Differential and Integral Equations Volume 18, Number 9 (2005), 1039-1054

INTEGRAL EQUATION METHODS FOR DIV-CURL PROBLEMS FOR PLANAR VECTOR FIELDS IN NONSMOOTH DOMAINS

Dorina Mitrea
Department of Mathematics, University of Missouri-Columbia
Columbia, MO 65211

(Submitted by: J.A. Goldstein)

1. Introduction and statement of main results

A classical problem arising in fluid dynamics and electrodynamics is the determination of a vector field with prescribed divergence and curl. Such a vector field can be considered in the whole plane/space or on domains with boundary when either its tangential component (Dirichlet condition) or its normal component (Neumann condition) is prescribed on the boundary. These problems are usually referred to as div-curl problems and have been studied in the literature both from an analytical point of view (see, e.g., [2], [12], [21]), as well as from a numerical analysis perspective (cf., e.g., [6], [18], [19], [20]), [17]. A nice exposition, which also deals with the case of mixed boundary conditions, can be found in [1].

In the broader context of partial differential equations, the importance of the div-curl system stems from the fact that this is the most basic linear elliptic system and, as such, it has stimulated a substantial body of work, particularly in relation to regularity aspects. Quite recently, some new, deep endpoint regularity results for div-curl problems have been announced by J. Bourgain and H. Brezis in [5].

In this paper, we treat planar div-curl problems, with prescribed Dirichlet or Neumann boundary conditions, on an arbitrary bounded domain with Lipschitz boundary. We are concerned with the regularity of the solution measured on Sobolev scales $H^{s,p}$, globally defined in the domain in question. In this analytic-geometric context, we derive estimates valid for optimal ranges of the exponents involved (p for integrability, and s for smoothness)

Accepted for publication: March 2005.
AMS Subject Classifications: 45E05, 35F15, 35J05, 35C15, 42B20.
Supported in part by a University of Missouri Summer Research Fellowship.

which extend and unify many of the earlier results in the literature. Indeed, variational methods tend to be restricted to the case $p = 2$ (when the spaces in question have a Hilbert structure), whereas methods based on locally flattening the boundary typically require that the boundary is at least $C^{1,1}$ smooth. However, for maximum applicability, it is desirable to allow both nonsmooth boundaries and the widest possible range of indices. Here we develop an approach based on the progress made in [15] where sharp invertibility results for layer potentials on Besov spaces on Lipschitz curves have been obtained. Hence, the theory of singular integral operators of Calderón-Zygmund type plays a prominent role in this paper.

In order to state our main results we shall need some notation and terminology. Recall that a domain $\Omega \subset \mathbb{R}^2$ is called Lipschitz provided its boundary can be described, locally, by means of graphs of Lipschitz functions in suitable systems of coordinates. Given such a domain Ω, we set $\Omega_+ := \Omega$ and $\Omega_- := \mathbb{R}^2 \setminus \overline{\Omega}$. Also, we denote by $\vec{\nu} = \{\nu_1, \nu_2\}$ the outward unit normal to Ω which is defined almost everywhere with respect to the arclength measure $d\sigma$ on $\partial\Omega$. Accordingly, $\vec{t} := \{-\nu_2, \nu_1\}$, the unit tangent vector to $\partial\Omega$ is also well defined almost everywhere on $\partial\Omega$.

Let 'dot' stand for the usual inner product of vectors and set $\nabla := \{\partial_1, \partial_2\}$ and $\nabla^t := \{\partial_2, -\partial_1\}$, acting on scalar-valued functions. If $\vec{u} = \{u_1, u_2\}$ is a vector field with locally integrable components in Ω, set $\operatorname{div} \vec{u} := \partial_1 u_1 + \partial_2 u_2$, $\operatorname{curl} \vec{u} := \partial_1 u_2 - \partial_2 u_1$. Throughout the paper, all derivatives are considered in the sense of distributions.

For $1 < p < \infty$, we denote by L^p the space of p-integrable functions (which will be defined either over Ω or $\partial\Omega$), and by $L^p_1(\partial\Omega)$ the space of functions in $L^p(\partial\Omega)$ whose tangential derivatives are $L^p(\partial\Omega)$. For each $0 < s < 1$, $1 < p, q < \infty$, $B^{p,q}_s(\partial\Omega)$ is the usual scale of Besov spaces over $\partial\Omega$ with smoothness s. By real interpolation $(L^p(\partial\Omega), L^p_1(\partial\Omega))_{\theta,q} = B^{p,q}_\theta(\partial\Omega)$, with $0 < \theta < 1$, $1 < p, q < \infty$. Furthermore, the spaces $B^{p,q}_s(\partial\Omega)$ with $-1 < s < 0$ are defined by duality. Also, $H^{s,p}(\Omega)$, $s \in \mathbb{R}$, $1 < p < \infty$, denotes the usual scale of L^p-based Sobolev (or Bessel potential) spaces on Ω. We will also work with Besov spaces over Ω, in which case, $\big(L^p(\Omega), H^{1,p}(\Omega)\big)_{\theta,q} = B^{p,q}_\theta(\Omega)$. For more details see [3], [4], [22].

Going further, we denote by $H^{s,p}_0(\Omega)$, the space of distributions in $H^{s,p}(\mathbb{R}^2)$ with support in $\bar{\Omega}$. It turns out that for $\frac{1}{p} < s < 1 + \frac{1}{p}$ the space $H^{s,p}_0(\Omega)$ can be identified with the closure of $C^\infty_0(\Omega)$ (the space of all C^∞ functions, compactly supported in Ω), in the norm of $H^{s,p}(\Omega)$. For the same range, we also have $H^{s,p}_0(\Omega) = \{v \in H^{s,p}(\Omega) : \operatorname{Tr} v = 0\}$ (see Proposition 3.3 in [10]), where Tr stands for the trace operator on $\partial\Omega$. Regarding the latter, it is well known that

$$\mathrm{Tr}: H^{s,p}(\Omega) \longrightarrow B^{p,p}_{s-\frac{1}{p}}(\partial\Omega), \quad 1<p<\infty, \;\; \tfrac{1}{p}<s<\tfrac{1}{p}+1, \tag{1.1}$$

is bounded and has a right inverse (i.e., an extension operator)

$$\mathrm{Ext}: B^{p,p}_{s-\frac{1}{p}}(\partial\Omega) \longrightarrow H^{s,p}(\Omega), \quad 1<p<\infty, \;\; \tfrac{1}{p}<s<\tfrac{1}{p}+1. \tag{1.2}$$

We will also need the fact that

$$H^{s,p}_0(\Omega)=H^{s,p}(\Omega) \quad \text{for} \quad 1<p<\infty \quad \text{and} \quad \frac{1}{p}-1<s<\frac{1}{p}. \tag{1.3}$$

See [23] for a proof.

For a generic normed space of distributions in Ω, say $(X(\Omega),\|\cdot\|_{X(\Omega)})$, we define $X(\Omega,\mathbb{R}^2):=\{\vec{u}=(u_1,u_2): u_1,u_2\in X(\Omega)\}$ and equip it with the norm $\|\vec{u}\|_{X(\Omega,\mathbb{R}^2)}:=\|u_1\|_{X(\Omega)}+\|u_2\|_{X(\Omega)}$. Throughout the paper, $\langle\cdot,\cdot\rangle$ will stand for various natural parings between spaces of functions/distributions and their duals.

The spaces of harmonic vector fields with vanishing normal and tangential traces are, respectively,

$$\mathcal{H}^2_{\tan}(\Omega,\mathbb{R}^2):=\Big\{\vec{u}\in L^2(\Omega,\mathbb{R}^2): \operatorname{div}\vec{u}=0,\, \operatorname{curl}\vec{u}=0 \text{ in } \Omega,\, \vec{\nu}\cdot\vec{u}=0\Big\}, \tag{1.4}$$

$$\mathcal{H}^2_{\mathrm{nor}}(\Omega,\mathbb{R}^2):=\Big\{\vec{u}\in L^2(\Omega,\mathbb{R}^2): \operatorname{div}\vec{u}=0,\, \operatorname{curl}\vec{u}=0 \text{ in } \Omega,\, \vec{t}\cdot\vec{u}=0\Big\}. \tag{1.5}$$

The dimension of each of these spaces is equal to $b_1(\Omega)$, the first Betti number of Ω (i.e., the number of cuts necessary to render Ω simply connected). See Chapter IX in [8] for a proof of this fact when Ω is smooth, and [15] when Ω is Lipschitz. We shall also work with

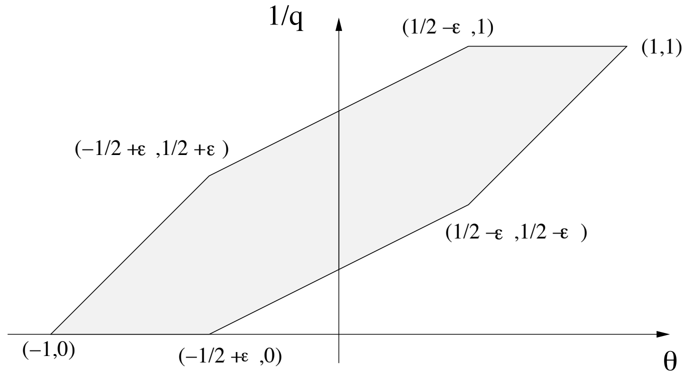
$$\mathbb{R}_{\Omega_\pm}:=\Big\{\sum\lambda_i\,\chi_{\mathcal{O}_i}: \lambda_i\in\mathbb{R},\, \mathcal{O}_i \text{ bounded connected component of } \Omega_\pm\Big\}, \tag{1.6}$$

where $\chi_{\mathcal{O}}$ stands for the characteristic function of the set $\mathcal{O}$ (cf. [14]).

To state the main result of our paper, for each $0\le\varepsilon\le\frac{1}{2}$ consider the following three conditions:

$$\begin{aligned} &(i) && 0<\tfrac{1}{q}\le\tfrac{1}{2}-\varepsilon \;\text{ and }\; \tfrac{1}{q}-1<\theta<\tfrac{2}{q}-\tfrac{1}{2}+\varepsilon;\\ &(ii) && \tfrac{1}{2}-\varepsilon<\tfrac{1}{q}<\tfrac{1}{2}+\varepsilon \;\text{ and }\; \tfrac{1}{q}-1<\theta<\tfrac{1}{q};\\ &(iii) && \tfrac{1}{2}+\varepsilon\le\tfrac{1}{q}<1 \;\text{ and }\; \tfrac{2}{q}-\tfrac{3}{2}-\varepsilon<\theta<\tfrac{1}{q}. \end{aligned} \tag{1.7}$$

The set of points $(\theta, \frac{1}{q})$ verifying (i), (ii), or (iii) above determines a region in the plane which we will denote by $\mathcal{Q}_\varepsilon$.



The region Q_ε

Note that the larger ε the larger $\mathcal{Q}_\varepsilon$ and that when $\varepsilon = \frac{1}{2}$ this region becomes $\{(s, \frac{1}{q}) : 1 < q < \infty, -1 + \frac{1}{q} < s < \frac{1}{q}\}$.

Theorem 1.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$. Then there exists a parameter $\varepsilon = \varepsilon(\partial\Omega) \in (0, \frac{1}{2}]$ such that for each $(\theta, \frac{1}{q}) \in \mathcal{Q}_\varepsilon$ the boundary-value problem*

$$(BVP_1) \qquad \begin{cases} \vec{u} \in H^{\theta,q}(\Omega, \mathbb{R}^2), \\ \mathrm{curl}\, \vec{u} = j \in H^{\theta-1,q}(\Omega), \\ \mathrm{div}\, \vec{u} = k \in H_0^{\theta-1,q}(\Omega), \\ \vec{\nu} \cdot \vec{u} = f \in B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega), \end{cases}$$

has a solution if and only if

$$\langle f, \chi|_{\partial\Omega}\rangle = \langle k, \chi\rangle, \quad \text{for all } \chi \in \mathbb{R}_{\Omega_+}. \tag{1.8}$$

The space of null solutions for (BVP_1) is precisely $\mathcal{H}^2_{\tan}(\Omega, \mathbb{R}^2)$.

Similar results are valid for

$$(BVP_2) \qquad \begin{cases} \vec{u} \in H^{\theta,q}(\Omega, \mathbb{R}^2), \\ \mathrm{curl}\, \vec{u} = j \in H_0^{\theta-1,q}(\Omega), \\ \mathrm{div}\, \vec{u} = k \in H^{\theta-1,q}(\Omega), \\ \vec{t} \cdot \vec{u} = f \in B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega), \end{cases}$$

the compatibility condition this time being

$$\langle f, \chi|_{\partial\Omega}\rangle = -\langle j, \chi\rangle, \quad \textit{for all } \chi \in \mathbb{R}_{\Omega_+}, \tag{1.9}$$

in which case, the space of null solutions is $\mathcal{H}^2_{\rm nor}(\Omega, \mathbb{R}^2)$.

These well-posedness results are sharp in the class of Lipschitz domains. If, however, $\partial\Omega$ *is of class* C^1*, then we may choose* $\varepsilon = \frac{1}{2}$.

A word of caution is in order here. Strictly speaking, in (BVP_1), $\vec{u}$ is a distribution in Ω while k is a given distribution in $\mathbb{R}^2$, so the third condition in (BVP_1) is understood as $\operatorname{div}\vec{u} = k|_\Omega$. A similar interpretation applies to the second condition in (BVP_2).

The special case $\theta = 0$ is worth stating separately. Recall that a boundary problem is called Fredholm solvable if it has a finite-dimensional space of null solutions and if the data must satisfy finitely many linearly independent compatibility conditions (the difference of these two numbers being the index of the problem).

Corollary 1.2. *For each bounded Lipschitz domain* Ω *in* $\mathbb{R}^2$ *there exists a small* $\varepsilon = \varepsilon(\partial\Omega) > 0$ *such that the boundary-value problems*

$$\begin{aligned}&\vec{u} \in L^q(\Omega, \mathbb{R}^2),\ \operatorname{curl}\vec{u} = j \in H^{-1,q}(\Omega),\ \operatorname{div}\vec{u} = k \in H_0^{-1,q}(\Omega),\\ &\vec{\nu}\cdot\vec{u} = f \in B^{q,q}_{-\frac{1}{q}}(\partial\Omega),\end{aligned}$$

and

$$\begin{aligned}&\vec{u} \in L^q(\Omega, \mathbb{R}^2),\ \operatorname{curl}\vec{u} = j \in H_0^{-1,q}(\Omega),\ \operatorname{div}\vec{u} = k \in H^{-1,q}(\Omega),\\ &\vec{t}\cdot\vec{u} = f \in B^{q,q}_{-\frac{1}{q}}(\partial\Omega),\end{aligned}$$

are each Fredholm solvable, of index zero, if $\frac{4}{3} - \varepsilon < q < 4 + \varepsilon$.

A number of important estimates for the $\operatorname{div} - \operatorname{curl}$ system are implicit in the fact that (BVP_1) and (BVP_2) are well posed and a few are singled out below, in a series of corollaries.

Corollary 1.3. *For each* $\Omega \subset \mathbb{R}^2$*, bounded Lipschitz domain, there exists* $\varepsilon = \varepsilon(\partial\Omega) \in (0, \frac{1}{2}]$ *with the following significance. Let* $\vec{\omega}_1, \cdots, \vec{\omega}_{b_1(\Omega)}$ *and* $\vec{\psi}_1, \cdots, \vec{\psi}_{b_1(\Omega)}$ *be two arbitrary, fixed bases for* $\mathcal{H}^2_{\rm tan}(\Omega, \mathbb{R}^2)$ *and* $\mathcal{H}^2_{\rm nor}(\Omega, \mathbb{R}^2)$*, respectively. Then, for each* $(\theta, \frac{1}{q}) \in \mathcal{Q}_\varepsilon$,

$$\|\vec{u}\|_{H^{\theta,q}(\Omega,\mathbb{R}^2)} \le C\|\operatorname{div}\vec{u}\|_{H_0^{\theta-1,q}(\Omega)} + C\|\operatorname{curl}\vec{u}\|_{H^{\theta-1,q}(\Omega)} \tag{1.10}$$

$$+ C\|\vec{\nu}\cdot\vec{u}\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)} + C\sum_{l=1}^{b_1(\Omega)} |\langle \vec{u},\vec{\omega}_l\rangle|,$$

holds for vector fields $\vec{u}\in H^{\theta,q}(\Omega,\mathbb{R}^2)$ *such that* $\operatorname{div}\vec{u}\in H_0^{\theta-1,q}(\Omega)$. *Furthermore,*

$$\|\vec{u}\|_{H^{\theta,q}(\Omega,\mathbb{R}^2)} \le C\|\operatorname{div}\vec{u}\|_{H^{\theta-1,q}(\Omega)} + C\|\operatorname{curl}\vec{u}\|_{H_0^{\theta-1,q}(\Omega)} \tag{1.11}$$
$$+ C\|\vec{t}\cdot\vec{u}\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)} + C\sum_{l=1}^{b_1(\Omega)} |\langle \vec{u},\vec{\psi}_l\rangle|,$$

holds for vector fields $\vec{u}\in H^{\theta,q}(\Omega,\mathbb{R}^2)$ *such that* $\operatorname{curl}\vec{u}\in H_0^{\theta-1,q}(\Omega)$.

Corollary 1.4. *Retain the hypotheses of Corollary* 1.3 *and, in addition, assume that* Ω *is simply connected. Then there exists* $C>0$ *such that*

$$\|\vec{u}\|_{H^{\theta,q}(\Omega,\mathbb{R}^2)} \le C\|\operatorname{div}\vec{u}\|_{H_0^{\theta-1,q}(\Omega)} + C\|\operatorname{curl}\vec{u}\|_{H^{\theta-1,q}(\Omega)} + C\|\vec{\nu}\cdot\vec{u}\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)}, \tag{1.12}$$

$$\|\vec{u}\|_{H^{\theta,q}(\Omega,\mathbb{R}^2)} \le C\|\operatorname{div}\vec{u}\|_{H^{\theta-1,q}(\Omega)} + C\|\operatorname{curl}\vec{u}\|_{H_0^{\theta-1,q}(\Omega)} + C\|\vec{t}\cdot\vec{u}\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)}, \tag{1.13}$$

uniformly for $\vec{u}\in H^{\theta,q}(\Omega,\mathbb{R}^2)$ *such that* $\operatorname{div}\vec{u}\in H_0^{\theta-1,q}(\Omega)$ *and* $\operatorname{curl}\vec{u}\in H_0^{\theta-1,q}(\Omega)$, *respectively.*

Corollary 1.5. *Let* $\Omega\subset\mathbb{R}^2$ *be a bounded Lipschitz domain and let* $\varepsilon=\varepsilon(\partial\Omega)\in(0,\frac{1}{2}]$ *be as in Theorem 1.1. Finally, assume that* $(\theta,\frac{1}{q}),(s,\frac{1}{p})\in\mathcal{Q}_\varepsilon$ *are such that*

$$\theta > s \quad \text{and} \quad \tfrac{1}{p}-\tfrac{s}{2}\ge\tfrac{1}{q}-\tfrac{\theta}{2}. \tag{1.14}$$

Then there exists $C>0$ *such that*

$$\|\vec{u}\|_{H^{\theta,q}(\Omega,\mathbb{R}^2)} \le C\Big(\|\vec{u}\|_{H^{s,p}(\Omega,\mathbb{R}^2)} + \|\operatorname{div}\vec{u}\|_{H_0^{\theta-1,q}(\Omega)}$$
$$+ \|\operatorname{curl}\vec{u}\|_{H^{\theta-1,q}(\Omega)} + \|\vec{\nu}\cdot\vec{u}\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)}\Big),$$
$$\|\vec{u}\|_{H^{\theta,q}(\Omega,\mathbb{R}^2)} \le C\Big(\|\vec{u}\|_{H^{s,p}(\Omega,\mathbb{R}^2)} + \|\operatorname{div}\vec{u}\|_{H^{\theta-1,q}(\Omega)}$$
$$+ \|\operatorname{curl}\vec{u}\|_{H_0^{\theta-1,q}(\Omega)} + \|\vec{t}\cdot\vec{u}\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)}\Big),$$

uniformly for $\vec{u}\in H^{\theta,q}(\Omega,\mathbb{R}^2)$ *such that* $\operatorname{div}\vec{u}\in H_0^{\theta-1,q}(\Omega)$ *and* $\operatorname{curl}\vec{u}\in H_0^{\theta-1,q}(\Omega)$, *respectively.*

Estimates of this type have a rich history, with many predecessors. For the purpose of this introduction we recall that the Friedrichs type estimates

$$\|\vec{u}\|_{H^{1,2}(\Omega,\mathbb{R}^2)} \leq C\Big(\|\vec{u}\|_{L^2(\Omega,\mathbb{R}^2)} + \|\mathrm{div}\,\vec{u}\|_{L^2(\Omega)} + \|\mathrm{curl}\,\vec{u}\|_{L^2(\Omega)} + \|\vec{\nu}\cdot\vec{u}\|_{H^{\frac{1}{2},2}(\partial\Omega)}\Big), \tag{1.15}$$

$$\|\vec{u}\|_{H^{1,2}(\Omega,\mathbb{R}^2)} \leq C\Big(\|\vec{u}\|_{L^2(\Omega,\mathbb{R}^2)} + \|\mathrm{div}\,\vec{u}\|_{L^2(\Omega)} + \|\mathrm{curl}\,\vec{u}\|_{L^2(\Omega)} + \|\vec{t}\cdot\vec{u}\|_{H^{\frac{1}{2},2}(\partial\Omega)}\Big), \tag{1.16}$$

uniformly for $\vec{u} \in H^{1,2}(\Omega,\mathbb{R}^2)$, play a fundamental role in electromagnetism. For a discussion regarding the validity of (1.15)-(1.16) in the case when Ω is either a convex polygon or has a boundary of class $C^{1,1}$, see pages 212-213 in Chapter IX of [8]. Cf. also [13] for a discussion of the relationship between (1.15)-(1.16) and the $H^{2,2}$-regularity of the solutions for the Poisson problem for the Laplacian with Dirichlet and Neumann boundary conditions. Let us also point out that a related version of (1.15)-(1.16), i.e.,

$$\begin{aligned}\|\vec{u}\|_{H^{1/2,2}(\Omega,\mathbb{R}^2)} \quad \leq \quad & C\|\vec{u}\|_{L^2(\Omega,\mathbb{R}^2)} + C\|\mathrm{div}\,\vec{u}\|_{L^2(\Omega)} + C\|\mathrm{curl}\,\vec{u}\|_{L^2(\Omega)} \\ & + C\,\min\left\{\|\vec{\nu}\cdot\vec{u}\|_{L^2(\partial\Omega)}\,,\,\|\vec{t}\cdot\vec{u}\|_{L^2(\partial\Omega)}\right\},\end{aligned} \tag{1.17}$$

has been established in [16], for arbitrary Lipschitz domains.

Note that (1.15)-(1.16) formally correspond to taking $\theta = 1$, $s = 0$ and $p = q = 2$ in the estimates in Corollary 1.5 though, of course, the condition $(1, \frac{1}{2}) \in \mathcal{Q}_\varepsilon$ is not expected to hold for an arbitrary Lipschitz domain Ω. Thus, the estimates in Corollary 1.5 can be thought of as the appropriate versions of Friedrichs inequalities (1.15)-(1.16) for optimal ranges of indices (smoothness and integrability) and for domains whose boundaries are only Lipschitz (thus, less regular than those typically considered in the literature).

We close this section by explaining how the (boundary) distributions $\vec{\nu}\cdot\vec{u}$ and $\vec{t}\cdot\vec{u}$ in (BVP_1) and (BVP_2) are defined. To see this, assume that $\Omega \subset \mathbb{R}^2$ is a bounded Lipschitz domain and fix $1 < q < \infty$ and $-1 + \frac{1}{q} < \theta < \frac{1}{q}$ (note that if $(\theta, 1/q) \in \mathcal{Q}_\varepsilon$ then these conditions are automatically satisfied). Also, assume that $\vec{u} \in H^{\theta,q}(\Omega,\mathbb{R}^2)$ has $\mathrm{div}\,\vec{u} \in H_0^{\theta-1,q}(\Omega)$ (i.e., $\mathrm{div}\,\vec{u}$, originally a distribution in Ω, is extended to a distribution in $H_0^{\theta-1,q}(\Omega)$). Recalling the extension operator (1.2), we define $\vec{\nu}\cdot\vec{u}$ as the functional

$$\langle\vec{\nu}\cdot\vec{u},\varphi\rangle := \int_\Omega (\mathrm{Ext}\,\varphi)\,\mathrm{div}\,\vec{u} + \vec{u}\cdot\nabla\,(\mathrm{Ext}\,\varphi), \quad \text{for all } \varphi \in B^{p,p}_{1-\frac{1}{p}-\theta}(\partial\Omega), \quad \tfrac{1}{p} + \tfrac{1}{q} = 1. \tag{1.18}$$

This definition is meaningful since the pairings in the right-hand side of (1.18) are well defined. Indeed, $\mathrm{Ext}\,\varphi \in H^{1-\theta,p}(\Omega)$ and $H_0^{\theta-1,q}(\Omega) = (H^{1-\theta,p}(\Omega))^*$, while based on (1.3) we have

$$H^{-\theta,p}(\Omega) = (H_0^{\theta,q}(\Omega))^* = (H^{\theta,q}(\Omega))^* \quad \text{for} \quad \tfrac{1}{q} - 1 < \theta < \tfrac{1}{q}. \tag{1.19}$$

Thus, granted the current assumptions on $\vec{u}$, we get $\vec{\nu} \cdot \vec{u} \in (B^{p,p}_{\frac{1}{q}-\theta}(\partial\Omega))^* = B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)$, as desired (observe that $-1 < \theta - \frac{1}{q} < 0$). These arguments also imply

$$\|\vec{\nu} \cdot \vec{u}\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)} \le C\Big(\|\vec{u}\|_{H^{\theta,q}(\Omega,\mathbb{R}^2)} + \|\operatorname{div} \vec{u}\|_{H_0^{\theta-1,q}(\Omega)}\Big). \tag{1.20}$$

Next, for a vector field $\vec{u} \in H^{\theta,q}(\Omega,\mathbb{R}^2)$ with the property that $\operatorname{curl} \vec{u} \in H_0^{\theta-1,q}(\Omega)$ we define $\vec{t} \cdot \vec{u} \in B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega) = (B^{p,p}_{1-\frac{1}{p}-\theta}(\partial\Omega))^*$, $1/p + 1/q = 1$, by

$$\langle \vec{t} \cdot \vec{u}, \psi\rangle := \int_\Omega (\operatorname{Ext} \psi) \operatorname{curl} \vec{u} - \vec{u} \cdot \nabla^t (\operatorname{Ext} \psi), \quad \text{for all} \quad \psi \in B^{p,p}_{1-\frac{1}{p}-\theta}(\partial\Omega), \quad \tfrac{1}{p} + \tfrac{1}{q} = 1. \tag{1.21}$$

Moreover,

$$\|\vec{t} \cdot \vec{u}\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)} \le C\Big(\|\vec{u}\|_{H^{\theta,q}(\Omega,\mathbb{R}^2)} + \|\operatorname{curl} \vec{u}\|_{H_0^{\theta-1,q}(\Omega)}\Big). \tag{1.22}$$

Remark I. Similar considerations apply to the case of Ω_-.

Remark II. Our definitions (1.18), (1.21) with $\theta = 0$ and $q = 2$ should be compared with the trace theorems for vector fields from Theorems 1-2 on page 204 of [8]. In these theorems, in order to define $\vec{\nu} \cdot \vec{u} \in H^{-1/2,2}(\partial\Omega)$ and $\vec{t} \cdot \vec{u} \in H^{-1/2,2}(\partial\Omega)$, the authors assume that $\vec{u} \in L^2(\Omega,\mathbb{R}^2)$ satisfies $\operatorname{div} \vec{u} \in L^2(\Omega)$ and $\operatorname{curl} \vec{u} \in L^2(\Omega)$, respectively. In our case, we hypothesize that $\operatorname{div} \vec{u}$ and $\operatorname{curl} \vec{u}$, respectively, belong to $H_0^{-1,2}(\Omega)$ which is a less stringent demand on the smoothness of the vector field $\vec{u} \in L^2(\Omega,\mathbb{R}^2)$ though, at the same time, it is a significant constraint on the supports of these distributions.

2. PRELIMINARIES

We begin by discussing some integration by parts formulas of independent interest.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and assume that $1 < q < \infty$, $-1 + \frac{1}{q} < \theta < \frac{1}{q}$. Then the integration by parts formula*

$$\int_\Omega \Phi \operatorname{div} \vec{u} = \langle \vec{\nu} \cdot \vec{u}\,,\, \operatorname{Tr} \Phi\rangle - \int_\Omega \vec{u} \cdot \nabla \Phi \tag{2.1}$$

holds for any $\vec{u} \in H^{\theta,q}(\Omega,\mathbb{R}^2)$ with $\operatorname{div} \vec{u} \in H_0^{\theta-1,q}(\Omega)$ and any $\Phi \in H^{1-\theta,p}(\Omega)$, $\frac{1}{p} + \frac{1}{q} = 1$. Furthermore,

$$\int_\Omega \Psi \operatorname{curl} \vec{u} = \langle \vec{t} \cdot \vec{u}\,,\, \operatorname{Tr} \Psi\rangle + \int_\Omega \vec{u} \cdot \nabla^t \Psi, \tag{2.2}$$

for any $\vec{u} \in H^{\theta,q}(\Omega,\mathbb{R}^2)$ *with* $\operatorname{curl}\vec{u} \in H_0^{\theta-1,q}(\Omega)$ *and any* $\Psi \in H^{1-\theta,p}(\Omega)$, $\frac{1}{p}+\frac{1}{q}=1$.

Proof. Consider (2.1). Given $\vec{u}$, Φ, as in the statement above, set

$$\varphi := \operatorname{Tr}\Phi \in B^{p,p}_{1-\frac{1}{p}-\theta}(\partial\Omega), \quad w := \Phi - \operatorname{Ext}\varphi \in H_0^{1-\theta,p}(\Omega). \tag{2.3}$$

By (1.18) and linearity, proving (2.1) comes down to showing that $\int_\Omega w \operatorname{div}\vec{u} = -\int_\Omega \vec{u}\cdot\nabla w$. To justify this, recall that it is possible to choose a sequence of functions $w_j \in C_0^\infty(\Omega)$ such that $w_j \to w$ in $H^{1-\theta,p}(\Omega)$. In particular, $\nabla w_j \to \nabla w$ in $H^{-\theta,p}(\Omega,\mathbb{R}^2)$. Consequently,

$$\int_\Omega w \operatorname{div}\vec{u} = \lim_{j\to\infty}\int_\Omega w_j \operatorname{div}\vec{u} = -\lim_{j\to\infty}\int_\Omega \nabla w_j\cdot\vec{u} = -\int_\Omega \vec{u}\cdot\nabla w. \tag{2.4}$$

Above, the first equality utilizes $\operatorname{div}\vec{u} \in H_0^{\theta-1,q}(\Omega) = (H^{1-\theta,p}(\Omega))^*$, the second equality is just the distributional definition of the divergence operator, and the third is based on the fact that $\nabla w_j \to \nabla w$ in $H^{-\theta,p}(\Omega,\mathbb{R}^2) = (H_0^{\theta,q}(\Omega,\mathbb{R}^2))^* = (H^{\theta,q}(\Omega,\mathbb{R}^2))^*$.

The proof of (2.2) is similar and we omit the details. □

Next, we summarize the properties of the layer potential operators which are going to play an important role in the proof of Theorem 1.1. Recall that a fundamental solution for the Laplacian $\Delta := \partial_1^2 + \partial_2^2$ is

$$\Gamma(x) := \frac{1}{2\pi}\ln|x|, \quad x \in \mathbb{R}^2\setminus\{0\}, \tag{2.5}$$

and set $\Gamma(x,y) := \Gamma(x-y)$, for $x,y\in\mathbb{R}^2$, $x\neq y$. Then the single layer (harmonic) potential operator associated with Ω is

$$\mathcal{S}f(x) := \int_{\partial\Omega}\Gamma(x,y)f(y)\,d\sigma(y), \qquad x\in\mathbb{R}^2\setminus\partial\Omega, \tag{2.6}$$

for $f:\partial\Omega\to\mathbb{R}$. We also set

$$Sf := \mathcal{S}f\Big|_{\partial\Omega}. \tag{2.7}$$

If $g:\partial\Omega\to\mathbb{R}$, the principal-value operator

$$K^*g(x) := p.v.\int_{\partial\Omega}\frac{\partial\Gamma(x,y)}{\partial\nu(x)}g(y)\,d\sigma(y), \quad \text{for a.e. } x\in\partial\Omega, \tag{2.8}$$

appears naturally when considering the nontangential traces

$$\vec{\nu}\cdot(\nabla\mathcal{S}f)\Big|_{\partial\Omega_\pm} = \left(\mp\tfrac{1}{2}I + K^*\right)f, \tag{2.9}$$

where "I" is the identity operator. For $k>0$, K_k^* is defined as in (2.8), with $\Gamma(x,y)$ replaced by $-\frac{i}{4}H_0^{(1)}(k|x-y|)$, the fundamental solution corresponding to the Helmholtz operator $\Delta - k^2$. Here, $H_0^{(1)}$ is the first Hankel function which has a singularity of type $\ln|x|$ at $x=0$.

For later use we also note that

$$\vec{t}\cdot(\nabla\mathcal{S}f)\Big|_{\partial\Omega_+}=\vec{t}\cdot(\nabla\mathcal{S}f)\Big|_{\partial\Omega_-}. \tag{2.10}$$

Finally, the Newtonian potential associated with Ω is defined by

$$\Pi u(x):=\int_\Omega \Gamma(x,y)u(y)\,dy, \qquad x\in\Omega, \tag{2.11}$$

for $u:\Omega\to\mathbb{R}$. We set (cf. [14])

$$\mathbb{R}_{\partial\Omega_\pm}:=\Big\{\sum\lambda_i\,\chi_{\partial\mathcal{O}_i}:\lambda_i\in\mathbb{R},\,\mathcal{O}_i \text{ bounded connected component of } \Omega_\pm\Big\}. \tag{2.12}$$

The main properties of the operators listed above which are relevant for us here are collected in the following theorem.

Theorem 2.2. *For each bounded Lipschitz domain Ω in $\mathbb{R}^2$, the following hold.*

(1) *The Newtonian potential*

$$\Pi: H_0^{-s,q}(\Omega)\to H^{2-s,q}(\Omega) \quad \textit{is bounded for each } 1<q<\infty,\ 0\le s\le 2. \tag{2.13}$$

(2) *For each $1<q<\infty$ and $0<s<1$,*

$$\mathcal{S}:B^{q,q}_{-s}(\partial\Omega)\to H^{1-s+\frac{1}{q},q}(\Omega), \tag{2.14}$$

$$S:B^{q,q}_{-s}(\partial\Omega)\to B^{q,q}_{1-s}(\partial\Omega), \tag{2.15}$$

$$K^*:B^{q,q}_{s-1}(\partial\Omega)\to B^{q,q}_{s-1}(\partial\Omega), \tag{2.16}$$

are bounded operators.

(3) *There exists $\varepsilon=\varepsilon(\partial\Omega)\in(0,\frac12]$ such that the operators*

$$\pm\tfrac12 I+K^* \quad : \quad \Big\{f\in B^{q,q}_{\theta-\frac1q}(\partial\Omega):\langle f,\chi\rangle=0,\ \textit{for all } \chi\in\mathbb{R}_{\partial\Omega_\mp}\Big\}\longrightarrow \tag{2.17}$$
$$\Big\{f\in B^{q,q}_{\theta-\frac1q}(\partial\Omega):\langle f,\chi\rangle=0,\ \textit{for all } \chi\in\mathbb{R}_{\partial\Omega_\mp}\Big\}$$

are isomorphisms whenever $(\theta,\frac1q)\in\mathcal{Q}_\varepsilon$.

Moreover, for each $k>0$, the operators

$$\pm\tfrac12 I+K^*_k:B^{q,q}_{\theta-\frac1q}(\partial\Omega)\longrightarrow B^{q,q}_{\theta-\frac1q}(\partial\Omega) \tag{2.18}$$

are also invertible granted that $(\theta,\frac1q)\in\mathcal{Q}_\varepsilon$. The invertibility region $\mathcal{Q}_\varepsilon$ for the operators (2.18) *is sharp in the class of Lipschitz domains. If, however, $\partial\Omega\in C^1$, then we may take $\varepsilon=\frac12$.*

(4) *Assume that $0<\theta<1$, $1<q<\infty$, $f\in B^{q,q}_{\theta-\frac1q}(\partial\Omega)$ and set $\vec{u}_\pm:=\nabla\mathcal{S}f$ in $\Omega_\pm$. Then*

$$\vec{\nu}\cdot\vec{u}_\pm=\big(\mp\tfrac12 I+K^*\big)f \quad \textit{in} \ \ B^{q,q}_{\theta-\frac1q}(\partial\Omega), \tag{2.19}$$

$$\vec{t}\cdot\vec{u}_+ = \vec{t}\cdot\vec{u}_- \quad in \;\; B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega). \tag{2.20}$$

Proof. The fact that the Newtonian potential is a bounded operator in the context of (1) follows from classical Calderón-Zygmund theory, duality and interpolation. The operators (2.14)-(2.16) have been discussed in [9], whereas the invertibility of the operators (2.17)-(2.18) has been proved in [15]. Finally, the jump-relations (2.19)-(2.20) can be proved based on (2.9)-(2.10) and a density argument, given that all the quantities involved depend continuously on f. □

Remark. Strictly speaking, the conclusions in (3) of Theorem 2.2 may require that some attention is paid to the logarithmic capacity of Ω. However, since in all applications of this result the underlying domain may be suitably dilated in order to avoid potential complications caused by the logarithmic capacity, we shall safely ignore this aspect in the future.

3. The proof of the main results and corollaries

We start with the

Proof of Theorem 1.1. Let $\varepsilon > 0$ be as in Theorem 2.2 and fix $(\theta, \frac{1}{q}) \in \mathcal{Q}_\varepsilon$. The fact that (1.8) is a necessary condition follows from integrations by parts. Next, suppose that (1.8) holds, and we will prove that (BVP_1) has a solution. As a preliminary step, pick some $g \in H^{\theta+1,q}(\Omega)$ such that $\Delta g = j$ in Ω. Such a function can be constructed by extending the given j to $\mathbb{R}^2$ with compact support and preservation of smoothness, then applying a suitable Newtonian potential.

Next, for some $h \in B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)$ (to be specified later), define

$$\vec{u} := -\nabla^t g + \nabla\Pi k + \nabla\mathcal{S}h \quad \text{in} \;\; \Omega. \tag{3.1}$$

Thanks to Theorem 2.2, $\vec{u} \in H^{\theta,q}(\Omega, \mathbb{R}^2)$. Also, by a direct calculation, $\vec{u}$ verifies the interior conditions in (BVP_1).

There remains checking the boundary condition which, given (3.1), reduces to ensuring that

$$(-\tfrac{1}{2}I + K^*)\, h = f + \vec{t}\cdot\nabla g - \vec{\nu}\cdot\nabla\Pi k. \tag{3.2}$$

Integrations by parts based on Proposition 2.1 and the compatibility condition (1.8) imply that $f + \vec{t}\cdot\nabla g - \vec{\nu}\cdot\nabla\Pi k$ belongs to the space

$$\{\xi \in B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega) : \langle \xi, \chi|_{\partial\Omega}\rangle = 0, \text{ for all } \chi \in \mathbb{R}_{\Omega_+}\}.$$

Using the invertibility properties of the operator $-\frac{1}{2}I + K^*$ (cf. (3) in Theorem 2.2), we may then take

$$\begin{aligned} h &:= \big(-\tfrac{1}{2}I + K^*\big)^{-1}(f + \vec{t}\cdot\nabla g - \vec{\nu}\cdot\nabla\Pi k) \\ &\in \Big\{\xi \in B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega) : \; \langle \xi, \chi|_{\partial\Omega}\rangle = 0, \text{ for all } \chi \in \mathbb{R}_{\Omega_+}\Big\}, \end{aligned}$$

so that (3.2) holds. For this choice of h, the vector field $\vec{u}$ given in (3.1) is a solution of (BVP_1). This completes the proof of the existence part for (BVP_1).

Turning our attention to uniqueness, we recall that it was proved in [15] that for $(\theta, \frac{1}{q}) \in \mathcal{Q}_\varepsilon$ we have

$$\Big\{\vec{u} \in H^{\theta,q}(\Omega,\mathbb{R}^2) : \operatorname{div}\vec{u} = 0,\ \operatorname{curl}\vec{u} = 0 \text{ in } \Omega,\ \vec{\nu}\cdot\vec{u} = 0\Big\} = \mathcal{H}^2_{\tan}(\Omega,\mathbb{R}^2). \tag{3.3}$$

Consequently, the space of null solutions for (BVP_1) is $\mathcal{H}^2_{\tan}(\Omega,\mathbb{R}^2)$.

The results corresponding to (BVP_2) follow from what we have proved so far. Specifically, direct computations imply that $\vec{u}$ is a solution of (BVP_1) if and only if $*\vec{u} := \{-u_2, u_1\}$ is a solution of (BVP_2) with $\operatorname{curl}(*\vec{u}) = \operatorname{div}\vec{u}$, $\operatorname{div}(*\vec{u}) = -\operatorname{curl}\vec{u}$, $\vec{t}\cdot(*\vec{u}) = \vec{\nu}\cdot\vec{u}$.

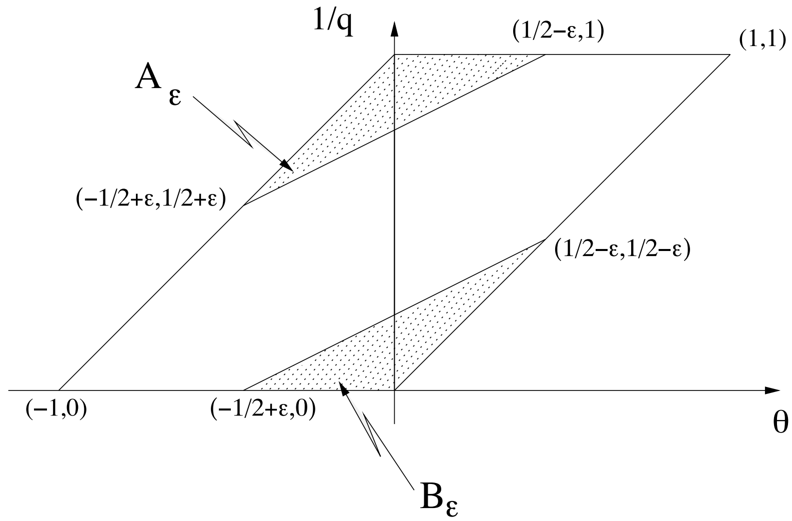
Finally, we are left with showing that the range of solvability $\mathcal{Q}_\varepsilon$ for (BVP_1) (and (BVP_2)) is sharp. The restriction $\frac{1}{q} - 1 < \theta < \frac{1}{q}$ is required in order to define $\vec{\nu}\cdot\vec{u}$. The only regions contained in $\frac{1}{q} - 1 < \theta < \frac{1}{q}$, $0 < \frac{1}{q} < 1$ and not contained in $\mathcal{Q}_\varepsilon$ are

$$\mathcal{A}_\varepsilon := \Big\{(\theta, \tfrac{1}{q}) : \tfrac{1}{2} + \varepsilon \le \tfrac{1}{q} < 1,\ \tfrac{1}{q} - 1 < \theta \le \tfrac{2}{q} - \tfrac{3}{2} - \varepsilon\Big\} \tag{3.4}$$

and

$$\mathcal{B}_\varepsilon := \Big\{(\theta, \tfrac{1}{q}) : 0 < \tfrac{1}{q} \le \tfrac{1}{2} - \varepsilon,\ \tfrac{2}{q} - \tfrac{1}{2} + \varepsilon \le \theta < \tfrac{1}{q}\Big\} \tag{3.5}$$

as seen below.



The regions A_ε and B_ε

To complete the proof of Theorem 1.1, it suffices to show that for any $(\theta, \frac{1}{q}) \in \mathcal{A}_\varepsilon \cup \mathcal{B}_\varepsilon$ the problem (BVP_1) is not well posed. We will prove this by reasoning by contradiction. To this end, fix $(\theta, \frac{1}{q}) \in \mathcal{A}_\varepsilon \cup \mathcal{B}_\varepsilon$ such that (BVP_1) is well posed. Under these assumptions, our goal is to show that the operator $-\frac{1}{2}I + K_k^*$, $k > 0$, is invertible on $B^{q,q}_{\theta-\frac{1}{q}}(\Omega)$. This, in turn, will contradict the optimality of the region $\mathcal{Q}_\varepsilon$ in (3) of Theorem 2.2.

To proceed, let $g \in B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)$ and set $\vec{u}_+ := \nabla\mathcal{S}g$ in Ω and $\vec{u}_- := \nabla\mathcal{S}g$ in $\Omega_R := B_R \setminus \bar{\Omega}$, where B_R is a ball of a large, fixed radius R, and containing $\bar{\Omega}$.

Then $\vec{u}_+$ solves (BVP_1) for $j = k = 0$ and boundary datum $(-\frac{1}{2}I + K^*)g$. Moreover, $\vec{u}_-$ solves (BVP_2) in the domain $\Omega_R := B_R \setminus \bar{\Omega}$ for $j = k = 0$ and for the boundary datum $\vec{t} \cdot (\nabla\mathcal{S}g)|_{\Omega_R}$ (defined in the sense of (1.18)). We may then write

$$\begin{aligned}
\|g\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)} &\leq \|\vec{\nu}\cdot\vec{u}_+\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)} + \|\vec{\nu}\cdot\vec{u}_-\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)} \qquad (3.6)\\
&\leq \|(-\tfrac{1}{2}I + K^*)g\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)} + C\|\vec{u}_-\|_{H^{\theta,q}(\Omega_R,\mathbb{R}^2)} + \|\mathrm{Comp}(\vec{u})\|\\
&\leq \|(-\tfrac{1}{2}I + K^*)g\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)} + C\|\vec{t}\cdot\vec{u}_-\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)} + \|\mathrm{Comp}(\vec{u})\|\\
&\leq \|(-\tfrac{1}{2}I + K^*)g\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)} + C\|\vec{t}\cdot\vec{u}_+\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)} + \|\mathrm{Comp}(\vec{u})\|\\
&\leq \|(-\tfrac{1}{2}I + K^*)g\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)} + C\|\vec{u}_+\|_{H^{\theta,q}(\Omega,\mathbb{R}^2)} + \|\mathrm{Comp}(\vec{u})\|\\
&\leq \|(-\tfrac{1}{2}I + K^*)g\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)} + C\|\vec{\nu}\cdot\vec{u}_+\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)} + \|\mathrm{Comp}(\vec{u})\|\\
&= C\|(-\tfrac{1}{2}I + K^*)g\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)} + \|\mathrm{Comp}(\vec{u})\|.
\end{aligned}$$

The first inequality relies on (2.19) and the triangle inequality. In the second inequality we have used the version of (1.20) corresponding to Ω_R, and the fact that the mapping $g \mapsto (\vec{\nu}\cdot\nabla\mathcal{S}g)\Big|_{\partial B_R}$ is a compact operator (its kernel has no singularity). For the third inequality we used the second estimate in Corollary 1.5 (whose proof is independent of the current considerations). Here $\|\vec{u}\|_{H^{s,p}(\Omega,\mathbb{R}^2)} = \|\mathrm{Comp}(\vec{u})\|$ since the embedding $H^{\theta,q}(\Omega,\mathbb{R}^2) \hookrightarrow H^{s,p}(\Omega,\mathbb{R}^2)$ is compact. The fourth inequality is a consequence of (2.20), while in the fifth we have used (1.22). In the sixth inequality we have used the first estimate in Corollary 1.5 plus similar considerations as before. Finally, the last inequality uses the jump-formula (2.19) once again.

As a consequence, the estimate (3.6) implies that the operator $-\frac{1}{2}I + K^*$ has a finite-dimensional kernel and a closed range when acting on $B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)$. Since for each $k > 0$ the operator $K^* - K_k^*$ is compact on $B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)$, we conclude that $-\frac{1}{2}I + K_k^*$ has a finite-dimensional kernel and a closed range when acting on $B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)$.

To proceed from here we find it useful to analyze the cases when $(\theta, \frac{1}{q})$ belongs to the regions $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon$ separately.

First, if $(\theta, \frac{1}{q}) \in \mathcal{A}_\varepsilon$, from classical embedding theorems we have $L^2(\partial\Omega) \hookrightarrow B^{q',q'}_{\theta-\frac{1}{q'}}(\partial\Omega)$, for all $2 \geq q' \geq q$ (boundedness follows from [7] and invertibility from the techniques in [24]). Since $-\frac{1}{2}I + K_k^*$ is an isomorphism on $L^2(\partial\Omega)$, we conclude that $-\frac{1}{2}I+K_k^*$ has a dense range on $B^{q',q'}_{\theta-\frac{1}{q'}}(\partial\Omega)$, so it is onto. At this point we make use of Theorem 2.10 in [11] and (3) in Theorem 2.2 to conclude that $-\frac{1}{2}I + K_k^*$ is an isomorphism on $B^{q',q'}_{\theta-\frac{1}{q'}}(\partial\Omega)$ for all $2 \geq q' \geq q$.

Second, if $(\theta, \frac{1}{q}) \in \mathcal{B}_\varepsilon$, from embedding theorems we see that $B^{q'',q''}_{\theta-\frac{1}{q''}}(\partial\Omega) \hookrightarrow B^{2,2}_{\theta-\frac{1}{2}}(\partial\Omega)$, for any $q \geq q'' \geq 2$. Hence, using (3) in Theorem 2.2 it follows that $-\frac{1}{2}I + K_k^*$ is injective with closed range on $B^{q'',q''}_{\theta-\frac{1}{q''}}(\partial\Omega)$. One more use of Theorem 2.10 in [11] gives that $-\frac{1}{2}I + K_k^*$ is an isomorphism on $B^{q'',q''}_{\theta-\frac{1}{q''}}(\partial\Omega)$.

The conclusion is that if the pair $(\theta, \frac{1}{q}) \in \mathcal{A}_\varepsilon \cup \mathcal{B}_\varepsilon$ is such that (BVP_1) is well posed, then $-\frac{1}{2}I + K_k^*$ is an isomorphism on $B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)$. This contradicts (3) in Theorem 2.2 and finishes the proof of the fact that $\mathcal{Q}_\varepsilon$ is the sharp region of indices for which (BVP_1) is well posed. From this, we may also conclude that $\mathcal{Q}_\varepsilon$ is the sharp region of indices for which (BVP_2) is well posed given that the two problems transform into one another under the action of the $*$-operator. This completes the proof of Theorem 1.1. □

We now turn to the consequences of Theorem 1.1 stated in §1.

Proof of Corollary 1.2. This follows by taking $\theta = 0$ in Theorem 1.1 which, in turn, entails $\frac{4}{3+2\varepsilon} < q < \frac{4}{1-2\varepsilon}$; cf. the picture of the region $\mathcal{Q}_\varepsilon$. □

Proof of Corollary 1.3. We start by observing that, for a fixed $(\theta, \frac{1}{q}) \in \mathcal{Q}_\varepsilon$, the pairing $\langle \vec{w}, \vec{\omega} \rangle$ is meaningful for each $\vec{w} \in H^{\theta,q}(\Omega, \mathbb{R}^2)$ and each $\vec{\omega} \in \mathcal{H}^2_{\tan}(\Omega, \mathbb{R}^2)$. This can be seen by combining (3.3), (1.19), and the fact that $(\theta, \frac{1}{q}) \in \mathcal{Q}_\varepsilon$ if and only if $(-\theta, \frac{1}{p}) \in \mathcal{Q}_\varepsilon$, $\frac{1}{p} + \frac{1}{q} = 1$.

Next, consider $\vec{u}$ as in the statement of the corollary and set $j := \operatorname{curl} \vec{u} \in H^{\theta-1,q}(\Omega)$, $k := \operatorname{div} \vec{u} \in H_0^{\theta-1,q}(\Omega)$ and $f := \vec{\nu} \cdot \vec{u} \in B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)$. Then $\vec{u}$ is a solution of (BVP_1) and, from the proof of Theorem 1.1, we know that $\vec{u} = \vec{w} + \vec{\omega}$, for some $\vec{w}$ which has an integral representation as in (3.1) and $\vec{\omega} \in \mathcal{H}^2_{\tan}(\Omega, \mathbb{R}^2)$. Using (3.1) and the mapping properties of the operators in Theorem 2.2, we may therefore conclude that

$$\|\vec{w}\|_{H^{\theta,q}(\Omega,\mathbb{R}^2)} \leq C(\|k\|_{H_0^{\theta-1,q}(\Omega)} + \|j\|_{H^{\theta-1,q}(\Omega)} + \|f\|_{B^{q,q}_{\theta-\frac{1}{q}}(\partial\Omega)}). \tag{3.7}$$

In addition, if $\vec{\omega}_1, \cdots, \vec{\omega}_{b_1(\Omega)}$ is a basis for $\mathcal{H}^2_{\tan}(\Omega, \mathbb{R}^2)$ we can write

$$\vec{\omega} = \sum_{l=1}^{b_1(\Omega)} \left(\langle \vec{u}, \vec{\omega}_l \rangle - \langle \vec{w}, \vec{\omega}_l \rangle \right) \vec{\omega}_l. \tag{3.8}$$

Thus, $\|\vec{\omega}\|_{H^{\theta,q}(\Omega,\mathbb{R}^2)} \leq C \sum_{l=1}^{b_1(\Omega)} |\langle \vec{u}, \vec{\omega}_l \rangle| + C\|\vec{w}\|_{H^{\theta,q}(\Omega,\mathbb{R}^2)}$, which, in concert with (3.7), implies that $\vec{u}$ satisfies the estimate (1.10).

The proof of (1.11) is similar and this finishes the proof of Corollary 1.3. □

Proof of Corollary 1.4. This is a direct consequence of Corollary 1.3 given that $b_1(\Omega) = 0$ if Ω is simply connected. □

Proof of Corollary 1.5. This follows from Corollary 1.3 since, under the current hypotheses,

$$\begin{aligned}
&\sum_{l=1}^{b_1(\Omega)} |\langle \vec{u}, \vec{\omega}_l \rangle| + \sum_{l=1}^{b_1(\Omega)} |\langle \vec{u}, \vec{\psi}_l \rangle| \\
&\leq \|\vec{u}\|_{H^{s,p}(\Omega,\mathbb{R}^2)} \sum_{l=1}^{b_1(\Omega)} \left[\|\vec{\omega}_l\|_{H^{-s,p'}(\Omega,\mathbb{R}^2)} + \|\vec{\psi}_l\|_{H^{-s,p'}(\Omega,\mathbb{R}^2)} \right] \leq C\|\vec{u}\|_{H^{s,p}(\Omega,\mathbb{R}^2)}
\end{aligned} \tag{3.9}$$

where $1/p + 1/p' = 1$, since $H^{\theta,q}(\Omega, \mathbb{R}^2) \hookrightarrow H^{s,p}(\Omega, \mathbb{R}^2)$ by (1.14). □

REFERENCES

[1] G. Auchmuty and J.C. Alexander, *L^2 well-posedness of planar div-curl systems*, Arch. Ration. Mech. Anal., 160 (2001), 91–134.

[2] I.S. Aržanyh, *Resolvents of basic problems in field theory*, (Russian) Doklady Akad. Nauk SSSR (N.S.), 82 (1952), 545–548.

[3] C. Bennett and R. Sharpley, "Interpolation of Operators," Academic Press, Boston, 1988.

[4] J. Bergh and J. Löström, "Interpolation Spaces. An Introduction," Springer-Verlag, Berlin/New York, 1976.

[5] J. Bourgain and H. Brezis, *New estimates for the Laplacian, the div-curl, and related Hodge systems*, C. R. Math. Acad. Sci. Paris, 338 (2004), 539–543.

[6] J.H. Bramble and J.E. Pasciak, *A new approximation technique for div-curl systems*, Math. Comp., 73 (2004), 1739–1762.

[7] R. Coifman, A. McIntosh, and Y. Meyer, *L'intégrale de Cauchy definit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Annals of Math., 116 (1982), 361–388.

[8] R. Dautray and J.-L. Lions, "Mathematical Analysis and Numerical Methods for Science and Technology," Vol. 3, Spectral Theory and Applications, Springer-Verlag, Berlin, 1990.

[9] E. Fabes, O. Mendez and M. Mitrea, *Boundary layers on Sobolev-Besov spaces and Poisson's equation for the Laplacian in Lipschitz domains*, J. Funct. Anal., 159 (1998), 323–368.

[10] D. Jerison and C. E. Kenig, *The Inhomogeneous Dirichlet Problem in Lipschitz Domains*, J. Funct. Anal., 127, (1995), 204–234.

[11] N. Kalton and M. Mitrea, *Stability results on interpolation scales of quasi-Banach spaces and applications*, Trans. Amer. Math. Soc., 350, No. 10 (1998), 3903–3922.

[12] R. Kress, *Grundzüge einer Theorie der verallgemeinerten harmonischen Vektorfelder*, Methoden und Verfahren der Mathematischen Physics, 2 (1969), 49–83.

[13] M. Křížek and P. Neittaanmäki, *On the validity of Friedrichs' inequalities*, Math. Scand., 54 (1984), 17–26.

[14] D. Mitrea, *The method of layer potentials for non-smooth domains with arbitrary topology*, Int. Eq. Op. Th., 29 (1997), 320–328.

[15] D. Mitrea, *Layer potentials and Hodge decompositions in two dimensional Lipschitz domains*, Math. Ann., 322 (2002), 75–101.

[16] D. Mitrea, M. Mitrea and M. Taylor, *Layer Potentials, the Hodge Laplacian, and Global Boundary Problems in Nonsmooth Riemannian Manifolds*, Mem. Amer. Math. Soc., 150, no.713, 2001.

[17] M. Neudert and W. von Wahl, *Asymptotic behavior of the div-curl problem in exterior domains*, Adv. Differential Equations, 6 (2001), , no. 11, 1347–1376.

[18] R. Nicolaides, *Flow discretization by complementary volume techniques*, AIAA paper 89-1979, Proceedings of 9th AIAA CFD Meeting, Buffalo, NY, June 1989.

[19] R. Nicolaides, *Direct discretization of planar div-curl problems*, SIAM J. Numer. Anal. 29 (1992), , no. 1, 32–56.

[20] R. Nicolaides, *The covolume approach to computing incompressible flows*, in `Incompressible Computational Fluid Dynamics`, M. D. Gunzburger and R. A. Nicolaides eds., Cambridge University Press, Cambridge, 1993, 295–333.

[21] R. Nicolaides and D. Wang, *A higher-order covolume method for planar div-curl problems*, Tenth International Conference on Finite Elements in Fluids (Tucson, AZ, 1998), Internat. J. Numer. Methods Fluids 31 (1999), no. 1, 299–308.

[22] T. Runst and W. Sickel, "Sobolev Spaces of Fractional Order, Nemytskij Operators, and Nonlinear Partial Differential Operators," de Gruyter, Berlin, New York, 1996.

[23] H. Triebel, *Function spaces in Lipschitz domains and on Lipschitz manifolds*, Characteristic functions as pointwise multipliers, Rev. Mat. Complut. 15 (2002), no. 2, 475–524.

[24] G. Verchota, *Layer potentials and boundary value problems for Laplace's equation in Lipschitz domains*, J. Funct. Anal., 59 (1984), 572–611.