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TIME-INDEPENDENT ESTIMATES AND A COMPARISON THEOREM FOR A NONLINEAR INTEGROPARABOLIC EQUATION OF THE FOKKER-PLANCK TYPE

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(Submitted by: Daniele Andreucci)

to Enrico Magenes on his 80th birthday

Abstract. Time-independent bounds for the classical solutions of a certain nonlinear integroparabolic equation of the Fokker-Planck type are established. Such an equation describes the statistical time evolution of large populations of nonlinearly coupled random oscillators with inertia. The basic tool is deriving “energy-like” estimates. A comparison theorem is also proved to obtain estimates for the solutions in terms of some special solutions.

1. Introduction

Large populations of nonlinearly coupled random oscillators have been studied for a long time. This is due to the variety of phenomena, ranging from biology to physics, to neural networks, that they can describe. One of the most popular and successful models is the so-called Kuramoto (or Kuramoto-Sakaguchi) model, which depicts the dynamical behavior of populations of infinitely many “phase oscillators,” subject to a “mean-field” coupling mechanism, occurring when, so to say, each oscillator is affected by

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the whole of the others in the same way; see [6] and [10], e.g. A mathematical analysis of such a model has been presented in [7]. The Kuramoto model has been later generalized, so as to account for certain "inertial effects," in [3] and [1]. To be more precise, such a model is based on the nonlinear parabolic integrodifferential equation, of the Fokker-Planck type,

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial \omega^2} - \omega \frac{\partial \rho}{\partial \theta}$$

$$+\frac{\partial}{\partial \omega}\Big[\Big(\omega - \Omega - K\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\int_{0}^{2\pi} g(\Omega)\sin(\varphi-\theta)\rho(\varphi,\omega,t,\Omega)\,d\varphi\,d\omega\,d\Omega\Big)\rho\Big], \tag{1.1}$$

where $g(\Omega) > 0$ is a suitably given function and $K > 0$ is a given constant. Here t represents time, θ an angular variable, while ω and Ω are frequencies; $g(\Omega)$ has the meaning of distribution of the natural frequencies of the oscillators, whose space position is given by θ. The unknown function, ρ, however, is the "one-oscillator frequency distribution," considering that the present model refers to a population of infinitely many oscillators. The parameter K sizes the strength of nonlinearity, while another quantity, "the mass" of each oscillator, responsible for the aforementioned inertial effects, has been normalized to 1.

This equation should be solved along with some given data; that is, the initial value, $\rho(\theta, \omega, 0, \Omega)$, and suitable boundary data, typically the periodicity with respect to θ, are prescribed. The problem is pathological, that is, rather nonstandard, in that the equation is nonlinear and integrodifferential, fully degenerate in the space variable θ as it happens in the classical Fokker-Planck equation (no diffusion in θ), and periodicity in θ is required. Moreover, the problem is set on an ω-unbounded domain, and some coefficient of the equation (ω) is unbounded on this domain.

The corresponding basic qualitative analysis for such a problem has been developed in a series of papers; see [8], [9], and [4]. The first two of these papers, [8] and [9], concern the existence of *strong* solutions (which satisfy the equation almost everywhere in the considered domain) to such a problem, in suitable *anisotropic* Sobolev spaces, under rather general assumptions. In [4], existence and uniqueness of *classical* solutions have been established, under slightly stronger assumptions on the data. In any case, a key technique was the use of an associated *regularized* problem, obtained introducing in the right-hand side of the equation a term representing diffusion with respect to the space variable, θ, multiplied by a small parameter.

The long-time behavior, in particular the existence of a limit as time grows to infinity, of solutions to evolution equations is one of the core questions in the qualitative theory of partial differential equations. Indeed, the stabilization of certain processes, described for instance by parabolic equations, to steady-state, can be important in designing effective technological equipment in the chemical industry; see [5, 11, 12], e.g. To establish that the solution to a given equation does not grow to infinity when time increases is an important step in studying the stabilization problem. It seems however that, proceeding according to the scheme proposed in [9], *time independent* bounds for solutions to the problem being investigated here *cannot* be obtained. Therefore, in this paper we propose a certain modification of the existing comparison theorems in order to prove that estimates of solutions to the present problem, independent of the length of the time interval, do exist. Moreover, the long-time behavior of solutions to the Kuramoto equation has been studied numerically; see [2].

Here is the plan of the paper. In Section 2, we first formulate the problem and review the previously derived basic results. Then we establish time-independent estimates for solutions which do not depend on θ either (Section 3). These results are then used to obtain *time-independent estimates* for general (θ-dependent) solutions, by a new version of the *comparison theorem,* applicable to integrodifferential equations of the parabolic type (Section 4). In Section 5, finally, the main results of the paper are summarized.

2. Formulation of the problem and some auxiliary results

We are concerned with the following problem. For $(\theta, \omega, t, \Omega) \in Q_T = [0, 2\pi] \times \mathbf{R} \times [0, T] \times [-G, G]$, find a function $\rho(\theta, \omega, t, \Omega)$ which solves (in a suitable sense) the equation in (1.1), i.e.,

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial \omega^2} - \omega \frac{\partial \rho}{\partial \theta} + \frac{\partial}{\partial \omega}\left[(\omega - \Omega - \mathcal{S}_\rho(\theta, t))\, \rho\right] \tag{2.1}$$

where

$$\mathcal{S}_\rho(\theta, t) := K \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^{2\pi} g(\Omega)\, \sin(\varphi - \theta)\, \rho(\varphi, \omega, t, \Omega)\, d\varphi\, d\omega\, d\Omega \tag{2.2}$$

is 2π-periodic in θ,

$$\rho\,|_{\theta=0} = \rho\,|_{\theta=2\pi} \tag{2.3}$$

for $\omega \in \mathbf{R}$, $t \in [0, T]$, and $\Omega \in [-G, G]$, and satisfies the initial condition

$$\rho\,|_{t=0} = \rho_0(\theta, \omega, \Omega) \tag{2.4}$$

for $\theta \in [0, 2\pi]$, $\omega \in \mathbf{R}$, and $\Omega \in [-G, G]$, with ρ_0 normalized according to

$$\int_0^{2\pi} \int_{-\infty}^{+\infty} \rho_0(\theta, \omega, \Omega)\, d\omega\, d\theta = 1. \tag{2.5}$$

Here and below, the subscripts attached to the symbols $\mathcal{S}$ and $\mathcal{C}$, denoting integral terms, will *not* mean partial differentiation.

Below, we shall specify in which sense the function $\rho(\theta, \omega, t, \Omega)$ is required to satisfy the governing equation (2.1) as well as the initial and boundary conditions. In [4], it has been proved that such solutions are positive in Q_T, normalized as

$$\int_0^{2\pi} \int_{-\infty}^{+\infty} \rho(\theta, \omega, t, \Omega)\, d\omega\, d\theta = 1$$

for $t \in [0, T]$ and $\Omega \in [-G, G]$, the interval $[-G, G]$ being the support of g.

Let $l_0 \geq 0$ be an integer and $\alpha_0 \in (0, 1)$ a real constant. From now on, the data of the problem are supposed to be as follows:

Assumptions.

A) The initial profile, $\rho_0(\theta, \omega, \Omega)$, is the following: (a_1) a continuous function in all variables $(\theta, \omega, \Omega) \in Q = \mathbf{R} \times \mathbf{R} \times [-G, G]$, belonging to the Hölder space $C^{l_0+\alpha_0}(Q)$, for some fixed l_0 and α_0 as above; (a_2) 2π-periodic in θ; (a_3) positive in the whole of Q; (a_4) normalized for all $\Omega \in [-G, G]$ (see (2.5)); and (a_5) with an exponential decay in ω at infinity, along with some partial derivatives, according to the following estimate:

$$\sup_{\theta \in \mathbf{R},\, \Omega \in [-G, G]} \left| D^{l_1, l_2, l_3}_{\theta, \omega, \Omega} \rho_0(\theta, \omega, \Omega) \right| \leq C_0\, \mathrm{e}^{-M_0 \omega^2} \tag{2.6}$$

for $\omega \in \mathbf{R}$, $l_0 \geq 0$, and $l_1 + l_2 + l_3 \leq l_0$ (chosen above), $l_i \geq 0$ $(i = 1, 2, 3)$ being integers, and $C_0, M_0 > 0$ constants. No claim of optimality for the decay rate assumed for the initial profile as $|\omega| \to +\infty$ and implied by the estimate in (2.6) is made, such a condition being merely sufficient to prove Theorems 2.1 and 2.2, below. Here and in the sequel, D^l_ξ stands for the differential operator of order l_i with respect to the variable ξ_i, for each i, l and ξ being multi-indexes.

B) The frequency distribution density, $g(\Omega)$, is assumed to be as follows: (b_1) in the space $L^1(\mathbf{R})$ and (b_2) compactly supported on $[-G, G]$; (b_3) bounded; i.e., $g_0 := \sup_{\Omega \in \mathbf{R}} |g(\Omega)| < \infty$.

C) The coupling strength, $K > 0$, is a constant.

From now on, we use, for short, the notation D for any derivative $D^{l_1, l_2, l_3, l_4}_{\theta, \omega, t, \Omega}$ with $l_1 + l_2 + 2l_3 + l_4 \leq 2$.

It is understood that the variable Ω in (2.1) has to be picked up from the support of g (see (b_2)). Below, the L^1 norm of g,

$$A := \int_{-\infty}^{+\infty} |g(\Omega)|\, d\Omega, \tag{2.7}$$

will appear as a parameter.

In order to study the problem represented by equation (2.1) plus the associated data, we operate a *parabolic regularization* in the governing equation, (2.1). Moreover, *to avoid* the problem of coping with a coefficient, ω, *unbounded* in the slab Q_T, we replace ω with the bounded function

$$F_N(\omega) := \begin{cases} \omega, & \text{for} \quad |\omega| \le N, \\ (N+1)\,\mathrm{sgn}\,\omega, & \text{for} \quad |\omega| \ge N+5. \end{cases} \tag{2.8}$$

We assume that $F_N(\omega) \in C^{3+\alpha_0}(\mathbf{R})$ with the constant $\alpha_0 \in (0,1)$ appearing in (a_1), and is such that $F_N'(\omega) \ge 0$, and

$$\sup_{\omega\in\mathbf{R}} F_N'(\omega) \le 1, \qquad \sup_{\omega\in\mathbf{R}} \left|F_N''(\omega)\right| \le 1, \qquad \sup_{\omega\in\mathbf{R}} \left|F_N'''(\omega)\right| \le 1. \tag{2.9}$$

Therefore, instead of equation (2.1), we shall study first its *parabolic regularization,* represented by the family of equations

$$\begin{aligned} \frac{\partial \rho^{\varepsilon,N}}{\partial t} = \frac{\partial^2 \rho^{\varepsilon,N}}{\partial \omega^2} + \varepsilon\, \frac{\partial^2 \rho^{\varepsilon,N}}{\partial \theta^2} - F_N\, \frac{\partial \rho^{\varepsilon,N}}{\partial \theta} + \frac{\partial}{\partial \omega}\left(F_N \rho^{\varepsilon,N}\right) \\ - \Omega\, \frac{\partial \rho^{\varepsilon,N}}{\partial \omega} - \mathcal{S}_\rho^{\varepsilon,N}\, \frac{\partial \rho^{\varepsilon,N}}{\partial \omega}, \end{aligned} \tag{2.10}$$

satisfied by $\rho^{\varepsilon,N}(\theta,\omega,t,\Omega)$ in $Q_T \cap \{t > 0\}$, for any given $\varepsilon > 0$ and $N > 0$, with the initial profile in (2.4). The term $\mathcal{S}_\rho^{\varepsilon,N}$ is defined by formula (2.2) with the function $\rho^{\varepsilon,N}(\theta,\omega,t,\Omega)$ replacing $\rho(\theta,\omega,t,\Omega)$. Having added a second-derivative term in θ, we modify the periodic boundary conditions into

$$(\rho^{\varepsilon,N}, \rho_\theta^{\varepsilon,N})\,|_{\theta=0} = (\rho^{\varepsilon,N}, \rho_\theta^{\varepsilon,N})\,|_{\theta=2\pi} \tag{2.11}$$

for $\omega \in \mathbf{R}$, $t \in (0,T]$, and $\Omega \in [-G,G]$.

The problem above has been studied in [4], where the following existence theorem for the regularized problem (2.10), (2.11), (2.4), (2.5) has been proved:

Theorem 2.1. *Suppose the data of the problem* (2.10)*,* (2.11)*,* (2.4) *satisfy Assumptions* (*A*) *through* (*C*) (*with* $l_0 = 1$ *in* (A) *and the possible exception of* (b_3))*. Then, for every* $\varepsilon > 0$ *and* $N > 0$*, there exists a* classical *solution* $\rho^{\varepsilon,N}(\theta,\omega,t,\Omega)$ *to the problem* (2.10)*,* (2.11)*,* (2.4) *in* Q_T*. Such a solution*

1) *is a continuous function of all variables in* Q_T*, along with its partial derivatives,* $D^{l_1,l_2}_{\omega,\theta}\rho^{\varepsilon,N}$ *in* Q_T *for* $l_1+l_2=1$*, and* $D^{k,l_1,l_2}_{t,\omega,\theta}\rho^{\varepsilon,N}$ *in* $Q_T\cap\{t>0\}$*, for* $2k+l_1+l_2\le 3$;

2) *satisfies, in the classical sense, equation* (2.10) *in* $Q_T\cap\{t>0\}$*, the boundary data* (2.11) *in* Q_T*, along with the additional property* $\rho^{\varepsilon,N}_{\theta\theta}|_{\theta=0}=\rho^{\varepsilon,N}_{\theta\theta}|_{\theta=2\pi}$*, and the initial data* (2.4) *in* $Q_T\cup\{t=0\}$;

3) *is positive in* Q_T;

4) *is normalized as*

$$\int_0^{2\pi}\int_{-\infty}^{+\infty}\rho^{\varepsilon,N}(\theta,\omega,t,\Omega)\,d\omega\,d\theta=1,$$

for $t\in[0,T]$ *and* $\Omega\in[-G,G]$;

5) *has an exponential decay at infinity in* ω*, along with some derivatives, according to*

$$\sup_{\theta\in[0,2\pi],\,t\in[0,T],\,\Omega\in[-G,G]}\left|D^{l_1,l_2}_{\omega,\theta}\rho^{\varepsilon,N}(\theta,\omega,t,\Omega)\right|\le C\,\mathrm{e}^{-M\omega^2}$$

for $l_1+l_2\le 1$ *and* $\omega\in\mathbf{R}$*, where the constants* $C,M>0$ *depend only on* ε*,* N*,* G*,* T*,* KA*,* C_0*, and* M_0 (*defined in* (2.6)*, Assumption* (*A*))*; moreover,*

$$\sup_{\theta\in[0,2\pi],\,\Omega\in[-G,G]}\left|D^{k,l_1,l_2}_{t,\omega,\theta}\rho^{\varepsilon,N}(\theta,\omega,t,\Omega)\right|\le\frac{C}{\sqrt{t}}\,\mathrm{e}^{-M\omega^2}$$

for $2k+l_1+l_2=2$*,* $\omega\in\mathbf{R}$*, and* $t\in(0,T]$*, with the same constants* $C,M>0$ *introduced above*;

6) *is such that the integrals* $\mathcal{S}^{\varepsilon,N}_\rho(\theta,t)$ *are continuous functions for* $(\theta,t)\in[0,2\pi]\times[0,T]$*, along with their partial derivatives of arbitrary order with respect to* θ*, and are* uniformly bounded *with respect to* ε *and* N:

$$\sup_{\theta\in[0,2\pi],\,t\in[0,T]}\left|\mathcal{S}^{\varepsilon,N}_\rho(\theta,t)\right|+\sup_{\theta\in[0,2\pi],\,t\in[0,T]}\left|\frac{\partial}{\partial\theta}\,\mathcal{S}^{\varepsilon,N}_\rho(\theta,t)\right|\le 2KA;$$

7) *is such that, for* $l_1+l_2\le 1$*, the functions*

$$\int_0^{2\pi}\int_{-\infty}^{+\infty}\left(D^{l_1,l_2}_{\omega,\theta}\rho^{\varepsilon,N}\right)^2(\theta,\omega,t,\Omega)\,d\omega\,d\theta$$

are continuous for $(t,\Omega)\in[0,T]\times[-G,G]$.

In the sequel, we shall also use the decay properties of convolutions of functions with fundamental solutions to linear parabolic equations. Such properties have been established by the authors in [8], and hence their precise formulation will be omitted here.

The classical solvability for the problem under investigation has been established in [4], based on Theorem 2.1, provided that the Assumptions (A)–(C) above are satisfied with $l_0 = 4$ in (A). In fact, the following result has been proved in [4]:

Theorem 2.2. *Suppose the data of problem* (2.1)–(2.3) *satisfy all the assumptions in* (A)–(C) *with* $l_0 = 4$. *Then there exists a* classical *solution,* $\rho(\theta,\omega,t,\Omega)$, *to the problem* (2.1)–(2.3) *in* Q_T, *such that*

(1) $\rho(\theta,\omega,t,\Omega)$ *is a continuous, bounded function in* Q_T *along with its partial derivatives* $D^{l_1,l_2,l_3,l_4}_{\theta,\omega,t,\Omega}\rho(\theta,\omega,t,\Omega)$ *for* $l_1 + l_2 + 2l_3 + l_4 \leq 2$; *moreover, the derivatives* $D^{l_1,l_2,l_3,l_4}_{\theta,\omega,t,\Omega}\rho$ *with* $l_1 + l_2 + 2l_3 + l_4 \leq 2$ *belong to the* anisotropic Sobolev *space* $W_2^{2,3,1,2}(Q_T)$ *and to the* anisotropic Hölder *spaces* $C^{\lambda,\lambda,\frac{1}{12},\frac{1}{2}}(\mathcal{Q}_R)$ *for all* $\lambda \in (0,1)$, *where* $\mathcal{Q}_R := Q_T \cap \{\omega \in [-R,R]\}$, $R > 0$;

(2) $\rho(\theta,\omega,t,\Omega)$ *satisfies equation* (2.1) *in the classical sense in* Q_T, *and satisfies the boundary data in* (2.3) *and the initial data in* (2.4) *as a continuous function in* Q_T; *moreover,* $(\rho_\theta,\rho_{\theta\theta})|_{\theta=0} = (\rho_\theta,\rho_{\theta\theta})|_{\theta=2\pi}$ *in* Q_T;

(3) $\rho(\theta,\omega,t,\Omega) \geq 0$ *in* Q_T, *and it turns out to be normalized as*

$$\int_0^{2\pi}\int_{-\infty}^{+\infty} \rho(\theta,\omega,t,\Omega)\, d\omega d\theta = 1$$

for all $t \in [0,T]$ *and* $\Omega \in [-G,G]$;

(4) *for every value of the parameter* $M > 0$, *there exists a constant* $C = C(M) > 0$ *such that the estimate*

$$|D^{l_1,l_2,l_3,l_4}_{\theta,\omega,t,\Omega}\rho(\theta,\omega,t,\Omega)| \leq C\mathrm{e}^{-M|\omega|}$$

holds in Q_T *for* $l_1 + l_2 + 2l_3 + l_4 \leq 2$.

Finally, a classical *solution to the problem* (2.1)–(2.3) *in* Q_T, *satisfying the estimate in item* (4), *is* unique.

3. Time-independent bounds for θ-independent solutions

For simplicity, in this section we drop the multi-index (ε, N), and denote $\rho^{\varepsilon,N}$ and $\mathcal{S}^{\varepsilon,N}_\rho$ simply by ρ^ε and $\mathcal{S}_\rho$, respectively. We shall define, for convenience, the linear differential operators

$$L_\varepsilon\ \rho := \frac{\partial^2\rho}{\partial\omega^2} + \varepsilon\,\frac{\partial^2\rho}{\partial\theta^2} - F_N\,\frac{\partial\rho}{\partial\theta} + \frac{\partial}{\partial\omega}\,(F_N\rho) - \Omega\,\frac{\partial\rho}{\partial\omega}, \tag{3.1}$$

$$L\ \rho := \frac{\partial^2\rho}{\partial\omega^2} + \frac{\partial}{\partial\omega}\,(F_N\rho) - \Omega\,\frac{\partial\rho}{\partial\omega}. \tag{3.2}$$

By L_ε^f and L^f we shall denote the operators obtained applying the operators L_ε and L defined in (3.1) and (3.2), respectively, to the function $\rho = v\,\mathrm{e}^{f(\omega)}$ and then dropping the multiplier $\mathrm{e}^{f(\omega)}$; that is,

$$L_\varepsilon^f\, v := L_\varepsilon\, v + 2f'\, v_\omega + (f'' + f'^2 + F_N\, f' - \Omega f')\; v, \tag{3.3}$$

$$L^f\, v := L\, v + 2f'\, v_\omega + (f'' + f'^2 + F_N\, f' - \Omega f')\; v. \tag{3.4}$$

Here $f(\omega)$ is a smooth function that will be chosen later. The following abbreviations are also useful below:

$$\mathcal{C}_\rho(\theta, t) := K \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^{2\pi} g(\Omega)\, \cos(\varphi - \theta)\, \rho(\varphi, \omega, t, \Omega)\, d\varphi\, d\omega\, d\Omega, \tag{3.5}$$

$$\mathcal{S}_{\mathbf{v}}^f(\theta, t) := K \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^{2\pi} g(\Omega)\, \sin(\varphi - \theta)\, v(\varphi, \omega, t, \Omega)\, \mathrm{e}^{f(\omega)}\, d\varphi\, d\omega\, d\Omega, \tag{3.6}$$

$$\mathcal{C}_{\mathbf{v}}^f(\theta, t) := K \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_0^{2\pi} g(\Omega)\, \cos(\varphi - \theta)\, v(\varphi, \omega, t, \Omega)\, \mathrm{e}^{f(\omega)}\, d\varphi\, d\omega\, d\Omega \tag{3.7}$$

(cf. (2.2)). With such notation, the problem (2.10), (2.11), (2.4) can be rewritten as

$$\rho_t^\varepsilon = L_\varepsilon\, \rho^\varepsilon - \mathcal{S}_{\rho^\varepsilon}\, \rho_\omega^\varepsilon, \qquad (\rho^\varepsilon, \rho_\theta^\varepsilon)\,|_{\theta=0} = (\rho^\varepsilon, \rho_\theta^\varepsilon)\,|_{\theta=2\pi}, \qquad \rho^\varepsilon\,|_{t=0} = \rho_0. \tag{3.8}$$

As was proved in [8], there *exists* a *unique classical* solution to problem (3.8), and such a solution has the properties listed in Theorem 2.1. In [9], estimates of various kinds have been established for the solution $\rho^\varepsilon(\theta, \omega, t, \Omega)$. Such estimates are *uniform* with respect to both the regularizing parameters, $\varepsilon > 0$ and $N > 0$. Using these estimates, a strong solvability theorem (and then the classical solvability theorem) has been derived for the original problem in (2.1)–(2.4), by a compactness argument; see [4]. However, it seems that proceeding according to the scheme proposed in [9], *time independent* bounds for the solution, $\rho^\varepsilon(\theta, \omega, t, \Omega)$, *cannot* be obtained.

Here we prove such estimates, which do *not* depend on the length of the time interval, $T > 0$. The basic tool is a version of the comparison theorem. Such a theorem cannot be applied directly because of the integral term affecting the governing equation (2.1). In addition, this integral is extended over an *unbounded* domain in the "space variable" ω, $\omega \in \mathbf{R}$. Therefore, a special modification of the comparison theorem valid for purely differential equations needs to be proved. Using such a theorem, it will then be possible to establish *uniform* “a priori” estimates for the solutions to a number of *integroparabolic* equations containing integral terms extended over *unbounded domains*. We shall not discuss here all such possible generalizations.

Consider the bounded function $f(\omega) \in C^\infty(\mathrm{R})$, defined as

$$f(\omega) := \begin{cases} -\omega & \text{for} \quad \omega > 1 \\ \omega & \text{for} \quad \omega < -1, \end{cases} \tag{3.9}$$

and such that, in addition,

$$\operatorname{sgn}(\omega\, f'(\omega)) \le 0, \qquad \left\| f^{(k)}(\omega) \right\|_{C^\infty} \le c, \quad \text{for } k = 1, \dots, 4. \tag{3.10}$$

Clearly, we may and shall choose $f(\omega) < 0$ for all ω, a fact that will be used several times below.

Let us define the function

$$u(\theta, \omega, t, \Omega) := \mathrm{e}^{-f(\omega)}\, \rho^\varepsilon(\theta, \omega, t, \Omega), \tag{3.11}$$

related to the solution, ρ^ε, to the regularized problem in (3.8), which solves the problem

$$u_t = L^f_\varepsilon\, u - \mathcal{S}^f_{\mathbf{u}}\, (u_\omega + f'(\omega) u), \tag{3.12}$$

$$(u, u_\theta)\,|_{\theta=0} = (u, u_\theta)\,|_{\theta=2\pi}, \quad u\,|_{t=0} = u_0(\omega, \theta, \Omega) \equiv \rho_0(\omega, \theta, \Omega)\, \mathrm{e}^{-f(\omega)} \tag{3.13}$$

(see (3.3) and (3.6)). A number of properties of the function $u(\theta, \omega, t, \Omega)$, including its smoothness, positivity, and exponential decay as $|\omega| \to \infty$, hold according to the statements (1)–(7) in Theorem 2.1.

Consider now the special solution $R(\omega, t, \Omega)$ to equation in (3.8) (that is (2.10)) that is independent of θ. In this case, we have

$$\mathcal{S}_{\mathbf{R}}(\theta, t) \equiv 0. \tag{3.14}$$

Therefore, $R(\omega, t, \Omega)$ solves the Cauchy problem

$$R_t = L\, R, \qquad R\,|_{t=0} = R_0(\omega, \Omega), \tag{3.15}$$

for some given smooth function $R_0(\omega, \Omega)$, with

$$\sup_{\Omega \in [-G, G]} \left| D^{l_1, l_2}_{\omega, \Omega} R_0(\omega, \Omega) \right| \le c_0\, \mathrm{e}^{-m_0 \omega^2}.$$

Remark 3.1. Note that the solution $R(\omega, t, \Omega)$ to problem (3.15) does *not* depend on ε (the parameter multiplying the highest derivative in (3.8)), while it depends on N (the parameter of the “bounding function” in (2.8)). Therefore, all the constants appearing in this section do *not* depend on ε.

Lemma 3.1. *Suppose the data of the problem* (3.15) *satisfy Assumptions* (A)–(C) *in Section* 1 (*with* $l_0 = 1$ *in* (A) *and the possible exception of* (b_3)).

Then, for every $N > 0$, *there* exists *a* unique classical *solution,* $R(\omega, t, \Omega)$, *to problem* (3.15). *Such a solution*

1) is smooth;

2) is positive in Q_T;

3) is normalized as

$$\int_{-\infty}^{+\infty} R(\omega, t, \Omega)\, d\omega = \int_{-\infty}^{+\infty} R_0(\omega, \Omega)\, d\omega =: c_0, \tag{3.16}$$

for $t \in [0, T]$ *and* $\Omega \in [-G, G]$;

4) has an exponential decay at infinity in ω, *along with some derivatives, according to the estimate*

$$\sup_{t\in[0,T],\, \Omega\in[-G,G]} \left| D^l_\omega R(\omega, t, \Omega) \right| \le c\, \mathrm{e}^{-m\omega^2}$$

for $l \le 1$ *and* $\omega \in \mathbf{R}$, *where the constants* $c > 0$ *and* $m > 0$ *depend only on* N, G, T, KA, c_0, *and* m_0; *moreover,*

$$\sup_{\Omega\in[-G,G]} \left| D^{k,l}_{t,\omega} R(\omega, t, \Omega) \right| \le \frac{c}{\sqrt{t}}\, \mathrm{e}^{-m\omega^2}$$

for $2k + l = 2$, $\omega \in \mathbf{R}$, *and* $t \in (0, T]$, *with the same constants* $c, m > 0$ *introduced above*;

5) is uniformly square-integrable in ω; *i.e.,*

$$\int_{-\infty}^{+\infty} R^2(\omega, t, \Omega)\, d\omega \le c_1, \tag{3.17}$$

where c_1 *does depends neither on* N *nor on* T.

Proof. Proving existence and uniqueness of classical solutions to the Cauchy problem for the *linear* parabolic equation in (3.15) is standard. The properties (1), (2), and (4) could be easily established as was done in [8]. This part of the proof is based on two facts, namely, representation of solutions to parabolic problems as a convolution of the initial profile with the corresponding fundamental solution, and decay properties of such convolutions, established in [8]. The normalization property (3.16) follows immediately from equation (3.15) integrating both sides with respect to ω, since the solution $R(\omega, t, \Omega)$ is known to be smooth and to possess an appropriate decay rate as $|\omega| \to \infty$.

Hence, we need only to prove the uniformity of the estimate in (3.17) with respect to time. To do this, we multiply both sides of equation (3.15) by

$R(\omega, t, \Omega)$, integrate with respect to ω, and then integrate by parts. After simple calculations, we obtain

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} R^2(\omega,t,\Omega)\,d\omega + 2\int_{-\infty}^{+\infty} R_\omega^2(\omega,t,\Omega)d\omega = \int_{-\infty}^{+\infty} F_N'(\omega)R^2(\omega,t,\Omega)\,d\omega. \tag{3.18}$$

Note that, in view of the smoothness of $R(\omega, t, \Omega) > 0$ and its exponential decay as $|\omega| \to \infty$ for each $t > 0$ and each Ω, there exists a finite value of ω, say $\omega_1 = \omega_1(t, \Omega)$, such that

$$R(\omega_1(t,\Omega), t, \Omega) = \max_\omega R(\omega, t, \Omega) > 0. \tag{3.19}$$

According to the normalization identity in (3.16), on each unit interval of variation of ω there exists at least one point, such that $R(\omega, t, \Omega) \le c_0$. Otherwise, $R(\omega, t, \Omega) > 0$ for every value of its argument, and thus $\int_{-\infty}^{+\infty} R(\omega, t, \Omega)\,d\omega > c_0$, contradicting (3.16). In particular, there exists $\omega_2 = \omega_2(t, \Omega) \in [\omega_1(t, \Omega) - 1, \omega_1(t, \Omega))$, such that

$$R(\omega_2(t,\Omega), t, \Omega) \le c_0. \tag{3.20}$$

In view of the statement (2) of our lemma (the function R is positive), already established, we have for all $t \in [0, T]$

$$\begin{aligned}
&\int_{-\infty}^{+\infty} R^2(\omega,t,\Omega)\,d\omega \le R(\omega_1(t,\Omega),t,\Omega)\int_{-\infty}^{+\infty} R(\omega,t,\Omega)\,d\omega \qquad (3.21)\\
&= \left(R(\omega_2,t,\Omega) + \int_{\omega_2}^{\omega_1} \frac{\partial R}{\partial \omega}\,d\omega\right)\int_{-\infty}^{+\infty} R\,d\omega\\
&\le c_0^2 + c_0\int_{\omega_2}^{\omega_1} |R_\omega|\,d\omega \le \frac{3}{2}c_0^2 + \frac{1}{2}\int_{-\infty}^{+\infty} R_\omega^2\,d\omega
\end{aligned}$$

(see (3.16), (3.19), and (3.20)). From (3.18) and (3.21) (and (2.9)), we get the estimate

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} R^2(\omega,t,\Omega)\,d\omega + \frac{3}{2}\int_{-\infty}^{+\infty} R_\omega^2(\omega,t,\Omega)\,d\omega \le \frac{3}{2}c_0^2, \tag{3.22}$$

which produces the uniform estimate in (3.17). Now we follow the same scheme proposed in [7]. Consider the function

$$\Phi(t,\Omega) := \int_{-\infty}^{+\infty} R^2\,d\omega, \tag{3.23}$$

which is smooth in view of the properties of $R(\omega, t, \Omega)$. For all times $t > 0$ for which the inequality

$$\frac{\partial \Phi(t,\Omega)}{\partial t} \geq 0 \tag{3.24}$$

holds, we derive from (3.22)

$$\int_{-\infty}^{+\infty} R_\omega^2 \, d\omega \leq c_0^2. \tag{3.25}$$

Therefore, by (3.21), and for those values of t for which the inequality in (3.24) is satisfied, we have

$$\Phi(t,\Omega) = \int_{-\infty}^{+\infty} R^2 \, d\omega \leq \frac{3}{2} c_0^2 + \frac{1}{2} \int_{-\infty}^{+\infty} R_\omega^2 \, d\omega \leq 2c_0^2. \tag{3.26}$$

Thus, the smooth function $\Phi(t,\Omega)$ in (3.23) is uniformly bounded according to (3.26) for every Ω and at every time $t > 0$ when it does not decrease with t (see (3.24)). On the other hand, if $\frac{\partial \Phi}{\partial t} < 0$ on some interval $t \in (t_0, t_1)$, it follows that $\Phi(t,\Omega) \leq \Phi(t_0,\Omega)$; i.e., $\int_{-\infty}^{+\infty} R^2 \, d\omega \leq \int_{-\infty}^{+\infty} R^2|_{t=t_0} \, d\omega$. Note that either $t_0 = 0$, or relation (3.26) holds at that point. Therefore, the inequality in (3.17) holds with the constant $c := \max\left\{\int_{-\infty}^{+\infty} R_0^2 \, d\omega, 2c_0^2\right\}$, which is independent of T and N. □

We are now ready to establish a result similar to that in Lemma 3.1 for the function

$$v(\omega, t, \Omega) := \mathrm{e}^{-f(\omega)} R(\omega, t, \Omega), \tag{3.27}$$

where $R(\omega, t, \Omega)$ solves the problem (3.15), while the function $f(\omega)$ is defined according to (3.9). Recall that here it will be relevant that $f(\omega) < 0$. The point is that we do not know whether the exponential decay of $R(\omega, t, \Omega)$ as $|\omega| \to \infty$ is uniform or not with respect to the parameter N. On the other hand, by Lemma 3.1 we *do know* that the function $v(\omega, t, \Omega)$ in (3.27) is smooth, positive, and possesses an exponential decay as $|\omega| \to \infty$ for any given $\Omega \in [-G, G]$, $t > 0$, and $N > 0$. In fact, by item 4 of Lemma 3.1 and the fact that $f(\omega) < 0$, we infer that

$$v_t = \mathrm{e}^{-f(\omega)} R_t, \qquad v_\omega = \mathrm{e}^{-f(\omega)} R_\omega - f'(\omega) \mathrm{e}^{-f(\omega)} R;$$

hence, v enjoys the same decay properties as R, while (by the estimates in Lemma 3.1)

$$|v_\omega| = \left| R_\omega - f'(\omega) R \right| \, \mathrm{e}^{-f(\omega)} \leq (|R_\omega| + cR) \, e^{\omega} \leq \left(c e^{-m\omega^2} + c^2 e^{-m\omega^2} \right) e^{\omega}$$
$$\leq c(c+1) \, \mathrm{e}^{-m\omega^2 + \omega}.$$

Therefore, we shall prove that

Lemma 3.2. *Under the assumptions of Lemma* 3.1*, the function* $v(\omega,t,\Omega)$ *defined in* (3.27) *is square-integrable,*

$$\int_{-\infty}^{+\infty} v^2(\omega,t,\Omega)\,d\omega \le M_1, \tag{3.28}$$

uniformly *with respect to* N *and* T*. Note that the constant* M_1 *depends on* Ω*.*

Proof. As already mentioned, the idea of the proof is similar to that of Lemma 3.1. Clearly, the function $v(\omega,t,\Omega)$ solves the Cauchy problem

$$v_t = L^f\,v, \qquad v\,|_{t=0} = v_0(\omega,\Omega) := \mathrm{e}^{-f(\omega)}\,R_0(\omega,\Omega). \tag{3.29}$$

In fact, we have $\mathcal{S}^f_{\mathbf{v}} \equiv 0$ as v does not depend on θ. Multiplying both sides of equation (3.29) by $v(\omega,t,\Omega)$, and integrating with respect to ω, we get by a simple transformation the inequality

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} v^2\,d\omega + 2\int_{-\infty}^{+\infty} v_\omega^2\,d\omega \le 2\int_{-\infty}^{+\infty} F_N(\omega)\,f'(\omega)\,v^2\,d\omega + M_2\int_{-\infty}^{+\infty} v^2\,d\omega, \tag{3.30}$$

where the positive constant $M_2 := 1 + 2c^2 + 2c|\Omega|$ (since $|f'| \le c$) depends on $\|f'\|_{C^1}$ and Ω, while it is independent of N and T.

By the properties of $f(\omega)$ (defined in (3.9)) and of $F_N(\omega)$ (defined in (2.8)), we have $F_N(\omega)\,f'(\omega) \le 0$, and, moreover, $F_N(\omega)f'(\omega) \le -(M_2+1)$ for $|\omega| \ge M_2+1$, provided that $N > M_2+1$. Hence, we have

$$\begin{aligned} &2\int_{-\infty}^{+\infty} F_N(\omega)f'(\omega)v^2\,d\omega + M_2\int_{-\infty}^{+\infty} v^2\,d\omega + \int_{-\infty}^{+\infty} v^2\,d\omega \\ &\le -2(M_2+1)\int_{|\omega|\ge M_2+1} v^2\,d\omega + (M_2+1)\int_{-\infty}^{+\infty} v^2\,d\omega \qquad (3.31) \\ &= (M_2+1)\Big(-\int_{|\omega|\ge M_2+1} v^2 d\omega + \int_{|\omega|<M_2+1} v^2 d\omega\Big) < (M_2+1)\int_{|\omega|<M_2+1} v^2\,d\omega. \end{aligned}$$

Using (3.31), we obtain from (3.30)

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} v^2\,d\omega + 2\int_{-\infty}^{+\infty} v_\omega^2\,d\omega + \int_{-\infty}^{+\infty} v^2\,d\omega \le (M_2+1)\int_{-M_2-1}^{M_2+1} v^2\,d\omega. \tag{3.32}$$

According to the property (2.10) of the function $R(\omega,t,\Omega)$, we have

$$\int_{-M_2-1}^{M_2+1} v^2\,d\omega \equiv \int_{-M_2-1}^{M_2+1} \mathrm{e}^{-2f(\omega)}R^2\,d\omega \le M_3 \tag{3.33}$$

(see (3.27)), where M_3 is independent of N and T, but depends only on M_2. Using (3.32) and (3.33), we conclude that

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} v^2\,d\omega + 2\int_{-\infty}^{+\infty} v_\omega^2\,d\omega + \int_{-\infty}^{+\infty} v^2\,d\omega \le (M_2+1)M_3 =: M_4. \quad (3.34)$$

We use the same argument as in formulae (3.22)–(3.26); that is, if

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} v^2\,d\omega \ge 0,$$

then from (3.34) it follows that

$$\int_{-\infty}^{+\infty} v^2\,d\omega \le \frac{\partial}{\partial t}\int_{-\infty}^{+\infty} v^2\,d\omega + 2\int_{-\infty}^{+\infty} v_\omega^2\,d\omega + \int_{-\infty}^{+\infty} v^2\,d\omega \le M_5,$$

and hence

$$\int_{-\infty}^{+\infty} v^2\,d\omega \le M_5 \equiv M_5(L).$$

If, instead,

$$\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} v^2\,d\omega < 0$$

in some interval $t \in (t_0, t_1)$, it follows that

$$\int_{-\infty}^{+\infty} v^2\,d\omega \le \int_{-\infty}^{+\infty} v^2\big|_{t=t_0}\,d\omega = \int_{-\infty}^{+\infty} \mathrm{e}^{-2f(\omega)}\; R^2\big|_{t=t_0}\,d\omega =: \overline{M} \equiv \overline{M}(R_0),$$

wherefrom

$$\int_{-\infty}^{+\infty} v^2\,d\omega \le \max\{M_5(L), \overline{M}(R_0)\}.$$

Therefore, we obtain the estimate in (3.28) with the constant M_1 independent of N and T. □

Similarly to what has been done in Lemmas 3.1 and 3.2, we can establish an inequality stronger than (3.28); that is, not only is it $v \in L^2(\mathbf{R})$, but also $\sup|v| < \infty$.

Lemma 3.3. *Under the conditions of Lemma* 3.1*, the estimate*

$$\sup|v| + \int_{-\infty}^{+\infty} v_\omega^2\,d\omega \le M_6 \quad (3.35)$$

holds, where M_6 does not depend on N and T.

Hence, both $\sup|v| \le M_6$ and $\int_{-\infty}^{+\infty} v_\omega^2\,d\omega \le M_6$.

The proof is similar to that of Lemma 3.2 and just technical. Multiplying both sides of equation (3.29) by $v_{\omega\omega}(\omega,t,\Omega)$, after integrating with respect to ω and a number of integrations by parts, observing that

$$\int_{-\infty}^{+\infty} v_{\omega\omega}v_t\,d\omega = -\frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} v_\omega^2\,d\omega,$$

we derive the identity

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{-\infty}^{+\infty} v_\omega^2\,d\omega + \int_{-\infty}^{+\infty} v_{\omega\omega}^2\,d\omega \tag{3.36}$$

$$= -\int_{-\infty}^{+\infty}\Big[\frac{\partial}{\partial\omega}(F_N v) - \Omega v_\omega + 2f'v_\omega + \left(f'' + f'^2 + F_N f' - \Omega f'\right)v\Big]v_{\omega\omega}\,d\omega.$$

The two summands containing the term $F_N(\omega)$, unbounded as $N\to\infty$, can be handled as follows. For the first one, we have

$$-\int_{-\infty}^{+\infty}\frac{\partial}{\partial\omega}(F_N v)\,v_{\omega\omega}\,d\omega = \frac{3}{2}\int_{-\infty}^{+\infty} F_N'\,v_\omega^2\,d\omega - \frac{1}{2}\int_{-\infty}^{+\infty} F_N'''v^2\,d\omega. \tag{3.37}$$

Note that the multipliers F_N' and F_N''' do *not* depend on N. For the second one we obtain, integrating by parts,

$$\begin{aligned}-\int_{-\infty}^{+\infty} F_N f' v v_{\omega\omega}\,d\omega &= \int_{-\infty}^{+\infty} F_N f' v_\omega^2\,d\omega \\ &\quad + \int_{-\infty}^{+\infty} F_N f'' v v_\omega\,d\omega + \int_{-\infty}^{+\infty} F_N' f' v v_\omega\,d\omega.\end{aligned} \tag{3.38}$$

Note that $F_N f' \le 0$ in the first term of the right-hand side of (3.38). Moreover, the function $F_N(\omega)f''(\omega)$ is uniformly bounded with respect to N as $f''\equiv 0$ for $|\omega|>1$, while $F_N(\omega)\equiv\omega$ for $|\omega|<1$, $N>1$ (see (2.8) and (3.9)). Thus, as the first summand on the right-hand side of (3.38) is nonpositive, we obtain from (3.38), by the Cauchy-Schwarz inequality,

$$\begin{aligned}&-\int_{-\infty}^{+\infty} F_N f'\,v v_{\omega\omega}\,d\omega \\ &= \int_{-\infty}^{+\infty} F_N f'\,v_\omega^2\,d\omega + \int_{-\infty}^{+\infty} F_N f''\,v v_\omega\,d\omega + \int_{-\infty}^{+\infty} F_N' f'\,v v_\omega\,d\omega \\ &\le \int_{-\infty}^{+\infty} |F_n|\,\left|f''\right|\,|v v_\omega|\,d\omega + \int_{-\infty}^{+\infty}\left|F_N'\right|\,\left|f'\right|\,|v v_\omega|\,d\omega\end{aligned} \tag{3.39}$$

$$\equiv \int_{|\omega|\le 1} |F_N| \left|f''\right| |vv_\omega| \, d\omega + \int_{-\infty}^{+\infty} \left|F_N'\right| \left|f'\right| |vv_\omega| \, d\omega$$
$$\le c \int_{|\omega|\le 1} |\omega| \, |vv_\omega| \, d\omega + c \int_{-\infty}^{+\infty} |vv_\omega| \, d\omega$$
$$\le 2c \int_{-\infty}^{+\infty} |v| \, |v_\omega| \, d\omega \le M_7 \int_{-\infty}^{+\infty} v^2 \, d\omega + M_7 \int_{-\infty}^{+\infty} v_\omega^2 \, d\omega. \qquad (3.40)$$

Here $M_7 := 2c$ depends neither on $N > 0$ nor on $T > 0$. The remaining terms on the right-hand side of equation (3.36) can be estimated easily, by a rather lengthy though elementary procedure. Consider separately the two summands of the integrand which multiply $v_\omega v_{\omega\omega}$ and $vv_{\omega\omega}$, respectively. We obtain, integrating by parts,

$$\int_{-\infty}^{+\infty} (\Omega - 2f') \, v_\omega v_{\omega\omega} \, d\omega = \int_{-\infty}^{+\infty} f'' v_\omega^2 \, d\omega \le c \int_{-\infty}^{+\infty} v_\omega^2 \, d\omega; \qquad (3.41)$$

$$- \int_{-\infty}^{+\infty} (f'' + f'^2 - \Omega f') \, vv_{\omega\omega} \, d\omega$$
$$= \int_{-\infty}^{+\infty} v_\omega \left[v_\omega (f'' + f'^2 - \Omega f') + v \, (f''' + 2f'f'' - \Omega f'') \right] d\omega$$

$$\le c \, (1 + c + |\Omega|) \int_{-\infty}^{+\infty} v_\omega^2 \, d\omega + \frac{c}{2} \, (1 + 4c + |\Omega|) \int_{-\infty}^{+\infty} v^2 \, d\omega. \qquad (3.42)$$

Using (3.37), (3.39), (3.41), and (3.42), it is easy to derive from (3.36) the estimate

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} v_\omega^2 \, d\omega + 2 \int_{-\infty}^{+\infty} v_{\omega\omega}^2 \, d\omega \le M_8 \int_{-\infty}^{+\infty} v_\omega^2 \, d\omega + M_9 \int_{-\infty}^{+\infty} v^2 \, d\omega. \qquad (3.43)$$

Summing up side to side the inequalities in (3.34) and (3.43), multiplying first the former by the constant $(M_8 + 1)/2$, we obtain the estimate

$$\frac{\partial}{\partial t} \int_{-\infty}^{+\infty} \left(\frac{M_8 + 1}{2} v^2 + v_\omega^2 \right) d\omega + 2 \int_{-\infty}^{+\infty} v_{\omega\omega}^2 \, d\omega + \int_{-\infty}^{+\infty} v_\omega^2 \, d\omega \qquad (3.44)$$
$$\le M_{10} \int_{-\infty}^{+\infty} v^2 \, d\omega + M_{11},$$

where we set $M_{10} := \frac{M_8+1}{2} + M_9$ and $M_{11} := \frac{M_8+1}{2} M_5$. Now, following the same arguments adopted in Lemma 3.1 (see also [7]), it is easy to see that

$$\int_{-\infty}^{+\infty} v_\omega^2 \, d\omega + \frac{M_8+1}{2} \int_{-\infty}^{+\infty} v^2 \, d\omega \le M_{12}, \tag{3.45}$$

where M_{12} (as well as M_{10} and M_{11}) depends neither on N nor on T. Finally, we can evaluate the *uniform* norm, $\sup |v(\omega, t, \Omega)|$, proceeding as with relations (3.19)–(3.21). In fact, by the uniform estimate in (3.28), there exists, for every value of $\omega \in \mathbf{R}$, an $\omega_0 \in [\omega - 1, \omega)$, such that $v(\omega_0, t, \Omega) \le \sqrt{M_1}$. Therefore, by (3.45) using Hölder's inequality we get

$$\sup |v(\omega, t, \Omega)| = \sup \left| v(\omega_0, t, \Omega) + \int_{\omega_0}^{\omega} v_\omega(\omega, t, \Omega) \, d\omega \right| \le \sqrt{M_1} + \sqrt{M_{12}},$$

which, together with (3.45), proves the lemma. □

4. Time-independent estimates of solutions through a comparison theorem

Now we are ready to prove the main results of the paper. The time-independent estimates derived in the previous section for θ-independent solutions allow us to establish similar bounds for general (i.e., θ-dependent) solutions to both the regularized problem (equation (2.1) plus data) and the original "degenerate" problem (equation (2.10) plus data).

Theorem 4.1. *Suppose the data of the problem* (2.10), (2.11), (2.4) *satisfy Assumptions* (A)–(C) *in Section* 1 (*with* $l_0 = 2$ *in* (A) *and the possible exception of* (b_3)).

Then, the solution $\rho^\varepsilon(\theta, \omega, t, \Omega)$ *to the problem* (2.10), (2.11), (2.4) *satisfies the* time-independent *estimate*

$$\sup |\rho^\varepsilon(\theta, \omega, t, \Omega)| + \int_{-\infty}^{+\infty} \int_0^{2\pi} (\rho^\varepsilon)^2 \, d\theta \, d\omega \le M_{13}, \tag{4.1}$$

where the constant M_{13} *does not depend on* ε, N, *and* T, *but depends on* Ω.

Proof. We first compare the functions $u(\theta, \omega, t, \Omega)$ in (3.11) and $v(\omega, t, \Omega)$ in (3.27) (see also equations (3.12) and (3.29) along with (2.10) and (2.11)). As $v(\omega, t, \Omega)$ (defined in (3.27)) does not depend on θ, we have from (3.6)

$$\mathcal{S}_{\mathbf{v}}^f(t, \theta) = 0,$$

and thus we can also write the equation for $v(\omega, t, \Omega)$ as

$$v_t = L_\varepsilon^f \, v - \mathcal{S}_{\mathbf{v}}^f(t, \theta) \, \big(u_\omega + f'(\omega) \, u\big) \,. \tag{4.2}$$

Consider now the "weighted difference"

$$w(\theta,\omega,t,\Omega) := \mathrm{e}^{-\lambda t}\left[v(\omega,t,\Omega) - u(\theta,\omega,t,\Omega)\right], \tag{4.3}$$

where $\lambda > 0$ is a certain positive parameter to be chosen later. The function w in (4.3) solves the problem

$$w_t = L_\varepsilon^f\, w - \mathcal{S}_{\mathbf{w}}^f(\theta,t)\, b(\theta,\omega,t,\Omega) - \lambda\, w, \qquad w\,|_{t=0} = w_0 \tag{4.4}$$

with

$$w_0 = v_0 - u_0 \equiv \mathrm{e}^{-f(\omega)}\,(R_0(\omega,\Omega) - \rho_0^\varepsilon(\theta,\omega,\Omega)), \tag{4.5}$$

where we set for short

$$b(\theta,\omega,t,\Omega) := u_\omega(\theta,\omega,t,\Omega) + f'(\omega)\, u(\theta,\omega,t,\Omega). \tag{4.6}$$

Choosing the initial profile, $R_0(\omega,\Omega)$ in (3.15), not smaller than ρ_0^ε in (3.8), we obtain

$$w_0(\omega,\theta,\Omega) \ge 0. \tag{4.7}$$

Now, choose in (4.3) the parameter

$$\lambda := A_1 + A_f\, A_2 + 1, \tag{4.8}$$

where

$$A_1 := 1 + \sup_\omega |f''(\omega) + f'^2(\omega) - \Omega f'(\omega)|,$$
$$A_2 \equiv A_2(\varepsilon, N, t) := \sup_{\theta,\omega} |b(\theta,\omega,t,\Omega)| \equiv \sup_{\theta,\omega} |u_\omega + f'\, u|,$$
$$A_f := 2\pi \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(\Omega)\, \mathrm{e}^{f(\omega)}\, d\omega\, d\Omega.$$

As the functions $v(\omega,t,\Omega)$ in (3.27) and $u(\theta,\omega,t,\Omega)$ in (3.11) are both positive, we infer from (4.3) that, for every $t > 0$,

$$w(\theta,\omega,t,\Omega) \le \mathrm{e}^{-\lambda t} \sup_{\omega,\Omega} v(\omega,t,\Omega). \tag{4.9}$$

On the other hand, we can show that the inequality

$$w(\theta,\omega,t,\Omega) \ge -\mathrm{e}^{-\lambda t} \sup_{\omega,\Omega} v(\omega,t,\Omega) \tag{4.10}$$

holds. This can be proved by contradiction. Such a demonstration repeats in general terms the classical scheme followed when proving the maximum principle for parabolic equations. That is, assume the contradiction and consider the extremal point of the solution, and find out that the necessary properties required to be an extremum and the fact that our function solves the parabolic equation contradict each other. An additional problem in the present case is to evaluate the integral term, $\mathcal{S}^f_{\mathbf{w}}$, for which we use all available information, including the previously established inequality (4.9).

In fact, suppose that the inequality in (4.10) is wrong somewhere in Q_T. Let $(\theta^*, \omega^*, t^*, \Omega^*)$ be a point of Q_T (which depends on ε, N, and T) where $w(\theta, \omega, t, \Omega)$ attains its negative minimum. Such a point does exist with a finite value of ω^*, $|\omega^*| < K$, and $t^* > 0$, because T is finite and both v and u decay exponentially as $|\omega| \to \infty$, for any given λ, $\varepsilon > 0$, $N > 0$, and, moreover, (4.7) holds. As is well known, at this point of *negative* minimum the relations

$$w_t \le 0, \qquad L^f_\varepsilon\, w \ge (1 + f'' + f'^2 + F_N f' - \Omega f')\, w \ge A_1\, w \tag{4.11}$$

hold (cf. the definition of A_1 in (4.8) as well as the properties of $f(\omega)$ in (3.10) and $F_N(\omega)$ in (2.8), in particular that $F_N f' \le 0$ for all $\omega \in \mathbf{R}$). By the properties of $f(\omega)$, we also have

$$\left|\mathcal{S}^f_{\mathbf{w}}(\theta, t)\right| \le \sup|w| \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(\Omega)\mathrm{e}^{f(\omega)}\, d\omega\, d\Omega \cdot \int_0^{2\pi} |\sin(\varphi - \theta)|\; d\theta$$

$$\le \sup|w| \cdot 2\pi \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} g(\Omega)\, \mathrm{e}^{f(\omega)}\, d\omega\, d\Omega \equiv A_f \sup|w|$$

for $t = t^*$ (see (3.6)). Indeed, by inequality (4.9), the function $|w(\theta, \omega, t, \Omega)|$ attains its maximum value with respect to θ, ω, and Ω (for $t = t^*$), precisely at the minimum point $(\theta^*, \omega^*, t^*, \Omega^*)$ of w. Therefore, the estimate above could be rewritten as

$$|\mathcal{S}^f_{\mathbf{w}}(\theta, t)| \le A_f \cdot (-\min w) \equiv -A_f\, w(\theta^*, \omega^*, t^*, \Omega^*). \tag{4.12}$$

By the inequalities in (4.11) and (4.12) at the point $(\theta^*, \omega^*, t^*, \Omega^*)$ defined above, the left-hand side of equation (4.4) is nonpositive, while its right-hand side is bounded from below according to

$$L^f_\varepsilon\, w - \mathcal{S}^f_{\mathbf{w}}\, b - \lambda\, w \ge A_1\, w + A_f\, A_2\, w - \lambda w \equiv -w > 0. \tag{4.13}$$

This contradicts our assumption (that $w_t \le 0$), and therefore the relation (4.10) holds.

Now, using inequalities (4.9) and (4.10) together, we obtain by (3.35)

$$|w| = |u(\theta, \omega, t, \Omega) - v(\omega, t, \Omega)|\mathrm{e}^{-\lambda t} \le M_6 \mathrm{e}^{-\lambda t}; \tag{4.14}$$

i.e.,

$$\mathrm{e}^{-f(\omega)}|\rho^\varepsilon(\theta, \omega, t, \Omega) - R(\omega, t, \Omega)| \le M_6. \tag{4.15}$$

The function $e^{f(\omega)}$ being bounded ($f(\omega) < 0$; hence, $e^{f(\omega)} < 1$) and square-integrable (see (3.9)), and since $R = e^f v$, we infer finally from (3.35) and (4.15) the thesis of the theorem, (4.1). □

Remark 4.1. We proved Theorem 4.1 comparing the solutions to equation (2.10) with θ-independent solutions. Therefore, Theorem 4.1 can be regarded as a (version of the) comparison theorem for integroparabolic equations, where the space integrals are taken over unbounded domains.

Theorem 4.1 establishes that the solutions to the regularized problem (2.10), (2.11), (2.4) satisfy time-independent estimates, in a certain norm. This fact could be regarded as an additional property of the family of solutions constructed in [9]. An existence theorem for the solution to the limiting problem (2.1)–(2.4) has been proved in [9] by a compactness argument, that is, through the convergence of a sequence of solutions to the regularized problem. In view of Theorem 4.1, the aforementioned sequence posesses the additional property that all the constants in the estimates are time-independent. Therefore, we have, as an immediate corollary of results proved in [8, 9] and of Theorem 4.1, the following:

Theorem 4.2. *Suppose the data of the problem* (2.1)–(2.4) *satisfy all Assumptions* (A)–(C) *in Section* 1 (*with* $l_0 = 2$ *in (A)). Then, the solution* $\rho(\theta, \omega, t, \Omega)$ *to such a problem satisfies the* time-independent *estimate*

$$\sup |\rho(\theta, \omega, t, \Omega)| + \int_{-\infty}^{+\infty} \int_0^{2\pi} (\rho(\theta, \omega, t, \Omega))^2 \, d\theta \, d\omega \le M_{14}, \tag{4.16}$$

where the constant M_{14} *does not depend on* T*, but depends on* Ω.

This result represents the analog of that first obtained for the solution to the regularized problem.

5. Summary

Here we summarize the high points of the paper. First, time-independent estimates for solutions to a certain nonstandard integroparabolic equation over an unbounded domain have been established. Such estimates are important because long-time behavior of solutions have some interest for certain applications. Besides, they provide a basis for the numerical treatment of the aforementioned model equation. Furthermore, a version of the comparison theorem has been proved, suited to investigating parabolic equations containing space-integral terms. More precisely, we have shown that the general

(θ-dependent) solutions to problem (2.10), (2.11), (2.4) can be estimated in terms of θ-independent solutions; i.e.,

$$|\rho^\varepsilon(\theta,\omega,t,\Omega)| \leq \mathrm{e}^{f(\omega)}\,(|R(t,\omega,\Omega)| + M_6) \leq M'\,\mathrm{e}^{f(\omega)} \quad (< \infty)$$

(see (4.15)), wherefrom

$$(\rho^\varepsilon(\theta,\omega,t,\Omega))^2 \leq M''\mathrm{e}^{2f(\omega)},$$

and hence

$$\int_{-\infty}^{+\infty}\int_0^{2\pi} |\rho^\varepsilon(\theta,\omega,t,\Omega))|\; d\theta d\omega < 2\pi M''\int_{-\infty}^{+\infty} \mathrm{e}^{2f(\omega)}\,d\omega =: M''' \quad (< \infty).$$

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