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STRUCTURE OF DIRAC MATRICES AND INVARIANTS FOR NONLINEAR DIRAC EQUATIONS

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Abstract. We present invariants for nonlinear Dirac equations in space-time $\mathbb{R}^{n+1}$, by which we prove that a special choice of the Cauchy data yields free solutions. Our argument works for Klein-Gordon-Dirac equations with Yukawa coupling as well. Related problems on the structure of Dirac matrices are studied.

1. Introduction

In this paper we present invariants for nonlinear Dirac equations in space-time $\mathbb{R}^{n+1}$, by which we prove that a special choice of the Cauchy data yields free solutions. In [1] Chadam and Glassey considered a problem of that kind for Klein-Gordon-Dirac equations with Yukawa coupling in one and three dimensions. Their results as well as proofs, however, depend on particular representations of the Dirac matrices. The invariants are especially described in terms of components in the Dirac spinor field. Although their results hold in other cases by the unitary equivalence of the Dirac matrices, this fact seems to make it difficult to understand relations between the Dirac matrices and the invariants.

The purpose in this paper is to give a representation-free understanding of the problem and to generalize their results in several directions. Our assumptions (a1) and (a2) below are independent of particular representations of the Dirac matrices. Our argument works for Klein-Gordon-Dirac equations with Yukawa coupling as well.

This paper is organized as follows. In Section 2, we summarize basic notation and facts about Dirac matrices. Propositions 1 and 2 shall be used in Section 4. In Section 3, we present invariants for the nonlinear Dirac equation in Lemma 1 and Theorem 1 under the assumptions (a1) and (a2). As a corollary, we prove that the corresponding constraint on the Cauchy data is preserved and keeps nonlinear interaction null to yield free solutions. Up to Section 3, results are independent of space dimensions and of specific

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representations and degree of the Dirac matrices. In Section 4 we examine sufficient conditions for (a1) and (a2). For $n = 1$, the situation is very simple, as seen in Proposition 4. For $n \geq 2$, we introduce assumption (S) in Proposition 5. In Proposition 6, we prove that the assumption (S) is always satisfied in any space dimension if the degree of the Dirac matrices is chosen to be double the usual minimal choice $2^{[(n+1)/2]}$, where $[a]$ is the integral part of a nonnegative real number a. In Theorem 2, we prove that (S) is satisfied under a restrictive assumption on space dimensions if the degree of the Dirac matrices is minimal. In Section 5, we reproduce Chadam-Glassey's results in [1] as special cases of our results. In the Appendix, in connection with Proposition 2 we give an explicit representation of Dirac matrices with real components for $n = 8$ and $N = 16$ just for reference.

2. Dirac matrices

In this section we recall some basic facts about Dirac matrices. We refer the reader to [4], [5], and [7].

Definition 1. Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be matrices in $M_N(\mathbb{C})$. $(\alpha_0, \{\alpha_j\}_{j=1}^n)$ is an $(n+1)$-tuple of *Dirac matrices* when it satisfies the following:

(1) $\alpha_j^* = \alpha_j \quad$ for $j = 0, 1, \ldots, n$.

(2) (Anticommutation Relations)
$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I_N \quad$ for $j, k = 0, 1, \ldots, n$.

Here δ_{jk} is the Kronecker delta and I_N is the identity matrix in $M_N(\mathbb{C})$.

Remark 1. Let $(a_0, \{a_j\}_{j=1}^n)$ be an $(n+1)$-tuple of Dirac matrices, and let l be a nonnegative integer with $2l \leq n$. Put

$$\alpha_j = \begin{cases} a_j & \text{for} \quad j \leq 2l, \\ i^{l+1} a_0 \cdots a_{2l} a_j & \text{for} \quad j > 2l. \end{cases}$$

Then, $(\alpha_0, \{\alpha_j\}_{j=1}^n)$ becomes an $(n+1)$-tuple of Dirac matrices.

Proposition 1. *There is an $(n+1)$-tuple of Dirac matrices in $M_N(\mathbb{C})$ when $N = 2^{[\frac{n+1}{2}]}$.*

In the appendix of [4] an explicit construction of Dirac matrices is described. According to this method, for $n = 8l-1, 8l, 8l+1$ with $l \in \mathbb{N}$, we have an $(n+1)$-tuple of Dirac matrices $(\alpha_0, \{\alpha_j\}_{j=1}^n)$ which satisfies that

$$\overline{\alpha_j} = \begin{cases} a_j & \text{for} \quad j \leq 4l, \\ -a_j & \text{otherwise.} \end{cases}$$

Through Remark 1 this yields the following property.

Proposition 2. *Let* $N = 2^{[\frac{n+1}{2}]}$. *There exists an* $(n+1)$*-tuple of Dirac matrices in* $M_N(\mathbb{R})$ *when* $n \equiv 0,\ 1,\ 7$ *mod* 8.

See the Appendix for an explicit representation of Dirac matrices with real components for $N = 16$ and $n = 8$

3. Invariants of the nonlinear Dirac equation

Let $(\beta, \{\alpha_j\}_{j=1}^n)$ be Dirac matrices, and let ψ be a classical solution to the nonlinear Dirac equation,

$$\partial_t\psi + \alpha\cdot\nabla\psi + im\beta\psi = i\lambda(\beta\psi,\psi)\beta\psi, \tag{E}$$

where $\psi : \mathbb{R}\times\mathbb{R}^n \ni (t,x) \mapsto \psi(t,x) \in \mathbb{C}^N$, and m and $\lambda \in \mathbb{R}$. We use the notation

$$\alpha\cdot\nabla = \alpha_1\partial_1 + \cdots + \alpha_n\partial_n, \qquad \partial_j = \partial/\partial x_j,$$
$$(\phi,\psi) = \phi_1\bar{\psi}_1 + \cdots + \phi_N\bar{\psi}_N, \qquad |\phi|^2 = (\phi,\phi)$$

for $\phi = (\phi_1,\ldots,\phi_N)$, $\psi = (\psi_1,\ldots,\psi_N)$, and $x = (x_1,\ldots,x_n)$.

We suppose that ψ has sufficient regularity in space-time and decay at infinity in space to ensure formal calculations such as integration by parts and differentiation under the integral sign. Integrals without specific domains denote the Lebesgue integrals over $\mathbb{R}^n$. We drop the time notation unless otherwise specified. Regarding the Cauchy problem for (E) we refer the reader to [3], [6], and references therein.

Proposition 3.

$$\int |\psi|^2\,\mathrm{d}x = \textit{constant}.$$

Proof. Since $\psi = \psi(t)$ is a solution to (E),

$$\partial_t(\psi,\psi) \;=\; -2\Re(\alpha\cdot\nabla\psi,\psi) - 2\Re\left(i(m(\beta\psi,\psi) - \lambda(\beta\psi,\psi)^2)\right).$$

Since β is Hermitian, $(\beta\psi,\psi)$ is real. Hence

$$\partial_t(\psi,\psi) = -(\alpha\cdot\nabla\psi,\psi) - (\psi,\alpha\cdot\nabla\psi) = -\sum_k\left((\alpha_k\partial_k\psi,\psi) + (\psi,\alpha_k\partial_k\psi)\right)$$
$$= -\sum_k\left((\alpha_k\partial_k\psi,\psi) + (\alpha_k\psi,\partial_k\psi)\right) = -\sum_k\partial_k(\alpha_k\psi,\psi),$$

for α_k is Hermitian. By integrating with respect to x, the proof is completed.

Assumptions: Let A be an $N\times N$ matrix. We itemize assumptions as follows:

(a1) ${}^t\beta A\beta = -A$.
(a2) ${}^t\alpha_j A\alpha_j = A$ or ${}^tA$ for any $j = 1,\ldots,n$.

Remark 2. Let $(\tilde{\beta}, \{\tilde{\alpha}_j\}_{j=1}^n)$ be another $(n+1)$-tuple of Dirac matrices. It is well-known that there exists a unitary matrix such that $\tilde{\beta} = U^{-1}\beta U$ and $\tilde{\alpha}_j = U^{-1}\alpha_j U$. We define $\tilde{A} = U^{-1}AU$. Then $\tilde{A}$ satisfies (a1) and (a2) with $(\beta, \{\alpha_j\}_{j=1}^n)$ replaced by $(\tilde{\beta}, \{\tilde{\alpha}_j\}_{j=1}^n)$, provided that U is a real matrix. In this sense the assumptions above are described in a covariant way under orthogonal equivalence of representations of the Dirac matrices.

Lemma 1. *Assume* (a1) *and* (a2). *Then,*

$$\int \left(A\psi, \bar{\psi}\right) \mathrm{d}x = \textit{constant}.$$

Proof. Since ψ is a solution to (E),

$$\begin{aligned}\partial_t(A\psi, \overline{\psi}) &= -\sum_k (A\alpha_k\partial_k\psi, \overline{\psi}) - i(m - \lambda(\beta\psi, \psi))(A\beta\psi, \overline{\psi}) \\ &\quad -\sum_k (A\psi, \overline{\alpha}_k\partial_k\overline{\psi}) - i(m - \lambda(\beta\psi, \psi))(A\psi, \overline{\beta\psi}).\end{aligned}$$

By the assumption (a1),

$$(A\beta\psi, \overline{\psi}) = -({}^t\beta A\psi, \overline{\psi}) = -(A\psi, \overline{\beta\psi}).$$

Therefore,

$$\partial_t(A\psi, \overline{\psi}) = -\sum_k \left\{(A\alpha_k\partial_k\psi, \overline{\psi}) + (A\psi, \overline{\alpha}_k\overline{\psi})\right\}.$$

Let $\mathcal{S}$ be the set $\left\{k \in \mathbb{N} : 1 \le k \le n,\ {}^t\alpha_k A\alpha_k = A\right\}$. When $k \in \mathcal{S}$,

$$(A\psi, \overline{\alpha}_k\partial_k\overline{\psi}) = ({}^t\alpha_k A\psi, \partial_k\overline{\psi}) = (A\alpha_k\psi, \partial_k\overline{\psi}).$$

It follows that

$$(A\alpha_k\partial_k\psi, \overline{\psi}) + (A\psi, \overline{\alpha}_k\overline{\psi}) = \partial_k(A\alpha_k\psi, \overline{\psi}).$$

If $k \notin \mathcal{S}$, by the assumption (a2), ${}^t\alpha_k A\alpha_k$ equals ${}^tA$, so that $A\alpha_k$ is symmetric. Therefore there is a matrix $B \in M_N(\mathbb{C})$ such that $A\alpha_k = {}^tBB$. Hence

$$\partial_k(B\psi, \overline{B\psi}) = 2(B\partial_k\psi, \overline{B\psi}) = 2({}^tBB\partial_k\psi, \overline{\psi}) = 2(A\alpha_k\partial_k\psi, \overline{\psi}).$$

Similarly, since ${}^t\alpha_k A$ is symmetric, there is $C \in M_N(\mathbb{C})$ such that ${}^tCC = {}^t\alpha_k A$, which satisfies

$$\partial_k(C\psi, \overline{C\psi}) = 2(A\psi, \overline{\alpha}_k\partial_k\overline{\psi}).$$

Consequently,

$$\partial_t(A\psi, \overline{\psi}) = -\sum_k \left\{(A\alpha_k\partial_k\psi, \overline{\psi}) + (A\psi, \overline{\alpha}_k\overline{\psi})\right\}$$

$$= -\sum_{k\in\mathcal{S}} \partial_k(A\alpha_k\psi, \overline{\psi}) - \sum_{k\notin\mathcal{S}} \frac{1}{2}\left\{\partial_k(B_k\psi, \overline{B_k\psi}) + \partial_k(C_k\psi, \overline{C_k\psi})\right\},$$

where ${}^tB_kB_k = A\alpha_k$ and ${}^tC_kC_k = {}^t\alpha_kA$. By integrating with respect to x, the proof is completed. □

Theorem 1. *Let A be a unitary matrix of degree N, and let $\mu \in \mathbb{C}$. Assume* (a1) *and* (a2) *for A. Then,*

$$\int \left|\mu A\psi - \bar{\psi}\right|^2 \mathrm{d}x = \textit{constant.}$$

Proof. Since A is unitary,

$$\left|\mu A - \overline{\psi}\right|^2 = (1 + |\mu|^2)|\psi|^2 + 2\Re(A\psi, \overline{\psi}).$$

By virtue of Proposition 3 and Lemma 1, this completes the proof.

Remark 3. Let $\tilde{A}$ and $(\tilde{\beta}, \{\tilde{\alpha}_j\}_{j=1}^n)$ be as in Remark 2, and let U be orthogonal. Then the corresponding spinor field $\tilde{\psi}$ is given by $\tilde{\psi} = U^{-1}\psi$. In this setting we have $(\tilde{A}\tilde{\psi}, \overline{\tilde{\psi}}) = (A\psi, \overline{\psi})$ and $|\mu\tilde{A}\tilde{\psi} - \overline{\tilde{\psi}}|^2 = \left|\mu A\psi - \overline{\psi}\right|^2$, so that invariants in Lemma 1 and Theorem 1 are also invariant under orthogonal equivalence of representations of the Dirac matrices.

Corollary 1. *Assume* (a1) *and* (a2) *for a unitary matrix A and suppose that $A\psi(0) = \bar{\psi}(0)$. Then $(\beta\psi, \psi) = 0$ at any time. Namely, ψ is a free solution to* (E)*.*

Proof. If $A\psi(0) = \overline{\psi}(0)$, by Theorem 1, $A\psi = \overline{\psi}$ for all time. Hence

$$\begin{aligned}(\beta\psi, \psi) &= (\beta\overline{A\psi}, \psi) = -(\overline{A\beta\psi}, \psi) \quad \text{(by the assumption (a1))}\\ &= -(\overline{\beta\psi}, \overline{A^*}\psi) = -(\overline{\beta\psi}, \overline{A^*A\psi}) \quad \left(\text{by } \psi = \overline{A\psi}\right)\\ &= -(\overline{\beta\psi}, \overline{\psi}) = -\overline{(\beta\psi, \psi)}.\end{aligned}$$

Since β is Hermitian, $(\beta\psi, \psi)$ is real. This yields $(\beta\psi, \psi) \equiv 0$. □

4. Structure of Dirac matrices

In this section we shall be more specific about the main assumptions (a1) and (a2). A simple sufficient condition for (a1) and (a2) is given by (S) below. The condition (S) requires that only one of the Dirac matrices is pure imaginary and the rest of the Dirac matrices are real, which in turn provides a structural restriction on the space of Dirac matrices in connection with its degree N (Proposition 6) or its dimension n (Theorem 2). We start with Propositions 4 and 5. The proofs are simple and omitted.

Proposition 4. *Let $n = 1$. Let (β, α) be a 2-tuple of Dirac matrices, and suppose that β is a real matrix. Then, (β, α) satisfies* (a1) *and* (a2) *with $A = \alpha$.*

Proposition 5. *Let $(\beta, \{\alpha_j\}_{j=1}^n)$ be Dirac matrices. Assume the following property:*

(S) *$\overline{\beta} = \beta$ and there exists $j_0 \in \{1, \ldots, n\}$ such that*

$$\begin{aligned} \overline{\alpha_{j_0}} &= -\alpha_{j_0}, \\ \overline{\alpha_k} &= \alpha_k \quad \text{for all } k \text{ with } k \neq j_0. \end{aligned}$$

Set $A = i\beta\alpha_{j_0}$. Then, A is a real, symmetric, and orthogonal matrix which fulfills (a1) *and* (a2)*.*

Proposition 6. *Let $N = 2^{[\frac{n+1}{2}]+1}$. Then, for any n, there exists an $(n+1)$-tuple of Dirac matrices in $M_N(\mathbb{C})$ which satisfies* (S)*.*

Remark 4. The assumption (S) is preserved up to orthogonal changes of the Dirac matrices. See also Remarks 2 and 3.

Proof. Let $\left(b, \{a_j\}_{j=1}^n\right)$ be an $(n+1)$-tuple of Dirac matrices in $M_N(\mathbb{C})$ and let $T : M_N(\mathbb{C}) \to M_{2N}(\mathbb{R})$ be a mapping, which is real-linear, defined by

$$T(X) = \begin{pmatrix} \Re X & -\Im X \\ \Im X & \Re X \end{pmatrix}, \text{ where } \Re X = \frac{1}{2}\left(X + \overline{X}\right), \ \Im X = \frac{1}{2i}\left(X - \overline{X}\right).$$

Define $\left(\beta, \{\alpha_j\}_{j=1}^n\right)$ as

$$\beta = T(b), \qquad \alpha_j = \begin{cases} iT(ia_j) & \text{for } \ j = j_0, \\ T(a_j) & \text{otherwise.} \end{cases}$$

Then, $\left(\beta, \{\alpha_j\}_{j=1}^n\right)$ is an $(n+1)$-tuple of Dirac matrices which satisfies (S). By virtue of Proposition 1, this completes the proof. □

Theorem 2. *Let $N = 2^{[\frac{n+1}{2}]}$. Suppose that $n \equiv 1, 2, 3$ mod 8. Then, there exists an $(n+1)$-tuple of Dirac matrices in $M_N(\mathbb{C})$ which satisfies* (S)*.*

Proof. In the case of $n = 1, 2, 3$, the standard representation yields Dirac matrices with (S); see, *e.g.*, [4]. In other cases, by Proposition 2, there is an $(n-1)$-tuple of Dirac matrices $\{a_j\}_{j=0}^{n-2}$ in $M_N(\mathbb{R})$ with $N = \left[\frac{n-1}{2}\right]$. Let

$$\begin{aligned} \alpha_0 &= \begin{pmatrix} I_N & O \\ O & -I_N \end{pmatrix}, \qquad \alpha_j = \begin{pmatrix} O & a_{j-1} \\ a_{j-1} & O \end{pmatrix}, \quad j = 1, \ldots, n-1 \\ \alpha_n &= \begin{pmatrix} O & -iI_N \\ iI_N & O \end{pmatrix}. \end{aligned}$$

Then, $\{\alpha_j\}_{j=0}^n$ is an $(n+1)$-tuple of Dirac matrices which satisfies (S) with $j_0 = n$. $\square$

5. SPECIAL CASES

In this section we apply Corollary 1 and Proposition 5 to reproduce Chadam-Glassey's result on particular conditions which ensure free solutions in one and three space dimensions [1] (see also [2]) as special cases.

• $n = 1$, $N = 2$

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The following examples have the structure (S) with $j_0 = 2$.

• $n = 2$, $N = 2$

$$\alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

• $n = 3$, $N = 4$

$$\alpha_1 = \left(\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & 1 & & \\ 1 & & & \end{array}\right), \quad \alpha_2 = \left(\begin{array}{cc|cc} & & & -i \\ & & i & \\ \hline & -i & & \\ i & & & \end{array}\right),$$

$$\alpha_3 = \left(\begin{array}{cc|cc} & & 1 & \\ & & & -1 \\ \hline 1 & & & \\ & -1 & & \end{array}\right), \quad \beta = \left(\begin{array}{cc|cc} 1 & & & \\ & 1 & & \\ \hline & & -1 & \\ & & & -1 \end{array}\right),$$

$$A = \left(\begin{array}{cc|cc} & & & 1 \\ & & -1 & \\ \hline & -1 & & \\ 1 & & & \end{array}\right).$$

Here and hereafter, components in matrices without specific representations are understood to be zero.

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6. APPENDIX

The following are Dirac matrices in $M_{16}(\mathbb{R})$. See Proposition 2.

$$
\beta = \left(\begin{array}{cccc|cccc|cccc|cccc}
1 & & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & & & \\
\hline
 & & & & 1 & & & & & & & & & & & \\
 & & & & & 1 & & & & & & & & & & \\
 & & & & & & 1 & & & & & & & & & \\
 & & & & & & & 1 & & & & & & & & \\
\hline
 & & & & & & & & -1 & & & & & & & \\
 & & & & & & & & & -1 & & & & & & \\
 & & & & & & & & & & -1 & & & & & \\
 & & & & & & & & & & & -1 & & & & \\
\hline
 & & & & & & & & & & & & -1 & & & \\
 & & & & & & & & & & & & & -1 & & \\
 & & & & & & & & & & & & & & -1 & \\
 & & & & & & & & & & & & & & & -1
\end{array}\right)
$$

$$
\alpha_1 = \left(\begin{array}{cccc|cccc|cccc|cccc}
&&&&&&&&&&&&&&&1 \\
&&&&&&&&&&&&&&1& \\
&&&&&&&&&&&&&1&& \\
&&&&&&&&&&&&1&&& \\ \hline
&&&&&&&&&&&1&&&& \\
&&&&&&&&&&1&&&&& \\
&&&&&&&&&1&&&&&& \\
&&&&&&&&1&&&&&&& \\ \hline
&&&&&&&1&&&&&&&& \\
&&&&&&1&&&&&&&&& \\
&&&&&1&&&&&&&&&& \\
&&&&1&&&&&&&&&&& \\ \hline
&&&1&&&&&&&&&&&& \\
&&1&&&&&&&&&&&&& \\
&1&&&&&&&&&&&&&& \\
1&&&&&&&&&&&&&&&
\end{array}\right)
$$

$$
\alpha_2 = \left(\begin{array}{cccc|cccc|cccc|cccc}
&&&&&&&&&&&&&&1& \\
&&&&&&&&&&&&&&&-1 \\
&&&&&&&&&&&&1&&& \\
&&&&&&&&&&&&&-1&& \\ \hline
&&&&&&&&&&1&&&&& \\
&&&&&&&&&&&-1&&&& \\
&&&&&&&&1&&&&&&& \\
&&&&&&&&&-1&&&&&& \\ \hline
&&&&&&1&&&&&&&&& \\
&&&&&&&-1&&&&&&&& \\
&&&&1&&&&&&&&&&& \\
&&&&&-1&&&&&&&&&& \\ \hline
&&1&&&&&&&&&&&&& \\
&&&-1&&&&&&&&&&&& \\
1&&&&&&&&&&&&&&& \\
&-1&&&&&&&&&&&&&&
\end{array}\right)
$$

$$
\alpha_3 = \left(\begin{array}{cccc|cccc|cccc|cccc}
 & & & & & & & & & & & & 1 & & & \\
 & & & & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & & & & -1 & \\
 & & & & & & & & & & & & & & & -1 \\
\hline
 & & & & & & & & 1 & & & & & & & \\
 & & & & & & & & & 1 & & & & & & \\
 & & & & & & & & & & -1 & & & & & \\
 & & & & & & & & & & & -1 & & & & \\
\hline
 & & & & 1 & & & & & & & & & & & \\
 & & & & & 1 & & & & & & & & & & \\
 & & & & & & -1 & & & & & & & & & \\
 & & & & & & & -1 & & & & & & & & \\
\hline
1 & & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & & \\
 & & -1 & & & & & & & & & & & & & \\
 & & & -1 & & & & & & & & & & & &
\end{array}\right)
$$

$$
\alpha_4 = \left(\begin{array}{cccc|cccc|cccc|cccc}
 & & & & & & & & 1 & & & & & & & \\
 & & & & & & & & & 1 & & & & & & \\
 & & & & & & & & & & 1 & & & & & \\
 & & & & & & & & & & & 1 & & & & \\
\hline
 & & & & & & & & & & & & -1 & & & \\
 & & & & & & & & & & & & & -1 & & \\
 & & & & & & & & & & & & & & -1 & \\
 & & & & & & & & & & & & & & & -1 \\
\hline
1 & & & & & & & & & & & & & & & \\
 & 1 & & & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & & & \\
\hline
 & & & & -1 & & & & & & & & & & & \\
 & & & & & -1 & & & & & & & & & & \\
 & & & & & & -1 & & & & & & & & & \\
 & & & & & & & -1 & & & & & & & &
\end{array}\right)
$$

$$\alpha_5 = \left(\begin{array}{cccc|cccc|cccc|cccc}
 & & & & & & & & & -1 & & & & & & \\
 & & & & & & & & 1 & & & & & & & \\
 & & & & & & & & & & & 1 & & & & \\
 & & & & & & & & & & -1 & & & & & \\
\hline
 & & & & & & & & & & & & & -1 & & \\
 & & & & & & & & & & & & 1 & & & \\
 & & & & & & & & & & & & & & & 1 \\
 & & & & & & & & & & & & & & -1 & \\
\hline
 & 1 & & & & & & & & & & & & & & \\
-1 & & & & & & & & & & & & & & & \\
 & & & -1 & & & & & & & & & & & & \\
 & & 1 & & & & & & & & & & & & & \\
\hline
 & & & & & 1 & & & & & & & & & & \\
 & & & & -1 & & & & & & & & & & & \\
 & & & & & & & -1 & & & & & & & & \\
 & & & & & 1 & & & & & & & & & &
\end{array}\right)$$

$$\alpha_6 = \left(\begin{array}{cccc|cccc|cccc|cccc}
 & & & & & & & & & & & & & 1 & & \\
 & & & & & & & & & & & & -1 & & & \\
 & & & & & & & & & & & & & & & -1 \\
 & & & & & & & & & & & & & & 1 & \\
\hline
 & & & & & & & & & -1 & & & & & & \\
 & & & & & & & & 1 & & & & & & & \\
 & & & & & & & & & & & 1 & & & & \\
 & & & & & & & & & & -1 & & & & & \\
\hline
 & & & & & 1 & & & & & & & & & & \\
 & & & & -1 & & & & & & & & & & & \\
 & & & & & & & -1 & & & & & & & & \\
 & & & & & & 1 & & & & & & & & & \\
\hline
 & -1 & & & & & & & & & & & & & & \\
1 & & & & & & & & & & & & & & & \\
 & & & 1 & & & & & & & & & & & & \\
 & & -1 & & & & & & & & & & & & &
\end{array}\right)$$

$$\alpha_7 = \left(\begin{array}{cccc|cccc|cccc|cccc}
 & & & & & & & & & & -1 & & & & & \\
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$$\alpha_8 = \left(\begin{array}{cccc|cccc|cccc|cccc}
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