac{1}{2}-\frac{\mu+\beta_j-k_j}{n}$ and $\frac{1}{(\rho-[\alpha])q_0}=\frac{1}{2}-\frac{\mu}{n}$, so that $0\leq\beta_j<k_j+\frac{1}{\rho-1}$ are such that $\sum_{j=1}^{[\alpha]}\beta_j=\alpha-\mu$; then we get

$$\sum_{j=0}^{[\alpha]}\frac{1}{q_j}=\frac{[\alpha]-\alpha}{n}+(\rho-1)\left(\frac{1}{2}-\frac{\mu}{n}\right)+\frac{1}{2}=\frac{1}{\widetilde{q}},$$

since $\frac{1}{2}-\frac{\mu}{n}=\frac{1}{n(\rho-1)}$. Then by the Sobolev imbedding theorem we have

$$\left\||\nabla|^{k_j}u\right\|_{\mathbf{L}^{q_j}}\leq C\left\||\nabla|^{\mu+\beta_j}u\right\|_{\mathbf{L}^{2}}$$

and

$$\left\|u^{\rho-[\alpha]}\right\|_{\mathbf{L}^{q_0}}\leq C\|u\|_{\mathbf{L}^{(\rho-[\alpha])q_0}}^{\rho-[\alpha]}\leq C\left\||\nabla|^{\mu}u\right\|_{\mathbf{L}^{2}}^{\rho-[\alpha]}.$$

Therefore we obtain

$$\left\||\nabla|^{[\alpha]}\mathcal{N}(u)\right\|_{\mathbf{L}^{\widetilde{q}}}\leq C\left\||\nabla|^{\mu}u\right\|_{\mathbf{L}^{2}}^{\rho-[\alpha]}\prod_{j=1}^{[\alpha]}\left\||\nabla|^{\mu+\beta_j}u\right\|_{\mathbf{L}^{2}},$$

whence by Lemma 2.2

$$\langle t\rangle^{\eta}\left\||\nabla|^{[\alpha]}\mathcal{N}(u)\right\|_{\mathbf{L}^{\widetilde{q}}}\le C\left(\langle t\rangle^{\frac{\mu}{2}+\frac{n}{4}}\left\||\nabla|^{\mu}u\right\|_{\mathbf{L}^2}\right)^{\rho-[\alpha]}$$
$$\times\prod_{j=1}^{[\alpha]}\langle t\rangle^{\frac{\mu}{2}+\frac{\beta_j}{2}+\frac{n}{4}}\left\||\nabla|^{\mu+\beta_j}u\right\|_{\mathbf{L}^2}\le C\|u\|_{\mathbf{X}}^{\rho}. \qquad (2.11)$$

Now the second estimate of Lemma 2.3 yields

$$\langle t\rangle^{\frac{n}{2}(\rho-\frac{1}{r})}\left\|\mathcal{N}(u(t))\right\|_{\mathbf{L}^r}\le\left(\langle t\rangle^{\frac{n}{2}(1-\frac{1}{\rho r})}\left\|u(t)\right\|_{\mathbf{L}^{\rho r}}\right)^{\rho}\le C\|u\|_{\mathbf{X}}^{\rho} \qquad (2.12)$$

for all $1\le r\le\widetilde{q}$. Collecting together estimates (2.8) through (2.12) we obtain the result of the lemma. Lemma 2.5 is proved. □

The following lemma says that the asymptotic behavior of solutions to the linear Cauchy problem (2.1) is similar to that for the heat equation. Denote the heat kernel $G_0(t,x)=(4\pi t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t}}$, $\theta=(2\pi)^{-\frac{n}{2}}\int_{\mathbf{R}^n}\psi(x)\,dx$, and $\widetilde{\mathcal{G}}(t)=\left(\partial_t+\frac{1}{2}\right)\mathcal{G}(t)$.

Lemma 2.6. *The estimates*

$$\left\||\nabla|^{\alpha}\left(\mathcal{G}(t)\psi-\theta G_0(t)\right)\right\|_{\mathbf{L}^2}$$
$$\le C\langle t\rangle^{-\frac{\alpha}{2}-\frac{n}{4}-1}\|\psi\|_{\mathbf{L}^1}+C\langle t\rangle^{-\frac{\alpha}{2}-\frac{\delta}{2}}\left\|\langle\cdot\rangle^{\delta}\psi\right\|_{\mathbf{L}^2}+Ce^{-\frac{t}{2}}\left\||\nabla|^{\alpha}\langle\Delta\rangle^{-\frac{1}{2}}\psi\right\|_{\mathbf{L}^2}$$

$$\left\||\nabla|^{\alpha}\left(\widetilde{\mathcal{G}}(t)\psi-\theta G_0(t)\right)\right\|_{\mathbf{L}^2}\le C\langle t\rangle^{-\frac{\alpha}{2}-\frac{n}{4}-1}\|\psi\|_{\mathbf{L}^1}$$
$$+C\langle t\rangle^{-\frac{\alpha}{2}-\frac{\delta}{2}}\left\|\langle\cdot\rangle^{\delta}\psi\right\|_{\mathbf{L}^2}+Ce^{-\frac{t}{4}}\left\||\nabla|^{\alpha}\psi\right\|_{\mathbf{L}^2}$$

are true for all $t>0$, where $\frac{n}{2}<\delta<n$, $\alpha\ge0$, provided that the right-hand sides are finite.

Proof. Taking into account the estimates

$$\left\||\xi|^{\alpha}\left(e^{-\frac{t}{2}}L(t,\xi)-e^{-t\frac{\xi^2}{2}}\right)\right\|_{\mathbf{L}^r\left(|\xi|\le1\right)}\le C\langle t\rangle^{-\frac{\alpha}{2}-\frac{n}{2r}-1}$$

and

$$\left\|\langle\xi\rangle L(t,\xi)\right\|_{\mathbf{L}^{\infty}\left(|\xi|\ge1\right)}\le C$$

for all $t>0$, we get for $1\le q\le2$

$$\left\||\nabla|^{\alpha}\left(\mathcal{G}(t)\psi-\theta G_0(t)\right)\right\|_{\mathbf{L}^2}$$
$$\le Ce^{-\frac{t}{2}}\left\||\xi|^{\alpha}\left(L(t,\xi)-e^{-t\frac{(\xi^2-1)}{2}}\right)\widehat{\psi}(\xi)\right\|_{\mathbf{L}^2}+C\left\||\xi|^{\alpha}e^{-t\frac{\xi^2}{2}}\left(\widehat{\psi}(\xi)-\widehat{\psi}(0)\right)\right\|_{\mathbf{L}^2}$$

$$\leq Ce^{-\frac{t}{2}}\left\||\xi|^{\alpha}\left(L(t,\xi)-e^{-t\frac{(\xi^2-1)}{2}}\right)\right\|_{\mathbf{L}^{\frac{2q}{2-q}}(|\xi|\leq 1)}\left\|\widehat{\psi}(\xi)\right\|_{\mathbf{L}^{\frac{q}{q-1}}(|\xi|\leq 1)}$$
$$+Ce^{-\frac{t}{2}}\left\|\langle\xi\rangle L(t,\xi)\right\|_{\mathbf{L}^{\infty}(|\xi|\geq 1)}\left\||\xi|^{\alpha}\langle\xi\rangle^{-1}\widehat{\psi}(\xi)\right\|_{\mathbf{L}^{2}(|\xi|\geq 1)}$$
$$+C\langle t\rangle^{-\frac{\alpha}{2}-\frac{n}{4}}\left\|\langle\cdot\rangle^{\delta}\psi\right\|_{\mathbf{L}^2}$$
$$\leq C\langle t\rangle^{-\frac{\alpha}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{2})-1}\|\psi\|_{\mathbf{L}^q}+Ce^{-\frac{t}{2}}\||\nabla|^{\alpha}\langle\Delta\rangle^{-\frac{1}{2}}\psi\|_{\mathbf{L}^2}+C\langle t\rangle^{-\frac{\alpha}{2}-\frac{\delta}{2}}\|\langle\cdot\rangle^{\delta}\psi\|_{\mathbf{L}^2};$$

therefore, the first estimate of the lemma is true. Now since

$$\widetilde{L}(t,\xi)=(\partial_t+\frac{1}{2})(e^{-\frac{t}{2}}L(t,\xi))=e^{-\frac{t}{2}}\cos\left(t\sqrt{|\xi|^2-\tfrac{1}{4}}\right)$$

the estimates

$$\left\||\xi|^{\alpha}\left(\widetilde{L}(t,\xi)-e^{-t\frac{\xi^2}{2}}\right)\right\|_{\mathbf{L}^r(|\xi|\leq 1)}\leq C\langle t\rangle^{-\frac{\alpha}{2}-\frac{n}{2r}-1}$$

and

$$\left\|\widetilde{L}(t,\xi)\right\|_{\mathbf{L}^{\infty}(|\xi|\geq 1)}\leq Ce^{-\frac{t}{2}}$$

are valid for all $t>0$. Then the second estimate of the lemma follows in the same way as in the proof of the first one. Lemma 2.6 is proved. □

3. Proof of Theorem 1.1

We rewrite the Cauchy problem (1.1) in the form of the integral equation

$$u(t)=\widetilde{\mathcal{G}}(t)\varepsilon u_0+\mathcal{G}(t)\varepsilon u_1+\int_0^t\mathcal{G}(t-\tau)\mathcal{N}(u)(\tau)d\tau, \qquad (3.1)$$

where the Green's operator $\mathcal{G}(t)$ is defined in Section 2 and $\widetilde{\mathcal{G}}(t)=(\partial_t+\frac{1}{2})\mathcal{G}(t)$. We apply the contraction-mapping principle in a ball

$$\mathbf{X}_{\varepsilon}=\left\{v\in\mathbf{L}^{\infty}\left(0,\infty;\mathbf{L}^2(\mathbf{R})\right):\|v\|_{\mathbf{X}}\leq C\varepsilon\right\}.$$

Let us define the transformation

$$\mathcal{M}v(t)=\widetilde{\mathcal{G}}(t)\varepsilon u_0+\mathcal{G}(t)\varepsilon u_1+\int_0^t\mathcal{G}(t-\tau)\mathcal{N}(v)(\tau)\,d\tau. \qquad (3.2)$$

Applying estimates of Lemma 2.1 and Lemma 2.2 we get

$$\begin{aligned}
\left\||\nabla|^{\alpha}\mathcal{G}(t)\varepsilon u_1\right\|_{\mathbf{L}^2} &\leq C\varepsilon\langle t\rangle^{-\frac{\alpha}{2}-\frac{n}{4}}\left(\|u_1\|_{\mathbf{H}^{0,\delta}}+\|u_1\|_{\mathbf{H}^{\alpha-1,0}}\right),\\
\left\||\nabla|^{\alpha}\widetilde{\mathcal{G}}(t)\varepsilon u_0\right\|_{\mathbf{L}^2} &\leq C\varepsilon\langle t\rangle^{-\frac{\alpha}{2}-\frac{n}{4}}\left(\|u_0\|_{\mathbf{H}^{0,\delta}}+\|u_0\|_{\mathbf{H}^{\alpha,0}}\right),\\
\left\||\cdot|^{\delta}\mathcal{G}(t)\varepsilon u_1\right\|_{\mathbf{L}^2} &\leq C\varepsilon\langle t\rangle^{\frac{\delta}{2}-\frac{n}{4}}\|u_1\|_{\mathbf{H}^{0,\delta}},\\
\left\||\cdot|^{\delta}\widetilde{\mathcal{G}}(t)\varepsilon u_0\right\|_{\mathbf{L}^2} &\leq C\varepsilon\langle t\rangle^{\frac{\delta}{2}-\frac{n}{4}}\|u_0\|_{\mathbf{H}^{0,\delta}}.
\end{aligned}$$

Let $v \in \mathbf{X}_\varepsilon$ for sufficiently small $\varepsilon > 0$. Applying Lemma 2.4 and Lemma 2.5 we have

$$\|\mathcal{M}v\,(t)\|_{\mathbf{X}} \le C\varepsilon\left(\|u_1\|_{\mathbf{H}^{0,\delta}} + \|u_1\|_{\mathbf{H}^{\alpha-1,0}}\right) + C\varepsilon\left(\|u_0\|_{\mathbf{H}^{0,\delta}} + \|u_0\|_{\mathbf{H}^{\alpha,0}}\right)$$
$$+ \left\| \int_0^t \mathcal{G}(t-\tau)\mathcal{N}(v)(\tau)d\tau \right\|_{\mathbf{X}} \le C\varepsilon + C\,\|\mathcal{N}\,(v)\|_{\mathbf{Y}} \le C\varepsilon + C\,\|v\|_{\mathbf{X}}^{\rho} \le C\varepsilon.$$

Thus we see that $\mathcal{M}$ transforms $\mathbf{X}_\varepsilon$ into itself. In the same manner we estimate the difference

$$\|\mathcal{M}\,(v\,(t) - w\,(t))\|_{\mathbf{X}} \le \left\| \int_0^t \mathcal{G}\,(t-\tau)\,(\mathcal{N}\,(v)\,(\tau) - \mathcal{N}\,(w)\,(\tau))\,d\tau \right\|_{\mathbf{X}}$$
$$\le C\|\mathcal{N}(v) - \mathcal{N}(w)\|_{\mathbf{Y}} \le C(\|v\|_{\mathbf{X}}^{\rho-1} + \|w\|_{\mathbf{X}}^{\rho-1})\|v-w\|_{\mathbf{X}} \le C\varepsilon^{\rho-1}\|v-w\|_{\mathbf{X}}.$$

Thus we see that $\mathcal{M}$ is a contraction mapping in $\mathbf{X}_\varepsilon$. Hence there exists a unique solution $u \in \mathbf{C}\left([0,\infty)\,;\mathbf{H}^{\alpha,0} \cap \mathbf{H}^{0,\delta}\right)$ to the Cauchy problem (1.1).

Let us prove the asymptotics (1.2). By Lemma 2.6 and the Sobolev imbedding theorem we have for $2 \le p \le \frac{2n}{n-2\alpha}$ if $\alpha < \frac{n}{2}$; $2 \le p < \infty$ if $\alpha = \frac{n}{2}$; and $2 \le p \le \infty$ if $\alpha > \frac{n}{2}$

$$\begin{aligned}
&\left\|\mathcal{G}\,(t)\,\varepsilon u_1 - \varepsilon\theta_1 G_0\,(t)\,\right\|_{\mathbf{L}^p} \le C\left\|\left|\nabla\right|^{\alpha}\left(\mathcal{G}\,(t)\,\varepsilon u_1 - \varepsilon\theta_1 G_0\,(t)\right)\right\|_{\mathbf{L}^2} \\
\le\quad & C\,\langle t\rangle^{-\frac{n}{2}(1-\frac{2}{p})}\Big(\langle t\rangle^{-\frac{n}{4}-1}\left\|u_1\right\|_{\mathbf{L}^1} + C\langle t\rangle^{-\frac{\delta}{2}}\left\|u_1\right\|_{\mathbf{H}^{0,\delta}} \\
&+\langle t\rangle^{-\max(\frac{n}{4}+1,\frac{\delta}{2})}\left\|u_1\right\|_{\mathbf{H}^{\alpha-1,0}}\Big) \\
\le\quad & C\,\langle t\rangle^{-\frac{n}{2}(1-\frac{1}{p})}\Big(\langle t\rangle^{-1}\left\|u_1\right\|_{\mathbf{L}^1} + C\,\langle t\rangle^{-\frac{\delta}{2}+\frac{n}{4}}\left\|u_1\right\|_{\mathbf{H}^{0,\delta}} \\
&+\langle t\rangle^{-\max\left(1,\frac{\delta}{2}-\frac{n}{4}\right)}\left\|u_1\right\|_{\mathbf{H}^{\alpha-1,0}}\Big)
\end{aligned}$$

and

$$\begin{aligned}
\left\|\widetilde{\mathcal{G}}(t)\varepsilon u_0 - \varepsilon\theta_0 G_0(t)\right\|_{\mathbf{L}^p} &\le C\langle t\rangle^{-\frac{n}{2}(1-\frac{1}{p})}(\langle t\rangle^{-1}\left\|u_1\right\|_{\mathbf{L}^1} \\
&+ C\langle t\rangle^{-\frac{\delta}{2}+\frac{n}{4}}\left\|u_1\right\|_{\mathbf{H}^{0,\delta}} + \langle t\rangle^{-\max(1,\frac{\delta}{2}-\frac{n}{4})}\left\|u_0\right\|_{\mathbf{H}^{\alpha,0}})
\end{aligned}$$

for all $t > 0$, where $\theta_j = (2\pi)^{-\frac{n}{2}}\int_{\mathbf{R}^n} u_j(x)dx$, $j = 0,1$, and $G_0(t,x) = (4\pi t)^{-\frac{n}{2}}e^{-\frac{|x|^2}{4t}}$. Now consider the difference

$$\begin{aligned}
&\int_0^t \mathcal{G}\,(t-\tau)\,\mathcal{N}\,(u\,(\tau))\,d\tau - G_0\,(t,x)\int_0^\infty\int_{\mathbf{R}^n}\mathcal{N}\,(u\,(\tau,x))\,dxd\tau \\
&= \int_{\frac{t}{2}}^t \mathcal{G}\,(t-\tau)\,\mathcal{N}\,(u\,(\tau))\,d\tau
\end{aligned}$$

$$+\int_0^{\frac{t}{2}}\left(\mathcal{G}(t-\tau)\mathcal{N}(u(\tau))-G_0(t-\tau,x)\int_{\mathbf{R}^n}\mathcal{N}(u(\tau,x))\,dx\right)d\tau$$
$$+\int_0^{\frac{t}{2}}(G_0(t-\tau,x)-G_0(t,x))\int_{\mathbf{R}^n}\mathcal{N}(u(\tau,x))\,dxd\tau$$
$$-G_0(t,x)\int_{\frac{t}{2}}^{\infty}\int_{\mathbf{R}^n}\mathcal{N}(u(\tau,x))\,dxd\tau.$$

We use Lemma 2.1 with $\beta=[\alpha]$ and $q=\widetilde{q}$ in the first term to obtain

$$\left\|\int_{\frac{t}{2}}^{t}\mathcal{G}(t-\tau)\mathcal{N}(u(\tau))d\tau\right\|_{\mathbf{L}^p}\le C\left\|\mathcal{N}(u)\right\|_{\mathbf{Y}}\int_{\frac{t}{2}}^{t}\langle t-\tau\rangle^{-\frac{\alpha-[\alpha]}{2}-\frac{n}{2}(\frac{1}{\widetilde{q}}-\frac{1}{2})}\langle\tau\rangle^{-\eta}d\tau$$
$$\le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\min\left(1,\frac{n}{2}(\rho-1)-1\right)}$$

since $\widetilde{q}=\frac{2n}{n+2+2[\alpha]-2\alpha}$ and $\eta=\frac{\alpha}{2}+\frac{n}{2}\rho-\frac{n}{4}-\frac{1}{2}$ for $n\ge 2$, and $\widetilde{q}=2$, $\eta=\frac{\rho}{2}-\frac{1}{4}$ and $[\alpha]=0$ for $n=1$. We apply Lemma 2.6 to get for the second term

$$\begin{aligned}&\left\|\int_0^{\frac{t}{2}}\left(\mathcal{G}(t-\tau)\mathcal{N}(u(\tau))-G_0(t-\tau,x)\int_{\mathbf{R}^n}\mathcal{N}(u(\tau,x))\,dx\right)d\tau\right\|_{\mathbf{L}^p}\\
\le\;& C\int_0^{\frac{t}{2}}\Big(\langle t-\tau\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-1}\langle\tau\rangle^{-\frac{n}{2}(\rho-1)}\\
&+\langle t-\tau\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\frac{\delta}{2}+\frac{n}{4}}\langle\tau\rangle^{-\frac{n}{2}(\rho-1)+\frac{\delta}{2}-\frac{n}{4}}\Big)d\tau\\
\le\;& C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-1}+C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\frac{\delta}{2}+\frac{n}{4}}\le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\min\left(1,\frac{\delta}{2}-\frac{n}{4}\right)},\end{aligned}$$

where $2\le p\le\frac{2n}{n-2\alpha}$ if $\alpha<\frac{n}{2}$, $2\le p<\infty$ if $\alpha=\frac{n}{2}$, and $2\le p\le\infty$ if $\alpha>\frac{n}{2}$. We also have

$$\left\|\int_0^{\frac{t}{2}}(G_0(t-\tau,x)-G_0(t,x))\int_{\mathbf{R}^n}\mathcal{N}(u(\tau,x))\,dxd\tau\right\|_{\mathbf{L}^p}$$
$$\le C\int_0^{\frac{t}{2}}\langle t-\tau\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-1}\langle\tau\rangle^{1-\frac{n}{2}(\rho-1)}d\tau\le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\min\left(\frac{n}{2}(\rho-1)-1,1\right)}$$

and

$$\left\|G_0(t,x)\int_{\frac{t}{2}}^{\infty}\int_{\mathbf{R}^n}\mathcal{N}(u(\tau,x))\,dxd\tau\right\|_{\mathbf{L}^p}\le C\langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\frac{n}{2}(\rho-1)+1}.$$

Thus by the integral equation (3.1) we see that there exists a constant

$$A = \varepsilon\theta_1 + \varepsilon\theta_0 + \int_0^\infty \int_{\mathbf{R}^n} \mathcal{N}\left(u\left(\tau, x\right)\right) dx d\tau$$

such that the following asymptotics are valid:

$$\left\| u\left(t\right) - AG_0\left(t\right)\right\|_{\mathbf{L}^p} \leq C \left\langle t\right\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)-\min\left(1,\frac{\delta}{2}-\frac{n}{4},\frac{n}{2}(\rho-1)-1\right)}$$

for all $t > 0$, where $2 \leq p \leq \frac{2n}{n-2\alpha}$ if $\alpha < \frac{n}{2}$, $2 \leq p < \infty$ if $\alpha = \frac{n}{2}$, and $2 \leq p \leq \infty$ if $\alpha > \frac{n}{2}$. Theorem 1.1 is proved.

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