

ON THE REGULARITY OF THE SOLUTION OF THE DIRICHLET PROBLEM FOR HAMILTON-JACOBI EQUATIONS

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In memory of Stefano Benvenuti

Abstract. We consider the Dirichlet problem for a class of Hamilton-Jacobi equations and we show that if the viscosity solution is semiconcave near the boundary then it is globally semiconcave. As an application, we discuss the particular case of eikonal-type equations. Finally, we describe a sort of stability property for the singular set of the semiconcave viscosity solution of Hamilton-Jacobi equations.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be an open convex set and consider the Dirichlet problem

$$\begin{aligned} u(x) + H(x, \nabla u(x)) &= 0 && \text{in } \Omega \\ u &= g && \text{on } \partial\Omega. \end{aligned} \quad (1.1)$$

Here H and g are continuous functions. In general, the above problem does not admit global smooth (classical) solutions. We adopt the notion of weak solution introduced in [11] (see also [12]).

Definition 1.1. *We say that a continuous function $u : \bar{\Omega} \rightarrow \mathbb{R}$ is a viscosity subsolution of (1.1) if for every $\varphi \in C^1(\Omega)$ we have*

$$u(x) + H(x, \nabla\varphi(x)) \leq 0,$$

where $x \in \Omega$ denotes any local maximum point of $u - \varphi$. Analogously, u is a viscosity supersolution of (1.1) if for every $\varphi \in C^1(\Omega)$ and x a local minimum point of $u - \varphi$ then

$$u(x) + H(x, \nabla\varphi(x)) \geq 0.$$

Finally, we say that u is a viscosity solution if $u = g$ on $\partial\Omega$ and it is both a viscosity sub- and supersolution.

We deal with global regularity properties of the solution of (1.1). More precisely, we are interested in the semiconcavity of u on $\overline{\Omega}$ (see Definition 2.1).

In essence, the semiconcavity is an upper bound on the distributional second derivatives of the solution u .

We show that if a solution of equation (1.1) is semiconcave near the boundary then it is semiconcave on the whole $\overline{\Omega}$ (see Section 3).

From a technical point of view our approach is strongly inspired by [13]. We would like to point out that the convexity assumption for Ω is essential to our method of proof, even though there are related local results in the interior of Ω ([10]) that do not assume Ω is a convex set.

We recall that the structure of semiconcave functions is pretty well understood. A semiconcave function is locally Lipschitz continuous, hence, by a theorem of Rademacher, it is differentiable almost everywhere. We say that a point belongs to the *singular set* of u , $\Sigma(u)$, if u is not differentiable at such a point. Some precise estimates on the Hausdorff dimension of the singular set and its closure are available (see [18] and [16]). For instance, one can show that given a semiconcave function $u : \Omega \rightarrow \mathbb{R}$ the Hausdorff dimension of the singular set of u is less than or equal to $n - 1$ (we recall that $\Omega \subset \mathbb{R}^n$). Moreover, one can study the structure of the singular set around a given singular point x_0 . We say that the singularity at x_0 *propagates* if such a point is not isolated; i.e., x_0 is a cluster point for the singular set. There are several results on the propagation of singularities along Lipschitz curves and higher dimensional sets (see e.g. [4]).

As an application, in Section 4, we consider the special case of eikonal-type equations. We point out that the regularity of the solution of this equation can be analyzed by means of several different methods (see e.g. [14], [15], and [10]).

An easy consequence of the semiconcavity of the viscosity solution of a Hamilton-Jacobi equation is a sort of stability of the singularities. More precisely, let u be a semiconcave viscosity solution of the equation $F(x, u, \nabla u) = 0$ and suppose that u has a nonempty singular set. Consider some perturbed problems of the form $F_n(x, u_n, \nabla u_n) = 0$ with F_n converging to F (in L^∞), as $n \rightarrow \infty$. Then we show that, for n large enough, the singular sets of u_n are nonempty (see Section 5).

In an early stage of this work I benefited from many discussions with my colleague and friend Stefano Benvenuti. I would like to dedicate this paper to him as a sign of my deep gratitude, friendship, and a recollection of quite a number of great moments.

2. NOTATION AND PRELIMINARIES

For $p, q \in \mathbb{R}^n$, set $[p, q] := \{(1 - \lambda)p + \lambda q : \lambda \in [0, 1]\}$. Given $A \subset \mathbb{R}^n$ we define $d_A(x) := \inf\{|x - y| : y \in A\}$ ($x \in \mathbb{R}^n$). Let $\Omega \subset \mathbb{R}^n$; the symbol $\overline{\Omega}$ stands for the closure of Ω in \mathbb{R}^n .

As usual, we say that a function $u : \overline{\Omega} \rightarrow \mathbb{R}$ is Lipschitz continuous if

$$|u(x) - u(y)| \leq L_u |x - y| \quad \forall x, y \in \overline{\Omega},$$

for some positive constant L_u . In the sequel, we will refer to a constant L_u satisfying the above inequality as a Lipschitz constant for u on $\overline{\Omega}$.

Definition 2.1. A continuous function $u : \overline{\Omega} \rightarrow \mathbb{R}$ is said to be semiconcave on a set $A \subset \overline{\Omega}$ if and only if there exists a semiconcavity constant $C \in \mathbb{R}$ such that, for every $x \in A$ and for every $h \in \mathbb{R}^n$ such that $[x - h, x + h] \subset A$, we have

$$u(x + h) + u(x - h) - 2u(x) \leq C|h|^2. \quad (2.1)$$

We denote by $SC(\overline{\Omega})$ the set of all semiconcave functions on $\overline{\Omega}$.

Remark 2.1. It is easy to see that a semiconcave function is the sum of a concave function and a quadratic polynomial.

Let $u : \Omega \rightarrow \mathbb{R}$ be a continuous function and define the *superdifferential* of u at x as

$$D^+u(x) = \left\{ p \in \mathbb{R}^n : \limsup_{y \rightarrow x} \frac{u(y) - u(x) - p \cdot (y - x)}{|y - x|} \leq 0 \right\}.$$

Remark 2.2. (i) It is well known that, if u is a continuous function, $p \in D^+u(x)$ if and only if there exists a function $\varphi \in C^1$ such that x is a maximum point for $u - \varphi$ and $p = \nabla\varphi(x)$ (see e.g. [7]).

(ii) Let $u \in SC(\overline{\Omega})$ and let $\{u_j\}$ be a sequence of functions of class C^1 uniformly converging to the function u and, for some $x_0 \in \Omega$, let $p \in D^+u(x_0)$. Then, one can show that there exists a sequence of points $\{x_j\}$ converging to x_0 such that

$$p = \lim_{j \rightarrow \infty} \nabla u_j(x_j).$$

3. THE SEMICONCAVITY RESULT

In this section we assume that the Hamiltonian H is a continuous function satisfying the following conditions

(H1) for every $L > 0$ there exists a positive number c_0 such that, for every $c \geq c_0$,

$$H(x+h, p+ch) + H(x-h, p-ch) - 2H(x, p) \geq -c|h|^2 \quad (3.1)$$

for every $x \in \overline{\Omega}$ and for every $p \in B(0, L)$.

The above condition (H1) is a sort of convexity assumption on H .

(H2) For every $x \in \overline{\Omega}$ and for every $p, q \in \mathbb{R}^n$,

$$|H(x, p) - H(x, q)| \leq \omega(|p - q|). \quad (3.2)$$

Here $\omega(\cdot)$ is a positive function such that $\omega(r) \rightarrow 0$ as $r \rightarrow 0$.

We define $\Omega_\rho = \{x \in \Omega : d_{\partial\Omega}(x) < \rho\}$.

(H3) There exists ρ_0 such that the viscosity solution u of the problem (1.1) is semiconcave in $\overline{\Omega}_{\rho_0}$. In order to fix the ideas, we denote by c_1 a semiconcavity constant for u in $\overline{\Omega}_{\rho_0}$.

Remark 3.1. An easy (strong) sufficient condition for (H3) is $C^{1,1}$ regularity of u near the boundary.

The following result holds.

Theorem 3.1. *Under Assumptions (H1), (H2), and (H3) let Ω be an open convex set and let u be a bounded, Lipschitz continuous viscosity solution of (1.1). Then u is semiconcave on $\overline{\Omega}$.*

The above statement can be rephrased by saying that, in some sense, the semiconcavity of u “propagates” from a neighbourhood of the boundary toward the interior of Ω .

Proof. From a technical point of view our proof is in essence the proof in [13].

(i) Ω bounded. In this case the assumption (H2) is unnecessary.

Let c_0, ρ_0 and c_1 be the numbers given by the assumptions (H1) and (H3). Take $c > \max\{c_0, c_1, 32\|u\|_\infty/\rho_0^2\}$ (this choice of c is motivated by Case 2 below). For $\alpha > 0$, set

$$\Phi(x, y, z; c, \alpha) := u(x) + u(y) - 2u(z) - \frac{1}{2\alpha}|x+y-2z|^2 - \frac{c}{2}(|x-z|^2 + |y-z|^2)$$

($x, y, z \in \overline{\Omega}$) and consider $\max_{x, y, z \in \overline{\Omega}} \Phi(x, y, z; c, \alpha)$. We denote by $(\overline{x}, \overline{y}, \overline{z})$ a maximizer of Φ over $\overline{\Omega}^3$. Clearly,

$$(\overline{x}, \overline{y}, \overline{z}) = (\overline{x}(c, \alpha), \overline{y}(c, \alpha), \overline{z}(c, \alpha));$$

moreover, for every $x \in \overline{\Omega}$ and for every $h \in \mathbb{R}^n$ such that $[x-h, x+h] \subset \overline{\Omega}$,

$$u(x+h) + u(x-h) - 2u(x) - c|h|^2 \leq \Phi(\overline{x}, \overline{y}, \overline{z}). \quad (3.3)$$

Now, we have that

$$\begin{aligned}\Phi(\bar{x}, \bar{y}, \bar{z}) &= u(\bar{x}) + u(\bar{y}) - 2u(\bar{z}) \\ &\quad - \frac{1}{2\alpha}|\bar{x} + \bar{y} - 2\bar{z}|^2 - \frac{c}{2}(|\bar{x} - \bar{z}|^2 + |\bar{y} - \bar{z}|^2) \geq \Phi(\bar{x}, \bar{x}, \bar{x}) = 0;\end{aligned}$$

hence,

$$|\bar{x} + \bar{y} - 2\bar{z}|^2 \leq 8\alpha\|u\|_\infty \quad \text{and} \quad |\bar{x} - \bar{z}|^2 + |\bar{y} - \bar{z}|^2 \leq \frac{8}{c}\|u\|_\infty. \quad (3.4)$$

Moreover, we have

$$\Phi(\bar{x}, \bar{y}, \bar{z}) \geq \Phi\left(\bar{x}, \bar{y}, \frac{\bar{x} + \bar{y}}{2}\right);$$

i.e.,

$$\begin{aligned}u(\bar{x}) + u(\bar{y}) - 2u(\bar{z}) - \frac{1}{2\alpha}|\bar{x} + \bar{y} - 2\bar{z}|^2 - \frac{c}{2}(|\bar{x} - \bar{z}|^2 + |\bar{y} - \bar{z}|^2) \\ \geq u(\bar{x}) + u(\bar{y}) - 2u\left(\frac{\bar{x} + \bar{y}}{2}\right) - \frac{c}{4}|\bar{x} - \bar{y}|^2\end{aligned}$$

and

$$\begin{aligned}2\left[u\left(\frac{\bar{x} + \bar{y}}{2}\right) - u(\bar{z})\right] &\geq 2\left[u\left(\frac{\bar{x} + \bar{y}}{2}\right) - u(\bar{z})\right] \\ &\quad - \frac{c}{2}(|\bar{x} - \bar{z}|^2 + |\bar{y} - \bar{z}|^2) + \frac{c}{4}|\bar{x} - \bar{y}|^2 \geq \frac{1}{2\alpha}|\bar{x} + \bar{y} - 2\bar{z}|^2.\end{aligned}$$

Hence,

$$2\alpha L_u \geq |\bar{x} + \bar{y} - 2\bar{z}|. \quad (3.5)$$

Here L_u is a Lipschitz constant for u . By the compactness of $\bar{\Omega}$, as α tends to 0, we obtain that (possibly passing to subsequences)

$$\bar{z} \rightarrow \hat{x}, \quad \bar{x} - \bar{z} \rightarrow \hat{h} \quad \text{and} \quad \bar{y} - \bar{z} \rightarrow -\hat{h} \quad (3.6)$$

with

$$|\hat{h}| \leq \sqrt{8\|u\|_\infty/c} \quad (3.7)$$

(by the second inequality in (3.4)). Moreover, Equation (3.5) yields that, once more possibly passing to subsequences,

$$\frac{\bar{x} + \bar{y} - 2\bar{z}}{\alpha} \rightarrow \hat{p} \quad (|\hat{p}| \leq L_u). \quad (3.8)$$

There are two cases. Either

Case 1: there is a sequence $\{\alpha_j\}$ converging to 0 such that $\bar{x}, \bar{y}, \bar{z} \in \Omega$

or

Case 2: for any sequence $\{\alpha_j\}$ converging to 0 either \bar{x} or \bar{y} or \bar{z} belongs to $\partial\Omega$ (if this is the case, to make things definite, we may assume that $\bar{x}(c, \alpha_j) \in \partial\Omega$).

In Case 1, using the definition of viscosity solution we obtain

$$u(\bar{x}) + u(\bar{y}) - 2u(\bar{z}) \leq 2H(\bar{z}, (\bar{x} + \bar{y} - 2\bar{z})/\alpha + c(\bar{x} + \bar{y} - 2\bar{z})/2) \\ - H(\bar{x}, (\bar{x} + \bar{y} - 2\bar{z})/\alpha + c(\bar{x} - \bar{z})) - H(\bar{y}, (\bar{x} + \bar{y} - 2\bar{z})/\alpha + c(\bar{y} - \bar{z})).$$

Passing to the limit in (3.3), as $\alpha_j \rightarrow 0$, using the above formula (3.6) and (3.8) we obtain

$$u(x+h) + u(x-h) - 2u(x) - c|h|^2 \\ \leq 2H(\hat{x}, \hat{p}) - H(\hat{x} + \hat{h}, \hat{p} + c\hat{h}) - H(\hat{x} - \hat{h}, \hat{p} - c\hat{h}) - c|\hat{h}|^2$$

and, by (H1), we conclude that

$$u(x+h) + u(x-h) - 2u(x) - c|h|^2 \leq 0.$$

In order to complete the proof it remains to consider Case 2.

Taking the limit, as $\alpha_j \rightarrow 0$, in formula (3.3), we have

$$u(x+h) + u(x-h) - 2u(x) - c|h|^2 \tag{3.9} \\ \leq u(\hat{x} + \hat{h}) + u(\hat{x} - \hat{h}) - 2u(\hat{x}) - c|\hat{h}|^2$$

with $\hat{x} + \hat{h} \in \partial\Omega$ and $\hat{x}, \hat{x} + \hat{h}, \hat{x} - \hat{h} \in \bar{\Omega}_{\rho_0}$, by (3.7). Indeed, by the definition of c ,

$$\sqrt{8\|u\|_\infty/c} < \rho_0/2.$$

Then, using (H3) in (3.9), we find

$$u(x+h) + u(x-h) - 2u(x) - c|h|^2 \leq (c_1 - c)|\hat{h}|^2 < 0.$$

Here, we used that $c > c_1$. This completes our proof in the case of Ω bounded.

(ii) Ω unbounded. In this case, set

$$\Phi(x, y, z; c, \alpha, \varepsilon) := u(x) + u(y) - 2u(z) \\ - \frac{1}{2\alpha}|x + y - 2z|^2 - \frac{c}{2}(|x - z|^2 + |y - z|^2) - \varepsilon\langle y \rangle$$

(as usual $\langle y \rangle = \sqrt{1 + |y|^2}$) and consider its maximum with respect to (x, y, z) over the unbounded set $\bar{\Omega}^3$. Repeating the computations done in the case of Ω bounded (plus some minor technicalities which, for instance, can be treated as in pages 70–71 of [7]) the proof is completed. \square

4. THE EIKONAL EQUATION

In this section we apply our regularity result to the eikonal equation. We consider the problem

$$\begin{cases} \sqrt{\langle A(x)\nabla d(x), \nabla d(x) \rangle} = 1 & x \in \Omega \\ d(x) = 0 & x \in \partial\Omega. \end{cases} \quad (4.1)$$

We assume that there exists a matrix-valued map of class C^2 , $\bar{\Omega} \ni x \mapsto \sigma(x)$, taking values in the $k \times n$ matrices, $k \leq n$, such that

$$A(x) = {}^T\sigma(x)\sigma(x) \quad (4.2)$$

(here ${}^T\sigma$ is the transposed matrix). As usual, we say that the boundary of Ω , $\partial\Omega$, is non-characteristic if

$$\langle A(x)\nu(x), \nu(x) \rangle \neq 0 \quad \forall x \in \partial\Omega$$

where $\nu(x)$ is the outward unit normal to $\partial\Omega$ at x .

Then the following result holds.

Theorem 4.1. *Under assumption (4.2), let Ω be an open bounded convex set with non-characteristic boundary of class C^2 and let d be the Lipschitz continuous viscosity solution of the equation (4.1). Then d is semiconcave on $\bar{\Omega}$.*

We begin with the following elementary result.

Lemma 4.1. *Let d be a Lipschitz continuous nonnegative function on $\bar{\Omega}$ and set $u(x) := 1 - e^{-d(x)}$ ($x \in \bar{\Omega}$). Then*

- (i) *u is Lipschitz continuous on $\bar{\Omega}$;*
- (ii) *if u is semiconcave on $\bar{\Omega}$ then also d is semiconcave on $\bar{\Omega}$.*

The proof of the above result is well known; in spite of this we present it since it is short and makes the exposition self-contained.

Proof. Let $x, y \in \bar{\Omega}$ then

$$|u(x) - u(y)| = |e^{-d(x)} - e^{-d(y)}| \leq L_d|x - y|,$$

and (i) follows.

In order to prove (ii), we observe that

$$d(x) = -\log(1 - u(x))$$

and we consider

$$d(x+h) + d(x-h) - 2d(x)$$

with $x, x+h, x-h \in \overline{\Omega}$ (the only interesting case occurs if the above quantity is positive). Then

$$\begin{aligned} d(x+h) + d(x-h) - 2d(x) &\leq C[(1-u(x))^2 - (1-u(x+h))(1-u(x-h))] \\ &= C[u(x+h) + u(x-h) - 2u(x) + u(x)^2 - (u(x+h)-u(x))(u(x-h)-u(x)) \\ &\quad - u(x+h)u(x) - u(x-h)u(x) + u(x)^2] \\ &\leq C[(1-u(x))(u(x+h) + u(x-h) - 2u(x)) + L_u^2|h|^2] \leq C_1|h|^2 \end{aligned}$$

for suitable positive constants C, C_1 (independent of x and h). This completes our proof. \square

Proof of Theorem 4.1. First, we observe that the regularity of the data and the fact that the boundary $\partial\Omega$ is non-characteristic yield that there exists a positive number ρ_0 such that the viscosity solution of (4.1) is in $C^2(\overline{\Omega}_{\rho_0})$ (e.g. see Theorem 2.1 in [1]). Let us consider the Kruřkov transform of d :

$$u(x) = 1 - e^{-d(x)} \quad (x \in \overline{\Omega}).$$

If d is a viscosity solution of (4.1) then u solves the Dirichlet problem (1.1) with

$$H(x, p) := \sqrt{\langle A(x)p, p \rangle} - 1.$$

Hence, because of Lemma 4.1, the proof reduces to showing that assumptions (H1) and (H3) of Section 3 are fulfilled. The fact that $d \in C^2(\overline{\Omega}_{\rho_0})$ yields that u satisfies (H3) and it remains to show that (H1) holds. We have, for some α with $|\alpha| = 1$,

$$\begin{aligned} &H(x+h, p+ch) + H(x-h, p-ch) - 2H(x, p) \\ &= |\sigma(x+h)(p+ch)| + |\sigma(x-h)(p-ch)| - 2|\sigma(x)p| \\ &\geq \langle \sigma(x+h)(p+ch) + \sigma(x-h)(p-ch) - 2\sigma(x)p, \alpha \rangle \\ &= \langle (\sigma(x+h) + \sigma(x-h) - 2\sigma(x))p, \alpha \rangle \\ &\quad + c\langle (\sigma(x+h) - \sigma(x-h))h, \alpha \rangle \geq -(c_0|p| + cL_\sigma)|h|^2 \end{aligned}$$

where c_0 is a bound for the Lipschitz constant of the first derivatives of σ and L_σ is a Lipschitz constant of σ . Now, if $L_\sigma < 1/2$ then the conclusion follows since, for c large enough,

$$(c_0|p| + 2cL_\sigma) \leq c.$$

Finally, if $L_\sigma \geq 1/2$, in order to conclude, it is enough to consider instead of (4.1) the equivalent problem

$$\begin{cases} \frac{1}{1+2L_\sigma} \sqrt{\langle A(x)\nabla d(x), \nabla d(x) \rangle} = \frac{1}{1+2L_\sigma} & x \in \Omega \\ d(x) = 0 & x \in \partial\Omega. \end{cases}$$

This completes our proof. \square

5. A REMARK ON THE SINGULAR SET

Although this section does not have a central role in this paper we think that Theorem 5.2 below might shed some light on our result.

Theorem 5.2. *Let Ω be an open bounded set and let F_n be a sequence of real-valued continuous functions on $\Omega \times \mathbb{R} \times \mathbb{R}^n$ converging locally uniformly to F . Let $\{u_n\}$ be a sequence of continuous functions such that*

$$F_n(x, u_n, \nabla u_n) = 0 \quad \text{in} \quad \Omega$$

(in the viscosity sense) and assume that the sequence $\{u_n\}$ converges locally uniformly to a semiconcave function u satisfying (still in the viscosity sense)

$$F(x, u, \nabla u) = 0 \quad \text{in} \quad \Omega.$$

If there exists $x_0 \in \Sigma(u)$ such that, for some $p_0 \in D^+u(x_0)$,

$$F(x_0, u(x_0), p_0) < 0, \tag{5.1}$$

then, for n large enough, $\Sigma(u_n) \neq \emptyset$.

Remark 5.1. We observe that, in the case of evolutive Hamilton-Jacobi equations of the form $u_t + H(t, x, u, \nabla u) = 0$ with $p \mapsto H(t, x, u, p)$ strictly convex, condition (5.1) reduces to $(t_0, x_0) \in \Sigma(u)$ (see e.g. [5]).

Proof. We argue by contradiction. Suppose that for some subsequence (still denoted by u_n)

$$\Sigma(u_n) = \emptyset. \tag{5.2}$$

By Remark 2.2, it follows that there exists a function $\varphi \in C^1(\Omega)$ such that $u - \varphi$ attains a local maximum at x_0 and $\nabla\varphi(x_0) = p_0$, with p_0 given by (5.1). Now, Remark 2.2 (ii) yields the existence of a sequence $x_n \in \Omega$ such that

$$\lim_{n \rightarrow \infty} x_n = x_0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \nabla u_n(x_n) = p_0. \tag{5.3}$$

In the last identity we used (5.2). The result follows by contradiction:

$$0 = F_n(x_n, u_n(x_n), \nabla u_n(x_n)) \rightarrow F(x_0, u(x_0), p_0) < 0.$$

Remark 5.2. The above proof is based on a couple of ingredients: the stability property of the viscosity solutions and the upper semicontinuity of the superdifferential. It is easy to see that the same result holds for semiconcave functions with general semiconcavity modulus (see [2] for the definition and for further properties of the singular set).

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