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EXPONENTIAL INTEGRABILITY OF TEMPERATURE IN THE THERMISTOR PROBLEM

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Abstract. We consider weak solutions to the initial- boundary-value problem for the system $\frac{\partial u}{\partial t} - \mathrm{div}(K(u)\nabla u) = \sigma(u)|\nabla\varphi|^2$, $\mathrm{div}\,(\sigma(u)\nabla\varphi) = 0$ in the case where $K(u)$ and $\sigma(u)$ may both tend to 0 as $u \to \infty$. It is established that u in the solution belongs to some Orlicz space under certain conditions. This implies that u is exponentially integrable in some cases.

1. Introduction.

The purpose of this paper is to present some new contributions to the mathematical analysis of the following problem:

$$\frac{\partial u}{\partial t} - \mathrm{div}\ (K(u)\nabla u) = \sigma(u)|\nabla\varphi|^2 \ \text{ in } \ \Omega_T \equiv \Omega \times (0,T), \tag{1.1}$$

$$\mathrm{div}(\sigma(u)\nabla\varphi) = 0 \ \text{ in } \ \Omega_T, \tag{1.2}$$

$$u(x,t) = u_0(x,t) \text{ on } \partial_p\Omega_T, \tag{1.3}$$

$$\frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \Sigma_N \equiv \Gamma_N \times (0,T), \tag{1.4}$$

$$\varphi(x,t) = \varphi_0(x,t) \text{ on } \Sigma_D \equiv \Gamma_D \times (0,T). \tag{1.5}$$

Here $T > 0$, Ω is a bounded domain in R^N with smooth boundary $\partial\Omega$, Γ_D is a nonempty open subset of $\partial\Omega$, $\Gamma_N = \partial\Omega \setminus \overline{\Gamma_D}$, $\partial_p\Omega_T = \partial\Omega \times (0,T) \cup \Omega \times \{0\}$ is the parabolic boundary of Ω_T, and $u_0(x,t)$, $\varphi_0(x,t)$, $K(u)$, and $\sigma(u)$ are known functions of their arguments.

In [10], the author establishes the following theorem.

Theorem A. *Assume the following:*

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(H1) *$K(u)$ and $\sigma(u)$ are continuous and positive. Moreover,*

$$\sigma \leq M \text{ for some } M > 0, \text{ and} \tag{1.6}$$

$$\int_0^\infty K(s)ds = \infty. \tag{1.7}$$

(H2) *$u_0 \in W^{1,\infty}(\Omega_T)$ and $\varphi_0 \in L^\infty(0,T; W^{1,\infty}(\Omega))$.*

Then for each $T > 0$ there is a capacity solution to (1.1)–(1.5).

A capacity solution is a weak solution in which u is allowed to be unbounded. This gives rise to the possibility that the distributional derivatives of φ with respect to the spatial variables may be a pure distribution. The notion of a capacity solution provides an answer to the question of in what sense the partial differential equations and the boundary conditions are satisfied in this situation. We refer the reader to [10, 11] for its precise definition.

A result of [12] asserts that if we further have

$$0 < m \leq K(u) \leq M \text{ for some } m, M$$

in addition to the assumptions of Theorem A, then there exists a positive number c such that

$$\fint_{Q(z,r)} (u - u_{z,r})^2 dy\, d\tau \leq c$$

for all $z \in \Omega_T$ and $r > 0$ with $Q(z,r) \subset \Omega_T$, where

$$\begin{aligned} z &= (x,t), \\ Q(z,r) &= \{(y,\tau) : |y - x| < r,\ t - r^2 < \tau < t\}, \\ u_{z,r} &= \fint_{Q(z,r)} u\, dy\, d\tau. \end{aligned}$$

That is, $u \in BMO(\Omega_T)$ in the parabolic sense.

In this paper, we shall develop a result along this line. To be more precise, we have

Theorem B. *Let the assumptions of Theorem A hold. Assume*

$$K(u) = \sigma(u)g(u), \tag{1.8}$$

where $g(u)$ satisfies the differential inequality

$$g'(u) \leq \frac{g^2(u)}{2(\|\varphi_0\|_{\infty,\Omega_T})^2} \text{ on } [0,\infty). \tag{1.9}$$

Then we have

$$\int_\Omega \exp\left(\alpha \int_0^{u(x,t)} g(s)ds\right) dx \leq c \text{ for some } \alpha > 0. \tag{1.10}$$

For simplicity, we assume that $u \geq 0$ throughout this paper. This can easily be achieved by assuming that $u_0 \geq 0$.

The class of functions g satisfying (1.9) is big enough for many applications [6]. If we take $g = c$, then we have the exponential integrabilty of u.

The proof of Theorem B will be presented in Section 2, while Section 3 is concerned with the case where σ is a constant.

Problem (1.1)–(1.5) is often called the thermistor problem, and it arises in the study of electrical heating of a conductor (see [7, 1]). In this situation u is the temperature of the conductor, and φ the electrical potential. The first equation describes the diffusion of heat in the presence of the Joule heating, which is the rate of energy generation associated with electrical current flow, while the second equation represents the conservation of electrical charges. The boundary conditions describe how the conductor is connected electrically and thermally to its surroundings. The functions $K(u)$ and $\sigma(u)$ are the thermal conductivity and the electrical conductivity, respectively. Their precise forms are determined by the particular physical application one has in mind. See, e.g., [6] for various forms suggested for K and σ in industrial applications.

2. Proof of Theorem B

The proof of Theorem B is divided into several lemmas.

Let (u, φ) be given as in Theorem A. We may assume that

$$(u, \varphi) \in S_T \equiv L^\infty(\Omega_T) \cap L^2(0, T; W^{1,2}(\Omega)) \times L^\infty(\Omega_T) \cap L^\infty(0, T; W^{1,2}(\Omega)).$$

This is because a capacity solution is constructed as the limit of a sequence of approximate solutions in S_T. See [10, 11] for details.

Lemma 2.1. *Let $f \in C^1(\mathbf{R})$ be such that*

$$f > 0, \ f' > 0 \quad \text{on} \quad \mathbf{R}. \tag{2.1}$$

Suppose that there is an $l \geq l_0 \equiv \|u_0\|_{\infty,\Omega_T}$ with the property

$$(\|\varphi_0\|_{\infty,\Omega_T})^2 \frac{f'(s)}{f(s)} \leq \varepsilon \frac{K(s)}{\sigma(s)} \quad \text{on} \ [l, \infty) \tag{2.2}$$

for some $\varepsilon \in (0, 1)$. Then we have

$$\begin{aligned} &\int_\Omega \Big(\int_l^{u(y,t)} f(s) ds \Big)^+ dy + (1 - \varepsilon) \int_{u>l} K(u) f'(u) |\nabla u|^2 dy\, d\tau \\ &\leq 2f(l) \int_{u>l} \sigma(u) |\nabla \varphi|^2 dy\, d\tau, \end{aligned} \tag{2.3}$$

$$\int_{u>l} f(u)\sigma(u)|\nabla\varphi|^2 dy\, d\tau \leq \frac{2-\varepsilon}{1-\varepsilon} f(l) \int_{u>l} \sigma(u)|\nabla\varphi|^2 dy\, d\tau. \tag{2.4}$$

Proof. Let f be given as in the lemma. Then for each $l \geq l_0$, we have

$$(f(u) - f(l))^+|_{\partial_p \Omega_T} = 0.$$

Note from (1.1) and (1.2) that

$$u_t \in L^2(0,T; W^{-1,2}(\Omega)), \tag{2.5}$$

$$(f(u) - f(l))^+ \in L^2(0,T; W_0^{1,2}(\Omega)). \tag{2.6}$$

On account of the chain rule, we have

$$\frac{d}{dt}\int_\Omega \int_0^{u(x,t)} (f(s) - f(l))^+ ds dx = (u_t, (f(u) - f(l))^+), \tag{2.7}$$

where $(\cdot, \cdot)$ denotes the duality pairing between $W^{-1,2}(\Omega)$ and $W^{1,2}(\Omega)$. Keep this in mind and use $(f(u) - f(l))^+$ as a test function in (1.1) to get

$$\begin{aligned} &\int_\Omega \int_0^{u(y,t)} (f(s) - f(l))^+ ds dy + \int_{u>l} K(u) f'(u)|\nabla u|^2 dy\, d\tau \\ &= \int_{u>l} \sigma(u)|\nabla\varphi|^2 (f(u) - f(l))^+ dy\, d\tau. \end{aligned} \tag{2.8}$$

By using $\varphi(f(u) - f(l))^+$ as a test function in (1.2), we derive

$$\begin{aligned} &\int_{u>l} \sigma(u)|\nabla\varphi|^2 (f(u) - f(l))^+ dy\, d\tau = -\int_{u>l} \sigma(u)\nabla\varphi\varphi f'(u)\nabla u\, dy\, d\tau \qquad (2.9) \\ &\leq \frac{1}{2}\int_{u>l} \sigma(u)|\nabla\varphi|^2 f(u) dy\, d\tau + \frac{1}{2}\int_{u>l} \sigma(u)\varphi^2 \frac{(f'(u))^2}{f(u)}|\nabla u|^2 dy\, d\tau \end{aligned}$$

from which it follows that

$$\begin{aligned} &\int_{u>l} \sigma(u)|\nabla\varphi|^2 f(u) dy\, d\tau \qquad (2.10) \\ &\leq \int_{u>l} \sigma(u)\varphi^2 \frac{(f'(u))^2}{f(u)}|\nabla u|^2 dy\, d\tau + 2f(l)\int_{u>l} \sigma(u)|\nabla\varphi|^2 dy\, d\tau. \end{aligned}$$

Plug this into (2.8) to get

$$\begin{aligned} &\int_\Omega \int_0^{u(y,t)} (f(s) - f(l))^+ ds dy + \int_{u>l} K(u) f'(u)|\nabla u|^2 dy\, d\tau \qquad (2.11) \\ &\leq (\|\varphi_0\|_{\infty,\Omega_T})^2 \int_{u>l} \sigma(u)\frac{(f'(u))^2}{f(u)}|\nabla u|^2 dy\, d\tau + f(l)\int_{u>l} \sigma(u)|\nabla\varphi|^2 dy\, d\tau. \end{aligned}$$

It is not difficult to show [12] that

$$\int_\Omega (u-l)^+ dy \le \int_{u>l} \sigma(u)|\nabla\varphi|^2 dy\, d\tau. \tag{2.12}$$

Use (2.2) in (2.11) to get (2.3), while (2.4) follows from (2.10) and (2.3).

One can easily verify that a solution to (2.2) is given by

$$f(u) = \exp\Big(\frac{\varepsilon}{(\|\varphi_0\|_{\infty,\Omega_T})^2}\int_0^u \frac{K(s)}{\sigma(s)}ds\Big). \tag{2.13}$$

Fix t_0 in $(0,T]$. Define $\Gamma(x,t)$ to be the solution of the following backward problem:

$$\begin{aligned} \frac{\partial\Gamma}{\partial t} + \operatorname{div}(K(u)\nabla\Gamma) &= 0 \text{ in } \Omega_{t_0} \equiv \Omega\times(0,t_0), &(2.14)\\ \Gamma &= 0 \text{ on } \partial\Omega\times(0,t_0), &(2.15)\\ \Gamma|_{t=t_0} &= h(x), &(2.16)\end{aligned}$$

where $h(x)$ is any measurable function with the property

$$h\ge 0, \qquad \int_0^h \ln(1+s)ds \in L^1(\Omega). \tag{2.17}$$

Lemma 2.2. *Let f be given as in Lemma 2.1. Then*

$$\begin{aligned} &\int_\Omega\int_0^{u(x,t_0)} f(s)ds\, h(x)dx + \int_{\Omega_{t_0}} K(u)f'(u)|\nabla u|^2\Gamma\, dx\, dt\\ &\le \int_{\Omega_{t_0}} \sigma(u)f(u)|\nabla\varphi|^2\Gamma\, dx\, dt + \|h\|_{1,\Omega}\Big\|\int_0^{u_0} f(s)ds\Big\|_{\infty,\Omega_T}. \end{aligned}\tag{2.18}$$

Proof. Without loss of generality, we may assume that

$$\Gamma \in L^\infty(\Omega\times(0,t_0))\cap L^2(0,t_0;W^{1,2}(\Omega)) \tag{2.19}$$

because we can also view Γ as the limit of a sequence of solutions of (2.14) in the space. On account of this, we can conclude that $f(u)\Gamma$ is a legitimate test for (1.1). Upon using it, we obtain

$$\begin{aligned} &(\frac{\partial u}{\partial t}, f(u)\Gamma) + \int_\Omega K(u)f'(u)|\nabla u|^2\Gamma dx\\ &+\int_\Omega K(u)f(u)\nabla u\nabla\Gamma\, dx = \int_\Omega \sigma(u)f(u)|\nabla\varphi|^2\Gamma\, dx. \end{aligned}\tag{2.20}$$

Now let w be the solution of the following problem:

$$\frac{\partial w}{\partial t} - \operatorname{div}(K(u)\nabla w) = 0 \text{ in } \Omega_T, \tag{2.21}$$

$$w = \int_0^{u_0} f(s)ds \text{ on } \partial_p\Omega_T. \tag{2.22}$$

Use $\int_0^u f(s)ds - w$ as a test function in (2.14) to get

$$\Big(\frac{\partial\Gamma}{\partial t}, \int_0^u f(s)ds - w\Big) - \int_\Omega K(u)\nabla\Gamma(f(u)\nabla u - \nabla w)dx = 0. \tag{2.23}$$

Observe from (2.21) that

$$\int_\Omega K(u)\nabla\Gamma\nabla w\, dx = -(\frac{\partial w}{\partial t}, \Gamma). \tag{2.24}$$

We can conclude from (1.1) that

$$\frac{\partial}{\partial t}\int_0^u f(s)ds \in L^2(0,T;W^{-1,2}(\Omega)) + L^1(\Omega_T). \tag{2.25}$$

This, together with (2.19) and a standard approximation argument, implies

$$(\frac{\partial u}{\partial t}, f(u)\Gamma) = \Big(\frac{\partial}{\partial t}\int_0^u f(s)ds, \Gamma\Big), \tag{2.26}$$

$$\begin{aligned}&\frac{d}{dt}\int_\Omega \Big(\int_0^u f(s)ds - w\Big)\Gamma\, dx \\ &= \Big(\frac{\partial}{\partial t}\Big(\int_0^u f(s)ds - w\Big), \Gamma\Big) + \Big(\frac{\partial}{\partial t}\Gamma, \int_0^u f(s)ds - w\Big).\end{aligned} \tag{2.27}$$

Use (2.26) in (2.20), then add the resulting equation to (2.23), and thereby obtain

$$\begin{aligned}&\Big(\frac{\partial}{\partial t}\int_0^u f(s)ds, \Gamma\Big) + \Big(\frac{\partial}{\partial t}\Gamma, \int_0^u f(s)ds - w\Big) \\ &+ \int_\Omega K(u)f'(u)|\nabla u|^2\Gamma\, dx + \int_\Omega K(u)\nabla w\nabla\Gamma\, dx = \int_\Omega \sigma(u)f(u)|\nabla\varphi|^2\Gamma\, dx.\end{aligned}$$

This, along with (2.24) and (2.27), yields

$$\frac{d}{dt}\int_\Omega \Big(\int_0^u f(s)ds - w\Big)\Gamma\, dx + \int_\Omega K(u)f'(u)|\nabla u|^2\Gamma\, dx = \int_\Omega \sigma(u)f(u)|\nabla\varphi|^2\Gamma\, dx. \tag{2.28}$$

Integrate the equation over $(0,t_0)$ to obtain

$$\begin{aligned}&\int_\Omega \Big(\int_0^{u(x,t_0)} f(s)ds - w(x,t_0)\Big)h(x)dx + \int_{\Omega_{t_0}} K(u)f'(u)|\nabla u|^2\Gamma\, dx\, dt \\ &= \int_{\Omega_{t_0}} \sigma(u)f(u)|\nabla\varphi|^2\Gamma\, dx\, dt.\end{aligned} \tag{2.29}$$

The lemma follows from the maximum principle for w.

Lemma 2.3. *Let f be given as in Lemma* 2.1. *Then*

$$\int_\Omega \sigma(u)|\nabla\varphi|^2 f(u)\Gamma\, dy \le 2\int_\Omega \sigma(u)\varphi^2\frac{(f'(u))^2}{f(u)}|\nabla u|^2\Gamma\, dy \tag{2.30}$$
$$+\frac{1}{2}\int_\Omega \sigma(u)|\nabla\varphi|^2 f(u)dy + 2\int_\Omega \sigma(u)\varphi^2 f(u)\frac{|\nabla\Gamma|^2}{1+\Gamma}dy\,.$$

Proof. Let f be given as in the lemma. By using $\varphi f(u)\Gamma$ as a test function in (1.2), we derive

$$\int_\Omega \sigma(u)|\nabla\varphi|^2 f(u)\Gamma\, dy \tag{2.31}$$
$$= -\int_\Omega \sigma(u)\nabla\varphi\varphi f'(u)\nabla u\Gamma\, dy - \int_\Omega \sigma(u)\nabla\varphi\varphi f(u)\nabla\Gamma\, dy$$
$$\le \frac{1}{4}\int_\Omega \sigma(u)|\nabla\varphi|^2 f(u)\Gamma\, dy + \int_\Omega \sigma(u)\varphi^2\frac{(f'(u))^2}{f(u)}|\nabla u|^2\Gamma\, dy$$
$$+\frac{1}{4}\int_\Omega \sigma(u)|\nabla\varphi|^2 f(u)(1+\Gamma)dy + \int_\Omega \sigma(u)\varphi^2 f(u)\frac{|\nabla\Gamma|^2}{1+\Gamma}dy\,,$$

from which the lemma follows.

Lemma 2.4. *There exist two positive numbers c_1 and c_2 such that*

$$\int_\Omega\int_0^{u(x,t_0)} g(s)ds\, h(x)dx \le c_1 + c_2\int_\Omega (1+h(x))\ln(1+h(x))dx \tag{2.32}$$

for all h satisfying (2.17).

Proof. Use $\ln(1+\Gamma)$ as a test function in (2.14) to get

$$\frac{d}{dt}\int_\Omega\int_0^{\Gamma(x,t)} \ln(1+s)dsdx - \int_\Omega K(u)\frac{|\nabla\Gamma|^2}{1+\Gamma}dx = 0. \tag{2.33}$$

Integrate this equation over $(0,t_0)$ to obtain

$$\int_{\Omega_{t_0}} K(u)\frac{|\nabla\Gamma|^2}{1+\Gamma}dx \le \int_\Omega\int_0^{h(x)} \ln(1+s)ds\, dx \le \int_\Omega (1+h(x))\ln(1+h)ds\, dx. \tag{2.34}$$

Now we take

$$f = \frac{K}{\sigma} = g.$$

Condition (1.9) implies that the second integral in (2.30) is bounded by the second integral in (2.29). Clearly, (2.2) is a consequence of (1.9), and thus the third integral in (2.30) is bounded by known data due to Lemma 2.1. Combining (2.34), (2.30), and (2.29), we obtain the lemma.

Lemma 2.5. *Let ψ be a nonnegative function in $L^1(\Omega)$ such that*

$$\int_\Omega \psi(x)h(x)dx \le c_1 + c_2 \int_\Omega (1+h(x))\ln(1+h(x))dx \tag{2.35}$$

for all h satisfying (2.17), *where c_1 and c_2 are two positive constants. Then we have*

$$\int_\Omega \exp(\alpha\psi(x))dx < \infty \tag{2.36}$$

for some positive number α depending only on c_2.

This lemma is implied in the book [8] on Orlicz spaces. We are not able to find a theorem in [8] that can be applied to our case directly . So we offer a proof here.

Proof. We first assume that ψ is bounded. Then let

$$h = \frac{e^{\beta\psi(x)} - 1}{\psi(x)},$$

where β is a positive number to be determined later. Clearly, $0 \le h \in L^\infty(\Omega)$. Plug h into (2.35) to get

$$\int_\Omega (e^{\beta\psi(x)} - 1)dx \le c_1 + c_2 \int_\Omega (e^{\beta\psi(x)} - 1 + \psi(x)) \frac{\ln \frac{(e^{\beta\psi(x)} - 1 + \psi(x))}{\psi(x)}}{\psi(x)} dx. \tag{2.37}$$

It is elementary to show that

$$\lim_{s\to\infty} \frac{\ln \frac{(e^{\beta s} - 1 + s)}{s}}{s} = \beta.$$

Thus there exists an $l > 0$ such that

$$\frac{\ln \frac{(e^{\beta s} - 1 + s)}{s}}{s} < 2\beta \text{ on } [l, \infty).$$

We estimate

$$\begin{aligned} &\int_\Omega (e^{\beta\psi(x)} - 1 + \psi(x)) \frac{\ln \frac{(e^{\beta\psi(x)} - 1 + \psi(x))}{\psi(x)}}{\psi(x)} dx \\ &= \int_{\psi \le l} \frac{(e^{\beta\psi(x)} - 1 + \psi(x))}{\psi(x)} \ln \frac{(e^{\beta\psi(x)} - 1 + \psi(x))}{\psi(x)} dx \end{aligned}$$

$$
\begin{aligned}
& + \int_{\psi > l} (e^{\beta\psi(x)} - 1 + \psi(x)) \frac{\ln \frac{(e^{\beta\psi(x)} - 1 + \psi(x))}{\psi(x)}}{\psi(x)} dx \\
\leq \quad & c(l,\beta) + 2\beta \int_\Omega (e^{\beta\psi(x)} - 1 + \psi(x)) dx.
\end{aligned}
$$

Choose β so that $2\beta c_2 < 1$. Then use the above estimate in (2.37) to get the desired result. The case where ψ is unbounded can be handled by approximation. The proof is complete.

3. $\sigma =$ A CONSTANT

In this section we consider the case where σ is a constant. That is, $\sigma = A$, where A is a positive number. Then the second equation in the system no longer involves u. We have

$$
\Delta\varphi = 0 \text{ in } \Omega_T, \tag{3.1}
$$

$$
\frac{\partial\varphi}{\partial\nu} = 0 \text{ on } \Sigma_N, \tag{3.2}
$$

$$
\varphi = \varphi_0 \text{ on } \Sigma_D. \tag{3.3}
$$

A result in [5, p. 22] asserts that

$$
\varphi \in L^\infty(0,T; W^{1,\infty}_{loc}(\Omega)). \tag{3.4}
$$

If $\partial\Gamma_D$ is regular enough, then it also holds [4] that

$$
\text{ess sup}_{0\leq t\leq T} \|\nabla\varphi(\cdot,t)\|_{p,\Omega} \leq c \tag{3.5}
$$

for some $p > 2$ and

$$
\varphi \in L^\infty(0,T; C^\alpha(\overline{\Omega}))
$$

for some $\alpha \in (0,1)$.

Theorem C. *Let* $(H2)$ *be satisfied. Assume the following:*

(H3) $K(s)$ *is a continuous function on* R *with the property*

$$
\frac{c}{s^\beta} \leq K(s) \quad on \ [l_0,\infty), \tag{3.6}
$$

where $l_0 = \|u_0\|_{\infty,\Omega_T}$, *and* β *and* c *are positive numbers.*

(H4) $\partial\Omega$ *and* Γ_D *are such that for each* $p > 2$ *there is a* c *such that* (3.5) *holds.*

Then no matter how big β *is there is a bounded solution to* (1.1)–(1.5).

We must point out that one can construct $\partial\Omega$ and Γ_D in such a way so that (H4) does not hold. The precise conditions under which (H4) does hold are not clear. Unfortunately, we did not emphasize this point in [12]. Also note that we do not impose an up-bound on β. This means that (1.7) may not hold. Thus the existence of a classical weak solution must rely on the boundedness of u. If $\beta = 1, \Omega = R^N$, and $\varphi \equiv 0$, the resulting problem is considered in [2].

Proof. We once again assume $(u, \varphi) \in S_T$ in our calculations. First we show that for each $r > 0$ there is a positive number $c = c(r, T)$ such that

$$\|(u-l)^+\|_{r+1,\Omega_T} \leq c\||\nabla\varphi|^2\|_{r+1,\Omega_T} \tag{3.7}$$

for $l \geq l_0$. To see this, we use $[(u-l)^+]^r$ as a test function in (1.1) to get

$$\begin{aligned}
&\int_{\Omega\times\{t_0\}} \frac{1}{r+1}[(u-l)^+]^{r+1} dx \qquad (3.8)\\
&\leq A\int_{\Omega_{t_0}} |\nabla\varphi|^2[(u-l)^+]^r dx\, d\tau \\
&\leq \frac{r}{r+1}\int_{\Omega_{t_0}} [(u-l)^+]^{r+1} dx\, d\tau + \frac{1}{r+1}A^{r+1}\int_{\Omega_{t_0}} |\nabla\varphi|^{2(r+1)} dx\, d\tau.
\end{aligned}$$

Then an application of Gronwall's inequality yields (3.7).

For each nonnegative integer n, set

$$l_n = l_0 + (1 - \frac{1}{2^n})l,$$

where l is a positive number to be determined later. Fix $r > 0$. Multiply (1.1) by the test function $[(u-l_{n+1})^+]^r$ and integrate over Ω_{t_0} to obtain

$$\begin{aligned}
&\frac{1}{r+1}\int_{\Omega} [(u-l_{n+1})^+]^{r+1} dx \qquad (3.9)\\
&\quad + \int_{\Omega_{t_0}} rK(u)[(u-l_{n+1})^+]^{r-1}|\nabla u|^2 dx\, d\tau \\
&= \int_{\Omega_{t_0}} A|\nabla\varphi|^2[(u-l_{n+1})^+]^r dx\, d\tau.
\end{aligned}$$

Set

$$m = \min_{\Omega_T} K(u) \geq \frac{c}{(\max_{\Omega_T} u)^\beta}. \tag{3.10}$$

By the embedding theorem [3, p. 7], we derive

$$\begin{aligned}
&\int_{\Omega_T} \left\{[(u-l_{n+1})^+]^{r+1}\right\}^{\frac{N+2}{N}} dx\, dt \qquad (3.11)\\
&\leq c\int_{\Omega_T} \left|\frac{r+1}{2}[(u-l_{n+1})^+]^{\frac{r-1}{2}}\nabla u\right|^2 dx\, dt
\end{aligned}$$

$$\cdot\left(\text{ess sup}_{0\leq t\leq T}\int_\Omega [(u-l_{n+1})^+]^{r+1}dx\,dt\right)^{2/N}$$
$$\leq \frac{c(r)}{m}\left(\int_{\Omega_T}|\nabla\varphi|^2[(u-l_{n+1})^+]^r dx\,dt\right)^{\frac{N+2}{N}}.$$

Let

$$Y_n=\int_{\Omega_T}[(u-l_n)^+]^{r+1}dx\,dt.$$

Then

$$Y_{n+1}\leq\left(\int_{\Omega_T}\left\{[(u-l_{n+1})^+]^{r+1}\right\}^{\frac{N+2}{N}}dx\,dt\right)^{\frac{N}{N+2}}|\{u>l_{n+1}\}|^{\frac{2}{N+2}}. \quad (3.12)$$

On the other hand, we have

$$Y_n\geq\int_{u>l_{n+1}}[(u-l_n)^+]^{r+1}dx\,dt\geq\frac{l^{r+1}}{2^{(n+1)(r+1)}}|\{u\geq l_{n+1}\}|. \quad (3.13)$$

We conclude from (3.12), (3.11), and (3.13) that

$$\begin{aligned}
Y_{n+1} &\leq \frac{cb^n}{m^{\frac{N}{N+2}}l^{(r+1)\frac{2}{N+2}}}Y_n^{\frac{2}{N+2}}\int_{\Omega_T}|\nabla\varphi|^2[(u-l_{n+1})^+]^r dx\,dt \quad (3.14)\\
&\leq \frac{cb^n}{m^{\frac{N}{N+2}}l^{(r+1)\frac{2}{N+2}}}Y_n^{\frac{2}{N+2}}\left(\int_{\Omega_T}[(u-l_{n+1})^+]^{r+1}dx\,dt\right)^{\frac{r}{r+1}}\\
&\quad\cdot\left(\int_{\Omega_T}(|\nabla\varphi|^2)^{r+1}dx\,dt\right)^{\frac{1}{r+1}}\\
&\leq \frac{cb^n}{m^{\frac{N}{N+2}}l^{(r+1)\frac{2}{N+2}}}Y_n^{\frac{2}{N+2}+\frac{r}{r+1}}\||\nabla\varphi|^2\|_{r+1.\Omega_T},
\end{aligned}$$

where $b=2^{(r+1)\frac{2}{N+2}}$. Now choose r so that

$$\frac{r}{r+1}+\frac{2}{N+2}>1.$$

This puts us in a position to apply Lemma 4.1 in [3, p. 12]. By the calculations in [2] and with the aid of (3.10) and (3.7), we are eventually led to the following estimate:

$$\begin{aligned}
\max_{\Omega_T}u &\leq l_0+l\leq l_0+c\frac{\left(\||\nabla\varphi|^2\|_{r+1,\Omega_T}\right)^{\frac{N+2}{2(r+1)}}\left(\|u-l_0\|_{r+1,\Omega_T}\right)^{\frac{2(r+1)-N-2}{2(r+1)}}}{m^{\frac{N}{2(r+1)}}}\\
&\leq l_0+c(\max_{\Omega_T}u)^{\frac{\beta N}{2(r+1)}}\||\nabla\varphi|^2\|_{r+1,\Omega_T}.
\end{aligned}$$

So no matter how big β is, we can always select r so that $\frac{\beta N}{2(r+1)} < 1$. This implies the desired result.

It is easy to see from our proof that if $|\nabla\varphi| \in L^\infty(\Omega_T)$ then u is bounded even without the assumption (3.6).

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