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# FRONTS ON A LATTICE

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**Abstract.** Motivated by a model of a system of many particles at low densities, we consider a lattice differential equation with two uniform steady states and we investigate the existence of travelling fronts connecting them. This leads to a two-point boundary-value problem for a nonlinear delay-differential equation. We replace the original parabolic nonlinearity by a piecewise-linear function, where explicit computations are possible. We find monotone and nonmonotone fronts. Finally we also describe all the fronts such that the $\alpha$-limit is the unstable uniform state. For different values of the wave speed $c$ of the front we find bounded and unbounded as well as eventually periodic orbits, i.e., orbits $u_c(x)$ that are periodic for $x \geqslant x_{\mathrm{per}}(c)$ for some $x_{\mathrm{per}}(c) \in \mathbb{R}$.

## 1. Introduction

In this paper we study the propagation of fronts in the Cauchy problem for a sequence of functions $\{u_n(t)\}_{-\infty}^{\infty}$ defined on a one-dimensional infinitely extended lattice:

$$u_n' = -\,u_n + u_{n-1}^2, \qquad n \in \mathbb{Z}, \quad t > 0, \tag{1.1a}$$

$$u_n(0) = a_n \qquad n \in \mathbb{Z}, \tag{1.1b}$$

where $a_n \in [0,1]$ for every $n \in \mathbb{Z}$.

Problem (1.1) arises in the context of a clock model for a dilute gas of $N$ particles with short-range interactions, in which every particle carries a clock with a discrete time $k \in \mathbb{Z}$ which is advanced at every collision. This happens according to the following rule: when two particles collide, they *both* reset their respective clock values, say $k$ and $\ell$, to either $k+1$ or $\ell+1$,

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whichever is the largest. Thus, if we denote the number of particles with clock value $k$ by $N_k$, we obtain the following dynamical equation:

$$\frac{dN_k}{dt} = -\sum_{\substack{\ell=-\infty \\ \ell\neq k}}^{\infty} R_{k,\ell} - 2R_{k,k} + 2\sum_{\ell=-\infty}^{k-1} R_{k-1,\ell},$$

where $R_{k,\ell}$ denotes the rate by which collisions occur between particles with clock values $k$ and $\ell$. We assume this rate to be proportional to $N_k N_\ell / N^2$ when $k \neq \ell$ and to $N_k^2/(2N^2)$ when two particles with equal clock value $k$ collide. Then, writing $f_k = N_k/N$ and scaling the time appropriately, we are led to the equation

$$\frac{df_k}{dt} = -f_k + f_{k-1}^2 + 2f_{k-1}C_{k-2} = -f_k + C_{k-1}^2 - C_{k-2}^2,$$

where we have set

$$C_k = \sum_{\ell=-\infty}^{k} f_\ell.$$

Adding the equations for $f_\ell$ for all values of $\ell \leqslant k$ then yields

$$\frac{dC_k}{dt} = -C_k + C_{k-1}^2. \tag{1.2}$$

For further details of this model we refer to [17]. First equations such as (1.2) also arise in the context of cellular neural networks. For details we refer to [2] and [10] and the references cited there.

We can place equation (1.1) in the framework of a family of classical equations by writing it as

$$u_n' = -u_n + u_{n-1} - f(u_{n-1}), \tag{1.3}$$

where

$$f(s) = s(1-s). \tag{1.4}$$

The first two terms on the right of (1.3) can then be viewed as a "convection" term, and we can draw an analogy between (1.3) and the first-order PDE

$$w_t = -w_x - w(1-w). \tag{1.5}$$

It is well known that this equation has monotone travelling fronts of the form

$$u(x,t) = \varphi(\xi), \qquad \xi = x - ct, \qquad c \in \mathbb{R},$$

between the two constant solutions $w = 0$ and $w = 1$ such that

$$\varphi(-\infty) = 0 \qquad \text{and} \qquad \varphi(+\infty) = 1, \tag{1.6}$$

for wave speeds above a critical speed $c^* = 1$. This warrants the conjecture that equation (1.1) also has increasing fronts for a half-infinite interval of wave speeds $c$. Such fronts would be of the form

$$u_n(t) = \varphi(\xi), \qquad \xi = n - ct. \tag{1.7}$$

This is also observed numerically (see [4]).

A great deal of work has been done on the study of travelling waves on discrete versions of the Fisher equation

$$w_t = w_{xx} - f(w), \tag{1.8}$$

in which the second derivative is replaced by the discrete Laplacian and $f$ is given by (1.4). The discretization of the Nagumo equation, i.e., the discretized equation (1.8) with $f$ given by the cubic function

$$f(s) = s(1-s)(s-a), \qquad \text{for some } a \in (0,1), \tag{1.9}$$

has also been studied. We mention in particular the work by Zinner *et al.* [14, 15, 16], by Hsu *et al.* [9, 10, 11, 12], and that of Mallet-Paret *et al.* [1, 13], and the references given there. However, the assumptions in there do not cover equation (1.1a).

In our first result we assume the existence of a monotone front and establish a lower bound for the wave speed $c$. Specifically we prove that

$$c > \frac{1}{\log 2}. \tag{1.10}$$

In this paper we are concerned with the existence, uniqueness, and qualitative properties of travelling fronts of equation (1.3) in which the nonlinearity $f$ has been replaced by a piecewise-linear function,

$$f(s) = \begin{cases} s & \text{for} \quad s \leqslant \frac{1}{2}, \\ 1-s & \text{for} \quad s \geqslant \frac{1}{2}. \end{cases} \tag{1.11}$$

In this context we also mention the work of Elmer and Van Vleck, [5] and [6], on a piecewise rendering of the Nagumo equation. For the original variable $u_n$ this yields the equation

$$u_n' = -u_n + g(u_{n-1}), \tag{1.12}$$

where

$$g(s) = \begin{cases} 0 & \text{for} \quad s \leqslant \frac{1}{2}, \\ 2s-1 & \text{for} \quad s \geqslant \frac{1}{2}. \end{cases} \tag{1.13}$$

When we look for traveling waves, and make the substitution (1.7), we obtain

$$u_n'(t) = -c\,\varphi'(\xi) \quad \text{and} \quad u_{n-1}(t) = \varphi(\xi - 1),$$

so that the function $\varphi(\xi)$ needs to satisfy the delay-differential equation

$$c\varphi'(\xi) = \varphi(\xi) - g\big(\varphi(\xi - 1)\big), \qquad \xi \in \mathbb{R}. \tag{1.14}$$

We shall seek a solution $(c, \varphi)$ of equation (1.14) which connects the constant states, i.e.,

$$\varphi(-\infty) = 0 \qquad \text{and} \qquad \varphi(+\infty) = 1. \tag{1.15}$$

Before stating the main existence theorem, we introduce a *critical wave speed* $c_0$: it is the unique positive root of the equation

$$c\,e^{1/c} = 2\,e \qquad (c_0 = 4.31107\dots). \tag{1.16}$$

**Theorem 1.1.** *For each* $c \geqslant c_0$, *equation* (1.3) *with* $f$ *given by* (1.11) *has a unique monotone travelling wave* $u_n(t) = \varphi(n - ct)$, *where* $\varphi(\xi)$ *is a solution of problem* (1.14), (1.15).

In Figure 1 we show fronts for different wave speeds $c > c_0$.

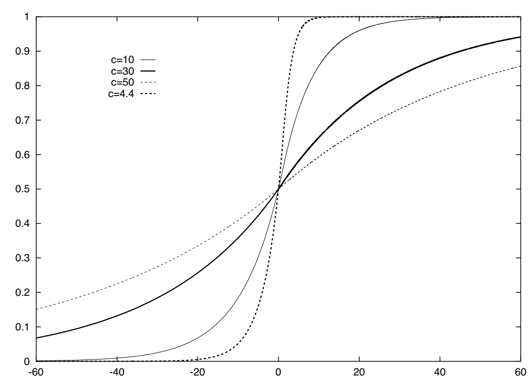



FIGURE 1. Monotone fronts for some values of $c > c_0$

A natural question to ask is whether there exist fronts for speeds below the critical speed $c_0$. Employing a mixture of analytical and numerical methods we find fronts for speeds in an interval $(c_{\text{unb}}, c_0)$, where $c_{\text{unb}} > 0$. In this regime the fronts are no longer monotone. In addition, we find two further critical speeds, $c_{\text{unb}} < c_{\text{per}} < c_{\text{bif}} < c_0$, where the shape of the front changes. In particular, for speeds in the interval $(c_{\text{unb}}, c_{\text{per}})$ numerical results exhibit fronts which tend to $\varphi = 0$ as $\xi \to -\infty$, but which are periodic for $\xi$ large enough. In Figure 2 we show a front for $c \in (c_{\text{bif}}, c_0)$ (Figure 2(a)) and an eventually periodic front for $c \in (c_{\text{unb}} < c_{\text{per}})$ (Figure 2(b)).

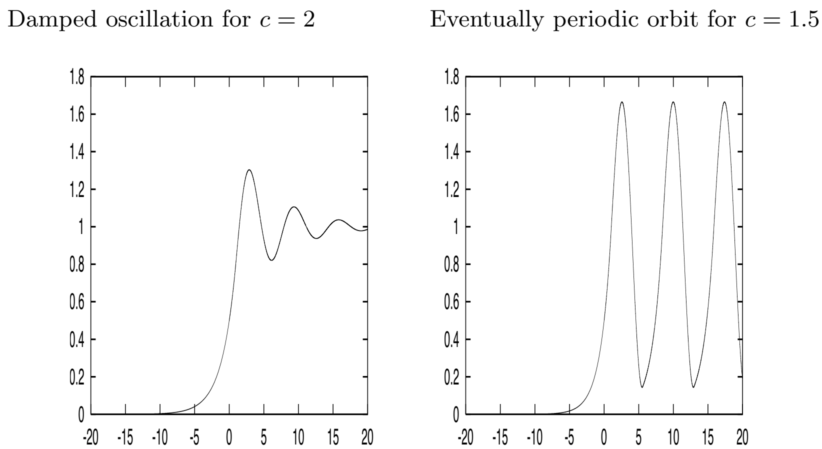



Fig. 2(a) Fig. 2(b)

FIGURE 2. Graphs of orbits for different values of $c$

Let us briefly describe the proof of Theorem 1.1. Since the function $f$ given by (1.11) is piecewise linear and the monotone travelling wave $\varphi$ increases from 0 to 1, there is no loss of generality when we fix $\varphi(0) = \frac{1}{2}$. Then, for $\xi < 1$, the function $\varphi(\xi)$ satisfies a linear ODE, which we can solve explicitly, and for $\xi > 1$ it satisfies a linear DDE which we will analyze in detail using the Laplace transform. Theorem 1.1 is then proved by means of an analysis of $\varphi'(\xi)$.

The plan of the paper is the following: in Section 2 we first prove the lower bound (1.10). In Section 3 we give a detailed outline of the proof Theorem 1.1. The roots of the characteristic equation of the linearized equation at $u = 1$ will play a central role in the analysis; this equation will be discussed in Section 4. In Section 5 we discuss the corresponding fundamental solution. In Section 6 we prove a monotonicity property when $c \geqslant c_0$, and in Section 7 we wrap up the proof of Theorem 1.1 and we present an explicit expression for the front for $c \geqslant c_0$.

The existence of waves established by Theorem 1.1 is the starting point of a study of fronts for $c < c_0$. This is done in Section 8. Using analytical arguments and numerical computations, we discuss the behaviour of *all* the fronts for $0 < c < c_0$ which satisfy equation (1.3), such that $\varphi(-\infty) = 0$.

There we show the existence of different types of orbits: nonmonotone fronts such that $\varphi(\infty) = 1$, as in Figure 2(a), bounded orbits where $\varphi(\infty)$ is not defined, *eventually periodic orbits*, as in Figure 2(b), and unbounded orbits.

## 2. A LOWER BOUND FOR THE WAVE SPEED C

In this section we consider increasing travelling-wave solutions of the equation

$$u_n' = -u_n + u_{n-1}^2,$$

and establish a lower bound for the wave speed. As explained in the Introduction we therefore need to study the problem

$$cu'(x) = u(x) - u^2(x-1) \qquad \text{for} \quad x \in \mathbb{R}, \tag{2.1a}$$

$$u(-\infty) = 0, \quad u(0) = \frac{1}{2}, \quad u(+\infty) = 1, \tag{2.1b}$$

where we have eliminated the translation invariance by pinning the solution at the origin.

We first prove a few bounds and estimates for increasing solutions of problem (2.1).

**Theorem 2.1.** *Let $u$ be an increasing solution of problem* (2.1)*. Then*

(1)

$$u(x) \leqslant \frac{e^{x/c}}{1+e^{x/c}} \qquad \textit{for} \quad x < 0,$$

*and hence $u(x) = O(e^{x/c})$ as $x \to -\infty$.*

(2)

$$1 - u(x) \leqslant \frac{1}{1+e^{x/c}} \qquad \textit{for} \quad x > 0,$$

*and hence $1 - u(x) = O(e^{-x/c})$ as $x \to +\infty$.*

(3)

$$\lim_{x \to -\infty} u(x)\, e^{-x/c} \stackrel{\text{def}}{=} \ell \in (\tfrac{1}{2}, 1).$$

**Proof.** The main ingredients of the proof are the monotonicity and the bounds $0 < u < 1$. From the differential equation we obtain

$$cu'(x) = u(x) - u^2(x-1) > u(x) - u^2(x).$$

Let $y > x$. Then, integrating this inequality over the interval $(x, y)$ yields

$$\frac{u(y)}{1-u(y)}\, e^{-y/c} > \frac{u(x)}{1-u(x)}\, e^{-x/c}. \tag{2.2}$$

For $x < 0$ and $y = 0$ we obtain Part 1, and for $x = 0$ and $y > 0$ we obtain Part 2.

To prove Part 3, we transform to the variable $v(x) = u(x)e^{-x/c}$. Plainly,

$$v'(x) = \{u'(x) - \frac{1}{c}u(x)\}\, e^{-x/c} = -\frac{1}{c}\, u^2(x-1)\, e^{-x/c} < 0.$$

Hence, $v$ is a decreasing function. By the first inequality $v$ is bounded above by 1. Therefore,

$$\lim_{x\to-\infty} v(x) = \ell_1 > v(0) = \frac{1}{2}.$$

Next, we define the function

$$w(x) = \frac{u(x)}{1-u(x)}\, e^{-x/c}.$$

We have shown in (2.2) that $w$ is a strictly increasing function. Therefore,

$$\lim_{x\to-\infty} w(x) = \ell_2 < w(0) = 1, \quad \text{and} \quad \lim_{x\to-\infty} v(x) = \lim_{x\to-\infty} w(x) < 1,$$

as asserted in Part 3.

We are now ready to prove the lower bound for the wave speed.

**Theorem 2.2.** *Let $u$ be an increasing solution of problem* (2.1). *Then* $c > 1/\log 2$.

**Proof.** We return to the function $v(x) = u(x)e^{-x/c}$. Because $u < 1$, we have the upper bound

$$v(x) < e^{-x/c} \qquad \text{for} \qquad x \in \mathbb{R}. \tag{2.3}$$

An elementary computation shows that $v$ is a solution of the integral equation

$$v(x) = \frac{1}{2} - \int_0^x \frac{1}{c}\, e^{(t-2)/c}\, v^2(t-1)dt, \tag{2.4}$$

and that it is bounded above by

$$v_+(x) = \begin{cases} 1 & x \leqslant -c\log 2 \\ \frac{1}{2}\, e^{-x/c} & -c\log 2 < x < 0 \\ \frac{1}{2} & 0 \leqslant x \leqslant c\log 2 \\ e^{-x/c} & x > c\log 2. \end{cases}$$

When we substitute this upper bound for $v$ into the integral equation (2.4), we obtain the following lower bound on the half line $x > 1 + c\log 2$:

$$v(x) > e^{-x/c} + \tfrac{1}{2} - \tfrac{3}{2}e^{-1/c} + e^{-2/c}. \tag{2.5}$$

From (2.3) and (2.5) we conclude that

$$\tfrac{1}{2} - \tfrac{3}{2}e^{-1/c} + e^{-2/c} < 0,$$

which implies that $c > \frac{1}{\log 2}$. This completes the proof of the theorem.

## 3. Outline

In order to prove Theorem 1.1 we need to find a *strictly increasing* solution of the two-point boundary-value problem

$$u'(x) = \tfrac{1}{c}\big\{u(x) - g\big(u(x-1)\big)\big\}, \qquad x \in \mathbb{R}, \tag{3.1a}$$

$$u(-\infty) = 0, \tag{3.1b}$$

$$u(+\infty) = 1. \tag{3.1c}$$

Since $u$ is strictly increasing from 0 to 1, there exists a unique value of $x$, where its graph crosses the line $u = \frac{1}{2}$. Without loss of generality we may choose this value to be the origin. Then $u(x-1) < \frac{1}{2}$ for $x < 1$, and we find that $u(x)$ must satisfy

$$u'(x) = \tfrac{1}{c}\, u(x) \qquad \text{for} \quad -\infty < x < 1, \tag{3.2a}$$

$$u(-\infty) = 0 \quad \text{and} \quad u(0) = \tfrac{1}{2}. \tag{3.2b}$$

Therefore,

$$u(x) = \tfrac{1}{2}\, e^{x/c} \qquad \text{for} \quad -\infty < x < 1. \tag{3.3}$$

Since it is required that $u(x) \to 0$ as $x \to -\infty$, it is evident that $c$ needs to be *positive*.

For $x > 1$ we have $u(x-1) > \frac{1}{2}$, and we must solve the problem

$$u'(x) = \tfrac{1}{c}\big\{u(x) + 1 - 2u(x-1)\big\} \qquad \text{for} \quad 1 < x < \infty, \tag{3.4a}$$

$$u(x) = \tfrac{1}{2}\, e^{x/c}, \qquad \text{for} \quad 0 \leqslant x \leqslant 1, \tag{3.4b}$$

$$u(\infty) = 1. \tag{3.4c}$$

It will be convenient to transform this problem, and introduce the variables

$$t = x - 1 \qquad \text{and} \qquad y(t) = 1 - u(x). \tag{3.5}$$

Problem (3.4) then becomes

$$y'(t) = \tfrac{1}{c}\big\{y(t) - 2y(t-1)\big\} \qquad \text{for} \quad 0 < t < \infty, \tag{3.6a}$$

$$y(t) = \eta(t) \qquad \text{for} \quad -1 \leqslant t \leqslant 0, \tag{3.6b}$$

$$y(\infty) = 0, \tag{3.6c}$$

where

$$\eta(t) \stackrel{\text{def}}{=} 1 - u(1+t) = 1 - \tfrac{1}{2}\, e^{(1+t)/c}. \tag{3.7}$$

We now need to identify values of $c > 0$ for which problem (3.6) has a solution with the property

$$y(t) < \tfrac{1}{2} \qquad \text{for} \quad 0 < t < \infty. \tag{3.8}$$

Then $u(x-1) > \frac{1}{2}$ for $x > 1$, as assumed at the outset.

Problem (3.6) is a linear problem which can be solved by means of the Laplace transform. Thus, we write formally

$$\hat{y}(s) = \mathcal{L}(y)(s) \stackrel{\text{def}}{=} \int_0^\infty y(t)\, e^{-st}\, dt.$$

Transformation of equation (3.6a) then yields

$$\begin{aligned}
\{s\hat{y}(s) - y(0)\} &= \frac{1}{c}\,\hat{y}(s) - \frac{2}{c}\int_0^\infty y(t-1)\, e^{-st}\, dt \\
&= \frac{1}{c}\,\hat{y}(s) - \frac{2}{c}\int_0^1 y(t-1)\, e^{-st}\, dt - \frac{2}{c}\int_1^\infty y(t-1)\, e^{-st}\, dt \\
&= \frac{1}{c}\,\hat{y}(s) - \frac{2}{c}\int_0^\infty \psi(t)\, e^{-st}\, dt - \frac{2}{c}\, e^{-s}\hat{y}(s),
\end{aligned}$$

where

$$\psi(t) = \begin{cases} \eta(t-1) = 1 - e^{t/c}, & 0 < t < 1 \\ 0, & \text{otherwise.} \end{cases} \tag{3.9}$$

Thus,

$$\Big(\frac{2}{c}e^{-s} + s - \frac{1}{c}\Big)\,\hat{y}(s) = \; y(0) - \frac{2}{c}\hat{\psi}(s). \tag{3.10}$$

Since

$$y(0) = \eta(0) = 1 - \frac{1}{2}e^{1/c}, \tag{3.11}$$

this yields the expression

$$\hat{y}(s) = \Big(1 - \frac{1}{2}e^{1/c}\Big)\hat{\Gamma}(s) - \frac{2}{c}\hat{\psi}(s)\hat{\Gamma}(s), \tag{3.12}$$

where

$$\hat{\Gamma}(s) = \frac{1}{\frac{2}{c}e^{-s} + s - \frac{1}{c}}. \tag{3.13}$$

Assuming for the moment that we can carry out the inverse Laplace transform, we formally obtain for $y(t)$

$$y(t) = \Big(1 - \frac{1}{2}e^{1/c}\Big)\,\Gamma(t) - \frac{2}{c}(\Gamma * \psi)(t), \tag{3.14}$$

where $*$ denotes the convolution product

$$(\Gamma * \psi)(t) = \int_0^t \psi(s)\ \Gamma(t-s)\, ds.$$

If it can be shown that $y(t)$ is a decreasing function, then substitution into (3.5) yields an increasing solution $u(x)$ of Problem (3.1).

## 4. The characteristic equation

The Laplace transform $\hat{\Gamma}(s)$ of the fundamental solution $\Gamma(t)$ has poles at the zeros of the *characteristic equation*,

$$\frac{2}{c}\, e^{-\lambda} + \lambda - \frac{1}{c} = 0. \tag{4.1}$$

In this section we investigate these zeros. Note that if $\lambda$ is a root of equation (4.1) then so is its complex conjugate $\overline{\lambda}$. Hence, we need only consider zeros with nonnegative imaginary part.

The complex equation (4.1) for $\lambda = a + ib$ is equivalent to the following system:

$$\tfrac{2}{c}\, e^{-a} \cos b = \tfrac{1}{c} - a, \tag{4.2a}$$

$$\tfrac{2}{c}\, e^{-a} \sin b = b. \tag{4.2b}$$

We note that (4.2b) is trivially satisfied for $b = 0$. For $a$ we then obtain from (4.2a) the equation

$$\frac{2}{c}\, e^{-a} + a - \frac{1}{c} = 0. \tag{4.3}$$

An elementary analysis leads to the following theorem about the real roots of equation (4.1):

**Theorem 4.1.** *There exist two constants $c_- \in (0, 2)$ and $c_0 > 2$ satisfying*

$$c\, e^{1/c} = 2e \qquad (c_0 = 4.31107\ldots, \quad c_- = 0.37336\ldots)$$

*such that*

(a) *For $c < 0$, equation* (4.3) *has exactly one positive solution of multiplicity one.*
(b) *For $c = 0$, equation* (4.3) *has exactly one positive solution* ($\lambda = \log 2$) *of multiplicity two.*
(c) *For $0 < c < c_-$, equation* (4.3) *has exactly two positive solutions.*
(d) *For $c = c_-$, equation* (4.3) *has exactly one positive solution of multiplicity two.*
(e) *For $c_- < c < c_0$, equation* (4.3) *has no real solution.*

(f) *For $c = c_0$, equation* (4.3) *has exactly one negative solution of multiplicity two:*

$$a_0 = -1 + \frac{1}{c_0} = -\log\left(\frac{c_0}{2}\right) \qquad (a_0 = -0.768\ldots). \tag{4.4}$$

(g) *For $c > c_0$, equation* (4.3) *has exactly two negative solutions $a_1(c)$ and $a_2(c)$ which satisfy $a_2(c) < a_0 < a_1(c) < 0$ and*

$$a_1(c) = -\frac{1}{c-2} + O\Big(\frac{1}{c^3}\Big) \qquad \textit{as} \quad c \to \infty.$$

In the next theorem we discuss the location of the complex roots of the characteristic equation. We make two preliminary observations:

($a$) From (4.2b) we deduce the inequality

$$e^a = \frac{2}{c}\frac{\sin b}{b} < \frac{2}{c}.$$

Hence, for $c > 2$ all the complex solutions have negative real part. Actually, this lower bound can be improved.

**Definition.** The critical wave speed $c_{\rm bif}$ will be defined as the biggest value for which the characteristic equation has purely imaginary roots.

We see from (4.2) that

$$c_{\rm bif} = \frac{3\sqrt{3}}{\pi} \approx 1.654. \tag{4.5}$$

In this section, as well as in Sections 5, 6, and 7, we will always consider $c > c_{\rm bif}$. In Section 8 we shall explore the range $c \in (0, c_{\rm bif})$.

($b$) Since by assumption $c > 0$ and $a < 0$, it follows from (4.2a) that $\cos(b) > 0$ and from (4.2b) that $\sin(b) > 0$. Hence

$$b \in I_k = \Big(2(k-2)\pi, 2(k-2)\pi + \frac{\pi}{2}\Big) \qquad \text{for some} \quad k \in \{2,3,4\ldots\}.$$

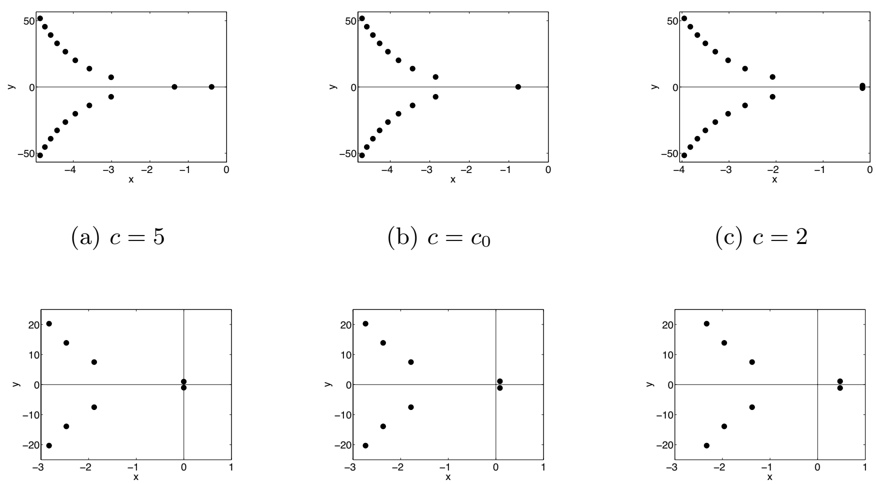
Elementary analysis shows that for $k = 2$, there exists a nonreal root if and only if $c < c_0$.

**Theorem 4.2.** *For each $k \geqslant 3$ there exists a unique nonreal complex root $\lambda_k = (a_k, b_k)$ of equation* (4.1) *with positive imaginary part such that if $c \geqslant c_0$, then $a_k(c) < a_2(c) < 0$, where $a_2(c)$ is the smallest real solution of* (4.1) *found in Theorem* 4.1, *and*

$$\begin{aligned} -\log\Big(2(k-2)\pi + \frac{\pi}{2}\Big) - \log(\sin b_k) < \quad & a_k + \log\left(\tfrac{c}{2}\right) \quad < -\log\big(2(k-2)\pi\big), \\ 2(k-2)\pi < \quad & \qquad b_k \qquad < 2(k-2)\pi + \frac{\pi}{2} \end{aligned}$$

*for* $k = 3, 4, 5, \ldots$. *In particular* $a_k = O(-\log k)$ *as* $k \to \infty$.

In Figure 3 we show the diagram of the eigenvalues of equation (4.1) for different values of wave speed $c$.




(a) $c = 5$ (b) $c = c_0$ (c) $c = 2$

(d) $c = c_{\text{bif}}$ (e) $c = 1.5$ (f) $c = 1$

FIGURE 3. Location of the eigenvalues for different values of $c$.

## 5. THE EXPANSION OF THE FUNDAMENTAL SOLUTION

In the last section we have investigated the poles of $\hat{\Gamma}(s)$ that we will need, in order to invert the Laplace operator. The inversion formula is given by

$$\Gamma(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{-iR}^{iR} e^{st}\, \hat{\Gamma}(s) ds \tag{5.1}$$

if $e^{st}\hat{\Gamma}(s)$ is integrable. In Lemma 5.1 it is shown that we may apply the residues formula, and express the integral in (5.1) as a series of residues computed at the poles of $\hat{\Gamma}(s)$, i.e., at the zeros $(a_k, \pm b_k)$ of the characteristic function

$$h(s) = \frac{2}{c}e^{-s} + s - \frac{1}{c}.$$

For convenience, let us introduce the following notation: for each $c > 0$, we will denote the set of *all* the roots of the characteristic function $h(s)$ by $\{z_k = z_k(c) : k = 1, 2, \ldots\}$. In last section we have found that they correspond to

$$\{a_1, a_2, a_k \pm ib_k : k = 3, 4, \ldots\} \qquad \text{for} \quad c > c_0,$$
$$\{a_k \pm ib_k : k = 2, 3, \ldots\} \qquad \text{for} \quad c < c_0.$$

**Lemma 5.1.** *Let $\{z_k\}$ be the sequence of all the roots of the function $h$. We have*

$$\Gamma(t) = \sum_{k=1}^{\infty} \operatorname{Res}\bigl(h^{-1}(s), s = z_k\bigr)\, e^{z_k t}. \tag{5.2}$$

**Proof.** We consider the sequence of semicircles $S_n$ of radius $R_n$:

$$S_n = \Bigl\{R_n e^{i\theta} : \frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2}\Bigr\}.$$

Here $R_n \to \infty$ as $n \to \infty$. In order to prove (5.2) we need to check that the integral taken along $S_n$ tends to zero when $n \to \infty$.

From Theorem 4.2 we know that $b_k \in \bigl(2(k-3)\pi, 2(k-3)\pi + \frac{\pi}{2}\bigr)$, $k = 3, 4, 5 \ldots$. By choosing the radii $R_k$ appropriately, we can ensure that on the semicircles $S_k$ the denominator of the integrand $h(s)$ is bounded away from zero. Specifically, we choose $R_k = (2k+1)\pi$. Then

$$|R_k - \lambda_k| \geqslant \bigl|R_k - |\lambda_k|\bigr| = \bigl|(2k+1)\pi - |b_k|\sqrt{1+\theta_k}\,\bigr|,$$

where

$$\theta_k = \frac{a_k^2}{b_k^2} \sim \Bigl(\frac{\log k}{2\pi k}\Bigr)^2 \qquad \text{as} \quad k \to \infty.$$

Therefore,

$$\liminf_{k\to\infty}\ |R_k - \lambda_k| \geqslant \frac{\pi}{2}.$$

With this choice of radii $R_n$, and with $n$ large enough, each successive semidisk contains one more zero and on $S_n$ the denominator is bounded away from zero as $n \to \infty$.

The next step is to estimate the integrand on $S_n$ and to show that it tends to zero as $n \to \infty$ sufficiently fast. To that end we divide $S_n$ into two parts:

$$S_n^1 = \{s \in S_n : \operatorname{Re}(s) \leqslant -\mu \log|s|\},$$
$$S_n^2 = \{s \in S_n : \operatorname{Re}(s) > -\mu \log|s|\},$$

where $\mu$ is a positive constant.

We take $\mu > 1$. Then on $S_n^1$ we have

$$|se^s| \leqslant e^{(1-\mu)\log|s|} \to 0 \quad \text{as} \quad |s| \to \infty,$$

so that

$$|e^{-s}h^{-1}(s)| = \frac{c}{|2 + cse^s - e^s|} \to \frac{c}{2} \quad \text{as} \quad |s| \to \infty.$$

Therefore,

$$\sup_{s\in S_n^1} |h^{-1}(s)| \sim \frac{c}{2} \sup_{s\in S_n^1} |e^s| = \frac{c}{2}\, e^{-\mu \log R_n} \quad \text{as} \quad R_n \to \infty.$$

Since the length of $S_n^1$ is less than $\pi R_n$, and $|e^{st}| \leqslant 1$ on $S_n^1$, it follows that

$$\Big| \int_{S_n^1} h^{-1}(s)e^{st}ds \Big| \leqslant O\big(e^{(1-\mu)\log R_n}\big) \quad \text{as} \quad R_n \to \infty.$$

Because $\mu > 1$ the integral over $S_n^1$ converges to zero uniformly for $t \geqslant 0$.

Next, we show that the integral over $S_n^2$ converges to zero as well. We write $S_n^2$ in complex notation:

$$S_n^2 = \Big\{R_n e^{\pm i\theta} : \frac{\pi}{2} < \theta < \arccos\Big(-\mu\frac{\log(R_n)}{R_n}\Big)\Big\}.$$

The Taylor expansion

$$\arccos(x) = \frac{\pi}{2} - x + O(x^3) \quad \text{as} \quad x \to 0$$

implies that, as $R_n \to \infty$, the length of $S_n^2$ is given asymptotically by

$$|S_n^2| = 2\mu\log(R_n) + O\Big(\frac{\log^3(R_n)}{R_n^2}\Big).$$

Since the exponential factor is bounded in modulus by one, and $h^{-1}(s) = O(R_n^{-1})$ on $S_n^2$, we conclude that

$$\Big| \int_{S_n^2} h^{-1}(s)e^{st}ds \Big| \leqslant O\Big(\frac{\log R_n}{R_n}\Big) \quad \text{as} \quad R_n \to \infty.$$

Thus, we have shown that

$$\lim_{R_n\to\infty} \int_{S_n} e^{st}\,\hat{\Gamma}(s)ds = 0,$$

and we can conclude by the residues formula that

$$\Gamma(t) = \frac{1}{2\pi i} \lim_{R\to\infty} \int_{-iR}^{iR} e^{st}\,\hat{\Gamma}(s)ds = \sum_{k=1}^{\infty} \text{Res}(h^{-1}(s), s = z_k)e^{z_k t}.$$

Since we have shown the validity of the residues formula, next we will compute the residues. We note that if $c > c_0$, then all the roots of $h(s)$ are simple and with negative real part and if $c = c_0$, then $h(s)$ has one double root. We will treat both cases separately.

**Lemma 5.2.** *Suppose that $c > c_0$. Then*

$$\Gamma(t) = \sum_{k=1}^{2} \frac{1}{h'(a_k)}\, e^{a_k t} + \sum_{k=3}^{\infty} \Big( \frac{1}{h'(\lambda_k)}\, e^{\lambda_k t} + cc \Big). \tag{5.3}$$

*Since*

$$h'(\lambda_k) = \lambda_k + 1 - \frac{1}{c},$$

*the expansion* (5.3) *can be written as*

$$\Gamma(t) = \sum_{k=1}^{2} \frac{e^{a_k t}}{a_k + 1 - \frac{1}{c}} + 2 \sum_{k=3}^{\infty} \frac{(a_k + 1 - \frac{1}{c}) \cos(b_k t) + b_k \sin(b_k t)}{(a_k + 1 - \frac{1}{c})^2 + b_k^2}\, e^{a_k t}. \tag{5.4}$$

**Proof.** The lemma is an immediate consequence of the residues formula (5.2). When $c > c_0$ the residues of $\hat{\Gamma}(s) = \frac{1}{h(s)}$ are given by $\frac{1}{h'(\lambda_k)}$.

**Lemma 5.3.** *Suppose that $c = c_0$. Then*

$$\Gamma(t) = 2\Big(t + \frac{1}{3}\Big)\, e^{a_0 t} + \sum_{k=3}^{\infty} \Big( \frac{1}{h'(\lambda_k)}\, e^{\lambda_k t} + \mathrm{cc} \Big), \tag{5.5}$$

*and the expansion* (5.5) *can be written as*

$$\Gamma(t) = 2\Big(t + \frac{1}{3}\Big)\, e^{a_0 t} + 2 \sum_{k=3}^{\infty} \frac{(a_k + 1 - \frac{1}{c_0}) \cos(b_k t) + b_k \sin(b_k t)}{(a_k + 1 - \frac{1}{c_0})^2 + b_k^2}\, e^{a_k t}. \tag{5.6}$$

**Proof.** When $c = c_0$, the computation made for the zeros $\lambda_k$ which are not real-valued is still valid since these zeros are all simple. However, in this case the zero $a_0$ (see Theorem 4.1) has multiplicity two and the computation is different.

Since $h$ is analytic we have the Taylor expansion

$$h(s) = \alpha_0 (s - a_0)^2 + \alpha_1 (s - a_0)^3 + \cdots, \tag{5.7}$$

which is valid in the disk with circumference $S_r = \{|s - a_0| = r\}$ for some sufficiently small $r$, and $\alpha_0 \neq 0$. Therefore

$$\frac{1}{h(s)} = \frac{1}{\alpha_0} \frac{1}{(s - a_0)^2} - \frac{\alpha_1}{\alpha_0^2} \frac{1}{s - a_0} + \text{ analytic function}$$

and, writing $w = s - a_0$,

$$\begin{aligned}\frac{1}{2\pi i}\int_{S_r}\frac{e^{st}ds}{h(s)} &= \frac{e^{a_0t}}{2\pi i}\int_{S_r}\frac{e^{wt}ds}{h(s)} = \frac{e^{a_0t}}{2\pi i}\frac{1}{\alpha_0}\int_{S_r}e^{wt}\Big(\frac{1}{w^2}-\frac{\alpha_1}{\alpha_0}\frac{1}{w}+\cdots\Big)\,ds\\ &= \frac{e^{a_0t}}{2\pi i}\frac{1}{\alpha_0}\int_{S_r}\Big(\frac{1}{w^2}+\frac{t}{w}-\frac{\alpha_1}{\alpha_0}\frac{1}{w}+\cdots\Big)\,ds\\ &= \frac{1}{\alpha_0}\Big(t-\frac{\alpha_1}{\alpha_0}\Big)e^{a_0t}.\end{aligned} \tag{5.8}$$

It remains to compute $\alpha_0$ and $\alpha_1$. We find

$$\alpha_0 = \lim_{s\to a_0}\frac{h(s)}{(s-a_0)^2} = \frac{1}{2}\,h''(a_0) = \frac{1}{2},$$

and

$$\alpha_1 = \lim_{s\to a_0}\frac{h(s)-a_0(s-a_0)^2}{(s-a_0)^3} = \frac{1}{6}\,h'''(a_0) = -\,\frac{1}{6}.$$

If we use these expressions in (5.8) we obtain the desired expansions (5.5) and (5.6).

## 6. MONOTONICITY

We recall that we are interested in establishing monotonicity for two reasons. First, from the original application, $u_n$ denotes the sum of positive quantities, so it must be nondecreasing. On the other hand, it remains to check whether the solution $u(x)$ given in (3.14), with $\Gamma(t)$ given by the expansion formula in (5.3) or (5.5), satisfies the assumption that $u(x) > \frac{1}{2}$ for $x > 0$, which we made at the outset. Monotonicity of $u(x)$ automatically ensures that this is indeed the case.

As in Section 3, we transform to the variables $t$ and $y$ introduced in (3.5):

$$t = x - 1 \qquad \text{and} \qquad y(t) = 1 - u(x).$$

We shall prove the following monotonicity result.

**Theorem 6.1.** *For every $c \geqslant c_0$ the solution $y(t)$ of the problem*

$$y'(t) = \tfrac{1}{c}\{y(t) - 2y(t-1)\} \qquad \textit{for} \quad 0 < t < \infty \tag{6.1a}$$

$$y(t) = \eta(t) = 1 - \tfrac{1}{2}e^{(1+t)/c} \qquad \textit{for} \quad t \leqslant 0 \tag{6.1b}$$

*is strictly decreasing.*

**Proof.** The proof is carried out again by means of the Laplace transform, as in Section 3. We begin with a preliminary result.

**Lemma 6.1.** *Let* $y(t)$ *be the solution of problem* (6.1)*. Then*

$$y'(t) = -\frac{1}{2c}\,\Gamma(t+1) \qquad \textit{for} \quad t > 0. \tag{6.2}$$

**Proof.** We multiply equation (6.1a) by $e^{-st}$, integrate over $t \in (0, \infty)$, and express all the integrals in terms of $\int_{-1}^{\infty} e^{-st} y'(t)dt$. After an elementary computation this yields

$$h(s) \int_{-1}^{\infty} e^{-st} y'(t)dt = -\frac{1}{2c}\, e^{s}.$$

Since

$$\int_{-1}^{\infty} e^{-st} y'(t)dt = \int_{-1}^{0} e^{-st} \eta'(t)dt + \mathcal{L}(y')(s),$$

we obtain for the Laplace transform of $y'$

$$\mathcal{L}(y')(s) = -\frac{1}{2c}\,\frac{e^{s}}{h(s)} - \int_{-1}^{0} e^{-st}\eta'(t)dt = -\frac{1}{2c} I_1(s) - I_2(s). \tag{6.3}$$

An elementary computation yields for $I_2(s)$

$$I_2(s) = \int_{-1}^{0} e^{-st}\eta'(t)dt = -\frac{1}{2c}\,\frac{e^{s} - e^{(1/c)}}{s-(1/c)}. \tag{6.4}$$

To find an expression for $I_1(s)$, we compute the Laplace transform of $\Gamma(t+1)$:

$$\mathcal{L}(\Gamma(t+1))(s) = \int_{0}^{\infty} \Gamma(t+1)e^{-st}\,dt = e^{s}\Big(\int_{0}^{\infty} - \int_{0}^{1}\Big)\Gamma(\tau)e^{-s\tau}\,d\tau.$$

From [8], pages 19 and 20, we know that $\Gamma(t)$ is the solution of equation (6.1a) with initial data on $[-1, 0]$ given by

$$y(t) = 0 \quad \text{for} \quad -1 \leqslant t < 0, \quad \text{and} \quad y(0) = 1.$$

Therefore, $\Gamma(t) = e^{t/c}$ for $0 < t < 1$, and hence

$$\mathcal{L}(\Gamma(1+t))(s) = \frac{e^{s}}{h(s)} - \frac{e^{s} - e^{1/c}}{s-(1/c)} = \frac{e^{s}}{h(s)} + 2cI_2(s). \tag{6.5}$$

Thus, putting (6.4) and (6.5) into (6.3) we can conclude

$$\mathcal{L}\big(y'(\cdot)\big)(s) = \mathcal{L}\big(\Gamma(1+\cdot)\big)(s), \tag{6.6}$$

and (6.2) follows.

Thus, it suffices to show that $\Gamma(t)$ is positive for $t > 1$. We will again treat the cases $c > c_0$ and $c = c_0$ separately.

(a) **Positivity of $\mathbf{\Gamma(t)}$ for the case $\mathbf{c > c_0}$**

We recall from (5.3) that $\Gamma(t)$ is given by

$$\Gamma(t) = \frac{e^{a_1 t}}{a_1 + 1 - 1/c} + \frac{e^{a_2 t}}{a_2 + 1 - 1/c} + 2\sum_{k=3}^{\infty} A_k(t) \frac{1}{|h'(\lambda_k)|}\, e^{a_k t},$$

where $a_1$ and $a_2$ are such that $a_1 + 1 - 1/c > 0$ and $a_2 + 1 - 1/c < 0$, and $|A_k(t)| \leqslant 1$ for every $k \geqslant 3$. To show that $\Gamma(t)$ is positive we need a bound on the third term $R(t)$. By Theorem 4.2, the exponent $a_k t$ can be estimated by

$$a_k t = -t\log\left(\frac{c}{2}\right) + t\log\frac{\sin b_k}{b_k} \leqslant -t\log\left(\frac{c}{2}\right) - t\log\big(2(k-2)\pi\big).$$

Hence

$$\begin{aligned} |R(t)| &\leqslant 2\sum_{k=3}^{\infty} \frac{1}{2(k-2)\pi}\, e^{a_k t} \leqslant \sum_{k=3}^{\infty} \frac{1}{(k-2)\pi}\, e^{-t\log(c/2) - t\log(2(k-2)\pi)} \\ &= \frac{1}{\pi}\, e^{-t\log(\pi c)} \sum_{k=3}^{\infty} \left(\frac{1}{k-2}\right)^{t+1}. \end{aligned}$$

We are interested in the range $t \geqslant 1$. Then

$$|R(t)| \leqslant \frac{1}{\pi}\, e^{-t\log(\pi c)} \sum_{k=3}^{\infty} \left(\frac{1}{k-2}\right)^2 = \frac{\pi}{6}\, e^{-t\log(\pi c)},$$

so that

$$\Gamma(t) \geqslant \frac{e^{a_1 t}}{a_1 + 1 - 1/c} + \frac{e^{a_2 t}}{a_2 + 1 - 1/c} - \frac{\pi}{6}\, e^{-t\log(\pi c)} \stackrel{\text{def}}{=} Q(t,c) \qquad \text{for} \quad t \geqslant 1.$$

**Proposition 6.1.** *If $c > c_0$, then $Q(t,c) > 0$ for $t \geqslant 1$.*

**Proof.** We consider the equivalent problem of showing the positivity of the function $Q(t,c)e^{-a_1 t}$, i.e., of

$$Q(t,c)e^{-a_1 t} = \frac{1}{a_1 + 1 - 1/c} + \frac{e^{(a_2 - a_1)t}}{a_2 + 1 - 1/c} - \frac{\pi}{6}\, e^{-t(a_1 + \log(\pi c))}$$

for $c > c_0$ and $t \geqslant 1$. This function is seen to be increasing with respect to $t$. Therefore, its minimum is reached at $t = 1$. Hence, it remains to show that the function

$$Q(1,c) = \frac{1}{a_1 + 1 - 1/c} + \frac{e^{a_2 - a_1}}{a_2 + 1 - 1/c} - \frac{\pi}{6}\, e^{-(a_1 + \log(\pi c))}$$

is positive for every $c > c_0$. This can easily be checked by the use of a computer program.

(b) **Positivity of $\Gamma(t)$ for the case $c = c_0$**

We first consider the range $t \geqslant 1$. There

$$\Gamma(t) \geqslant 2\Big(t + \frac{1}{3}\Big)e^{a_0 t} - \frac{\pi}{6}\, e^{-t\log(\pi c_0)} \stackrel{\text{def}}{=} Q_0(t).$$

As in the previous case, the function $e^{-a_0 t}Q_0(t)$ is seen to be increasing for $1 \leqslant t < \infty$, and hence $e^{-a_0 t}Q_0(t) \geqslant e^{-a_0}Q_0(1) > 0$. Thus $\Gamma(t) > 0$, and we conclude that $y(t)$ is a decreasing function for $t \geqslant 0$.

## 7. The final solution

The solution of Problem (3.1) can be expressed in the form of an expansion.

**Theorem 7.1.** *If $c > c_0$, the solution $u(x)$ of Problem* (3.1) *can be written as*

$$u(x) = \begin{cases} \frac{1}{2}\, e^{x/c} & \text{for} \quad x \leqslant 1, \\ 1 - \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{(-z_k)(c + cz_k - 1)}\, e^{z_k x} & \text{for} \quad x > 1, \end{cases} \tag{7.1}$$

*where $\{z_k : k = 1, 2, 3, \ldots\}$ is the set of roots of the characteristic equation* (4.1).

*If $c = c_0$ we have*

$$u(x) = \begin{cases} \frac{1}{2}\, e^{x/c_0} & \text{for } x \leqslant 1, \\ 1 - \frac{1}{2}\big(x - b\big)e^{a_0 x} - \frac{1}{2}\sum_{k=2}^{\infty} \frac{1}{(-z_k)(c_0 + c_0 z_k - 1)}\, e^{z_k x} & \text{for } x > 1, \end{cases} \tag{7.2}$$

*where $\{z_k\}$ denotes all the nonreal complex roots of the characteristic function $h(s)$ and*

$$b = \frac{3c_0^2 + c_0 - 1}{3(c_0 - 1)^2} \approx 1.796.$$

**Proof.** Recall the change of variables used in (3.5). One way is to express the solution is via the equality (3.14). Another possibility is to use the equality (6.2) obtained in Lemma 6.1. We will use the latter.

Thus, in (6.2) we have transformed the original delay-differential equation (3.6) into an ordinary differential equation

$$\begin{cases} y'(t) = -\frac{1}{2c}\,\Gamma(t+1), & \text{for} \quad 0 < t < \infty, \\ y(0) = \eta(0) = 1 - \frac{1}{2}\, e^{1/c}. \end{cases} \tag{7.3}$$

The solution of (7.3) is given formally by

$$y(t) = \eta(0) - \frac{1}{2c}\int_0^t \Gamma(s+1)\,ds \qquad \text{for} \quad t \geqslant 0. \tag{7.4}$$

We make use of the expansion for $\Gamma$ obtained in (5.3) and (5.5) for the cases $c > c_0$ and $c = c_0$ respectively.

**The expansion for the case $c > c_0$.** We recall from (5.3) that $\Gamma(t)$ was given by

$$\Gamma(t) = \sum_{k=1}^{\infty} \frac{1}{h'(z_k)}\, e^{z_k t} = \sum_{k=1}^{\infty} \frac{1}{\frac{1}{c} - 1 - z_k}\, e^{z_k t}.$$

We substitute this expression into (7.4). An elementary computation shows that

$$y(t) = \eta(0) + \sum_{k=1}^{\infty} \alpha_k (e^{z_k t} - 1), \tag{7.5}$$

where the constants $\alpha_k$ are given by

$$\alpha_k = \frac{1}{z_k(1 - cz_k)(1 - c - cz_k)}. \tag{7.6}$$

We can simplify expression (7.5). We know that there exists a limit of (7.5) as $t$ tends to infinity, which can only be a steady-state solution of (3.6a), this is, zero. This means

$$\sum_{k=1}^{\infty} \alpha_k = \eta(0) = 1 - \frac{1}{2}\, e^{1/c}, \tag{7.7}$$

so that (7.5) becomes

$$y(t) = \sum_{k=1}^{\infty} \alpha_k e^{z_k t}. \tag{7.8}$$

Returning to the original variables $x$ and $u(x)$ we obtain (7.1) as required.

**The expansion for the case $c = c_0$.** We recall from (5.5) that $\Gamma(t)$ is given by

$$\Gamma(t) = 2\Big(t + \frac{1}{3}\Big)e^{a_0 t} + \sum_{k=2}^{\infty} \Big(\frac{1}{h'(z_k)}\, e^{z_k t}\Big).$$

When we put this expression into (7.5), with $c = c_0$,

$$y(t) = \eta(0) + \frac{1}{c_0}\Big(t - \frac{7c_0 - 4}{3(c_0 - 1)^2}\Big)\, e^{a_0 t} + \frac{7c_0 - 4}{3c_0(c_0 - 1)^2} + \sum_{k=2}^{\infty} \big\{\alpha_k (e^{z_k t} - 1)\big\}. \tag{7.9}$$

We can simplify the expression (7.9) for $y(t)$. We take the limit as $t$ tends to $\infty$, which will be zero. This yields the relation

$$\sum_{k=2}^{\infty} \alpha_k = \eta(0) + \frac{7c_0 - 4}{3c_0(c_0-1)^2} = 1 - \frac{e}{c_0} + \frac{7c_0 - 4}{3c_0(c_0-1)^2}, \tag{7.10}$$

and (7.9) becomes

$$y(t) = \frac{1}{c_0}\Big(t - \frac{7c_0 - 4}{3(c_0-1)^2}\Big)e^{a_0 t} + \sum_{k=2}^{\infty} \alpha_k e^{z_k t}. \tag{7.11}$$

Return to the original variables $x$ and $u(x)$ yields (7.2).

## 8. The range $c < c_0$

Having established the existence of monotone fronts for $c \geqslant c_0$, we now turn to the regime $c < c_0$. In describing the different types of fronts we observe, we distinguish a series of critical wave speeds:

$c_0$ : The smallest value of $c$ for which the characteristic equation (4.1) has real roots: two for $c > c_0$ and a double root for $c = c_0$ (cf. Theorem 4.1).

$c_{\text{bif}}$ : The value of $c$ for which the set of roots of the characteristic equation reaches the imaginary axis (cf. (4.5)).

For $c_{\text{bif}} < c < c_0$ the roots still have negative real part and the uniform state $u = 1$ is asymptotically stable. Thus, for $c_{\text{bif}} < c < c_0$ we may still expect fronts to exist, but with oscillatory tails as $x \to \infty$.

In the range $0 < c < c_{\text{bif}}$, we distinguish two further critical wave speeds: $c_{\text{unb}}$, below which fronts are unbounded, and $c_{\text{per}}$: for $c \in (c_{\text{unb}}, c_{\text{per}})$ solutions are eventually periodic. These values are ordered according to

$$0 < c_{\text{unb}} < c_{\text{per}} < c_{\text{bif}} < c_0.$$

The numerical values are found to be

$$c_{\text{unb}} = 1.455\ldots, \quad c_{\text{per}} = 1.618\ldots, \quad c_{\text{bif}} = 1.654\ldots, \quad c_0 = 4.311\ldots.$$

Our approach will be partly numerical and partly analytical. Results based in part or wholly on numerical simulations will be formulated in propositions.

Let us give an overview of the results that we will describe in this section. The goal is to obtain a qualitative description of the solution $u(x, c)$ of the problem

$$u'(x) = \tfrac{1}{c}\, u(x), \qquad \text{if } x > 1 \text{ and } u(x-1) \leqslant \tfrac{1}{2}, \tag{8.1a}$$

$$u'(x) = \tfrac{1}{c}\, \{u(x) + 1 - 2u(x-1)\} \qquad \text{if } x > 1 \text{ and } u(x-1) \geqslant \tfrac{1}{2}, \tag{8.1b}$$

$$u(x) = \tfrac{1}{2}\, e^{\frac{x}{c}} \qquad \text{for} \quad x \leqslant 1, \tag{8.1c}$$

when $c$ takes on different positive values, and we find the following:

(1) If $c \geqslant c_0$, then $u(x,c)$ is strictly increasing and $u(\infty,c) = 1$.

(2) If $c_{\text{bif}} < c < c_0$, then $u(x,c)$ is nonmonotone and $u(\infty,c) = 1$.

(3) If $c_{\text{unb}} < c \leqslant c_{\text{bif}}$, then $u(\cdot,c) > 0$ on $\mathbb{R}$ and bounded above. Moreover, if $c_{\text{unb}} < c \leqslant c_{\text{per}}$, then there exists $x_{\text{per}}(c) \leqslant 0$ and $T(c) > 0$ such that

$$u(x,c) = u(x + T(c), c) \qquad \text{for every} \qquad x \geqslant x_{\text{per}}(c).$$

(4) If $c = c_{\text{unb}}$, then there exists a constant $\tilde{x} > -1$ such that

$$u(\cdot, c_{\text{unb}}) > 0 \quad \text{on} \quad (-\infty, \tilde{x}) \quad \text{and} \quad u(\cdot, c_{\text{unb}}) = 0 \quad \text{on} \quad [\tilde{x}, \infty).$$

(5) If $0 < c < c_{\text{unb}}$, then

$$\lim_{x\to\infty} u(x,c) = -\infty.$$

Part 1 has been proved in the previous sections.

**The range $\boldsymbol{c_{\text{bif}} < c < c_0}$.** For $c < c_0$, but close to $c_0$, small damped oscillations around the uniform state $u = 1$ appear. However the solution remains above the value $u = \frac{1}{2}$ for all $x > 1$. As we reduce $c$ further, we find that the amplitude of the oscillations increases and there exists a value $c_1 \in (c_{\text{bif}}, c_0)$ for which the graph of $u$ touches the line $u = \frac{1}{2}$ ($c_1 \approx 1.664$). Specifically,

(1) For $c_1 < c < c_0$ we find a nonmonotone solution of the delay-differential equation (8.1) which tends to 1 as $x \to \infty$, such that $u(x,c) > \frac{1}{2}$ for every $x > 1$.

(2) For $c = c_1$ we find a nonmonotone solution of the delay-differential equation (8.1) converging to 1 and a point $x_1 > 1$ ($x_1 \approx 5.667$) such that $u(x_1, c_1) = \frac{1}{2}$ and $u(x, c_1) > \frac{1}{2}$ for every $x > 1$, $x \neq x_1$.

(3) For $c < c_1$ the condition $u(x,c) \geqslant \frac{1}{2}$ does not hold for some $x \in [1, \infty)$.

A natural question is the following: How much smaller can we choose $c$ for there to exist a front-type solution, i.e., a heteroclinic orbit which connects $u = 0$ to $u = 1$? Notice that the difference between both $c_1$ and $c_{\text{bif}}$ is small ($c_1 - c_{\text{bif}} \approx 0.01$); thus, the regime where the question remains open is rather narrow. For values of $c$ *close* to $c_{\text{bif}}$, the amplitude of the oscillations around $u = 1$ is seen to increase and the number of times that $u(x,c)$ crosses the line $u = \frac{1}{2}$ also increases. Motivated by numerical experiments, we formulate the following conjecture:

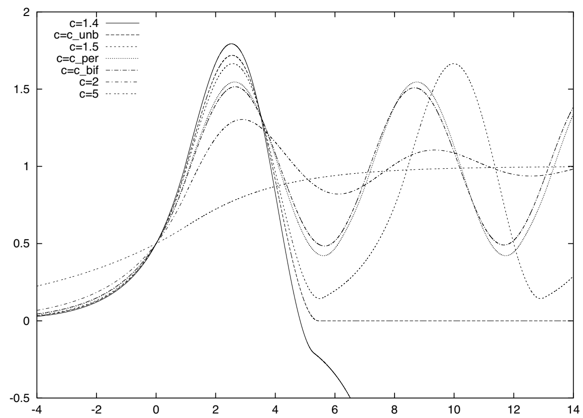



FIGURE 4. Fronts for different values of $c$

**Conjecture 8.1.**

(1) *There exists a sequence of values* $c_1 > c_2 > \cdots > c_n > \cdots \geqslant c_{\text{bif}}$ *such that for* $c_n > c > c_{n+1}$*, the graph of the solution* $u$ *crosses the line* $u = \frac{1}{2}$ *transversally, exactly* $2n$ *times. For* $c = c_n$*,* $u$ *crosses the line* $u = \frac{1}{2}$ *transversally, exactly* $2(n-1)$ *times, and once it is tangent to the line.*

(2) *The sequence of values* $c_n$ *satisfies that*

$$c_\infty \stackrel{\text{def}}{=} \lim_{n\to\infty} c_n = c_{\text{bif}}.$$

(3) *For* $c_{\text{unb}} < c \leqslant c_\infty$*, the number of points where the orbit* $u(x,c)$ *crosses the line* $u = \frac{1}{2}$ *is infinity. Each time the crossing is transversal.*

**Remark.** Conjecture 8.1 implies that for every $c > c_{\text{bif}}$, Problem (8.1) has a front-type solution, because there exists a point $x_c > 1$ such that $u(\cdot, c)$ solves the linear DDE (8.1b) on $(x_c, \infty)$, and hence

$$\lim_{x\to\infty} u(x,c) = 1. \qquad (8.2)$$

Part 3 of the conjecture means that the set of values of $c$ for which the boundary condition (8.2) holds is precisely $c > c_\infty$, and by Part 2, $c_\infty = c_{\text{bif}}$.

**The range $0 < c \leqslant c_{\text{bif}}$.** Numerical experiments suggest that the solution is bounded for $c_{\text{unb}} < c < c_{\text{bif}}$ and unbounded for $0 < c < c_{\text{unb}}$. For speeds in the range of bounded solutions we find another critical speed $c_{\text{per}}$ such that for $c_{\text{unb}} < c \leqslant c_{\text{per}}$, bounded orbits are eventually periodic; i.e., there exists a point $x_{\text{per}}(c)$ such that $u(\cdot, c)$ is periodic for $x > x_{\text{per}}(c)$.

In the following lemma we show how positivity and boundedness are connected. Let $u(x, c)$ be the solution of problem (8.1), and let

$$\xi(c) \stackrel{\text{def}}{=} \sup\{x > 1 : u(\cdot, c) > 0 \text{ on } (-\infty, x)\}. \tag{8.3}$$

Plainly, since $u > 0$ on $\mathbb{R}^-$, $\xi$ is well defined. If $u(x, c) > 0$ for all $x \in \mathbb{R}$, then $\xi(c) = \infty$. We shall show that if $\xi(c) < \infty$, then there are two alternatives:

(a) $u(\cdot, c) = 0$ on $\big[\xi(c), \infty\big)$;

(b) $u(\cdot, c) < 0$ on $\big(1 + \xi(c), \infty\big)$ and $u(x, c) \to -\infty$ as $x \to \infty$.

**Lemma 8.1.** *Suppose that $\xi(c) < \infty$. Then, by definition,*

$$u(\xi(c), c) = 0 \qquad \textit{and} \qquad u(x, c) > 0 \quad \textit{for} \quad -\infty < x < \xi(c),$$

*and we may distinguish two alternatives:*

(1) *We have*

$$u(x, c) \leqslant \tfrac{1}{2} \qquad \textit{for} \qquad x \in [\xi(c) - 1, \xi(c)].$$

*In that case,*

$$u(x, c) = 0 \qquad \textit{for every} \qquad x \geqslant \xi(c).$$

(2) *We have*

$$u(x, c) > \tfrac{1}{2} \qquad \textit{for some} \qquad x \in [\xi(c) - 1, \xi(c)].$$

*Then $u(x, c)$ is nonincreasing for every $x \geqslant \xi(c)$.*

*Moreover, if $u(\xi(c) - 1, c) < \frac{1}{2}$, then $u(x, c)$ is strictly decreasing for $x \geqslant \xi$.*

*Define $A(c) \stackrel{\text{def}}{=} -u\big(1 + \xi(c), c\big)$. Then $A(c) > 0$ and for $x \geqslant 1 + \xi(c)$ we have*

$$u(x, c) = -A(c) \exp\Big(\frac{x - 1 - \xi(c)}{c}\Big). \tag{8.4}$$

*In particular, $\lim_{x\to\infty} u(x, c) = -\infty$.*

**Proof.** We write the proof in terms of the variables $t = x - 1$ and $y = 1 - u$, and we put $\tau = \xi - 1$.

By assumption, $\tau < \infty$, and by the definition of $\xi$, we have $y < 1$ on $(-\infty, \tau)$ and $y(\tau) = 1$. The equation now becomes

$$cy'(t) = y(t) - 1 \qquad \text{if} \quad t > 0 \quad \text{and} \quad y(t-1) \geqslant \tfrac{1}{2}, \tag{8.5}$$

$$cy'(t) = y(t) - 2y(t-1) \qquad \text{if} \quad t > 0 \quad \text{and} \quad y(t-1) \leqslant \tfrac{1}{2}, \tag{8.6}$$

so that

$$cy'(t) \geqslant y(t) - 1 \qquad \text{for} \qquad t \geqslant \tau. \tag{8.7}$$

Remembering that $y(\tau) = 1$, we conclude that $y(t) \geqslant 1$ for all $t \geqslant \tau$.

In Case (2a) we have $y \geqslant \frac{1}{2}$ on $(\tau - 1, \tau)$, so that we have equality in (8.7), and

$$y(t) = 1 \qquad \text{for all} \qquad t \geqslant \tau. \tag{8.8}$$

On the other hand, in Case (2b) we have $y < \frac{1}{2}$ on some subinterval of $(\tau - 1, \tau)$. Thus, if we write

$$cy'(t) = y(t) - 2y(t-1) = y(t) - 1 + f(t),$$

where

$$f(t) = 1 - 2y(t-1),$$

we see that $f(t) \geqslant 0$ and $f(t) > 0$ on some subset of $(\tau - 1, \tau)$. Therefore, $y(\tau + 1) > 1$.

For $t \geqslant 1 + \tau(c)$, we know that $y(t)$ obeys the ODE (8.5). Solving it with the initial condition $y(1 + \tau(c), c) = 1 + A(c)$ gives us the expression (8.4).

In the following proposition we summarize the results of a numerical investigation, identifying the values of the speed $c$ with the three cases (a), (b), and (c) enumerated above.

**Proposition 8.1.** *There exists a critical speed* $c_{\text{unb}} \in (0, c_{\text{bif}})$ ($c_{\text{unb}} \approx 1.455$) *such that*

(1) *For* $c > c_{\text{unb}}$, $u(x, c) > 0$ *for every* $x > 1$.
(2) *For* $c = c_{\text{unb}}$, *there exists a point* $\tilde{x} > 1$ ($\tilde{x} \approx 5.449$) *such that* $u(x, c_{\text{unb}}) > 0$ *for* $x < \tilde{x}$ *and* $u(x, c_{\text{unb}}) = 0$ *for* $x \geqslant \tilde{x}$.
(3) *For* $0 < c < c_{\text{unb}}$, *there exists a point* $x_c < \tilde{x}$ *such that* $u(x_c, c) = \frac{1}{2}$ *and* $u(1 + x_c, c) < 0$. *By Proposition* 8.1, *this implies that there exists a constant* $B(c) > 0$ *such that*

$$u(x, c) = -B(c) \exp\left(\frac{x - (x_c + 1)}{c}\right) \qquad \textit{for} \quad x \geqslant 1 + x_c.$$

*In particular,* $\lim_{x \to \infty} u(x, c) = -\infty$.

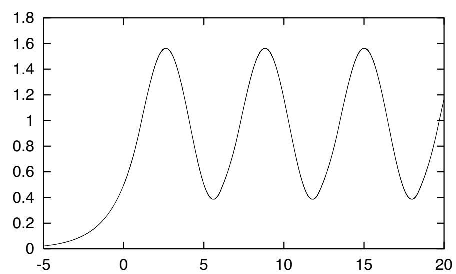
**The range $c_{\text{unb}} < c \leqslant c_{\text{bif}}$.** We now focus on the interval $(c_{\text{unb}}, c_{\text{bif}}]$. So far we can only say that $u(x, c)$ remains positive for all $x \in \mathbb{R}$. In the following we show that under certain conditions, which we can verify numerically, for some values of $c$ in this interval, there exists a point $x_{\text{per}}(c)$ such that the orbit $u(x, c)$ is periodic for $x \geqslant x_{\text{per}}(c)$.

We introduce the points

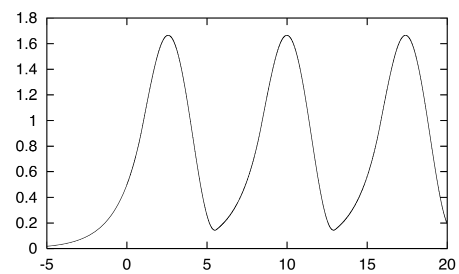
$$x_1(c) \stackrel{\text{def}}{=} \sup\Big\{x > 1 : u(\cdot, c) > \frac{1}{2} \text{ on } (1, x)\Big\}, \tag{8.9}$$

$$x_2(c) \stackrel{\text{def}}{=} \sup\Big\{x > x_1(c) : u(\cdot, c) < \frac{1}{2} \text{ on } (x_1(c), x)\Big\}, \tag{8.10}$$

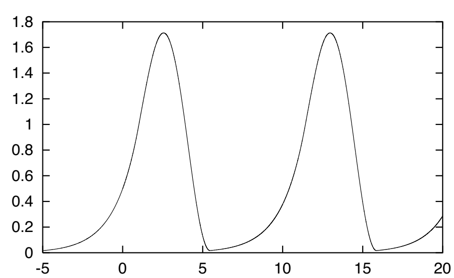
whenever they exist, and the constant $L(c) \stackrel{\text{def}}{=} u\big(1 + x_1(c), c\big)$. The points $x_1(c)$ and $x_2(c)$ are the first two positive values of $x$ where $u(x, c)$ crosses the line $u = \frac{1}{2}$. On the basis of numerical evidence—see also Conjecture 8.1—the point $x_1(c)$ exists if $c \leqslant c_1$ and $x_2$ exists if $c < c_1$. As $c \leqslant c_{\text{bif}}$, we know that $x_1(c)$ is a well-defined positive number.



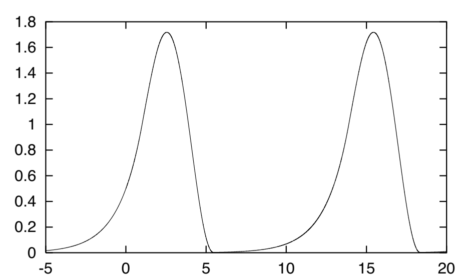

(a) $c = 1.6$




(b) $c = 1.5$




(c) $c = 1.46$




(d) $c = 1.456$

FIGURE 5. Eventually periodic orbits for different values of $c$.

In the following proposition, we prove the existence of the announced *eventually periodic solutions* under the condition that

$$x_2(c) - x_1(c) \geqslant 1. \tag{8.11}$$

Note that this condition implies that $L(c) \leqslant \frac{1}{2}$. We characterize numerically the values of $c \in (c_{\text{unb}}, c_{\text{bif}})$ for which this condition is satisfied, and in Figure 5 we show some eventually periodic orbits for different values of $c$. Note that the amplitude and the period both increase as $c$ decreases. Some of those properties are described in Theorem 8.1.

**Theorem 8.1.** *Let $u(x,c)$ be the solution of problem* (8.1), *and let $c$ be such that*

(a) *The points $x_1(c)$ and $x_2(c)$ exist.*

(b) *The distance between $x_1(c)$ and $x_2(c)$ is large enough; i.e.,* (8.11) *is satisfied.*

*Then there exists a point $x_{\text{per}}(c) \leqslant 0$ such that*

$$u(x,c) = \frac{1}{2}\, e^{x/c} \qquad \textit{for} \qquad x \leqslant x_{\text{per}}$$

*and $u(x,c)$ is periodic for $x \geqslant x_{\text{per}}(c)$. The period $T(c)$ is given by*

$$T(c) = 1 + x_1(c) + c\log\left(\frac{1}{2L(c)}\right). \tag{8.12}$$

**Proof.** We write $t_i = x_i - 1$ $(i = 1, 2)$. Then, by assumption (8.11),

$$t_2(c) - t_1(c) \geqslant 1 \tag{8.13}$$

Moreover, writing $y(t) = 1 - u(t)$, we obtain

$$y(1 + t_1(c), c) = 1 - u(1 + x_1(c), c) = 1 - L. \tag{8.14}$$

Since $y(t) \geqslant \frac{1}{2}$ for $t_1 \leqslant t \leqslant t_2$, we can find $y(t)$ on $(t_1 + 1, t_2 + 1)$ by solving the problem

$$\begin{cases} cy'(t) = y(t) - 1 & \text{for} \quad t_1 + 1 < t < t_2 + 1, \\ y(t_2) = \frac{1}{2}. \end{cases} \tag{8.15}$$

Note that because of assumption (8.11)

$$t_1 + 1 \leqslant t_2 < t_2 + 1.$$

The solution of problem (8.14) is given by

$$y(t) = 1 - \frac{1}{2}\exp\left(\frac{t - t_2}{c}\right) \qquad \text{for} \quad t_1 + 1 \leqslant t \leqslant t_2 + 1.$$

Since $y(t_1 + 1) = 1 - L$ by (8.14), we obtain the following relation between $t_1$, $t_2$, and $L$:

$$t_2 = 1 + t_1 + c\log\left(\frac{1}{2L}\right). \tag{8.16}$$

Now, consider the following auxiliary function:

$$z(t) = y(t + t_2 + 1) \qquad \text{for} \qquad t \geqslant -1. \tag{8.17}$$

Then $z(t)$ satisfies the same delay-differential equation (8.6) as $y(t)$, and

$$z(t) = y(t) \qquad \text{for} \qquad -1 \leqslant t \leqslant 0.$$

Hence, by uniqueness this implies that

$$z(t) = y(t + t_2 + 1) = y(t).$$

In other words, $y(t)$ is periodic and the period is given by $T = t_2 + 1$. From (8.16) we obtain that the period is given as in (8.12).

To complete the proof, we need to determine $t_{\text{per}}(c)$. This is found to be the value which satisfies $y(t_{\text{per}}) = 1 - L$. A simple computation shows that $t_{\text{per}}$ is given by

$$t_{\text{per}}(c) = -1 - c \log\left(\frac{1}{2L(c)}\right) < -1. \tag{8.18}$$

**Remark.** Note that the condition (8.11) is essential. If it is not satisfied, we can still conclude is that $z(t)$ and $y(t)$ satisfy the same DDE (8.6). However, the initial condition is different. Indeed, in this case what we know is that there exists an interval of length $t_2 - t_1 < 1$ on which $y(t)$ and $z(t)$ coincide. Specifically, $z = y$ on $[0, t_2 - t_1]$ , which is not enough for uniqueness.

To finish this section, it remains to characterize numerically values of $c$ for which condition (8.11) is satisfied and hence can apply to Theorem 8.1.

**Proposition 8.2.** *There exists* $c_{\text{unb}} < c_{\text{per}} < c_{\text{bif}}$ ($c_{\text{per}} \approx 1.618$) *such that condition* (8.11) *is satisfied if* $c_{\text{unb}} < c \leqslant c_{\text{per}}$. *By Theorem* 8.1, *this implies that there exists a point* $x_{\text{per}}(c) \leqslant 0$ *given by* (8.18) *and a constant* $T(c) > 0$ *given by* (8.12) *such that*

$$u(x, c) = u(x + T(c), c) \qquad \textit{for every} \quad x \geqslant x_{\text{per}}(c).$$

*Moreover,*

$$\lim_{c \downarrow c_{\text{unb}}} T(c) = \infty, \qquad \lim_{c \uparrow c_{\text{per}}} T(c) = 1 + x_1(c_{\text{per}}), \tag{8.19}$$

$$\lim_{c \downarrow c_{\text{unb}}} x_{\text{per}}(c) = -\infty, \qquad \lim_{c \uparrow c_{\text{per}}} x_{\text{per}}(c) = 0. \tag{8.20}$$

**Discussion.** In the interval $I \stackrel{\text{def}}{=} (c_{\text{unb}}, c_{\text{bif}})$ the value $x_1(c)$ given by (8.9) can be expected to be well defined and finite. This can be seen in two ways: by using Part 3 of Conjecture 8.1 or by recalling that the spectrum of the operator associated to the linear DDE (8.1b) contains eigenvalues with positive real part. Hence, the amplitude of the oscillations around the uniform steady state $u = 1$ increases and eventually the orbit crosses $u = \frac{1}{2}$.

On $I$ the point $x_2(c)$ defined by (8.10) is also well defined and finite. The argument works as follows: the orbit crosses the line $u = \frac{1}{2}$ at $x = x_1$, and also $u > 0$ on $\mathbb{R}$ because $c > c_{\mathrm{unb}}$, by Proposition 8.1. Suppose that the orbit satisfies that $0 < u(\cdot, c) < \frac{1}{2}$ on $(x_1(c), \infty)$. Then, eventually it must satisfy the ODE (8.1a). The ODE forces the orbit to cross the line $u = \frac{1}{2}$, which contradicts our assumption. Hence, $x_2(c)$ is well defined and finite.

Thus, we have that $x_1(c)$ and $x_2(c)$ are well defined on $I$. By using the continuity of solutions with respect to the parameter $c$, we know that

$$\lim_{c \downarrow c_{\mathrm{unb}}} x_2(c) - x_1(c) = \infty$$

because $x_2(c)$ tends to infinity. In particular, we can find a small right-neighbourhood of $c_{\mathrm{unb}}$ where condition (8.11) is satisfied. Specifically, there exists some $\varepsilon > 0$ such that condition (8.11) holds on $(c_{\mathrm{unb}}, c_{\mathrm{unb}} + \varepsilon]$. We define

$$c_{\mathrm{per}} \stackrel{\mathrm{def}}{=} \sup\{c \in I : \text{condition (8.11) holds on } (c_{\mathrm{unb}}, c)\}. \tag{8.21}$$

This definition gives us the criteria to find $c_{\mathrm{per}}$: it must satisfy the equation

$$u\big(1 + x_1(c), c\big) = \frac{1}{2}. \tag{8.22}$$

The limits in (8.19) are obtained from (8.12) and (8.16) by using the fact that

$$\lim_{c \downarrow c_{\mathrm{unb}}} L(c) = 0 \qquad \text{and} \qquad \lim_{c \uparrow c_{\mathrm{per}}} L(c) = \frac{1}{2}.$$

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