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AMBROSETTI-PRODI-TYPE PROBLEMS FOR QUASILINEAR ELLIPTIC PROBLEMS

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Abstract. We consider nonlinear perturbations of the p-Laplacian depending on a real parameter and subject to zero Dirichlet boundary data; we establish results which guarantee the existence of at least one and at least two solutions for certain parameter ranges.

1. Introduction

In this paper we consider the Dirichlet boundary-value problem

$$\begin{cases} \Delta_p u + f(u) = |t|^{p-2} t \hat{h} + h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded, open set with the smooth boundary in $\mathbb{R}^N$, $p > 1$, and Δ_p is the p-Laplace operator defined by $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$; f is a continuously differentiable function which satisfies the growth conditions

$$\lim_{s\to-\infty} \frac{f(s)}{|s|^{p-2}s} = \alpha, \qquad \text{and} \qquad \lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = \beta;$$

and $\hat{h}$ and h are assumed to be in the space $C\left(\overline{\Omega}\right)$, $\hat{h} \geq 0$ in $\overline{\Omega}$, $\alpha < \lambda_1$, and $\beta > \lambda_1$, where λ_1 is the smallest eigenvalue of

$$\begin{cases} \Delta_p u + \lambda |u|^{p-2} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

We first demonstrate the existence of a number $t_h >> 1$ such that if $t \geq t_h$, (1.1) has a negative solution. If we restrict $\hat{h}$ to be strictly positive in $\overline{\Omega}$ we shall prove that there exists t_l such that (1.1) has no solution if $t \leq t_l$ by using sub-supersolution theorems. We then show that if $\hat{h} > 0$ in $\overline{\Omega}$, then there exist t_1 and t_2, $t_1 \leq t_2$, such that if $t_1 \leq t \leq t_2$, (1.1) has at least one solution, at least two solutions if $t > t_2$, and no solutions if $t < t_1$. We further show that multiple solutions are possible in the case $\hat{h} \geq 0$ in $\overline{\Omega}$ for a sufficiently large t by using a limiting argument.

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The result obtained in this paper, in some sense, extends the result established by Ambrosetti and Prodi [1]. They consider the case $p = 2$, i.e.,

$$\begin{cases} \Delta u + f(u) = g & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $g \in C^{0,\alpha}(\overline{\Omega})$, and their main theorem shows that there exists in the space $C^{0,\alpha}(\overline{\Omega})$, a closed, connected C^1 manifold M of codimension 1 such that $C^{0,\alpha}(\overline{\Omega}) \setminus M$ consists of exactly two connected components, A_1 and A_2, such that the following hold: (i) if $g \in A_1$, (1.3) has no solution; (ii) if $g \in A_2$, (1.3) has exactly 2 solutions; and (iii) if $g \in M$, (1.3) has a unique solution. This result, when $g = t\phi + h$, where ϕ is a positive principal eigenfunction of Δ subject to zero Dirichlet boundary data, and $\int_\Omega \phi h = 0$, gives the manifold M as

$$M = \{g : g = t(h)\phi + h\},$$

where the map $h \mapsto t(h)$ is a C^1 mapping.

For related results and methods we refer to [4] and [16] and their references.

2. Preliminary results

2.1. **The p-Laplace operator and its inverse.** It has been shown in [13] that for any $g \in L^q(\Omega)$, $q = \frac{p}{p-1}$, the problem

$$\begin{cases} -\Delta_p u = g(x) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in W_0^{1,p}(\Omega)$. We define $u =: R_p(g)$. Furthermore, using regularity results of [5, 11, 18] it follows that if $g \in L^\infty(\Omega)$, then $u = R_p(g) \in C_0^{1,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$. Hence the map $R_p : C(\overline{\Omega}) \to C_0^{1,\alpha}(\overline{\Omega})$ is well-defined. It follows that

Proposition 2.1. *The map $R_p : C(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ is completely continuous.*

2.2. **The eigenvalue problem and Poincaré's inequality.** The eigenvalue problem

$$\begin{cases} \Delta_p u + \lambda |u|^{p-2} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

is known to have a principal eigenvalue, $0 < \lambda_1$, which is isolated, and there are no smaller numbers $\lambda_0 < \lambda_1$ for which (2.1) has a nontrivial solution. An eigenfunction associated with this smallest eigenvalue λ_1, w_1, is known to be in $C_0^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0,1)$, $\frac{\partial w_1}{\partial \nu} \neq 0$ (throughout $\frac{\partial}{\partial \nu}$ denotes the inward-normal derivative along the boundary of Ω), and w_1 is one-signed and does not vanish in Ω. Also, all other eigenfunctions associated with λ_1 are constant multiples of w_1; i.e., one may say that λ_1 is a simple eigenvalue. It is also known that if w is an eigenfunction associated with another eigenvalue different from λ_1, then w has to change sign as follows from the following proposition proved in [12].

Proposition 2.2. *Let $\lambda_1 \leq \lambda'$, where λ' is an eigenvalue of* (1.2), *and $w > 0$ be an eigenfunction associated with λ'; then $\lambda' = \lambda_1$.*

(For these results see [2, 12].) Hence, from here on, we take $w_1(x) > 0$ in Ω and $\frac{\partial w_1}{\partial \nu} > 0$ on $\partial\Omega$. The principal eigenvalue λ_1 may be defined by the formula

$$\lambda_1 = \inf_{\substack{u \in W_0^{1,p}(\Omega) \\ \int_\Omega |u|^p = 1}} \int_\Omega |\nabla u|^p.$$

From this characterization immediately follows Poincaré's inequality in $W_0^{1,p}(\Omega)$ (see, e.g., [6]),

$$\lambda_1 \int_\Omega |u|^p \leq \int_\Omega |\nabla u|^p,$$

for all $u \in W_0^{1,p}(\Omega)$.

2.3. **Sub- and supersolutions.** Under suitable assumptions on f, for $u_0 \in W^{1,p}(\Omega)$, the existence of a solution to a general problem

$$\begin{cases} -\Delta_p u = f(x,u) & \text{in } \Omega \\ u = u_0 & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

may be proven using sub-supersolution techniques (see e.g. [3, 8, 9, 10]).

Definition 2.3. *A weak sub-* (*super-*) *solution of* (2.2) *is a function $u \in W^{1,p}(\Omega)$ such that $u \leq (\geq)\ u_0$ on $\partial\Omega$ in the sense of traces and*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi \ \leq (\geq)\ \int_\Omega f(x,u)\phi,$$

for all $\phi \in W_0^{1,p}(\Omega)$ and $\phi \geq 0$ almost everywhere in Ω.

Theorem 2.4 (Sub-Supersolution Theorem)**.** *Assume that for the boundary-value problem* (2.2) *there exist $\underline{u}$ and $\overline{u}$, sub- and supersolutions, respectively, such that*

$$-\infty < C_1 \leq \underline{u} \leq \overline{u} \leq C_2 < \infty, \ \text{a.e. } \Omega,$$

where C_1 and C_2 are constants, and assume that f is a Carathéodory function which also satisfies

$$|f(x,u)| \leq C_3 \ \text{a.e. for } (x,u) \in \Omega \times [C_1, C_2].$$

Then the boundary-value problem (2.2) *has a weak solution* $u \in W^{1,p}(\Omega)$ *with* $\underline{u} \le u \le \overline{u}$ *almost everywhere in* Ω*; i.e.,* $u = u_0$ *on* $\partial\Omega$ *in the sense of traces and*

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi \ = \ \int_\Omega f(x,u)\phi$$

for all $\phi \in W_0^{1,p}(\Omega)$*.*

2.4. **Uniqueness.** Henceforth, let us denote by

$$u^+ := \max\{u, 0\}, \ u^- := -\min\{u, 0\}.$$

We have the following uniqueness result:

Proposition 2.5. *Suppose* $\alpha < \lambda_1 < \beta$*. Then, the problem*

$$\begin{cases} -\Delta_p u &= \beta|u|^{p-2}u^+ - \alpha|u|^{p-2}u^- & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega \end{cases} \tag{2.3}$$

has only the trivial weak solution.

Proof. Suppose $u \in W_0^{1,p}(\Omega)$ is a nontrivial weak solution of the problem and $u^- \not\equiv 0$. Then since u satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi = \int_\Omega \beta|u|^{p-2}u^+\phi - \int_\Omega \alpha|u|^{p-2}u^-\phi,$$

for all $\phi \in W_0^{1,p}(\Omega)$, and $u^- \in W_0^{1,p}(\Omega)$ ([7]), by letting $\phi = u^-$, we have

$$\int_\Omega |\nabla u|^{p-2}\nabla u\nabla(u^-) = -\int_\Omega \alpha|u|^{p-2}u^-u^-,$$

or

$$\int_\Omega |\nabla u^-|^p = \int_\Omega \alpha|u^-|^p. \tag{2.4}$$

On the other hand, by Poincaré's inequality, and since $\alpha < \lambda_1$,

$$\int_\Omega |\nabla u^-|^p \ge \lambda_1 \int_\Omega |u^-|^p > \alpha \int_\Omega |u^-|^p,$$

contradicting (2.4). Hence $u^- \equiv 0$ almost everywhere in Ω. Thus, u is a solution of

$$\begin{cases} -\Delta_p u &= \beta|u|^{p-2}u & \text{in } \Omega \\ u &= 0 & \text{on } \partial\Omega, \end{cases}$$

and $u \ge 0$ in Ω. Therefore, u satisfies the inequalities

$$\begin{cases} \Delta_p u &\le 0 \\ u &\ge 0 \end{cases}$$

in Ω weakly. Then, by the strong maximum principle (see [15]), if there exists $x_0 \in \Omega$ such that $u(x_0) = 0$, it will follow that $u \equiv 0$ in Ω, but we assumed that u is a nontrivial weak solution. Thus $u > 0$ in Ω. Hence, by Proposition 2.2, $\beta = \lambda_1$, which is a contradiction to $\lambda_1 < \beta$. Therefore (2.3) has only the trivial solution. □

3. Existence of solutions for large t

In this section we establish, for large $t \in \mathbb{R}$, the existence of a solution to the following Dirichlet boundary-value problem:

$$\begin{cases} \Delta_p u + f(u) = |t|^{p-2} t\hat{h} + h & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $f \in C^1(\mathbb{R})$ satisfies

$$\lim_{s\to-\infty} \frac{f(s)}{|s|^{p-2}s} = \alpha < \lambda_1 \tag{3.2}$$

and

$$\lim_{s\to+\infty} \frac{f(s)}{s^{p-1}} = \beta > \lambda_1. \tag{3.3}$$

We assume $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is bounded with a sufficiently smooth boundary, $\hat{h}, h \in C(\overline{\Omega})$, and $\hat{h} \geq 0$ in $\overline{\Omega}$. The following is the main theorem in this section; it will be proved using several auxiliary results.

Theorem 3.1. *Suppose f satisfies conditions* (3.2)–(3.3) *given above, $\hat{h}, h \in C(\overline{\Omega})$, $\hat{h} \geq 0$, and $\hat{h} \not\equiv 0$ in $\overline{\Omega}$. Then there exists $t_h >> 1$ such that for each $t \geq t_h$, the problem* (3.1) *has a solution.*

3.1. **Some lemmas and the proof of the theorem.** By the conditions (3.2) and (3.3), we can write f as

$$f(u) = \beta |u|^{p-2} u^+ - \alpha |u|^{p-2} u^- + g(u), \tag{3.4}$$

where g is a continuous function which is smooth, except possibly at the origin, and $\lim_{s\to\pm\infty} g(s)/(|s|^{p-2}s) = 0$. Fixing $t > 0$, we replace f in (3.1) by (3.4), and let $v = \frac{u}{t}$. We thus obtain the equivalent problem

$$\begin{cases} \Delta_p v + \beta |v|^{p-2} v^+ - \alpha |v|^{p-2} v^- + \frac{g(tv)}{t^{p-1}} = \hat{h} + \frac{h}{t^{p-1}} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Thus a weak solution u to problem (3.1) (for $t > 0$) is given by tv, for $t > 0$, where v is a weak solution to (3.5).

Consider the problem

$$\begin{cases} \Delta_p v + \beta |v|^{p-2} v^+ - \alpha |v|^{p-2} v^- = \hat{h} & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Since $\overline{v} = 0$ satisfies

$$\int_\Omega |\nabla \overline{v}|^{p-2} \nabla \overline{v} \nabla \phi \geq \int_\Omega (\beta |\overline{v}|^{p-2} \overline{v}^+ - \alpha |\overline{v}|^{p-2} \overline{v}^- - \hat{h})\phi$$

for all $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq 0$ almost everywhere in Ω, we see that $\overline{v} = 0$ is a weak supersolution for the problem (3.6).

Lemma 3.2. *The problem* (3.6) *has a negative weak subsolution,* $\underline{v} \in C_0^1(\overline{\Omega})$ *with* $\frac{\partial v}{\partial \nu} < 0$ *on* $\partial\Omega$.

Proof. Consider the functional $J : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$J(u) := \int_\Omega \Big(\frac{|\nabla u|^p}{p} - \alpha \frac{|u|^p}{p} - 2\hat{h} u \Big). \tag{3.7}$$

This functional is bounded below, coercive (because $\alpha < \lambda_1$), and weakly lower semicontinuous, and hence will assume its minimum (see, e.g., [17]). Since the Euler-Lagrange equation associated with the energy functional J is

$$\begin{cases} -\Delta_p u &= \alpha |u|^{p-2} u + 2\hat{h} \quad \text{in } \Omega \\ \quad u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

if u is a minimizer of (3.7) in $W_0^{1,p}(\Omega)$, it is a solution to (3.8).

Suppose u is a weak solution to (3.8). Then

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi = \int_\Omega \alpha |u|^{p-2} u \phi + \int_\Omega 2\hat{h}\phi$$

for all $\phi \in W_0^{1,p}(\Omega)$. Suppose $u^- \not\equiv 0$. Then, by taking $\phi = u^-$,

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla u^- = \int_\Omega \alpha |u|^{p-2} u u^- + \int_\Omega 2\hat{h} u^-,$$

and

$$-\int_\Omega |\nabla u^-|^p + \int_\Omega \alpha |u^-|^p = \int_\Omega 2\hat{h} u^-. \tag{3.9}$$

Since $\alpha < \lambda_1$, by Poincaré's inequality, the left-hand side of (3.9) is negative while the right-hand side is nonnegative. Therefore $u^- \equiv 0$ in Ω. Hence, if u is a weak solution to (3.8), then $u \geq 0$ in Ω, and u satisfies

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla \phi \quad = \quad \int_\Omega (\alpha |u|^{p-2} u + 2\hat{h})\phi \geq 0$$

for all $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq 0$ almost everywhere in Ω. Therefore, since $u \in C_0^1(\overline{\Omega})$ and since $\Delta_p u = -\alpha |u|^{p-2} u - 2\hat{h} \leq 0$ weakly, and $u \geq 0$ in $\overline{\Omega}$, it follows from maximum principles (see again [15]) that $u > 0$ in Ω, and $\frac{\partial u}{\partial \nu} > 0$ on $\partial\Omega$ since $\hat{h} \not\equiv 0$ implies $u \not\equiv 0$.

We now let $\underline{v} = -u$. Then $\underline{v}$ is a negative weak solution for

$$\Delta_p \underline{v} = -\alpha|\underline{v}|^{p-2}\underline{v} + 2\hat{h}.$$

Hence, since $\underline{v} < 0$ in Ω, it solves

$$\Delta_p \underline{v} + \beta|\underline{v}|^{p-2}\underline{v}^+ - \alpha|\underline{v}|^{p-2}\underline{v}^- = 2\hat{h}$$

in a distributional sense. Thus we have

$$-\int_\Omega |\nabla \underline{v}|^{p-2}\nabla \underline{v}\nabla\phi + \int_\Omega (\beta|\underline{v}|^{p-2}\underline{v}^+ - \alpha|\underline{v}|^{p-2}\underline{v}^-)\phi = \int_\Omega 2\hat{h}\phi \geq \int_\Omega \hat{h}\phi,$$

for all $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq 0$ almost everywhere in Ω. Therefore $\underline{v}$ is a negative strict weak subsolution of (3.6). A similar existence result can be found in Huang [8]. In addition to the existence of a positive solution to (3.8), he also proves the uniqueness of the solution in [8]. □

From here on, we choose $\underline{v}$ to be the negative weak subsolution found above.

Define the map $\Psi : C_0^1(\overline{\Omega}) \to C(\overline{\Omega})$ by

$$\Psi(v) := \beta|v|^{p-2}v^+ - \alpha|v|^{p-2}v^- - \hat{h},$$

and $\tilde{\Psi} : C_0^1(\overline{\Omega}) \to C(\overline{\Omega})$ by

$$\tilde{\Psi}(v) := \begin{cases} \Psi(0) & \text{if } v \geq 0 \\ \Psi(v) & \text{if } \underline{v} < v < 0 \\ \Psi(\underline{v}) & \text{if } v \leq \underline{v}, \end{cases}$$

and the subset D of $C_0^1(\overline{\Omega})$ by

$$D := \left\{ u \in C_0^1(\overline{\Omega}) : \underline{v} < u < 0 \text{ in } \Omega,\ \frac{\partial \underline{v}}{\partial \nu} < \frac{\partial u}{\partial \nu} < 0 \text{ on } \partial\Omega,\ \|u\|_{C_0^1(\overline{\Omega})} < M \right\},$$

where $M > C$, and C is defined by the following straightforward lemma.

Lemma 3.3. *Solutions for* (3.6) *such that*

$$\underline{v} \leq u \leq 0 \text{ in } \Omega$$

are uniformly bounded in $C_0^1(\overline{\Omega})$.

It is also straightforward to verify that $\Psi : C_0^1(\overline{\Omega}) \to C(\overline{\Omega})$ maps bounded sets into bounded sets, and $\tilde{\Psi} : C_0^1(\overline{\Omega}) \to C(\overline{\Omega})$ maps any set into a bounded set, and both are continuous. Furthermore, the map $R : C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ defined as $R(u) = R_p(\Psi(u))$ is completely continuous, as is the mapping $\tilde{R} : C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ defined as $\tilde{R}(u) := R_p(\tilde{\Psi}(u))$. The next lemma may be proved using variational principles and also follows from a result in [8].

Lemma 3.4. *Suppose $\hat{h} \not\equiv 0$ in $\overline{\Omega}$. If $v \in C_0^1(\overline{\Omega})$ is a weak solution to* (3.6) *such that $\underline{v} \le v \le 0$ in Ω and $\frac{\partial \underline{v}}{\partial \nu} \le \frac{\partial v}{\partial \nu} \le 0$ on $\partial\Omega$, then $\underline{v} < v < 0$ in Ω and $\frac{\partial \underline{v}}{\partial \nu} < \frac{\partial v}{\partial \nu} < 0$ on $\partial\Omega$.*

Lemma 3.5. *Solving the problem* (3.6) *on the set D is equivalent to solving*

$$\begin{cases} -\Delta_p v &= \tilde{\Psi}(v) \quad \text{in } \Omega \\ \quad v &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.10}$$

Proof. If v solves (3.6) on D, then $\underline{v} < v < 0$ in Ω; hence, it solves (3.10).

Suppose v solves (3.10), and the Lebesgue measure of

$$\Omega' := \{x \in \Omega : v(x) \ge 0\}$$

is positive. Since v satisfies

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla \phi = \int_\Omega \tilde{\Psi}(v)\phi, \tag{3.11}$$

and

$$\int_\Omega \Psi(0)\phi \le 0 \tag{3.12}$$

for all $\phi \in W_0^{1,p}(\Omega)$, $\phi \ge 0$ almost everywhere in Ω, by adding (3.11) and (3.12), and reorganizing the terms in the resulting inequality, we get

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla \phi \le \int_\Omega (\tilde{\Psi}(v) - \Psi(0))\phi, \tag{3.13}$$

for all $\phi \in W_0^{1,p}(\Omega)$, $\phi \ge 0$ almost everywhere in Ω. If we let $\phi = v^+$ in (3.13), then

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla v^+ = \int_{\Omega'} |\nabla v|^p \ge 0$$

and

$$\int_\Omega (\tilde{\Psi}(v) - \Psi(0))v^+ \quad = \quad \int_{\Omega'} (\tilde{\Psi}(v) - \Psi(0))v^+ = 0.$$

Thus

$$\int_{\Omega'} |\nabla v|^p = 0;$$

hence, $v \le 0$ in Ω.

To show that $v \ge \underline{v}$ in Ω, suppose v is a solution to (3.10), and the Lebesgue measure of $\Omega'' := \{x \in \Omega : v \le \underline{v}\}$ is positive. Then v and $\underline{v}$, respectively, satisfy

$$\int_\Omega |\nabla v|^{p-2} \nabla v \nabla \phi = \int_\Omega \tilde{\Psi}(v)\phi, \tag{3.14}$$

$$\int_\Omega |\nabla \underline{v}|^{p-2}\nabla \underline{v}\nabla \phi \leq \int_\Omega \Psi(\underline{v})\phi, \tag{3.15}$$

for all $\phi \in W_0^{1,p}(\Omega)$, $\phi \geq 0$ almost everywhere in Ω. By subtracting (3.14) from (3.15), we have

$$\int_\Omega (|\nabla \underline{v}|^{p-2}\nabla \underline{v} - |\nabla v|^{p-2}\nabla v)\nabla \phi \leq \int_\Omega (\Psi(\underline{v}) - \tilde{\Psi}(v))\phi,$$

for all $\phi \in W_0^{1,p}(\Omega)$, $\phi \geq 0$ almost everywhere in Ω. By letting $\phi = (\underline{v} - v)^+$, we have

$$\int_\Omega (|\nabla \underline{v}|^{p-2}\nabla \underline{v} - |\nabla v|^{p-2}\nabla v)\nabla (\underline{v} - v)^+ \leq \int_\Omega (\Psi(\underline{v}) - \tilde{\Psi}(v))(\underline{v} - v)^+.$$

Thus,

$$\begin{aligned} &\int_\Omega (|\nabla \underline{v}|^{p-2}\nabla \underline{v} - |\nabla v|^{p-2}\nabla v)\nabla (\underline{v} - v)^+ \\ = \; &\int_{\Omega''} (|\nabla \underline{v}|^{p-2}\nabla \underline{v} - |\nabla v|^{p-2}\nabla v)(\nabla \underline{v} - \nabla v) \geq 0. \end{aligned}$$

Also

$$\int_\Omega (\Psi(\underline{v}) - \tilde{\Psi}(v))(\underline{v} - v)^+ = \int_{\Omega''} (\Psi(\underline{v}) - \Psi(\underline{v}))(\underline{v} - v)^+ = 0.$$

Therefore, we have

$$\int_{\Omega''} (|\nabla \underline{v}|^{p-2}\nabla \underline{v} - |\nabla v|^{p-2}\nabla v)(\nabla \underline{v} - \nabla v) = 0.$$

It therefore follows, after some computation, that

$$\int_{\Omega''} |\nabla \underline{v}|^p = \int_{\Omega''} |\nabla v|^p,$$

and further that

$$\int_{\Omega''} |\nabla \underline{v}|^{p-2}\nabla \underline{v} \cdot \nabla v - |\nabla \underline{v}|^{p-1}|\nabla v| = 0;$$

i.e.,

$$\nabla \underline{v} \cdot \nabla v = |\nabla \underline{v}||\nabla v| \text{ a.e. in } \Omega''.$$

One quickly concludes that Ω'' has measure zero, and therefore $v \geq \underline{v}$ in Ω. Some of the arguments and computations used in this step can be found in [10]. □

The next lemma is easy to verify.

Lemma 3.6. *The set D is open in $C_0^1(\overline{\Omega})$.*

Lemma 3.7. *There does not exist* $u \in \partial D$ *such that* $u = R(u)$ *or* $u = \tilde{R}(u)$.

Proof. Suppose $u \in \overline{D}$ such that $u = R(u)$. Then $\underline{v} \le u \le 0$ in Ω, $\frac{\partial \underline{v}}{\partial \nu} \le \frac{\partial u}{\partial \nu} \le 0$ on $\partial\Omega$, and $\|u\|_{C_0^1(\overline{\Omega})} \le M$. If $\underline{v} \le u \le 0$ in Ω, by Lemma 3.3, $\|u\|_{C_0^1(\overline{\Omega})} < M$. Also, by Lemma 3.4, if u satisfies $\underline{v} \le u \le 0$ in Ω and $\frac{\partial \underline{v}}{\partial \nu} \le \frac{\partial u}{\partial \nu} \le 0$ on $\partial\Omega$, then $\underline{v} < u < 0$ in Ω and $\frac{\partial \underline{v}}{\partial \nu} < \frac{\partial u}{\partial \nu} < 0$ on $\partial\Omega$. By Lemma 3.5, if $u = \tilde{R}(u)$, then $u = R(u)$ on $\overline{D}$; hence, $u \notin \partial D$. □

Lemma 3.8. *The Leray-Schauder degree*

$$\deg(id - R(\cdot), D, 0) \tag{3.16}$$

is defined in $C_0^1(\overline{\Omega})$ *where id is the identity map in* $C_0^1(\overline{\Omega})$.

Proof. The mapping $id - R(\cdot)$ is a completely continuous perturbation of the identity. By Lemma 3.6, D is open in $C_0^1(\overline{\Omega})$. Lemma 3.7 shows that there are no solutions of $u - R(u) = 0$ on the boundary of D. Therefore, the degree in (3.16) is defined. □

Lemma 3.9. *The Leray-Schauder degree*

$$\deg(id - \tilde{R}(\cdot), D, 0) \tag{3.17}$$

is defined in $C_0^1(\overline{\Omega})$.

Proof. The degree in (3.17) is defined, since $R \equiv \tilde{R}$ on D. □

Lemma 3.10. *The Leray-Schauder degree given by* (3.17) *is 1.*

Proof. Let U be any bounded, open set such that $\overline{D} \subset U$. If $u = \tilde{R}(u)$, then $u \in D$; hence, $u \notin \overline{U} \setminus D$. Thus the Leray-Schauder degree

$$\deg(id - \tilde{R}(\cdot), U, 0)$$

is defined in $C_0^1(\overline{\Omega})$. By the excision property of the Leray-Schauder degree, it follows that (see [14])

$$\deg(id - \tilde{R}(\cdot), U, 0) = \deg(id - \tilde{R}(\cdot), D, 0).$$

For example, since $\overline{D} \subset B_M(0)$, the ball of radius M centered at 0, we observe that all solutions of the problem (3.10) are contained in D by Lemma 3.7; hence, there are no solutions on the boundary of $B_{M'}(0)$ for all $M' \ge M$. Thus, $\deg(id - \tilde{R}(\cdot), B_{M'}(0), 0)$ is defined in $C_0^1(\overline{\Omega})$, and constant for all M' sufficiently large.

The mapping defined by $(\xi, u) \mapsto \xi \tilde{R}(u)$ is completely continuous from $[0,1] \times C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ and is zero free on $\partial B_{M'}(0)$. We hence obtain (using the homotopy property of the degree (see again [14])) the result. □

Lemma 3.11.

$$\deg(id - R(\cdot), D, 0) = 1.$$

Proof. By Lemmas 3.5, 3.8, and 3.9,

$$\deg(id - R(\cdot), D, 0) = \deg(id - \tilde{R}(\cdot), D, 0).$$

By Lemma 3.10,

$$\deg(id - R(\cdot), D, 0) = \deg(id - \tilde{R}(\cdot), B_{M'}(0), 0) = \deg(id, B_{M'}(0), 0).$$

Hence, the conclusion follows. □

Next, one may easily show, using the continuous perturbation property of the Leray-Schauder degree ([14]) that there exists $t_h >> 1$ such that if $t \geq t_h$,

$$\deg(id - R(\cdot), D, 0) = \deg(id - R'(\cdot), D, 0), \tag{3.18}$$

where $R'(u) := R_p\left(\beta|u|^{p-2}u^+ - \alpha|u|^{p-2}u^- + g(tu)/t^{p-1} - \hat{h} - h/t^{p-1}\right)$.

Lemma 3.12. *There exists $t_h >> 1$ such that for $t \geq t_h$,* (3.18) *holds.*

The above auxiliary results now allow us to prove Theorem 3.1.

Proof. By Lemma 3.11 and Lemma 3.12, there exists t_h such that for all $t \geq t_h$, we have

$$\deg(id - R'(\cdot), D, 0) = 1.$$

Hence, there exists a solution $v \in D$ of (3.6) for each $t \geq t_h$. Therefore, (3.5) has a solution $u = tv$ for each $t \geq t_h$. □

4. SOLUTION BOUNDS

In this section we first show that if $\hat{h} > 0$ in $\overline{\Omega}$, and if (t, u) solves (3.1), then $t \geq t_l$ for some $t_l < 0$. Next, we show that if t is in some compact interval $[a, b]$, and (t, u) solves (3.1), then there exists a constant $M_0 = M_0([a, b]) > 0$ such that $\|u\|_{C_0^1(\overline{\Omega})} < M_0$.

4.1. Nonexistence of solutions for large negative t.

In this section $\hat{h} > 0$ in $\overline{\Omega}$. In Section 3 it was only required that $\hat{h} \geq 0$ in $\overline{\Omega}$ to show the existence of a solution for a large positive t.

We first establish

Proposition 4.1. *Assume that f satisfies conditions* (3.2)–(3.3), *that $h, \hat{h} \in C(\overline{\Omega})$, and $\hat{h} > 0$ in $\overline{\Omega}$. Then there exists $t_- \in \mathbb{R}$ such that for all $t \leq t_-$, weak solutions $u \in C_0^1(\overline{\Omega})$ of* (3.1) *must satisfy $u \geq 0$ in Ω.*

Proof. The requirements imposed on f imply that there exist α' and a positive constant $C = C(\alpha')$ such that $\alpha < \alpha' < \lambda_1$ and

$$f(s) \geq \alpha' |s|^{p-2} s - C, \ \forall s \in \mathbb{R}. \tag{4.1}$$

Suppose that the conclusion of the proposition does not hold. Then there exist sequences $\{u_n\}_{n=1}^{\infty} \subset C_0^1(\overline{\Omega})$ and $\{t_n\}_{n=1}^{\infty} \subset \mathbb{R}$ such that $t_n \to -\infty$, (u_n, t_n) satisfies (3.1) in a distributional sense, and $u_n^- \not\equiv 0$ for each $n = 1, 2, 3, \ldots$. Therefore

$$-\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \phi + \int_\Omega f(u_n)\phi = \int_\Omega (|t_n|^{p-2} t_n \hat{h} + h)\phi, \tag{4.2}$$

for all $\phi \in W_0^{1,p}(\Omega)$. Hence, by (4.1) and (4.2), we have

$$-\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla \phi + \int_\Omega \alpha' |u_n|^{p-2} u_n \phi \leq \int_\Omega (|t_n|^{p-2} t_n \hat{h} + h + C)\phi, \tag{4.3}$$

for all $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq 0$ almost everywhere in Ω. Since $\hat{h} > 0$ in $\overline{\Omega}$, there exists $\epsilon > 0$ such that $\hat{h} \geq \epsilon$ in $\overline{\Omega}$. Then,

$$|t_n|^{p-2} t_n \hat{h} \leq |t_n|^{p-2} t_n \epsilon.$$

We also have

$$h + C \leq \max_{\overline{\Omega}} h + C \leq C_1$$

in $\overline{\Omega}$ for some constant C_1, and there exists $N_0 > 0$ such that for all $n \geq N_0$,

$$|t_n|^{p-1} \epsilon \geq C_1.$$

Thus, for those n,

$$|t_n|^{p-2} t_n \hat{h} + h + C \leq 0$$

in $\overline{\Omega}$, and

$$\int_\Omega (|t_n|^{p-2} t_n \hat{h} + h + C)\phi \leq 0,$$

for all $n \geq N_0$ and for all $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq 0$ almost everywhere in Ω.

Hence, for each $n \geq N_0$, u_n satisfies

$$-\int_\Omega |\nabla u_n|^{p-2} \nabla u_n \nabla \phi + \int_\Omega \alpha' |u_n|^{p-2} u_n \phi \leq 0,$$

for all $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq 0$ almost everywhere in Ω. Therefore, u_n is a weak supersolution for

$$\begin{cases} \Delta_p u + \alpha' |u|^{p-2} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

We now fix $n \geq N_0$.

Suppose w is a negative eigenfunction associated with the principal eigenvalue, λ_1, of the eigenvalue problem (2.1). Then, by the homogeneity of the p-Laplacian, discussed in Section 2.2, any positive constant multiple of w is a negative eigenfunction associated with λ_1. Also, since $\frac{\partial w}{\partial \nu} < 0$ on $\partial\Omega$, we can find a constant multiple of w, γw, such that $\gamma w \leq u_n$ in Ω and $\gamma w = 0 = u_n$ on $\partial\Omega$. Since $\alpha' < \lambda_1$, γw satisfies

$$\begin{aligned} 0 &= -\int_\Omega |\nabla(\gamma w)|^{p-2}\nabla(\gamma w)\nabla\phi + \int_\Omega \lambda_1|(\gamma w)|^{p-2}(\gamma w)\phi \\ &\leq -\int_\Omega |\nabla(\gamma w)|^{p-2}\nabla(\gamma w)\nabla\phi + \int_\Omega \alpha'|(\gamma w)|^{p-2}(\gamma w)\phi, \end{aligned}$$

for all $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq 0$ almost everywhere in Ω. Thus γw is a weak subsolution for the problem (4.4). Since we have $\gamma w \leq u_n$ in Ω, by using Theorem 2.4, there exists a solution for (4.4), u, such that $\gamma w \leq u \leq u_n$ in Ω. As a consequence, since $u_n^- \not\equiv 0$ in Ω, we have $u^- \not\equiv 0$.

Now, since u satisfies

$$-\int_\Omega |\nabla u|^{p-2}\nabla u\nabla\phi + \int_\Omega \alpha'|u|^{p-2}u\phi = 0,$$

for all $\phi \in W_0^{1,p}(\Omega)$, and since $u^- \in W_0^{1,p}(\Omega)$ (see [6]), by letting $\phi = u^-$, we have

$$-\int_\Omega |\nabla u|^{p-2}\nabla u\nabla u^- + \int_\Omega \alpha'|u|^{p-2}uu^- = 0.$$

Then u^- satisfies

$$-\int_\Omega |\nabla(-u^-)|^{p-2}\nabla(-u^-)\nabla u^- + \int_\Omega \alpha'|-u^-|^{p-2}(-u^-)u^- = 0;$$

hence, we have

$$\int_\Omega |\nabla u^-|^p - \int_\Omega \alpha'|u^-|^p = 0.$$

However, if $u^- \not\equiv 0$, then, since $\alpha' < \lambda_1$, we obtain

$$\int_\Omega |\nabla u^-|^p = \int_\Omega \alpha'|u^-|^p < \int_\Omega \lambda_1|u^-|^p,$$

which contradicts the Poincaré inequality. Consequently, we must have $u^- \equiv 0$. Therefore, (4.4) has only the trivial solution, which is a contradiction to $u^- \not\equiv 0$ and hence to the assumption. Therefore, there exists $t_-(= t_{N_0}) < 0$ such that if $t \leq t_-$, and u is a solution for (3.1) corresponding to such t, it must be the case that $u \geq 0$ in Ω. □

Theorem 4.2. *Suppose f satisfies the conditions* (3.2)–(3.3), $h, \hat{h} \in C(\overline{\Omega})$, *and $\hat{h} > 0$ in $\overline{\Omega}$; then there exists $t_l \in \mathbb{R}$ such that for all $t \leq t_l$,* (3.1) *has no weak solution.*

Proof. First suppose $t'_- < 0$ is such that if $t \leq t'_-$,

$$|t|^{p-2}t \quad < \quad \frac{f(0) - \max_{\overline{\Omega}} h}{\max_{\overline{\Omega}} \hat{h}}.$$

Then, since $\hat{h} > 0$,

$$|t|^{p-2}t < \frac{f(0) - \max_{\overline{\Omega}} h}{\max_{\overline{\Omega}} \hat{h}} \leq \frac{f(0) - h(x)}{\hat{h}(x)},$$

for all $x \in \overline{\Omega}$; i.e.,

$$|t|^{p-2}t\hat{h}(x) + h(x) - f(0) < 0,$$

for all $x \in \overline{\Omega}$. Then, $u \equiv 0$ cannot be a solution for (3.1).

Let $\epsilon > 0$ be such that $\beta - \epsilon > \lambda_1$. Since $\lim_{s\to\pm\infty} \frac{g(s)}{|s|^{p-2}s} = 0$, for this ϵ, there exists $\delta > 0$ such that

$$-g(s) - \epsilon s^{p-1} < 0$$

for all $s > \delta$, and for ϵ and δ, there exists $C > 0$ such that

$$-g(s) - \epsilon s^{p-1} \leq C$$

for all $0 \leq s \leq \delta$ since g is piecewise smooth.

For this C, since $\hat{h} > 0$ in $\overline{\Omega}$, there exists $t''_- < 0$ such that

$$\int_\Omega (|t|^{p-2}t\hat{h} + h + C)\phi \leq 0, \tag{4.5}$$

for all $t \leq t''_-$, and $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq 0$ almost everywhere in Ω.

Proposition 4.1 implies that, if $\alpha < \lambda_1$, then there exists t_- such that for all $t \leq t_-$, (3.1) has only nonnegative solutions. Suppose $t \leq \min\{t_-, t'_-, t''_-\}$ and let $u \geq 0$ be a solution to (3.1). Then, (u, t) satisfies

$$-\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi + \beta \int_\Omega u^{p-1}\phi = \int_\Omega (|t|^{p-2}t\hat{h} + h)\phi - \int_\Omega g(u)\phi; \tag{4.6}$$

thus, by (4.5) and (4.6), u satisfies

$$-\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\phi + \beta \int_\Omega u^{p-1}\phi \leq \int_\Omega (-g(u) - C)\phi, \tag{4.7}$$

for all $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq 0$ almost everywhere in Ω. By subtracting $\int_\Omega \epsilon u^{p-1}\phi$ from both sides of the equation in (4.7), we obtain

$$-\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi + (\beta - \epsilon)\int_\Omega u^{p-1}\phi \leq \int_\Omega (-g(u) - \epsilon u^{p-1} - C)\phi. \quad (4.8)$$

If $\Omega_1 := \{x \in \Omega : u(x) > \delta\}$ and $\Omega_2 := \{x \in \Omega : 0 \leq u(x) \leq \delta\}$,

$$\begin{aligned} & \int_\Omega (-g(u) - \epsilon u^{p-1} - C)\phi \\ = & \int_{\Omega_1} (-g(u) - \epsilon u^{p-1} - C)\phi + \int_{\Omega_2} (-g(u) - \epsilon u^{p-1} - C)\phi \leq 0. \end{aligned}$$

Hence,

$$-\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi + (\beta - \epsilon)\int_\Omega u^{p-1}\phi \leq 0,$$

and thus u is a weak supersolution of

$$\begin{cases} \Delta_p u + (\beta - \epsilon)|u|^{p-2}u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.9)$$

Also, since $u \geq 0$ in Ω, and

$$-\int_\Omega |\nabla u|^{p-2}\nabla u \nabla \phi \leq -(\beta - \epsilon)\int_\Omega u^{p-1}\phi \leq 0,$$

for all $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq 0$ almost everywhere in Ω, $\Delta_p u \leq 0$ weakly. Then it follows from maximum principles that $u > 0$ in Ω, and that $\frac{\partial u}{\partial \nu} > 0$ on $\partial\Omega$ since $u \not\equiv 0$.

Suppose w is a positive eigenfunction associated with the principal eigenvalue of (1.2). Then, since any constant multiple of w is an eigenfunction, we can find $\eta > 0$ such that $u \geq \eta w > 0$ in Ω. Then u is a positive supersolution, and ηw is a positive subsolution of (4.9) such that $u \geq \eta w > 0$ in Ω. Thus, by Theorem 2.4, there exists a positive solution, $\tilde{u}$, such that $u \leq \tilde{u} \leq \eta w$ in Ω for (4.9) and $\tilde{u}$ is a nontrivial weak solution to

$$\begin{cases} \Delta_p u + (\beta - \epsilon)|u|^{p-2}u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

contradicting Proposition 2.2. Therefore, if $t \leq t'_-$, (3.1) does not possess nonnegative solutions. Thus, if $t \leq t_l := \min\{t_-, t'_-, t''_-\}$, then (3.1) does not have a solution. □

4.2. **Solution bounds for** $t \in [a, b]$**.**

Theorem 4.3. *Given a compact interval* $[a, b]$ *and a number* $C > 0$*, there exists a constant* $M_0 = M_0([a, b], C)$ *such that if* (t, u) *is a solution of* (3.1) *with* $t \in [a, b]$ *and* $\|\hat{h}\|_{C(\overline{\Omega})} \leq C$*, then* $\|u\|_{C_0^1(\overline{\Omega})} \leq M_0$*.*

In order to prove the theorem, we first note the following lemma.

Lemma 4.4. *Let* $\{u_n\}_{n=1}^{\infty}$ *be a sequence in* $C_0^1(\overline{\Omega})$ *such that* $\|u_n\|_{C_0^1(\overline{\Omega})} \to \infty$ *as* $n \to \infty$*,* $v_n = u_n/\|u_n\|_{C_0^1(\overline{\Omega})}$*, and* $\{t_n\}_{n=1}^{\infty} \subset [a, b]$ *for some* a *and* b *such that* $a \leq b$*, where* (t_n, u_n) *solves* (3.1) *for each* $n = 1, 2, 3, \ldots$*. Assume that* $\hat{h}, h \in C(\overline{\Omega})$*,* $\|\hat{h}\|_{C(\overline{\Omega})} \leq C$*, and that* $g \in C^1(\mathbb{R})$ *is such that* $\lim_{|s| \to \infty} \frac{g(s)}{|s|^{p-2}s} = 0$*. Then the sequence* $\{\tilde{v}_n\}_{n=1}^{\infty}$ *given by*

$$\tilde{v}_n := \beta|v_n|^{p-2}v_n^+ - \alpha|v_n|^{p-2}v_n^- + \frac{g(t_n v_n)}{\|u_n\|_{C_0^1(\overline{\Omega})}^{p-1}} - \frac{|t_n|^{p-2}t_n}{\|u_n\|_{C_0^1(\overline{\Omega})}^{p-1}}\hat{h} - \frac{h}{\|u_n\|_{C_0^1(\overline{\Omega})}^{p-1}}$$

is bounded in $C(\overline{\Omega})$*.*

We now give the proof of Theorem 4.3.

Proof. Suppose the conclusion of Theorem 4.3 does not hold. Then there exist sequences $\{t_n\}_{n=1}^{\infty} \subset [a, b]$, $\{\hat{h}_n\}_{n=1}^{\infty} \subset C(\overline{\Omega})$, and $\{u_n\}_{n=1}^{\infty} \subset C_0^1(\overline{\Omega})$ such that $\|\hat{h}_n\|_{C(\overline{\Omega})} \leq C$ for some $C > 0$ and $\|u_n\|_{C_0^1(\overline{\Omega})} \to +\infty$ as $n \to +\infty$, and that u_n is a weak solution of the problem (3.1) corresponding to $t = t_n$, and $\hat{h} = \hat{h}_n$. Thus, u_n satisfies

$$u_n = R_p(\beta|u_n|^{p-2}u_n^+ - \alpha|u_n|^{p-2}u_n^- + g(u_n) - |t_n|^{p-2}t_n\hat{h}_n - h). \tag{4.10}$$

By dividing both sides of (4.10) by $\|u_n\|_{C_0^1(\overline{\Omega})}$ and using the homogeneity of R_p, we obtain that the sequence $\{v_n\}$, where $v_n = u_n/\|u_n\|_{C_0^1(\overline{\Omega})}$, by passing to a subsequence, if necessary, converges, say, $v_n \to v_0$, for $v_0 \in C_0^1(\overline{\Omega})$, which must be a solution of

$$\begin{cases} -\Delta_p v = \beta|v|^{p-2}v^+ - \alpha|v|^{p-2}v^- & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

which has only the trivial solution, by Proposition 2.5. Since $\|v_n\|_{C_0^1(\overline{\Omega})} = 1$ for all $n \in \mathbb{N}$, $\|v_0\|_{C_0^1(\overline{\Omega})} = 1$, contradicting that $v_0 = 0$ in $\overline{\Omega}$. Hence, the result follows. □

5. Multiplicity result: $\hat{h} > 0$ in $\overline{\Omega}$

The results of Sections 3 and 4 may be combined to show that, in the case $\hat{h} > 0$ in $\overline{\Omega}$, there are regions in $\mathbb{R} \times C_0^1(\overline{\Omega})$ where there are no solutions of (3.1), a region where there are multiple solutions possible, and a region where there is at least one solution. Using properties of the Leray-Schauder degree, and the following set of lemmas, we shall obtain such an existence and multiplicity result (cf. [16] for corresponding results and methods for ordinary differential equations).

Lemma 5.1. *Suppose f satisfies the conditions* (3.2)–(3.3), $\hat{h}, h \in C(\overline{\Omega})$, $\hat{h} \geq 0$, *and* $\hat{h} \not\equiv 0$ *in* $\overline{\Omega}$. *If* (3.1) *has a solution for* $t = \tau$ *for some* $\tau \in \mathbb{R}$, *then* (3.1) *has a solution for each* $t \geq \tau$.

Proof. Suppose (3.1) has a solution for $t = \tau$, and denote it by u_τ. Then, since $\hat{h} \geq 0$ for any $t \geq \tau$, we have

$$-\int_\Omega |\nabla u_\tau|^{p-2} \nabla u_\tau \nabla \phi + \int_\Omega f(u_\tau)\phi = \int_\Omega (|\tau|^{p-2}\tau \hat{h} + h)\phi \leq \int_\Omega (|t|^{p-2} t\hat{h} + h)\phi,$$

for all $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq 0$ almost everywhere in Ω. Therefore, u_τ is a supersolution for (3.1) for any $t \geq \tau$.

Now, define

$$\tilde{h} = \begin{cases} \max_{\overline{\Omega}} h & \text{if there exists } x \text{ such that } h(x) > 0 \text{ in } \Omega \\ 0 & \text{if } h(x) \leq 0 \text{ for all } x \in \Omega, \end{cases}$$

and consider the problem

$$\begin{cases} \Delta_p u + \alpha' |u|^{p-2} u = |t|^{p-2} t\hat{h} + \tilde{h} + C & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where α' and C are as defined in the proof of Proposition 4.1, i.e., such that

$$f(s) \geq \alpha' |s|^{p-2} s - C \text{ for all } s \in \mathbb{R}.$$

Then, since $\alpha' \in (\alpha, \lambda_1)$, and $|t|^{p-2} t\hat{h} + \tilde{h} + C \geq 0$ in Ω for $t \geq 0$, by replacing $2\hat{h}$ in the proof of Lemma 3.2 with $|t|^{p-2} t\hat{h} + \tilde{h} + C$, we obtain a negative solution for (5.1) for any $t \geq 0$. Fix $t \geq \tau$ and let $\tau_1 \geq t$. Then, if u_{τ_1} is a negative solution for (5.1) for $t = \tau_1$, u_{τ_1} satisfies

$$-\int_\Omega |\nabla u_{\tau_1}|^{p-2} \nabla u_{\tau_1} \nabla \phi + \int_\Omega \alpha' |u_{\tau_1}|^{p-2} u_{\tau_1} \phi = \int_\Omega (|\tau_1|^{p-2} \tau_1 \hat{h} + \tilde{h} + C)\phi, \tag{5.2}$$

for all $\phi \in W_0^{1,p}(\Omega)$. Also notice that by multiplying (5.2) by γ^{p-1} where $\gamma \geq 1$ and using the homogeneity of the p-Laplacian, we get

$$\begin{aligned} & -\int_\Omega |\nabla(\gamma u_{\tau_1})|^{p-2}\nabla(\gamma u_{\tau_1})\nabla\phi + \int_\Omega \alpha'|\gamma u_{\tau_1}|^{p-2}(\gamma u_{\tau_1})\phi \\ = \;& \gamma^{p-1}\int_\Omega (|\tau_1|^{p-2}\tau_1\hat{h} + \tilde{h} + C)\phi \geq \int_\Omega (|\tau_1|^{p-2}\tau_1\hat{h} + \tilde{h} + C)\phi, \end{aligned} \tag{5.3}$$

for all $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq 0$ almost everywhere in Ω. Then, we have

$$\begin{aligned} & -\int_\Omega |\nabla(\gamma u_{\tau_1})|^{p-2}\nabla(\gamma u_{\tau_1})\nabla\phi + \int_\Omega (\alpha'|\gamma u_{\tau_1}|^{p-2}(\gamma u_{\tau_1}) - C)\phi \\ \geq \;& \int_\Omega (|\tau_1|^{p-2}\tau_1\hat{h} + \tilde{h})\phi. \end{aligned}$$

Then, since

$$\begin{aligned} & -\int_\Omega |\nabla(\gamma u_{\tau_1})|^{p-2}\nabla(\gamma u_{\tau_1})\nabla\phi + \int_\Omega f(\gamma u_{\tau_1})\phi \\ \geq \;& -\int_\Omega |\nabla(\gamma u_{\tau_1})|^{p-2}\nabla(\gamma u_{\tau_1})\nabla\phi + \int_\Omega (\alpha'|\gamma u_{\tau_1}|^{p-2}(\gamma u_{\tau_1}) - C)\phi, \end{aligned} \tag{5.4}$$

and

$$\int_\Omega (|\tau_1|^{p-2}\tau_1\hat{h} + \tilde{h})\phi \geq \int_\Omega (|t|^{p-2}t\hat{h} + h)\phi,$$

for all $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq 0$ almost everywhere in Ω, we have

$$-\int_\Omega |\nabla(\gamma u_{\tau_1})|^{p-2}\nabla(\gamma u_{\tau_1})\nabla\phi + \int_\Omega f(\gamma u_{\tau_1})\phi \geq \int_\Omega (|t|^{p-2}t\hat{h} + h)\phi,$$

for all $\phi \in W_0^{1,p}(\Omega)$ such that $\phi \geq 0$ almost everywhere in Ω. Therefore, for any $\gamma \geq 1$, γu_{τ_1} is a negative subsolution for (3.1).

The inequalities above imply

$$-\int_\Omega |\nabla(\gamma u_{\tau_1})|^{p-2}\nabla(\gamma u_{\tau_1})\nabla\phi + \int_\Omega \alpha'|\gamma u_{\tau_1}|^{p-2}(\gamma u_{\tau_1})\phi \geq 0,$$

which can be used to show $u_{\tau_1} \leq 0$ in Ω as in the proof of Lemma 3.2. Also from this inequality above, we obtain

$$-\int_\Omega |\nabla(\gamma u_{\tau_1})|^{p-2}\nabla(\gamma u_{\tau_1})\nabla\phi \geq 0. \tag{5.5}$$

As in the proof of Lemma 3.2, we may use (5.5) and maximum principles to obtain $u_{\tau_1} < 0$ in Ω, and $\frac{\partial u_{\tau_1}}{\partial \nu} < 0$ on $\partial\Omega$ since $u_{\tau_1} = 0$ on $\partial\Omega$. Hence, we can find some $\gamma \geq 1$ such that $u_\tau \geq \gamma u_{\tau_1}$ in Ω. Therefore, by Theorem 2.4, (3.1) has a solution for $\tau \le t \le \tau_1$. However, the proof works for any $\tau_1 \ge 0$, and thus (3.1) has a solution for all $t \ge \tau$ if (3.1) has a solution for $t = \tau$. $\square$

Theorem 5.2. *Suppose f and h satisfy the conditions in Lemma* 5.1, *and $\hat{h} > 0$ in $\overline{\Omega}$. Then there exist t_1 and t_2, $t_1 \le t_2$, such that the problem* (3.1) *has no solution if $t < t_1$, at least one solution if $t_1 \le t \le t_2$, and at least two solutions if $t > t_2$.*

Proof. Define the mapping

$$F(u,t) := \begin{cases} R_1(u,t), & t \le 1 \\ R_2(u,t), & t \ge 1 \end{cases}$$
$$:= \begin{cases} R_p(\beta|u|^{p-2}u^+ - \alpha|u|^{p-2}u^- + g(u) - |t|^{p-2}t\hat{h} + h), \\ R_p\left(\beta|u|^{p-2}u^+ - \alpha|u|^{p-2}u^- + \frac{g(tu)}{t^{p-1}} - \hat{h} + \frac{h}{t^{p-1}}\right). \end{cases}$$

Then we may study the problem $u = F(u,t)$ instead of the problem (3.1), since the number of solutions to $u = F(u,t)$ is the same as the number of solutions to (3.1); viz., if (u,t) solves (3.1) for $t \le 1$, by definition, $u = R_1(u,t)$, and if (u,t) solves $u = R_2(u,t)$, then tu solves (3.1).

We first note that $F : C_0^1(\overline{\Omega}) \times \mathbb{R} \to C_0^1(\overline{\Omega})$ is a completely continuous map.

It follows from Theorem 4.2 that there exists t_l such that if $t \le t_l$, problem (3.1) does not have a solution. Then, by Theorem 3.1, there exists a number $t_h >> 1$ such that the problem (3.1) has a solution if $t \ge t_h$ by noting that the degree

$$\deg(id - R'(\cdot), D, 0) = 1,$$

where $R'(u) = R_p(\beta|u|^{p-2}u^+ - \alpha|u|^{p-2}u^- + \frac{g(tu)}{t^{p-1}} - \hat{h} + \frac{h}{t^{p-1}})$, and D is the subset of $C_0^1(\overline{\Omega})$ defined in Section 3; i.e.,

$$\deg(id - F(\cdot, t_h), D, 0) = 1.$$

Also, we have shown that solutions to (3.1) for all t in each compact interval $[a,b]$ are uniformly bounded (see Theorem 4.3); thus, there exists $C_1 > 0$ such that if u solves (3.1) for $t \in [t_l, t_h]$, then $\|u\|_{C_0^1(\overline{\Omega})} < C_1$. Since, if $t \ge 1$, solutions for (3.1) may be given by tv where (v,t) is a solution to $v = R_2(v,t)$, $\|tv\|_{C_0^1(\overline{\Omega})} < C_1$ implies $\|v\|_{C_0^1(\overline{\Omega})} < \frac{1}{t}C_1 \le C_1$. Therefore, all solutions to $u = F(u,t)$ for $t \le t_h$ are contained in $B_{C_1}(0) \times [t_l, t_h] \subset C_0^1(\overline{\Omega}) \times \mathbb{R}$; hence, $\overline{D} \subset B_{C_1}(0)$ and

$$\deg(id - F(\cdot, t), B_{C_1}(0), 0) = 0, \ t \le t_h.$$

Therefore, by the excision principle and the additivity and solution properties of the degree, another solution of (3.1) (outside D) must exist for $t = t_h$.

Now, by Lemma 5.1, we may define

$$t_1 := \inf\{t : \text{there exists a solution of (3.1)}\},$$

and

$$t_2 := \inf\{t_h : \text{there exist at least two solutions for (3.1) if } t \geq t_h\}.$$

Then we have the following:

1. There is no solution for (3.1) if $t < t_1$.
2. (3.1) has at least one solution if $t_1 \leq t \leq t_2$.
3. There are at least two solutions for (3.1) if $t > t_2$. □

6. Multiplicity result: $\hat{h} \geq 0$ in $\overline{\Omega}$

In Section 4, we found that multiple solutions are possible for (3.1) when $\hat{h} > 0$ in $\overline{\Omega}$. In this section, we address the same problem for the case $\hat{h} \geq 0$ by using a limiting argument.

Theorem 6.1. *Suppose f satisfies conditions* (3.2)–(3.3), $\hat{h}, h \in C(\overline{\Omega})$, *and* $\hat{h} \geq 0$, *but* $\hat{h} \not\equiv 0$ *in* $\overline{\Omega}$. *Then there exists* $\hat{t}$ *such that if* $t \geq \hat{t} \geq -\infty$, *the problem*

$$\begin{cases} \Delta_p u + f(u) &= |t|^{p-2}t\hat{h} + h \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{6.1}$$

has at least two solutions.

Proof. By Theorem 3.1, there exists $\hat{t}$ such that if $t \geq \hat{t}$, the problem

$$\begin{cases} \Delta_p u + \beta|u|^{p-2}u^+ - \alpha|u|^{p-2}u^- + \frac{g(tu)}{t^{p-1}} &= \hat{h} + \frac{h}{t^{p-1}} \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{6.2}$$

has a solution in the set

$$D = \left\{ u \in C_0^1(\overline{\Omega}) : \underline{v} < u < 0 \text{ in } \Omega,\ \frac{\partial \underline{v}}{\partial \nu} < \frac{\partial u}{\partial \nu} < 0 \text{ on } \partial\Omega,\ \|u\|_{C_0^1(\overline{\Omega})} < M \right\},$$

where $\underline{v}$ is a solution for

$$\begin{cases} \Delta_p u + \beta|u|^{p-2}u^+ - \alpha|u|^{p-2}u^- &= 2\hat{h} \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6.3}$$

which is a strict subsolution for

$$\begin{cases} \Delta_p u + \beta|u|^{p-2}u^+ - \alpha|u|^{p-2}u^- &= \hat{h} \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{6.4}$$

and M is as defined in Lemma 3.3.

Fix $t \in [\hat{t}, \infty)$ and let $\{\hat{h}_n\}_{n=1}^{\infty}$ be such that $\hat{h}_n > 0$ in $\overline{\Omega}$, $\hat{h}_n \to \hat{h}$ in $C(\overline{\Omega})$, and

$$R'_n(u) := R_p\left(\beta|u|^{p-2}u^+ - \alpha|u|^{p-2}u^- - \hat{h}_n + \frac{g(tu)}{t^{p-1}} - \frac{h}{t^{p-1}}\right).$$

Since $\hat{h}_n \to \hat{h}$ in $C(\overline{\Omega})$, for given $r > 0$, there exists $N_0 > 0$ such that

$$\|R'_n(u) - R'(u)\|_{C_0^1(\overline{\Omega})} < r$$

if $n \geq N_0$. Therefore, by Leray-Schauder degree properties

$$1 = \deg(id - R'(\cdot), D, 0) = \deg(id - R'_n(\cdot), D, 0) \tag{6.5}$$

for r small, and hence $u = R'_n(u)$ has a solution in D and no solutions on ∂D if $n \geq N_0$.

By applying the arguments in the proof of Theorem 5.2, we conclude that if $n \geq N_0$, then the problem

$$\begin{cases} \Delta_p u + \beta|u|^{p-2}u^+ - \alpha|u|^{p-2}u^- + \frac{g(tu)}{t^{p-1}} = \hat{h}_n + \frac{h}{t^{p-1}} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{6.6}$$

has at least two solutions, at least one in D and one in $C_0^1(\overline{\Omega}) \setminus \overline{D}$.

Let $n_j := N_0 - j + 1$ and let $\{u_{n_j}\}_{j=1}^{\infty}$ be such that $u_{n_j} \in C_0^1(\overline{\Omega}) \setminus \overline{D}$ is a solution for $u = R'_{n_j}(u)$ for each $j = 1, 2, 3, \ldots$. Then, u_{n_j} satisfies

$$u_{n_j} = R'_{n_j}(u_{n_j}). \tag{6.7}$$

Therefore, by Theorem 4.3, since t is fixed, solutions for (6.6) are uniformly bounded for all $j \in \mathbb{N}$.

Since R_p is completely continuous, by passing to a subsequence, if necessary, we may assume that $u_{n_j} \to \hat{u}$ as $j \to \infty$, for some $\hat{u} \in C_0^1(\overline{\Omega})$. By a limiting argument we obtain that $\hat{u}$ is a solution for (6.2).

We saw above that $\{u_{n_j}\}_{j=1}^{\infty} \subset C_0^1(\overline{\Omega}) \setminus \overline{D}$; hence, $\hat{u} \in C_0^1(\overline{\Omega}) \setminus \overline{D}$. Therefore, there exists $\hat{t}$ such that if $t \geq \hat{t}$, (6.2) has at least two solutions, one in D and another one in $C_0^1(\overline{\Omega}) \setminus \overline{D}$. Consequently, (6.1) has at least two solutions. □

In case that $\hat{h} \equiv 0$ in $\overline{\Omega}$, as the following example shows, (6.1) may not possess any solutions.

Proposition 6.2. *Suppose $k > 0$, $\beta > \lambda_1$, and $\alpha < \lambda_1$; then*

$$\begin{cases} \Delta_p u + \beta|u|^{p-2}u^+ - \alpha|u|^{p-2}u^- = -k & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{6.8}$$

has no solution.

Proof. We can use the same arguments as in the proof of Proposition 4.1 and that of Theorem 4.2. □

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