Differential and Integral Equations Volume 18, Number 11 (2005), 1257-1271

UNIQUENESS THEOREM OF THE CAUCHY PROBLEM FOR SCHRÖDINGER'S EQUATION IN WEIGHTED SOBOLEV SPACES

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Abstract. The present paper is concerned with the uniqueness of solutions to the Cauchy problem for the free Schrödinger equation. It is proved by the wellposedness of the associated adjoint problem that the solution to the Cauchy problem is unique in exponential weighted Sobolev spaces.

1. Introduction

Let us consider

$$i\frac{\partial}{\partial t}u(t,x) = -\frac{1}{2}\triangle u(t,x), \tag{1.1}$$

where i denotes the imaginary unit $\sqrt{-1}$, u a complex-valued unknown function of $t \in \mathbb{R}$ and $x \in \mathbb{R}^n$, and $\triangle$ the Laplacian in $\mathbb{R}^n$,

$$\triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}. \tag{1.2}$$

We investigate the Cauchy problem (1.1) with initial data

$$u(0,x) = u_0(x) \tag{1.3}$$

in exponential weighted Sobolev spaces. In general, it is called a Cauchy problem in that we consider the solutions to partial differential equations of evolution type with some initial condition. Let $I \subset \mathbb{R}$, X be a function space and $C^m(I;X)$ denote the set of all functions which are m-times continuously differentiable with respect to $t \in I$ in the topology of X. The Cauchy problem for (1.1) is said to be wellposed in X, if there exists a unique solution in $C^1(\mathbb{R};X)$ which satisfies the equation and the initial condition for any u_0 in X.

Accepted for publication: May 2005.
AMS Subject Classifications: 35Q40, 35S30, 35A05.

If $u_0 \in L^2(\mathbb{R}^n)$, we can solve the Cauchy problem for (1.1) by the method of Fourier transformation. In fact, the expression

$$u(t,x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{ix\cdot\xi - i\frac{t}{2}|\xi|^2} \hat{u}_0(\xi)\ d\xi \tag{1.4}$$

gives the solution to the Cauchy problem for (1.1) in $L^2(\mathbb{R}^n)$. In the expression, $x \cdot \xi = x_1\xi_1 + \cdots + x_n\xi_n$, $|\xi|^2 = \xi \cdot \xi$ and $\hat{u}_0$ stands for the Fourier transform of u_0, which is defined by

$$\hat{u}_0(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u_0(x)\ dx. \tag{1.5}$$

Plancherel's theorem implies the conservation law of the norm in $L^2(\mathbb{R}^n)$; that is,

$$\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} = \|u_0\|_{L^2(\mathbb{R}^n)} \tag{1.6}$$

for any $t \in \mathbb{R}$, where the notation for the norm in $L^2(\mathbb{R}^n)$

$$\|u\|_{L^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |u(x)|^2\ dx \tag{1.7}$$

is used for $u \in L^2(\mathbb{R}^n)$.

Let H^s denote the Sobolev space as a subset of squared-integrable functions in $\mathbb{R}^n$,

$$H^s = \{u \in L^2(\mathbb{R}^n);\ \langle\xi\rangle^s \hat{u} \in L^2(\mathbb{R}^n)\} \tag{1.8}$$

for $s \in \mathbb{R}$, where $\langle\xi\rangle = \sqrt{1+|\xi|^2}$ denotes a multiplication operator. The norm in H^s is defined by

$$\|u\|_s = \|\langle\xi\rangle^s \hat{u}\|_{L^2(\mathbb{R}^n)}. \tag{1.9}$$

Since the solution can be expressed by (1.4) for any u_0 in $L^2(\mathbb{R}^n)$, the Cauchy problem for (1.1) is wellposed in H^∞. More precisely, it follows for any $s \in \mathbb{R}$ that there exists a unique solution u in $C^0(\mathbb{R}; H^s) \cap C^1(\mathbb{R}; H^{s-2})$ when $u_0 \in H^s$.

Let H^s_σ denote the weighted Sobolev space, which is defined by

$$H^s_\sigma = \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^n);\ e^{-\sigma\langle x\rangle} u \in H^s\} \tag{1.10}$$

for a fixed $\sigma \in \mathbb{R}$ as a subset of locally square-integrable functions in $\mathbb{R}^n$; that is, $u \in L^2(K)$ for any compact set K in $\mathbb{R}^n$. By the definition, $H^s \subset H^s_\sigma$ when $\sigma > 0$. We also define the norm in H^s_σ,

$$\|u\|_{s,\sigma} = \|e^{-\sigma\langle x\rangle} u\|_s. \tag{1.11}$$

Although the existence and the uniqueness in $L^2(\mathbb{R}^n)$ hold in the Cauchy problem for (1.1), those are not trivial in the function space whose elements may increase exponentially along the space variables. To recognize the situation, we change unknown functions by $v(t,x) = e^{-\sigma\langle x\rangle}u(t,x)$. The Cauchy problem for (1.1) in the weighted Sobolev space is equivalent to the Cauchy problem

$$\begin{cases} i\dfrac{\partial}{\partial t}v(t,x) = \Big(-\dfrac{1}{2}\triangle - i\sigma\displaystyle\sum_{j=1}^{n}\dfrac{x_j}{\langle x\rangle}D_j + c(x)\Big)v(t,x) \\ v(0,x) = v_0(x) \end{cases} \tag{1.12}$$

in the usual Sobolev spaces, where $D_j = \frac{1}{i}\frac{\partial}{\partial x_j}$ and

$$c(x) = -\frac{1}{2}\Big\{\sigma^2 - \frac{\sigma}{\langle x\rangle} - \frac{\sigma^2}{\langle x\rangle^2} + \frac{(n+1)\sigma}{\langle x\rangle^3}\Big\}. \tag{1.13}$$

In other words, the Cauchy problem for (1.1) in H^s_σ is equivalent to the Cauchy problem for the Schrödinger-type equation (1.12) in H^s.

On the other hand, Ichinose has shown in [4] a necessary condition for the Cauchy problem for the Schrödinger-type equation to be wellposed in H^∞.

Proposition 1 (Ichinose)**.** *In order that*

$$\begin{cases} i\dfrac{\partial}{\partial t}v(t,x) = \Big(-\dfrac{1}{2}\triangle + \displaystyle\sum_{j=1}^{n} b_j(x)D_j + c(x)\Big)v(t,x) \\ v(0,x) = v_0(x) \end{cases} \tag{1.14}$$

be wellposed in H^∞, it is necessary that there exist positive constants M and N such that

$$\sup_{x\in\mathbb{R}^n}\sup_{\omega\in S^{n-1}}\Big|\sum_{j=1}^{n}\int_0^\rho Im\ b_j(x+2\theta\omega)\omega_j\ d\theta\Big| \le M\log(1+\rho) + N \tag{1.15}$$

for any $\rho \ge 0$, where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$.

According to Ichinose's result, the Cauchy problem (1.12) is not wellposed in H^∞. In fact, we can easily show that there is $L > 0$ such that

$$L\rho \le \int_0^\rho \mathrm{Im}\ b_j(x+2\theta\omega)\omega_j\ d\theta \tag{1.16}$$

for some $x \in \mathbb{R}^n$ and $\omega \in S^{n-1}$ chosen suitably when $b_j(x) = -i\sigma x_j/\langle x\rangle$. The estimate (1.16) violates the necessary condition so that the Cauchy problem (1.12) is not wellposed in the usual Sobovev spaces. Thus, we can not conclude immediately that the Cauchy problem (1.12) has a unique solution in $C^0(\mathbb{R};H^s)\cap C^1(\mathbb{R};H^{s-2})$ even when $u_0 \in H^s$.

Because of that reason, we will prove the uniqueness of solutions to the Cauchy problem (1.12), which is equivalent to that for (1.1), by a direct method in this paper.

Theorem 1 (uniqueness)**.** *Let $\sigma > 0$. If there is u in $C^0(\mathbb{R}; H^s_\sigma) \cap C^1(\mathbb{R}; H^{s-2}_\sigma)$ satisfying the Cauchy problem for (1.1) with $u_0 = 0$, then $u = 0$ in the topology of $C^0(\mathbb{R}; H^s_\sigma) \cap C^1(\mathbb{R}; H^{s-2}_\sigma)$.*

The key idea in the proof is that the existence of solutions to the adjoint problem implies the uniqueness of solutions to the Cauchy problem for (1.1). The adjoint problem for the Cauchy problem for (1.1) is wellposed in a certain dense subspace of $L^2(\mathbb{R}^n)$. We will follow this idea paying attention to precise estimates of constants to appear in our discussion.

The result is not devoted to the existence, but devoted to the uniqueness of solutions. Although the existence of solutions does not hold in general, Kajitani and the author have in [2, 3] constructed solutions by imposing some conditions on initial data.

Before we proceed to prove the uniqueness theorem, some notation used in the general theory of pseudo-differential operators is introduced. As is introduced by Hörmander, the symbol class of pseudo-differential operators is defined by

$$S^m_{\rho,\delta} = \left\{ p \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n);\ |p^{(\alpha)}_{(\beta)}(x,\xi)| \le C_{\alpha,\beta} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|} \right\} \tag{1.17}$$

for $m \in \mathbb{R}$, both ρ and δ between 0 and 1, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $|\beta| = \beta_1 + \cdots + \beta_n$ and $p^{(\alpha)}_{(\beta)}(x,\xi) = \partial^\alpha_\xi D^\beta_x p(x,\xi)$ with

$$\partial^\alpha_\xi = \partial^{\alpha_1}_{\xi_1} \cdots \partial^{\alpha_n}_{\xi_n} \tag{1.18}$$

$$D^\beta_x = (-i)^{|\beta|} \partial^{\beta_1}_{x_1} \cdots \partial^{\beta_n}_{x_n}, \tag{1.19}$$

for both $\alpha = (\alpha_1, \dots, \alpha_n)$ and $\beta = (\beta_1, \dots, \beta_n)$ in $\mathbb{N}^n$. Throughout the paper, we use $\mathbb{N} = \{0, 1, 2, \dots\}$ for the set of natural numbers.

The author thanks Professor K. Kajitani for his useful suggestion on this subject.

2. Fourier transform of analytic functions

In this section, we are going to define the notion of analyticity, then get an estimate of the Fourier transform of analytic functions. It becomes clear at the end of the section that the Fourier transform of analytic functions decays exponentially in the direction of space variables.

Lemma 1. *Let $p > 1$ and $l \in \mathbb{N}$. If ε satisfies*

$$0 < \varepsilon \leq \frac{1}{p\sqrt{(1+p^2)n}}, \tag{2.1}$$

then

$$|\partial_x^\alpha \langle x \rangle^{-2l}| \leq \left(\frac{p^2}{p^2-1}\right)^l \varepsilon^{-|\alpha|} |\alpha|! \langle x \rangle^{-2l} \tag{2.2}$$

for any $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{N}^n$.

Proof. We will discuss an estimate of $\langle x \rangle^{-2l}$ in the complex domain $\mathbb{C}^n$ by virtue of Cauchy's integral formula. Taking a contour Γ for the circle around x with the radius of ε; that is,

$$\Gamma = \{z \in \mathbb{C}^n; |z_j - x_j| = \varepsilon \ (j = 1, \ldots, n)\}, \tag{2.3}$$

then Cauchy's integral formula implies

$$\partial_x^\alpha \langle x \rangle^{-2l} = \frac{\alpha!}{(2\pi i)^n} \int_\Gamma \frac{\langle z \rangle^{-2l}}{\prod_{j=1}^n (z_j - x_j)^{\alpha_j + 1}} dz \tag{2.4}$$

for any $\alpha \in \mathbb{N}^n$. It should be remarked that $\langle z \rangle$ in the above integral means

$$\langle z \rangle = \sqrt{1 + z_1^2 + \cdots + z_n^2} \tag{2.5}$$

instead of $\sqrt{1 + \bar{z}_1 z_1 + \cdots + \bar{z}_n z_n}$, for the purpose of avoiding the loss of the holomorphic property of $\langle z \rangle$. In fact, although both of the definitions give the same value on the real line, the former is holomorphic in an open subset of $\mathbb{C}^n$ which contains $\mathbb{R}^n$, the latter is not holomorphic in any open subset of $\mathbb{C}^n$. In the integral, we change the variables from $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ to $\theta = (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ by $z = x + \varepsilon e^{i\theta}$; that is,

$$z_j = x_j + \varepsilon e^{i\theta_j} \quad (j = 1, \ldots, n) \tag{2.6}$$

where $0 \leq \theta_j < 2\pi$ for each $j = 1, \ldots, n$. Thus, we obtain

$$dz_j = i\varepsilon e^{i\theta_j} d\theta_j \quad (j = 1, \ldots, n) \tag{2.7}$$

or

$$dz = (i\varepsilon)^n e^{i(\theta_1 + \cdots \theta_n)} d\theta_1 \cdots d\theta_n. \tag{2.8}$$

Then, we have

$$\partial_x^\alpha \langle x\rangle^{-2l} = \frac{\varepsilon^{-|\alpha|}\alpha!}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\frac{\langle x+\varepsilon e^{i\theta}\rangle^{-2l}}{\prod_{j=1}^{n} e^{i\alpha_j\theta_j}}\,d\theta_1\cdots d\theta_n, \tag{2.9}$$

where we use $x+\varepsilon e^{i\theta} = (x_1+\varepsilon e^{i\theta_1},\dots,x_n+\varepsilon e^{i\theta_n})$ for simplicity.

On the other hand, the real part of $\langle x+\varepsilon e^{i\theta}\rangle^2$ satisfies

$$\mathrm{Re}\langle x+\varepsilon e^{i\theta}\rangle^2 = 1+\mathrm{Re}\sum_{j=1}^{n}\left(x_j+\varepsilon e^{i\theta_j}\right)^2 \geq 1+\sum_{j=1}^{n}\left(x_j^2-2|x_j|\varepsilon-\varepsilon^2\right). \tag{2.10}$$

Using Schwarz' inequality with interpolating parameter p,

$$2|x_j|\varepsilon = 2\cdot\frac{|x_j|}{p}\cdot p\varepsilon \leq \frac{x_j^2}{p^2}+p^2\varepsilon^2. \tag{2.11}$$

Thus, we obtain

$$\mathrm{Re}\langle x+\varepsilon e^{i\theta}\rangle^2 \geq \left(1-\frac{1}{p^2}\right)\langle x\rangle^2 \tag{2.12}$$

if $0<\varepsilon\leq \dfrac{1}{p\sqrt{(1+p^2)n}}$. Hence, if $p>1$,

$$\left|\langle x+\varepsilon e^{i\theta}\rangle^{-2l}\right| \leq \left(\mathrm{Re}\langle x+\varepsilon e^{i\theta}\rangle^2\right)^{-l} \leq \left(\frac{p^2}{p^2-1}\right)^l\langle x\rangle^{-2l}, \tag{2.13}$$

which shows that the absolute value of $\langle x+\varepsilon e^{i\theta}\rangle^{-2l}$ is bounded, so that we conclude

$$|\partial_x^\alpha\langle x\rangle^{-2l}| \leq \left(\frac{p^2}{p^2-1}\right)^l\varepsilon^{-|\alpha|}|\alpha|!\langle x\rangle^{-2l}. \tag{2.14}$$

This completes the proof of Lemma 1. □

Let $a(x)$ be a smooth function which belongs to $C^\infty(\mathbb{R}^n)$. We say that $a(x)$ is analytic if there exist $\rho_0>0$ and $C_0>0$ such that

$$\sup_{x\in\mathbb{R}^n}|\partial_x^\alpha a(x)| \leq C_0\rho_0^{-|\alpha|}|\alpha|! \tag{2.15}$$

for any $\alpha\in\mathbb{N}^n$.

According to the definition, $\langle x\rangle^{-2l}$ is analytic by the previous lemma.

Lemma 2. *Let $a(x)$ be analytic. If $l>\frac{n}{2}$, then there is $C>0$ independent of η such that*

$$\left|\int_{\mathbb{R}^n} e^{-iy\cdot\eta}\langle y\rangle^{-2l}a(x+y)\,dy\right| \leq \frac{C\rho_1}{\rho_1-n\rho}e^{-\rho|\eta|} \tag{2.16}$$

for any ρ *with* $0 < \rho < \dfrac{\rho_1}{n}$ *where* $\rho_1 = \min\{\varepsilon, \rho_0\}$ (ε *in Lemma* 1).

Proof. The identity $\partial_y^\alpha e^{-iy\cdot\eta} = (-i)^{|\alpha|}\eta^\alpha e^{-iy\cdot\eta}$ and integration by parts imply

$$\eta^\alpha \int_{\mathbb{R}^n} e^{-iy\cdot\eta}\langle y\rangle^{-2l} a(x+y)\,dy = \int_{\mathbb{R}^n} (i\partial_y)^\alpha e^{-iy\cdot\eta}\cdot\langle y\rangle^{-2l} a(x+y)\,dy$$
$$= (-i)^{|\alpha|}\int_{\mathbb{R}^n} e^{-iy\cdot\eta}\partial_y^\alpha\left[\langle y\rangle^{-2l} a(x+y)\right]dy.$$

By Leibniz's formula, we have

$$\left|\partial_y^\alpha\left[\langle y\rangle^{-2l} a(x+y)\right]\right| \le \sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}|\partial_y^{\alpha-\alpha'}\langle y\rangle^{-2l}|\cdot|\partial_y^{\alpha'} a(x+y)|.$$

Here, we apply Lemma 1 for $p=\sqrt{2}$, then the following estimate holds:

$$|\partial_y^\beta\langle y\rangle^{-2l}| \le C_1|\beta|!\varepsilon^{-|\beta|}\langle y\rangle^{-2l}$$

where $C_1 = 2^l$ and $0 < \varepsilon \le \frac{1}{\sqrt{6n}}$. Using this estimate and the assumption of this lemma,

$$\left|\partial_y^\alpha\left[\langle y\rangle^{-2l} a(x+y)\right]\right| \le C_0 C_1\langle y\rangle^{-2l}\sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}|\alpha-\alpha'|!|\alpha'|!\varepsilon^{-|\alpha-\alpha'|}\rho_0^{-|\alpha'|}.$$

Here we put ρ_1 and ρ_2, the minimum and the maximum of $\{\varepsilon, \rho_0\}$ respectively, where ε was present in Lemma 1. We can take ε such that $\varepsilon \ne \rho_0$, then

$$\sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}|\alpha-\alpha'|!|\alpha'|!\varepsilon^{-|\alpha-\alpha'|}\rho_0^{-|\alpha'|}$$
$$= \rho_1^{-|\alpha|}\sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}|\alpha-\alpha'|!|\alpha'|!\left(\frac{\rho_1}{\rho_2}\right)^{|\alpha'|}$$
$$= \rho_1^{-|\alpha|}\sum_{j=0}^{|\alpha|}\binom{|\alpha|}{j}(|\alpha|-j)!j!\left(\frac{\rho_1}{\rho_2}\right)^j$$
$$= \rho_1^{-|\alpha|}|\alpha|!\sum_{j=0}^{|\alpha|}\left(\frac{\rho_1}{\rho_2}\right)^j \le \frac{\rho_2}{\rho_2-\rho_1}\rho_1^{-|\alpha|}|\alpha|!.$$

Hence,

$$\left|\eta^\alpha \int_{\mathbb{R}^n} e^{-iy\cdot\eta}\langle y\rangle^{-2l} a(x+y)\ dy\right| \le \frac{\tilde{C}\rho_2}{\rho_2-\rho_1}|\alpha|!\rho_1^{-|\alpha|}$$

for any $\alpha \in \mathbb{N}^n$, where $\tilde{C} = C_0 C_1 \int_{\mathbb{R}^n} \langle y\rangle^{-2l}\ dy$ if $2l > n$. By using this result, we obtain

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^n} e^{-iy\cdot\eta}\langle y\rangle^{-2l} a(x+y)\ dy\Big| \\
&= e^{-\rho|\eta|}\sum_{k=0}^{\infty}\frac{\rho^k}{k!}\cdot|\eta|^k\cdot\Big|\int_{\mathbb{R}^n} e^{-iy\cdot\eta}\langle y\rangle^{-2l} a(x+y)\ dy\Big| \\
&\le e^{-\rho|\eta|}\sum_{k=0}^{\infty}\frac{\rho^k}{k!}\sum_{|\alpha|=k}\frac{k!}{\alpha!}\Big|\eta^\alpha\int_{\mathbb{R}^n} e^{-iy\cdot\eta}\langle y\rangle^{-2l} a(x+y)\ dy\Big| \\
&\le \frac{\tilde{C}\rho_2}{\rho_2-\rho_1}e^{-\rho|\eta|}\sum_{k=0}^{\infty}\left(\frac{\rho}{\rho_1}\right)^k\sum_{|\alpha|=k}\frac{k!}{\alpha!} \\
&= \frac{\tilde{C}\rho_2}{\rho_2-\rho_1}e^{-\rho|\eta|}\sum_{k=0}^{\infty}\left(\frac{n\rho}{\rho_1}\right)^k \le \frac{\tilde{C}\rho_1\rho_2}{(\rho_2-\rho_1)(\rho_1-n\rho)}e^{-\rho|\eta|}.
\end{aligned}$$

This completes the proof of Lemma 2. □

Corollary 1. *Let $a(x)$ be analytic. If $l > \frac{n}{2}$, then*

$$e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)}\Big|\int_{\mathbb{R}^n} e^{-iy\cdot\eta}\langle y\rangle^{-2l} a(x+y)\ dy\Big| \le \frac{C\rho_1}{\rho_1-n\rho}e^{-(\rho-\lambda)|\eta|} \tag{2.17}$$

for every ρ with $\lambda < \rho < \frac{\rho_1}{n}$ where $\rho_1 = \min\{\varepsilon, \rho_0\}$.

We can take $\varepsilon = \frac{1}{\sqrt{6n}}$ except for the case of $\rho_0 = \frac{1}{\sqrt{6n}}$. We use this result in the next section.

3. Exact estimate for $e^{\lambda\langle D\rangle}a(x)e^{-\lambda\langle D\rangle}$

In this section, we will estimate the operator $e^{\lambda\langle D\rangle}a(x)e^{-\lambda\langle D\rangle}$ when $a(x)$ is an analytic function. It is known that the symbol of $e^{\lambda\langle D\rangle}a(x)e^{-\lambda\langle D\rangle}$ belongs to $S^0_{1,0}$ if $a(x)$ is analytic. However, we need the exact estimate of the constant of the symbol which plays an important role at the stage that the L^2 boundedness theorem is applied in our proof of the uniqueness theorem.

Lemma 3. *For any $\alpha \in \mathbb{N}^n$, there exists $C_\alpha > 0$ such that*

$$|\partial_\xi^\alpha [\langle \xi + \eta \rangle - \langle \xi \rangle]| \leq C_\alpha \langle \xi \rangle^{-|\alpha|} \langle \eta \rangle \tag{3.1}$$

for any $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$ with $|\eta| \leq \frac{1}{2}|\xi|$.

Proof. It is an easy calculation that for any $\alpha \in \mathbb{N}^n$, there exists $C_\alpha^{(1)} > 0$ such that

$$\Big|\partial_\xi^\alpha \Big[\frac{\xi_j}{\langle \xi \rangle}\Big]\Big| \leq C_\alpha^{(1)} \langle \xi \rangle^{-|\alpha|} \tag{3.2}$$

for $j = 1, \ldots, n$. This estimate implies

$$\Big|\partial_\xi^\alpha \Big[\frac{\xi_j + \theta\eta_j}{\langle \xi + \theta\eta \rangle}\Big]\Big| \leq C_\alpha^{(2)} \langle \xi + \theta\eta \rangle^{-|\alpha|}, \tag{3.3}$$

where $\theta \in [0, 1]$. In general, the inequality $\langle \xi + \theta\eta \rangle^{-1} \leq 2\langle \xi \rangle^{-1}$ holds in the region $\{\eta \in \mathbb{R}^n; |\xi + \theta\eta| \geq \frac{1}{2}|\xi|\}$. Adding to this, the region $\{\eta \in \mathbb{R}^n; |\eta| \leq \frac{1}{2}|\xi|\}$ is always contained in $\{\eta \in \mathbb{R}^n; |\xi+\theta\eta| \geq \frac{1}{2}|\xi|\}$. Therefore, we conclude that the estimate $\langle \xi + \theta\eta \rangle^{-1} \leq 2\langle \xi \rangle^{-1}$ holds for any $\theta \in [0, 1]$ if $|\eta| \leq \frac{1}{2}|\xi|$. Next, we have

$$\langle \xi + \eta \rangle - \langle \xi \rangle = \sum_{j=1}^n \eta_j \int_0^1 \frac{\xi_j + \theta\eta_j}{\langle \xi + \theta\eta \rangle}\, d\theta \tag{3.4}$$

by the mean-value theorem. Differentiating both sides with respect to ξ,

$$\partial_\xi^\alpha [\langle \xi + \eta \rangle - \langle \xi \rangle] = \sum_{j=1}^n \eta_j \int_0^1 \partial_\xi^\alpha \Big[\frac{\xi_j + \theta\eta_j}{\langle \xi + \theta\eta \rangle}\Big] d\theta. \tag{3.5}$$

Finally, we obtain

$$\big|\partial_\xi^\alpha [\langle \xi + \eta \rangle - \langle \xi \rangle]\big| \leq \sum_{j=1}^n \langle \eta \rangle \int_0^1 \Big|\partial_\xi^\alpha \Big[\frac{\xi_j + \theta\eta_j}{\langle \xi + \theta\eta \rangle}\Big]\Big|\, d\theta \leq C_\alpha \langle \xi \rangle^{-|\alpha|} \langle \eta \rangle.$$

This completes the proof of Lemma 3. □

Lemma 4. *For any $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq 1$, there exists $C_\alpha > 0$ such that*

$$|\partial_\xi^\alpha e^{\lambda(\langle \xi+\eta \rangle - \langle \xi \rangle)}| \leq C_\alpha \langle \xi \rangle^{-|\alpha|} \langle \eta \rangle e^{\lambda(\langle \xi+\eta \rangle - \langle \xi \rangle)} \sum_{k=1}^{|\alpha|} \lambda^k \tag{3.6}$$

in the region $|\eta| \leq \frac{1}{2}|\xi|$.

Proof. In general theory, the differentiation formula of composite functions with multi-variables holds, such as,

$$\partial_\xi^\alpha e^{f(\xi)} = e^{f(\xi)} \sum_{k=1}^{|\alpha|} \sum_{\substack{\beta^{(1)}+\cdots+\beta^{(k)}=\alpha \\ |\beta^{(j)}|\geq 1 \ (j=1,\dots,k)}} \prod_{j=1}^{k} \left(\partial_\xi^{\beta^{(j)}} f\right)(\xi), \tag{3.7}$$

where f is a general function in $C^\infty(\mathbb{R}^n)$. Using this formula with $f(\xi) = \lambda(\langle\xi+\eta\rangle - \langle\xi\rangle)$ and Lemma 3 implies

$$\left|\partial_\xi^\alpha e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)}\right| \leq e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)} \sum_{k=1}^{|\alpha|} \lambda^k \sum_{\substack{\beta^{(1)}+\cdots+\beta^{(k)}=\alpha \\ |\beta^{(j)}|\geq 1 (j=1,\dots,k)}} \prod_{j=1}^{k} \left|\partial_\xi^{\beta^{(j)}} [\langle\xi+\eta\rangle - \langle\xi\rangle]\right|$$

$$\leq e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)} \left(\sum_{k=1}^{|\alpha|} C_k \lambda^k\right) \langle\xi\rangle^{-|\alpha|}\langle\eta\rangle \leq C_\alpha \langle\xi\rangle^{-|\alpha|}\langle\eta\rangle e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)} \sum_{k=1}^{|\alpha|} \lambda^k.$$

This completes the proof of Lemma 4. □

Lemma 5. *For any $\alpha \in \mathbb{N}^n$, there exists a polynomial $C_\alpha(\lambda)$ of λ with order $2l + |\alpha|$ such that*

$$\left|\partial_\xi^\alpha \left[(1-\triangle_\eta)^l e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)}\right]\right| \leq C_\alpha(\lambda)\langle\xi\rangle^{-|\alpha|}\langle\eta\rangle e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)} \tag{3.8}$$

in the region $|\eta| \leq \frac{1}{2}|\xi|$.

Proof. It is easy to show that there exists a function $Q_l(\lambda; \langle\xi+\eta\rangle)$ such that

$$(1-\triangle_\eta)^l e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)} = Q_l(\lambda; \langle\xi+\eta\rangle) e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)}$$

and that

$$Q_l(\lambda; \langle\xi+\eta\rangle) = q_0(\lambda) + \frac{q_1(\lambda)}{\langle\xi+\eta\rangle} + \cdots + \frac{q_{2l+1}(\lambda)}{\langle\xi+\eta\rangle^{2l+1}},$$

where $q_j(\lambda) = C_0^{(j)} + C_1^{(j)}\lambda + \cdots + C_{2l}^{(j)}\lambda^{2l}$ for $j = 0, 1, \dots, 2l+1$. In fact, the above identity follows from

$$\triangle_\eta\left(\langle\xi+\eta\rangle^{-m} e^{\rho\langle\xi+\eta\rangle}\right) = \Big\{\frac{\rho^2}{\langle\xi+\eta\rangle^m} + \frac{(n-2m-1)\rho}{\langle\xi+\eta\rangle^{m+1}} - \frac{m(n+m+2)+\rho^2}{\langle\xi+\eta\rangle^{m+2}} + \frac{(2m+1)\rho}{\langle\xi+\eta\rangle^{m+3}} + \frac{m(m+2)}{\langle\xi+\eta\rangle^{m+4}}\Big\} e^{\rho\langle\xi+\eta\rangle}. \tag{3.9}$$

Next, we use Leibniz's formula, then

$$\partial_\xi^\alpha\left[(1-\triangle_\eta)^l e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)}\right]=\sum_{\alpha'\le\alpha}\binom{\alpha}{\alpha'}\partial_\xi^{\alpha-\alpha'}Q_l(\lambda;\langle\xi+\eta\rangle)\cdot\partial_\xi^{\alpha'}e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)}.$$

Since $|\partial_\xi^\alpha Q_l(\lambda;\langle\xi+\eta\rangle)|\le C_\alpha(\lambda)\langle\xi+\eta\rangle^{-|\alpha|}$, we obtain from Lemma 4 that

$$\left|\partial_\xi^\alpha\left[(1-\triangle_\eta)^l e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)}\right]\right|\le C_\alpha(\lambda)\langle\xi\rangle^{-|\alpha|}\langle\eta\rangle e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)} \tag{3.10}$$

in the region $\{\eta\in\mathbb{R}^n;|\eta|\le\frac{1}{2}|\xi|\}$. This completes the proof of Lemma 5. □

Corollary 1 and Lemma 5 suggest the following proposition. This is the desired estimate of $e^{\lambda\langle D\rangle}a(x)e^{-\lambda\langle D\rangle}$.

Proposition 2. *Let $\lambda>0$ and $a(x)$ be analytic in $\mathbb{R}^n$. Put $\tilde{a}(x,D)=e^{\lambda\langle D\rangle}a(x)e^{-\lambda\langle D\rangle}$. If $0<\lambda<\rho$ (ρ in Corollary 1), then there exists a polynomial $C(\lambda)$ on $[0,\infty)$ such that*

$$\|\tilde{a}(x,D)u\|\le C(\lambda)\|u\| \tag{3.11}$$

and

$$C(\lambda)=\sum_{j=0}^{N_0}c_j\lambda^j \tag{3.12}$$

for some $N_0\in\mathbb{N}$ and $c_j>0$ for $j=0,1,\ldots,N_0$.

Proof. By the general theory of pseudo-differential operators, the product of two symbols, say $p_1(x,\xi)$ and $p_2(x,\xi)$, can be expressed as follows:

$$\frac{1}{(2\pi)^n}\text{Os-}\int\int_{\mathbb{R}^n\times\mathbb{R}^n}e^{-iy\cdot\eta}p_1(x,\xi+\eta)p_2(x+y,\xi)\,dy\,d\eta,$$

where we denote by Os-$\int\int$, the oscillatory integral with a suitable convergence factor, for example,

$$\lim_{\varepsilon\to+0}\int\int_{\mathbb{R}^n\times\mathbb{R}^n}e^{-iy\cdot\eta}e^{-\varepsilon(|y|^2+|\eta|^2)}p_1(x,\xi+\eta)p_2(x+y,\xi)\,dy\,d\eta.$$

In this case, the symbol of $e^{\lambda\langle D\rangle}a(x)e^{-\lambda\langle D\rangle}$ is denoted by

$$\begin{aligned}\tilde{a}(x,\xi)&=\frac{1}{(2\pi)^n}\text{Os-}\int\int e^{-iy\cdot\eta}e^{\lambda\langle\xi+\eta\rangle}a(x+y)e^{-\lambda\langle\xi\rangle}\,dy\,d\eta\\&=\frac{1}{(2\pi)^n}\text{Os-}\int\int(1-\triangle_\eta)^l e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)}e^{-iy\cdot\eta}\langle y\rangle^{-2l}a(x+y)\,dy\,d\eta\\&=\frac{1}{(2\pi)^n}\text{Os-}\int(1-\triangle_\eta)^l e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)}\check{a}(x,\eta)\,d\eta,\end{aligned}$$

where $\triangle_\eta$ stands for the Laplacian in $\mathbb{R}^n$ with respect to η and

$$\check{a}(x,\eta) = \int e^{-iy\cdot\eta}\langle y\rangle^{-2l} a(x+y)\ dy.$$

When we take an integer l greater than $\frac{n}{2}$, the integral with respect to y becomes the usual integral instead of the oscillatory integral because of the assumption of boundedness of a. Differentiating $\tilde{a}(x,\xi)$ with respect to x and ξ, then we obtain as an application of Corollary 1 with the assumptions on $a(x)$, and $(1-\triangle_\eta)^l e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)} = Q_l(\lambda;\langle\xi+\eta\rangle)e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)}$ in the proof of Lemma 5,

$$\left|\partial_\xi^\alpha D_x^\beta \int_{|\eta|\geq\frac{1}{2}|\xi|}(1-\triangle_\eta)^l e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)}\check{a}(x,\eta)\ d\eta\right| \leq C_1 e^{-(\rho-\lambda)|\xi|}$$

for some constant $C_1 > 0$. On the other hand, Lemma 5 is valid for $|\eta| \leq \frac{1}{2}|\xi|$, so that we also conclude

$$\left|\partial_\xi^\alpha D_x^\beta \int_{|\eta|\leq\frac{1}{2}|\xi|}(1-\triangle_\eta)^l e^{\lambda(\langle\xi+\eta\rangle-\langle\xi\rangle)}\check{a}(x,\eta)\ d\eta\right| \leq C_2(\lambda)\langle\xi\rangle^{-|\alpha|}.$$

Summing up this discussion, we obtain

$$\left|\partial_\xi^\alpha D_x^\beta \tilde{a}(x,\xi)\right| \leq C'(\lambda)\langle\xi\rangle^{-|\alpha|}.$$

This estimate makes it clear that $\tilde{a}(x,\xi)$ belongs to the symbol class $S^0_{1,0}$ (denote by $S^m_{\rho,\delta}$ the Hörmander class). Finally, the L^2 boundedness theorem implies

$$\|\tilde{a}(x,D)u\| \leq C(\lambda)\|u\| \tag{3.13}$$

with $C(\lambda) = \sum_{j=0}^{N_0} c_j\lambda^j$ for some $N_0 \geq 0$. This completes the proof of Proposition 2. □

4. Proof of the main theorem

In the present section, we prove the uniqueness of solutions to the Cauchy problem for the free Schrödinger equation in exponential weighted Sobolev spaces; namely, if there is u in $C^0([0,T];H^s_{-\sigma})\cap C^1([0,T];H^{s-2}_{-\sigma})$ such that

$$i\frac{\partial}{\partial t}u(t,x) = -\frac{1}{2}\triangle u(t,x), \quad u(0,x) = 0, \tag{4.1}$$

then $u = 0$.

Before we prove the uniqueness theorem, we introduce a proposition concerning wellposedness in a certain function class. It turns out that the uniqueness of the original Cauchy problem is obtained by considering the wellposedness of the adjoint Cauchy problem. For the purpose of that, we define

$$\hat{L}^2_\rho = \{\varphi \in L^2(\mathbb{R}^n); e^{\rho\langle\xi\rangle}\hat{\varphi}(\xi) \in L^2(\mathbb{R}^n_\xi)\} \tag{4.2}$$

for $\rho > 0$. Then, $\hat{L}^2_\rho$ is a subspace of $L^2(\mathbb{R}^n)$ and $\hat{L}^2_{\rho'} \subset \hat{L}^2_\rho$ if $\rho' < \rho$.

Proposition 3. *If the Cauchy problem*

$$\begin{cases} i\dfrac{\partial}{\partial t}w(t,x) = \dfrac{1}{2}\displaystyle\sum_{j=1}^n \left(D_j + i\sigma\frac{x_j}{\langle x\rangle}\right)^2 w(t,x) & \text{in } (0,T)\times\mathbb{R}^n \\ w(T,x) = w_T(x) & \text{on } \mathbb{R}^n \end{cases} \tag{4.3}$$

is wellposed in $\hat{L}^2_\rho$ uniformly in $t \in [0,T]$, then Theorem 1 is valid.

Proof. Put

$$L = i\frac{\partial}{\partial t} - \frac{1}{2}\sum_{j=1}^n \left(D_j - i\sigma\frac{x_j}{\langle x\rangle}\right)^2 \tag{4.4}$$

and

$$L^* = i\frac{\partial}{\partial t} - \frac{1}{2}\sum_{j=1}^n \left(D_j + i\sigma\frac{x_j}{\langle x\rangle}\right)^2. \tag{4.5}$$

When w is a solution of the Cauchy problem (4.3) in the assumption, integration by parts implies

$$\begin{aligned}
&\int_0^T (Lv(t,\cdot), w(t,\cdot))_{L^2(\mathbb{R}^n)}\ dt \\
&= \int_0^T (v(t,\cdot), L^*w(t,\cdot))_{L^2(\mathbb{R}^n)}\ dt + i\left[(v(t,\cdot), w(t,\cdot))_{L^2(\mathbb{R}^n)}\right]_0^T \\
&= (v(T,\cdot), w(T,\cdot))_{L^2(\mathbb{R}^n)} - (v(0,\cdot), w(0,\cdot))_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

If v satisfies

$$\begin{cases} Lv(t,x) = 0 & \text{in } (0,T)\times\mathbb{R}^n \\ v(0,x) = 0 & \text{on } \mathbb{R}^n, \end{cases} \tag{4.6}$$

then we have

$$(v(T,\cdot), w_T) = 0 \tag{4.7}$$

for any w_T in $\hat{L}^2_\rho$, which is dense in $L^2(\mathbb{R}^n)$. Consequently, we obtain

$$v(T) = 0 \tag{4.8}$$

in $L^2(\mathbb{R}^n)$. The discussion on the time interval $[0,T]$ implies the result on $\mathbb{R}$ by repeating the same procedure. In fact, if the uniqueness has been proved on $[0,T]$, then we give $u(T)$ as an initial data to the Cauchy problem on $[T, 2T]$. The same is true for the negative direction. Finally, we can extend the time interval to $\mathbb{R}$ in which the solution to the Cauchy problem is unique. $\square$

Now that all preparation has been done, we can prove the uniqueness theorem. In order that we obtain the uniqueness statement of the solution, it is sufficient that the Cauchy problem (4.3) is wellposed in $\hat{L}^2_\rho$; that is, there exists a solution to the Cauchy problem (4.3) for any $w_T \in \hat{L}^2_\rho$, where ρ is a positive constant with $0 < \rho < \frac{1}{n\sqrt{6n}}$.

Proof of Theorem 1. Put $\tilde{w}(t) = e^{\nu(T-t)\langle D\rangle} w(T-t)$ for $\nu > 0$ with $\nu T = \rho$. Hence, we obtain

$$\begin{cases} i\dfrac{\partial}{\partial t}\tilde{w}(t,x) \\ = \Big\{ -\dfrac{1}{2}\triangle + \Big(i\sigma \displaystyle\sum_{j=1}^{n} \tilde{b}_j(x,D)D_j - i\nu\langle D\rangle \Big) + \tilde{c}(x,D) \Big\} \tilde{w}(t,x) \\ \tilde{w}(0,x) = e^{\nu T\langle D\rangle} w_T(x) \end{cases} \tag{4.9}$$

where

$$\tilde{b}_j(x,D) = e^{\nu(T-t)\langle D\rangle} \frac{x_j}{\langle x\rangle} e^{-\nu(T-t)\langle D\rangle} \tag{4.10}$$

and

$$\tilde{c}(x,D) = e^{\nu(T-t)\langle D\rangle} c(x) e^{-\nu(T-t)\langle D\rangle}. \tag{4.11}$$

It should be remarked that both $x_j/\langle x\rangle$ and $c(x)$ are analytic with convergence radius one. We can find the solution $\tilde{w} \in C^1([0,T]; L^2(\mathbb{R}^n))$ to the Cauchy problem (4.9) if we give initial data $e^{\nu T\langle D\rangle} w_T \in L^2(\mathbb{R}^n)$. In fact, since

$$\operatorname{Re}\Big(\tilde{b}_j(x,D)D_j\tilde{w}(t,\cdot), \tilde{w}(t,\cdot)\Big)_{L^2(\mathbb{R}^n)} \le B_j(\nu,T)\|\langle D\rangle^{\frac{1}{2}}\tilde{w}(t,\cdot)\|^2_{L^2(\mathbb{R}^n)} \tag{4.12}$$

for $j = 1, \ldots, n$, where $B_j(\nu, T)$ is a positive constant which depends on ν and T, we obtain the energy estimate from the equation of the Cauchy problem (4.9) such as

$$\|\tilde{w}(t,\cdot)\|_{L^2(\mathbb{R}^n)} \le e^{Ct}\|e^{\nu T\langle D\rangle} w_T\|_{L^2(\mathbb{R}^n)} \quad (0 \le t \le T) \tag{4.13}$$

for some positive constant C, when $\nu(T-t) \le \rho < 1$ (See Corollary 1) and $\sigma\sum_{j=1}^n B_j(\nu,T) \le \nu$. From Proposition 2,

$$\sum_{j=1}^{n} B_j(\nu,T) \le C'(1+\nu T)^{N_0} \tag{4.14}$$

holds for a sufficiently large constant $C' > 0$, therefore when we choose ν and T such that $\nu T = \rho$ and $2\sigma C'(1 + \nu T)^{N_0} \leq \nu$ at the beginning of our discussion, then the proof has just completed. □

5. Application

When Kajitani and the author have constructed the solution under certain conditions in [2, 3], it was not clear that the Cauchy problem for the Schrödinger equation is wellposed in weighted Sobolev spaces. The estimate for the solution makes sense when the uniqueness of the solution is assured, because the estimate should be valid for all of the solutions. In other words, the estimate for the solution with the property of exponential time decay is essentially based on the unique solution to the Cauchy problem (1.1), which agrees with the solution in $L^2(\mathbb{R}^n)$. Consequently, we are certain to estimate the solution with the property of exponential time decay.

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