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CLASSIFICATION OF SYMMETRIC VORTICES FOR THE GINZBURG-LANDAU EQUATION

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Abstract. This work proposes a description of the set of symmetric vortices, defined as specific solutions of the Ginzburg-Landau equations for a superconducting cylinder with applied magnetic field. It is conducted through a two parameters shooting procedure which relates the behaviour of a symmetric vortex at the center to its behaviour at the boundary. The main result is that, for a given degree d, the set of parameters for which such a "shooting" leads to a "response" – i.e. admissible values for the radius $\bar{r}$ of the cylinder and the intensity h of the magnetic field – is a bounded subset in $\mathbb{R}^2$. This shows in particular that, for large intensities of the applied magnetic field, normal states do not appear as a limit of superconducting vortices of given degree, and that symmetric vortices are not equilibrium states of the system for too large or too low intensities of the applied magnetic field. Moreover, a simpler proof for the existence of bifurcations (a model for phase transitions) from the normal state to superconducting states, as studied in [11], is provided.

1. Introduction

1.1. **Symmetric vortices for the Ginzburg-Landau model of superconductivity.** The problem of existence and uniqueness of equilibrium states in the Ginzburg-Landau model of superconductivity, and their mathematical behaviour, is not yet completely understood, except perhaps in the special case of thin films (cf. [7] to [10]. For the cylinder case, one can refer to the work of É. Sandier and S. Serfaty, for instance [20], which presents a more or less complete picture at the adiabatic limit.

In the case of a superconducting cylinder of radius $\bar{r}$ and infinite length, with applied magnetic field of intensity h parallel to its axis, the first example of nontrivial critical states for the Ginzburg-Landau model is due to [6], in the limit case $\bar{r} = +\infty$ and $h = 0$. Later, those critical states of [6] were

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shown to be stable for type I materials (the penetration constant κ of the material is less than $1/\sqrt{2}$) but unstable for type II materials ($\kappa > 1/\sqrt{2}$) (cf. [1] and [16]).

Later, the authors of [11] have proved the existence of branches of nontrivial superconducting critical states for specific values of $\bar{r}$, and the stability of such branches when $\bar{r}$ and $\kappa\bar{r}$ are large enough.

In this paper, we deal with the existence of critical states and equilibrium states for a superconducting cylinder, and more precisely we investigate the structure of the set of all such states for all possible values of the parameters $\bar{r}$ and h, which are symmetric vortices of given degree d.

Let us consider the model of superconductivity proposed by the physicists G.L. Ginzburg and L.D. Landau ([14]), for a superconducting cylinder of radius $\bar{r}$, with applied magnetic field of intensity h and direction $\mathbf{e_3}$ parallel to the axis of the cylinder : a wave function (or order parameter) $\psi \in H^1(B_{\bar{r}}, \mathbb{C})$ is paired with a rescaled magnetic potential $A \in H^1(B_{\bar{r}}, \mathbb{R}^2)$ in the free energy functional

$$\mathcal{G}_{\bar{r},h}(\psi, A) = \frac{1}{2}\int_{B_{\bar{r}}} |i\nabla\psi + A|^2 dx + \frac{\kappa^2}{4}\int_{B_{\bar{r}}} \left(1 - |\psi|^2\right)^2 dx + \frac{1}{2}\int_{B_{\bar{r}}} |\text{curl } A - \kappa h \mathbf{e}_3|^2 \, dx, \tag{1.1}$$

where κ is a dimensionless parameter depending on the material.

Critical states are solutions of the equation $D\mathcal{G}_{\bar{r},h}(\psi, A) = 0$. They must satisfy in $B_{\bar{r}}$ the Ginzburg-Landau system

$$[GL] \qquad \begin{cases} \left(i\nabla + A\right)^*\left(i\nabla + A\right)\psi - \kappa^2\psi\left(1 - |\psi|^2\right) = 0 \\ \text{curl}^*\text{curl } A + \left(i\nabla\psi + A\psi\right)\cdot\psi = 0 \end{cases}$$

(where the dot $\cdot$ in the second line stands for a real valued scalar product), together with boundary conditions

$$\left(i\nabla\psi + A\psi\right) \cdot \mathbf{n}_{|\partial B_{\bar{r}}} = 0 \qquad \text{curl } A_{|\partial B_{\bar{r}}} = \kappa h \mathbf{e}_3 \, . \tag{1.2}$$

($\mathbf{n}$ is the normal outward unit vector.)

Equilibrium states will be defined here as local minimizers for the Ginzburg-Landau free energy functional. In particular, they are solutions of the Euler-Lagrange equation $D\mathcal{G}_{\bar{r},h}(\psi, A) = 0$ for which the quadratic form $D^2\mathcal{G}_{\bar{r},h}(\psi, A)$ on $H^1(B_{\bar{r}}, \mathbb{C} \times \mathbb{R}^2)$ is nonnegative.

We shall restrict our attention to pairs (ψ, A) which are of radial form of degree d ($d \in \mathbb{N}^*$); i.e.,

$$\psi(r,\theta) = f(r)e^{id\theta} \qquad A(r,\theta) = a(r)\mathbf{v} \ \text{ with } \ \mathbf{v} = \left| \begin{matrix} -\sin\theta \\ \cos\theta \end{matrix} \right. \tag{1.3}$$

for some real-valued functions f and a. This is the form of the states which appear in the bifurcation branches of [11] as well as in the vortices constructed in [6].

Remark 1.1. For (ψ, A) of radial form (1.3), the boundary conditions (1.2) can be written

$$f'(\bar{r}) = 0\ , \quad b'(\bar{r}) + \kappa\bar{r}h = 0, \tag{1.4}$$

and will be referred to as *Neumann boundary conditions at* $(\bar{r}, h)$.

From now on, we suppose that the values of the parameter κ and the degree d are fixed, while $\bar{r}$ and h are allowed to take any value in $\mathbb{R}_+^*$.

Note first, for any value of $\bar{r}$ and h, the existence of a solution, called a *normal state*, of the system $[GL]$ with boundary conditions (1.2):

$$\psi(r,\theta) \equiv 0 \qquad\qquad A(r,\theta) = \frac{\kappa h r}{2}\mathbf{v}. \tag{1.5}$$

It corresponds to a nonsuperconducting state of the system.

It will be shown that, for small values of $\bar{r}$ and independently of h, the normal solution is the unique solution of radial form for the equation $D\mathcal{G}_{\bar{r},h}(\psi, A) = 0$ (cf. also [3]). Above a critical value $\bar{r}_\mu$ depending on κ and d, such uniqueness is lost, and nonnormal symmetric vortices do appear (cf. Corollary 5.3). So the natural question is :

For which values of $\bar{r}$ and h do there exist nonnormal critical states (respectively equilibrium states) for $\mathcal{G}_{\bar{r},h}$ which are of radial form (1.3)?

In order to answer it, we shall adopt a two parameters shooting method (cf. [10] for the one-parameter case of thin films), based on the following elementary observation. The parameters $\bar{r}$ and h do not appear in the system of equations $[GL]$, so that it is natural to:

1. as a first step, investigate the set of all maximal solutions of the system $[GL]$ which are of radial form on a maximal disk in $\mathbb{R}^2$;
2. as a second step, seek for which of those solutions there can exist values of the parameters $\bar{r}$ and h such that the boundary conditions (1.2) are satisfied;
3. finally, try to determine which pairs $(\bar{r}, h)$ can be obtained in this way.

The consideration of the first step leads to the following definition:

Definition 1.2. *For given κ, d and h, one calls a maximal solution (ψ, A) of the Ginzburg-Landau system $[GL]$ satisfying* (1.3) *a* symmetric vortex (*of degree d*).

We denote by $\rho(\psi, A)$ the radius of the maximal disk on which this maximal solution is defined.

For a given pair $(\bar{r}, h)$ such that $\bar{r} < \rho(\psi, A)$, a symmetric vortex is called an $(\bar{r}, h)$-Neumann symmetric vortex *if it is a critical state for $\mathcal{G}_{\bar{r},h}$.*

It is called a simple $(\bar{r}, h)$-Neumann symmetric vortex *if, in addition, the wave component ψ does not vanish on $B_{\bar{r}} \backslash \{0\}$.*

In particular, an $(\bar{r}, h)$-Neumann symmetric vortex satisfies the boundary conditions (1.2) on the circle of radius $\bar{r}$.

Let us recall the following result of [21]:

Proposition 1.3. *Let $\kappa > 0$ and $d \geq 1$ be fixed. Then the set of all symmetric vortices of degree d is endowed with a natural topology. Moreover, the map which to (ψ, A) $\left[\left(\psi(r,\theta) = f(r)e^{id\theta}, A(r,\theta) = a(r)\mathbf{v}\right)\right]$ associates the two real numbers $\left(\alpha = a'(0), c = \frac{1}{d!}f^{(d)}(0)\right)$ defines a homeomorphism between the topological set of symmetric vortices of degree d, and the real plane $\mathbb{R}^2$.*

The reverse homeomorphism will be denoted

$$\mathbb{R}^2 \ni (\alpha, c) \mapsto (\psi_{\alpha,c}, A_{\alpha,c}). \tag{1.6}$$

In other words, the set of symmetric vortices of given degree appears as a connected metrizable locally compact topological space, homeomorphic to the plane. One will note that this homeomorphism relates deeply the topological features of this set to the behaviour of symmetric vortices at the center of the vortex. The "shooting" will consist in passing from the data at the center to the values at the boundary.

Remark 1.4. According to Proposition 1.3. through the parametrization (1.6) any subset of symmetric vortices can be represented as a subset of the plane, and any branch of symmetric vortices, as those constructed in [11], can be drawn as a curve in $\mathbb{R}^2$.

Our first classification is established in the following theorem:

Theorem 1. *Let (ψ, A) be a symmetric vortex of degree d. Then the function $f(r) = |\psi|(r)$, defined on $[0, \rho)$, has one and only one of the seven following properties*:

(1) *Normal solutions:* $\rho = +\infty$ *and* $f \equiv 0$. *In this case, there is some* $\alpha \in \mathbb{R}$ *such that* $b(r) = d - \alpha r^2$, *for all* $r \in [0, +\infty)$.

(2) *BC-vortices:* $\rho = +\infty$ *and* $f' > 0$ *on* $\mathbb{R}_+^*$. *In this case one has* $0 < f(r) < 1$, *for all* $r > 0$, *and*
   - $\lim_{r\to+\infty} f(r) = 1$, $\lim_{r\to+\infty} f'(r) = 0$;
   - $b'(r) < 0$, $\forall r > 0$ *and* $\lim_{r\to\infty} b(r) = \lim_{r\to\infty} b'(r) = 0$;
   - *the coresponding vortex* (ψ, A), *with* $\psi(r,\theta) = f(r)e^{id\theta}$ *and* $A(r,\theta) = a(r)\mathbf{v}$, *has finite energy:* $\mathcal{G}_{\bar{r},h}(\psi, A) < +\infty$ *with* $\bar{r} = +\infty$ *and* $h = 0$ *in formula (1.1).*

(3) *Monotone blowing-up vortices or* $\mathcal{D}$*-vortices:* $\rho < +\infty$ *and* $f'(r) > 0$ *on* $(0, \rho)$. *In this case, one has* $\lim_{r\uparrow\rho} f(r) = +\infty$.

(4) *Neumann vortices of first kind or* $\mathcal{N}_1$*-vortices:* $f > 0$ *for* $r > 0$ *close to* 0 *and* f *vanishes on* $(0, \rho)$. *If* z *is the first strictly positive zero of* f, *then* f' *has exactly one zero in* $(0, z)$, *denoted* $\widetilde{R}$.

(5) *Neumann vortices of second kind or* $\mathcal{N}_2$*-vortices:* $\rho = +\infty$, $f > 0$ *for* $r > 0$ *and* f' *vanishes on* $(0, +\infty)$. *Then* f' *has exactly one zero on* $\mathbb{R}_+^*$, *which will be denoted* $\widetilde{R}$. f *is increasing on* $[0, \widetilde{R}]$, *decreasing on* $[\widetilde{R}, +\infty)$, *and tends to* 0 *as* r *tends to* $+\infty$.

(6) *Neumann vortices of third kind or* $\mathcal{N}_3$**-vortices:** $\rho < +\infty$, $f > 0$ *on* $(0, \rho)$ *and* f' *vanishes exactly twice on* $(0, \rho)$. *Its first zero will be denoted* $\widetilde{R}$, *its second zero* $\widetilde{\widetilde{R}}$; f *increases on* $[0, \widetilde{R}]$ *and* $[\widetilde{\widetilde{R}}, \rho)$, *decreases on* $[\widetilde{R}, \widetilde{\widetilde{R}}]$. *One has* $\lim_{r\uparrow\rho} f(r) = +\infty$.

(7) *Neumann vortices of fourth kind or* $\mathcal{N}_4$**-vortices:** $\rho < +\infty$, *with* $\lim_{r\uparrow\rho} f(r) = +\infty$; f' *is nonnegative and has exactly one zero on* $(0, \rho)$, *denoted* $\widetilde{R}$, *such that* $f''(\widetilde{R}) = 0$.

The second step consists in seeking, for any given symmetric vortex (ψ, A), suitable values of the parameters $\bar{r}$ and h such that (ψ, A) is a critical state of $\mathcal{G}_{\bar{r},h}$ (provided such values exist).

From Theorem 1, one sees that, for a given ψ, there are at most two possible values for $\bar{r}$ such that $\partial_{\mathbf{n}}\psi_{|B_{\bar{r}}} = 0$, corresponding to local extrema or inflexion points of $|\psi|$, and such that ψ does not vanish on $B_{\bar{r}}\backslash\{0\}$: there is no such value in Cases 1, 2 and 3 of the theorem, one value in Cases 4, 5 and 7, two possible values for $\bar{r}$ in Case 6.

Figure 1 provides an illustration of Theorem 1. The possible values for $\bar{r}$, when they exist, are denoted $\widetilde{R}$ and (in Case 6) $\widetilde{\widetilde{R}}$.

Consider now the set $\mathcal{N}$ of the simple symmetric vortices of degree d, defined as the union of all admissible $(\bar{r}, h)$ of the simple $(\bar{r}, h)$-Neumann

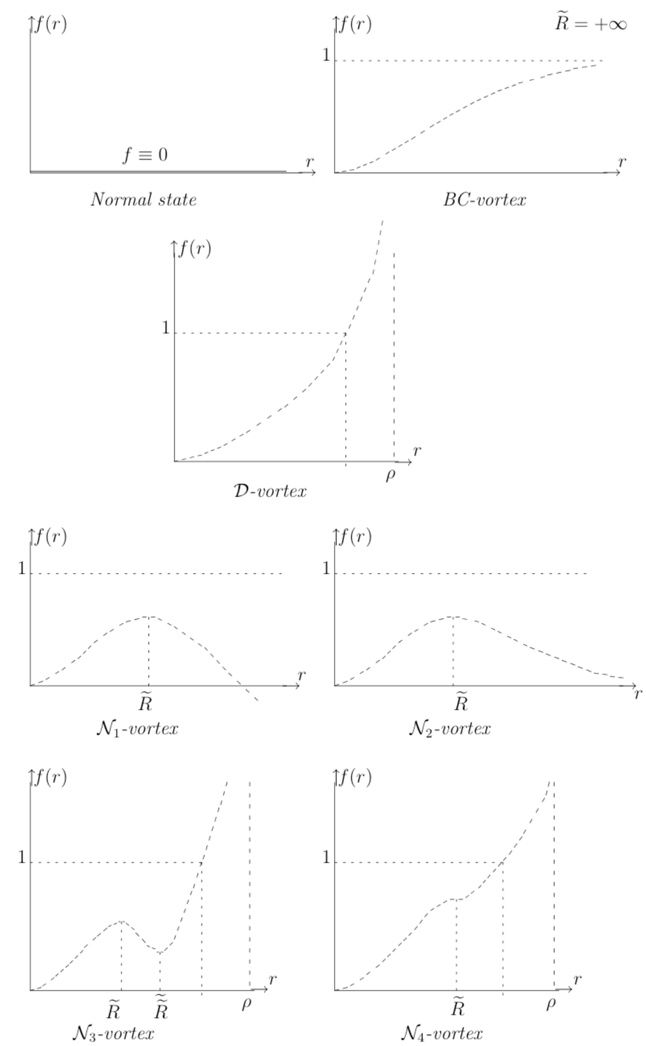

Figure 1. The seven possible behaviours for $f = |\psi|$.

symmetric vortices (Definition 1.2). From the physical point of view this set corresponds to the relevant vortices.

From a mathematical point of view, our main result will be that this set is relatively compact in the topological set of all symmetric vortices. Notation is those of Proposition 1.3 and (1.6).

Theorem 2. *Let κ and d be given. Then the set*

$$\mathcal{N} = \left\{(\alpha, c) \in (\mathbb{R}_+^*)^2 : \exists(\bar{r}, h) \in \mathbb{R}_+^2 : D\mathcal{G}_{\bar{r},h}(\psi_{\alpha,c}, A_{\alpha,c}) = 0\right\} \tag{1.7}$$

is a bounded subset of $\mathbb{R}_+^2$.

We draw a picture of the set $\mathcal{N}$. According to Remark 1.4, it is drawn as a subset of the plane, each symmetric vortex (ψ, A) being represented by the associated parameters (α, c) of Proposition 1.3.

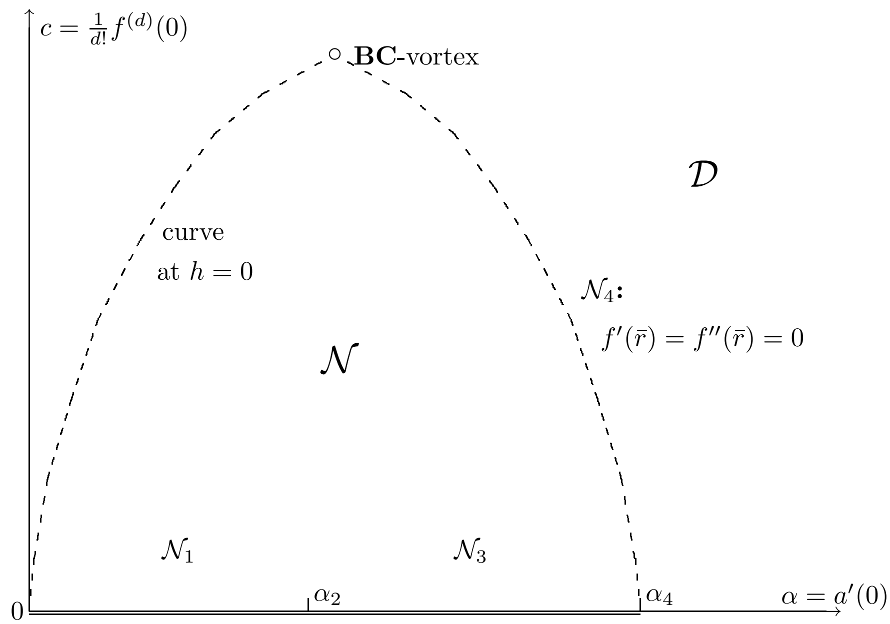

Figure 2. Configuration of $\mathcal{N}$.

This picture summarizes the first results of this work. To make it simpler, we have supposed that the BC-vortex is unique. (By a result of [1], this will be the case whenever the inequality $\kappa^2 \geq 2d^2$ is satisfied.)

The boundary curve at $h = 0$ is the set

$$\{(\alpha, c) \in (\mathbb{R}_+^*)^2 : \exists\bar{r},\ D\mathcal{G}_{\bar{r},0}(\psi_{\alpha,c}, A_{\alpha,c}) = 0\}.$$

It is a connected smooth analytic curve.

The boundary curve $\mathcal{N}_4$ is the part of the boundary consisting of $\mathcal{N}_4$-vortices of Theorem 1. It is an analytic curve which, for simplicity, is drawn as a smooth curve, as we show that it is in a neighbourhood of $(\alpha_4, 0)$.

The notation $\mathcal{N}_1$, $\mathcal{N}_3$ and $\mathcal{D}$ refer to the terminology of Theorem 1.

The horizontal axis $(c = 0)$ corresponds to normal states. The thicker segment $0 \leq \alpha \leq \alpha_4$ corresponds to normal states which can be reached as a limit of nonnormal simple symmetric vortices of degree d (cf. Consequence 3 below).

As a consequence of Theorem 2, one can provide some answers to the questions asked above about the pairs $(\bar{r}, h)$ which can be obtained as associated with symmetric vortices of given degree d :

1. There exists a minimal value $\bar{r}_\mu > 0$ of $\bar{r}$ such that, if a nonnormal symmetric vortex (ψ, A) is a critical state for $\mathcal{G}_{\bar{r},h}$, then $\bar{r} \geq \bar{r}_\mu$.

2. Assuming $\kappa > 1/\sqrt{2}$, there exists a minimal value h_μ and a maximal value h_M $(0 < h_\mu < h_M < +\infty)$ such that, if a nonnormal symmetric vortex (ψ, A) is an equilibrium state for $\mathcal{G}_{\bar{r},h}$, one has then $h_\mu \leq h \leq h_M$ (cf. Corollary 5.3 and Proposition 8.1).

3. The set of normal states which can appear as a limit of superconducting symmetric vortices corresponds to a bounded interval of values for h.

The last point is an important fact since one expects bifurcation branches of superconducting symmetric vortices to be issued from normal states for specific values of $\bar{r}$ and h: those bifurcation branches, which modelize a change of superconducting phase, have been intensively studied in [11], but they are also rather easy to build in our frame (cf. Section 4 and Remark 4.9).

1.2. **Organization of the paper.** Section 2 is devoted to the possible behaviours of the function $f(r) = |\psi|(r)$, which are of seven kinds (assuming $f \geq 0$ near 0) as explained in Theorem 1.

Section 3 is devoted to the first topological features of the set $\mathcal{N}$ of simple symmetric vortices considered in Theorem 2. In particular, it is shown that Cases 4, 5 and 6 of Theorem 1 correspond to (ψ, A) which are interior points of $\mathcal{N}$. In Section 4, the behaviour of $f = |\psi|$ is studied when the symmetric vortex is close to a normal state. The result is, for given κ and d, the existence of two limit values h_2 and h_4, $0 < h_2 < h_4$, such that for $h \in (0, h_2)$ (respectively $h \in (h_2, h_4)$, respectively $h > h_4$) and (ψ, A) close enough to the corresponding normal state, then (ψ, A) is a $\mathcal{N}_1$-vortex (respectively a $\mathcal{N}_3$-vortex, respectively a $\mathcal{D}$-vortex). Section 5 contains the main result of the paper, namely that the set $\mathcal{N}$ has a compact closure. What is shown is that the set of parameters (α, c) in $\mathbb{R}^2$, corresponding to $\mathcal{N}$ through the homeomorphism of Proposition 1.3, is a bounded subset of the plane. The first consequences of this boundedness are explored :

- behaviour of symmetric vortices for large values of $\bar{r}$;

- boundedness of the set of values of h such that the normal solution can be reached as a limit of superconducting vortices;

- reinterpretation, in terms of analytic curves in the plane, of the transition from the normal state to superconducting states, as observed by physicists and mathematically explored by [11];

- uniqueness of a radial critical state for small values of $\bar{r}$ (cf. [3]).

Relative compactness of $\mathcal{N}$ focuses the attention on its boundary $\partial\mathcal{N}$, which corresponds to four kinds of states:

a) Normal states corresponding to h in the bounded interval $[0, h_4]$.

b) BC-vortices: criteria for reaching them as limits of elements in $\mathcal{N}$ are given in Section 5.

c) Vortices corresponding to boundary conditions (1.2) with $h = 0$: when $\bar{r}$ varies, this defines in the plane of (α, c)'s a smooth analytic curve studied in Section 6.

d) (ψ, A)'s corresponding to $\mathcal{N}_4$-vortices. This defines also an analytic curve, studied in Section 7 of the paper.

In the last section (Section 8), it is shown that, for type II materials ($\kappa > 1/\sqrt{2}$), there cannot exist symmetric vortices which are equilibrium states for $\mathcal{G}_{\bar{r},h}$, either for small values or for large values of h.

2. A classification of symmetric vortices.

Setting $b(r) = d - ra(r)$ one checks easily that symmetric vortices (ψ, A), $\psi(r,\theta) = f(r)e^{id\theta}$, $A(r,\theta) = a(r)\mathbf{v}$, are in one-to-one correspondance with solutions (f, b), defined on some maximal interval $[0, \rho)$, of the system of ordinary equations

$$\begin{cases} f''(r) + \frac{1}{r}f'(r) = f(r)\Big(\frac{b(r)^2}{r^2} - \kappa^2(1 - f(r)^2)\Big) \\ b''(r) - \frac{1}{r}b'(r) = b(r)f(r)^2 \\ f(0) = 0, \quad b(0) = d, \end{cases} \tag{2.1}$$

while the boundary conditions (1.2) are equivalent to

$$f'(\bar{r}) = 0 \qquad \frac{b'(\bar{r})}{\bar{r}} = -\kappa h\ . \tag{2.2}$$

This section is focused on the proof of Theorem 1 stated in Section 1. Its proof requires a preliminary lemma:

Lemma 2.1. *The assumptions are those of the previous proposition.*

1. *On the interval* $(0,\rho)$*, the function* $r \to b(r)/r$ *has three possible behaviours*

- *either it is positive decreasing:* $(\frac{b(r)}{r})' < 0$, $\forall r \in (0,\rho)$*;*

- *or it reaches a unique minimum at* $r_0 \in (0,\rho)$*; in this case it is positive on* $(0,\rho)$*; one has* $(\frac{b(r)}{r})' < 0$ *on* $(0,r_0)$ *and* > 0 *on* (r_0,ρ)*;*

- *or it has a unique zero* $z > 0$*; in this case, one has* $(\frac{b(r)}{r})' < 0$ *on* $(0,\rho)$*.*

2. *On the interval* $(0,\rho)$*, the function* $r \to b(r)^2/r^2$ *has two possible behaviours:*

- *either it decreases:* $(\frac{b(r)^2}{r^2})' < 0$, $\forall r \in (0,\rho)$ *;*

- *or there exists* $r_0 \in (0,\rho)$ *such that* $(\frac{b(r)^2}{r^2})'$ *is* < 0 *on* $(0,r_0)$ *and* > 0 *on* (r_0,ρ)*.*

Proof of Lemma 2.1. The function $\gamma(r) = b(r)/r$ satisfies the differential equation

$$\gamma''(r) + \frac{1}{r}\gamma'(r) = \gamma(r)\big(f(r)^2 + \frac{1}{r^2}\big)\ . \tag{2.3}$$

By the maximum principle, it will have no positive maximum and no negative minimum.

For small r, γ is positive decreasing, so that γ has the three possible behaviours of the first alternative, which provides the two possible behaviours for γ^2, with $r_0 = z$ in the case when γ vanishes. □

Proof of Theorem 1. As f is analytic, one has either $f(r) = 0$, for all r, or $f(r) > 0$ for $r > 0$ close to 0. In the first situation, corresponding to item 1 of the theorem, the vortex (ψ, A) corresponding to (f,b) is a normal solution, with $a(r) = \alpha r$ and $b(r) = d - \alpha r^2$ for some $\alpha \in \mathbb{R}$.

In the second situation, one has $f'(r) > 0$ for $r > 0$ close to 0. There are then two possibilities: f' vanishes or does not vanish on $(0,\rho)$.

Suppose first that f' vanishes on $(0,\rho)$. Let $\widetilde{R}$ be the first strictly positive zero of f'; one has $f''(\widetilde{R}) \leq 0$. We shall distinguish two cases.

Case 1: $f''(\widetilde{R}) < 0$. For $r > \widetilde{R}$ close to $\widetilde{R}$, one has $f'(r) < 0$ and f decreases. There are then three possibilities beyond $\widetilde{R}$:

- Case 1.1: f' remains < 0 until f vanishes in turn, and we get a vortex as in item 4 of the theorem.

- Case 1.2: f' remains < 0 and f remains > 0 on its domain. Then, by results of [21] (Section 3), $\rho = +\infty$; i.e., f is defined on $\mathbb{R}_+$, bounded above by 1. Hence, f has a decreasing limit $\ell \in [0,1)$ when r tends to $+\infty$ and we claim that $\ell = 0$.

To prove it, note that b has two possible behaviours at infinity (cf. [21], Proposition 3.4): either it tends to 0, or it tends to $\pm\infty$. In the first case, one would have, when r tends to infinity, $(rf'(r))' \sim -r\kappa^2\ell(1-\ell)$ which, if $\ell \neq 0$, would imply $rf'(r) < -\delta r^2$ for some constant $\delta > 0$ and r large enough, hence $\lim_{r\to+\infty} f(r) = -\infty$ and gives a contradiction. In the second case, one has $|b(r)| \geq \delta r^2$ for some $\delta > 0$ and r large enough, and a similar argument leads to $\lim_{r\to+\infty} f(r) = +\infty$ and a contradiction.

Finally, we have $\ell = 0$ and a vortex corresponding to item 5 of the theorem.

- Case 1.3: f' has a second strictly positive zero $\widetilde{\widetilde{R}}$ and f is positive (hence strictly positive) on $(0, \widetilde{\widetilde{R}}]$.

We claim first $(\frac{b(r)^2}{r^2})' > 0$ at $r = \widetilde{\widetilde{R}}$. For if this was not true, then by Lemma 2.1 $\frac{b(r)^2}{r^2}$ would decrease on $[\widetilde{R}, \widetilde{\widetilde{R}}]$; as f is positive strictly decreasing on $(\widetilde{R}, \widetilde{\widetilde{R}})$, and $\frac{b(r)^2}{r^2} - \kappa^2(1-f^2)$ negative at $r = \widetilde{R}$, one would have $\frac{b(r)^2}{r^2} - \kappa^2(1-f^2) < 0$ on $(\widetilde{R}, \widetilde{\widetilde{R}})$, hence $rf'(r)' < 0$ on $(\widetilde{R}, \widetilde{\widetilde{R}})$. This gives a contradiction.

We claim now $f''(\widetilde{\widetilde{R}}) > 0$. Otherwise, one would have $f''(\widetilde{\widetilde{R}}) = 0$. Then, at $r = \widetilde{\widetilde{R}}$, $\frac{b(r)^2}{r^2} - \kappa^2(1-f^2)$ would be equal to 0 and would have a strictly positive derivative; hence, for $r < \widetilde{\widetilde{R}}$ close to $\widetilde{\widetilde{R}}$, one would have $(rf'(r))' < 0$ and $f'(r) > 0$. This contradicts the fact that $f'(r) < 0$ for $r > \widetilde{R}$ close to $\widetilde{R}$, and that $\widetilde{\widetilde{R}}$ is the first zero of f' beyond $\widetilde{R}$.

We have proved $f''(\widetilde{\widetilde{R}}) > 0$, which implies $f'(r) > 0$ for $r > \widetilde{\widetilde{R}}$ close to $\widetilde{\widetilde{R}}$. Let us show that f' does not vanish beyond $\widetilde{\widetilde{R}}$. Let R_3 be, if it exists, the first zero of f' which is strictly greater than $\widetilde{\widetilde{R}}$; on $(\widetilde{\widetilde{R}}, R_3]$, $\frac{b(r)^2}{r^2} - \kappa^2(1-f^2)$ is increasing, hence positive (since f^2 is increasing, and $b(r)^2/r^2$ is also increasing by the first claim above and Lemma 2.1) ; $rf'(r)$ increases, and we get a contradiction. So, $\widetilde{\widetilde{R}}$ is the last zero of f'.

As $\frac{b(r)^2}{r^2} - \kappa^2(1-f^2)$ and f increase beyond $\widetilde{\widetilde{R}}$, there will exist some $\delta > 0$ such that $(rf'(r))' \geq \delta r$ for r large enough, so that f reaches arbitrarily large values. By [21], Proposition 3.2, this implies that f reaches $+\infty$ at a finite distance, and we have got a vortex as in item 6 of the theorem.

Case 2: $f''(\widetilde{R}) = 0$. We have then $\frac{b(r)^2}{r^2} - \kappa^2(1-f^2) = 0$ at $r = \widetilde{R}$, and $f(r) = f(\widetilde{R}) + O((r-\widetilde{R})^3)$ near $\widetilde{R}$. Let $\beta = (\frac{b(r)^2}{r^2})'$ at $r = \widetilde{R}$.

Suppose first $\beta < 0$. Then one has $\frac{b(r)^2}{r^2} - \kappa^2(1-f^2) > 0$, $(rf'(r))' > 0$ and $f'(r) < 0$ for $r < \widetilde{R}$ close to $\widetilde{R}$, which contradicts the definition of $\widetilde{R}$ as the first strictly positive zero of f', since f' is positive for small r.
Suppose then $\beta = 0$. As $f(\widetilde{R}) < 1$, $\kappa^2(1-f(r)^2) = \frac{b(r)^2}{r^2}$ is not equal to 0. Referring to the proof of Lemma 2.1, $\widetilde{R}$ will be a strictly positive minimum for $\frac{b(r)}{r}$ and we shall have $(\frac{b(r)^2}{r^2})'' > 0$ at $r = \widetilde{R}$. From this we deduce that for $r < \widetilde{R}$ close to $\widetilde{R}$, one has $\frac{b(r)^2}{r^2} - \kappa^2(1-f^2) > 0$, hence $(rf'(r)' > 0$ and $f'(r) < 0$, which leads to the same contradiction.

So we have $\beta > 0$, and $f'''(\widetilde{R}) > 0$. Then, for $r > \widetilde{R}$ close to $\widetilde{R}$ we have $\frac{b(r)^2}{r^2} - \kappa^2(1-f^2) > 0$, $(rf'(r))' > 0$ and $f'(r) > 0$. One concludes, as in the analysis of the third possibility, that $\widetilde{R}$ is the last (and unique) zero of f', that f reaches arbitrarily large values and blows up at a finite distance. We have got a vortex as in item 7 of the theorem.

At that point, we have exhausted all the cases where f' vanishes at some $\widetilde{R} > 0$.
Suppose now $f'(r) > 0$ for all $r \in (0, \rho)$. If $\rho < +\infty$, then f tends to $+\infty$ as r tends increasingly to ρ, and we are in the situation of item 3 of the theorem.

The last case $f' > 0$ and $\rho = +\infty$, corresponding to item 2 of the theorem, is considered in [13] where it is shown that the assumptions $\rho = +\infty$, $f' \geq 0$ and f not identically 0 lead to $f'(r) > 0$ for all $r > 0$, f increasing from 0 to 1 and b decreasing from d to 0 as r runs from 0 to $+\infty$, with $\lim_{r\to+\infty} b'(r) = \lim_{r\to+\infty} rf'(r) = 0$ and $\mathcal{G}(\psi, A) < +\infty$.

This ends the proof of the theorem. □

Remark 2.2. The picture is the one drawn in Section 1 (Fig. 1). It will be an implicit consequence of the results of the next sections that the seven kind of vortices of Theorem 1 actually do exist (cf. Remark 4.8). Normal vortices satisfy boundary conditions (2.2) for $h = 2\alpha/\kappa$ and any value of $\bar{r}$. BC-vortices appear as the limit case $\bar{r} = +\infty$ and $h = 0$.

3. Topology on the set of symmetric vortices.

Let us recall in the next proposition ([21], Proposition 2.1), how symmetric vortices of given degree are parametrized by the plane $\mathbb{R}^2$. This will imply immediately Proposition 1.3.

Proposition 3.1. 1. *For any* $(\alpha, c) \in \mathbb{R}^2$*, there exists a unique solution* $(f_{\alpha,c}, b_{\alpha,c})$ *of system* (2.1)*, defined on a maximal interval* $[0, \rho_{\alpha,c})$ $(\rho_{\alpha,c} \in$

$(0, +\infty])$ *having expansion near* 0 *given by*

$$f_{\alpha,c}(r) = cr^d + O(r^{d+2}) \qquad b_{\alpha,c}(r) = d - \alpha r^2 + O(r^{2d+2})\,. \tag{3.1}$$

2. *The map* $(\alpha, c) \to \rho_{\alpha,c}$ *is l.s.c., and the map* $(\alpha, c, r) \to (f_{\alpha,c}(r), b_{\alpha,c}(r))$ *is analytic on the open set* $\{(\alpha, c, r) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}_+ : r < \rho_{\alpha,c}\}$.

3. *For any symmetric vortex* (ψ, A)*, there exists a unique* $(\alpha; c)$ *such that the corresponding pair* (f, b) *coincides with* $(f_{\alpha,c}, b_{\alpha,c})$.

In other words, for a given degree d and parameter κ, symmetric vortices are in one-to-one correspondance with pairs $(\alpha, c) \in \mathbb{R}^2$ provided by their behaviour at the center of the vortex.

The consideration of the three facts

- parametrization of symmetric vortices in Proposition 3.1,
- boundary conditions (2.2) at the beginning of section 2,
- classification in Theorem 1,

leads to the introduction of the following domains:

Definition 3.2. Let us define the following subsets of $\mathbb{R}_+^* \times \mathbb{R}_+^*$:

$$\mathcal{D} = \{(\alpha, c) \in \mathbb{R}_+^* \times \mathbb{R}_+^* : \rho_{\alpha,c} < +\infty \text{ and } f'_{\alpha,c}(r) > 0\ \forall\, r > 0\}$$

$$\widetilde{\mathcal{N}} = \left\{(\alpha, c) \in \mathbb{R}_+^* \times \mathbb{R}_+^* : \exists \bar{r} > 0 : f'_{\alpha,c}(\bar{r}) = 0 \right\}$$

$$\mathcal{N} = \left\{(\alpha, c) \in \widetilde{\mathcal{N}} : \exists \bar{r}, h > 0 : \begin{matrix} f'_{\alpha,c}(\bar{r}) = 0\ b'_{\alpha,c}(\bar{r}) = -\kappa h \bar{r} \\ f_{\alpha,c} > 0 \text{ on } (0, \bar{r}] \end{matrix}\right\}. \tag{3.2}$$

For j=1,...4 , $\widetilde{\mathcal{N}}_j$ (respectively $\mathcal{N}_j$) is defined as the subset of (α, c)'s in $\widetilde{\mathcal{N}}$ (respectively $\mathcal{N}$) such that the corresponding vortex is an $\mathcal{N}_j$-vortex. For $(\alpha, c) \in \widetilde{\mathcal{N}}$, one will denote by $\widetilde{R}(\alpha, c)$ the first strictly positive zero of $f'_{\alpha,c}$. For $(\alpha, c) \in \widetilde{\mathcal{N}}_3$, one will denote by $\widetilde{\widetilde{R}}(\alpha, c)$ the second strictly positive zero of $f'_{\alpha,c}$.

Remark 3.3. $\mathcal{N}$ is the set of (α, c)'s such that the corresponding vortex satisfies the boundary conditions (1.2) for some pair of parameters $(\bar{r}, h)$. Referring to the homeomorphism (1.6) of Proposition 1.3 and conditions (2.2), we say for short that $\mathcal{N}$ is *the set of symmetric vortices with Neumann conditions*.

Notice that, for (α, c) in $\mathcal{N}_1$, $\mathcal{N}_2$ or $\mathcal{N}_4$, the associated pair $(\bar{r}, h)$ is unique: $\bar{r} = \widetilde{R}(\alpha, c)$, $h = -\frac{b'_{\alpha,c}(\bar{r})}{\kappa \bar{r}}$; while, for $(\alpha, c) \in \mathcal{N}_3$, there are two possible pairs $(\bar{r}, h)$: $\bar{r} = \widetilde{R}(\alpha, c)$ or $\bar{r} = \widetilde{\widetilde{R}}(\alpha, c)$, with again $h = -\frac{b'_{\alpha,c}(\bar{r})}{\kappa \bar{r}}$.

The next proposition presents the first and easier properties of those sets:

Proposition 3.4. 1. *The sets* $\mathcal{D}$, $\widetilde{\mathcal{N}}_1$, $\widetilde{\mathcal{N}}_3$, $\widetilde{\mathcal{N}}_1 \cup \widetilde{\mathcal{N}}_2 \cup \widetilde{\mathcal{N}}_3$, $\mathcal{N}_1$, $\mathcal{N}_3$ *and* $\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$ *are open subsets of* $\mathbb{R}_+^* \times \mathbb{R}_+^*$.

2. $\widetilde{\mathcal{N}}_3 \cup \widetilde{\mathcal{N}}_4$ *is an open subset of* $\widetilde{\mathcal{N}}$; $\mathcal{N}_3 \cup \mathcal{N}_4$ *is an open subset of* $\mathcal{N}$.

3. $\widetilde{\mathcal{N}}_2$, $\widetilde{\mathcal{N}}_4$ *and* $\widetilde{\mathcal{N}}_1 \cup \widetilde{\mathcal{N}}_2$ *are closed subsets of* $\widetilde{\mathcal{N}}$; $\mathcal{N}_2$, $\mathcal{N}_4$ *and* $\mathcal{N}_1 \cup \mathcal{N}_2$ *are closed subsets of* $\mathcal{N}$.

4. *The map* $(\alpha, c) \to \widetilde{R}(\alpha, c)$ *is continuous on* $\widetilde{\mathcal{N}}$. *It is analytic on* $\widetilde{\mathcal{N}}_1 \cup \widetilde{\mathcal{N}}_2 \cup \widetilde{\mathcal{N}}_3 = \widetilde{\mathcal{N}} \backslash \widetilde{\mathcal{N}}_4$.

5. *The map* $(\alpha, c) \to \widetilde{\widetilde{R}}(\alpha, c)$ *is analytic on* $\widetilde{\mathcal{N}}_3$.

The map $(\alpha, c) \to \begin{cases} \widetilde{\widetilde{R}}(\alpha, c) & \text{if } (\alpha, c) \in \mathcal{N}_3 \\ \widetilde{R}(\alpha, c) & \text{if } (\alpha, c) \in \mathcal{N}_4 \end{cases}$ *is continuous on* $\mathcal{N}_3 \cup \mathcal{N}_4$.

Proof. The proof relies on Theorem 1, together with these three principles:

(a) Given $(\alpha_0, c_0) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$, there will exist some $r_0 > 0$ such that, for (α, c) close to (α_0, c_0) and $r \in (0, r_0]$, one has $f'_{\alpha,c}(r) > 0$. (This can be easily deduced from Lemma 4.2 (25) of [21].)

(b) If R is such that $f_{\alpha,c}(R) > 1$, one has $f'_{\alpha,c}(r) > 0$ for $r \geq R$ and $f_{\alpha,c}$ blows up at a finite distance: $\rho(\alpha, c) = +\infty$ (cf. Proposition 3.2. of [21]).

(c) On any interval $[r_0, R]$, with $R < \rho(\alpha_0, c_0)$, $f_{\alpha,c}$ converges uniformly to f_{α_0,c_0} and $f'_{\alpha,c}$ converges uniformly to f'_{α_0,c_0} as (α, c) tends to (α_0, c_0) (cf. Proposition 3.1).

From (a) and (c), one deduces that if $f_{\alpha_0,c_0}(r)$ (respectively f'_{α_0,c_0}) is > 0 for $0 < r \leq R$, then $f_{\alpha,c}(r)$ (respectively $f'_{\alpha,c}(r)$) is > 0 for $0 < r \leq R$ and (α, c) in a neighbourhood of (α_0, c_0).

Let $(\alpha_0, c_0) \in \mathcal{D}$, r_0 be as in (a) above and R such that $f_{\alpha_0,c_0}(R) > 1$. For (α, c) close to (α_0, c_0), one has first that $f'_{\alpha,c} > 0$ on $(0, R]$ as just said, and then that $f_{\alpha,c}(R) > 1$ and consequently, by principle (c), $f'_{\alpha,c}(r) > 0$ for $r \geq R$. We have proved that $\mathcal{D}$ is open.

Let $(\alpha_0, c_0) \in \widetilde{\mathcal{N}} \backslash \widetilde{\mathcal{N}}_4$. By Theorem 1, one has $f''_{\alpha_0,c_0}(\widetilde{R}(\alpha_0, c_0)) \neq 0$, and the implicit function theorem provides a neighbourhood $\mathcal{W}$ of (α_0, c_0), some $\varepsilon > 0$ and an analytic map $\mathcal{W} \ni (\alpha, c) \to \bar{r}(\alpha, c)$ such that

1. $f'_{\alpha,c}(\bar{r}(\alpha, c)) = 0$,
2. $f'_{\alpha,c}(r) \neq 0$ for $(\alpha, c) \in \mathcal{W}$, $|r - \widetilde{R}(\alpha_0, c_0)| \leq \varepsilon$ and $r \neq \bar{r}(\alpha, c)$.

As already noticed, for (α, c) close to (α_0, c_0) one has $f'_{\alpha,c} > 0$ on $(0, \widetilde{R}(\alpha_0, c_0) - \varepsilon]$. By 2., one has also $f'_{\alpha,c} > 0$ on $[\widetilde{R}(\alpha_0, c_0) - \varepsilon, \bar{r}(\alpha, c))$, so that $\bar{r}(\alpha, c)$ is the first zero of $f'_{\alpha,c}$. This means $\bar{r}(\alpha, c) = \widetilde{R}(\alpha, c)$. By continuity, one has

$f''_{\alpha,c}(\widetilde{R}(\alpha,c)) \neq 0$. We have shown that $\widetilde{\mathcal{N}} \backslash \widetilde{\mathcal{N}}_4 = \widetilde{\mathcal{N}}_1 \cup \widetilde{\mathcal{N}}_2 \cup \widetilde{\mathcal{N}}_3$ is an open subset of $\mathbb{R}_+^* \times \mathbb{R}_+^*$, on which $\widetilde{R}(\alpha,c)$ is analytic.

Note that $\mathcal{N}$ is defined as the set of (α,c) in $\widetilde{\mathcal{N}}$ such that $b'_{\alpha,c}(\widetilde{R}(\alpha,c)) < 0$, so that $\mathcal{N}_1 \cup \mathcal{N}_2 \cup \mathcal{N}_3$ is also an open subset of $\mathbb{R}_+^* \times \mathbb{R}_+^*$.

The fact that $\widetilde{\mathcal{N}}_3$ and $\mathcal{N}_3$ are open and that $\widetilde{\widetilde{R}}(\alpha,c)$ is analytic is proved with a similar argument.

$\widetilde{\mathcal{N}}_1$ and $\mathcal{N}_1$ are open, since they are defined by the condition : there exists $r > 0$ such that $f_{\alpha,c}(r) < 0$.

We have proved item 1 of the Proposition, and the analyticity assumptions in items 4 and 5. Let us prove now the continuity assumptions in 4 and 5.

Suppose $(\alpha_0, c_0) \in \widetilde{\mathcal{N}}_4$ and $f''_{\alpha_0,c_0}(R_0) = 0$. Let (α_n, c_n) be a sequence in $\widetilde{\mathcal{N}}$ with limit (α_0, c_0) ; one must show that $\widetilde{R}(\alpha_n, c_n)$ tends to R_0.

By principle (a), there will exist some $r_0 > 0$ such that $\widetilde{R}(\alpha_n, c_n) \geq r_0$ for n large enough. Moreover, one cannot have $\lim_{n\to+\infty} \widetilde{R}(\alpha_n, c_n) = +\infty$, for, if this was true, one would have $\rho(\alpha_n, c_n) \to +\infty$ and $\rho(\alpha_0, c_0) = +\infty$, which contradicts the properties of an $\mathcal{N}_4$-vortex established in Theorem 1. Similarly, no subsequence can tend to $+\infty$, so that $\widetilde{R}(\alpha_n, c_n)$ is a bounded sequence.

Any accumulation point R of this sequence will satisfy $R \geq r_0 > 0$ and $f'_{\alpha_0,c_0}(R) = 0$, which implies $R = R_0$ since, by Theorem 1, R_0 is the only strictly positive zero of f'_{α_0,c_0}. Thus, continuity of $\widetilde{R}(\alpha,c)$ in 4. of the proposition is proved.

The fact that $\widetilde{\widetilde{R}}(\alpha_n, c_n)$ tends to $\widetilde{R}(\alpha,c)$ if (α_n, c_n) is a sequence in $\mathcal{N}_3$ with limit (α,c) in $\mathcal{N}_4$, is proved with similar arguments.

All the remaining assumptions can be proved easily in a similar way. For instance, $\widetilde{\mathcal{N}}_1$ and $\mathcal{N}_1$ are open since they are defined by the condition that there exists $r_0 > 0$ such that $f_{\alpha,c}(r_0) < 0$. Closedness of $\widetilde{\mathcal{N}}_4$ in $\widetilde{\mathcal{N}}$ and of $\mathcal{N}_4$ in $\mathcal{N}$ comes from the condition $f''_{\alpha,c}(\widetilde{R}(\alpha,c)) = 0$. Closedness of $\widetilde{\mathcal{N}}_2$ in $\widetilde{\mathcal{N}}$ or $\mathcal{N}_2$ can be easily deduced from the three conditions $f_{\alpha,c} \geq 0$, $\rho(\alpha,c) = +\infty$ and the fact that there exists r_0 such that $f'_{\alpha,c}(r_0) < 0$. For the closedness of $\widetilde{\mathcal{N}}_1 \cup \widetilde{\mathcal{N}}_2$ in $\widetilde{\mathcal{N}}$: let (α_n, c_n) in $\widetilde{\mathcal{N}}_1$ tend to (α,c) in $\widetilde{\mathcal{N}}$; let z_n be the first strictly positive zero of f_{α_n,c_n}; if z_n has an accumulation point z, then $f_{\alpha,c}(z) = 0$ and $(\alpha,c) \in \widetilde{\mathcal{N}}_1$,; otherwise, $z_n \to +\infty$, and one checks that (α,c) corresponds to an $\mathcal{N}_2$-vortex, etc. □

4. Symmetric vortices close to a normal solution.

The purpose of this section is the study of $(f_{\alpha,c}, b_{\alpha,c})$ for small values of c. The results can be summarized in the following proposition:

Proposition 4.1. *For given $\kappa > 0$ and $d \geq 1$, there exist α_2 and α_4 in $\mathbb{R}_+^*$, with $\alpha_2 < \alpha_4$, such that*

- for $\alpha \in (0, \alpha_4)$ and $c > 0$ small enough, $f'_{\alpha,c}$ vanishes on $\mathbb{R}_+^*$ and $(\alpha, c) \in \mathcal{N}$;

- for $\alpha \in (0, \alpha_2)$ and $c > 0$ small enough, $f_{\alpha,c}$ vanishes on $\mathbb{R}_+^*$ and $(\alpha, c) \in \mathcal{N}_1$;

- for $\alpha \in (\alpha_2, \alpha_4)$ and $c > 0$ small enough, $f_{\alpha,c}$ does not vanish, $f'_{\alpha,c}$ vanishes twice and $(\alpha, c) \in \mathcal{N}_3$;

- for $\alpha \in (\alpha_4, +\infty)$ and $c > 0$ small enough, $f'_{\alpha,c}$ no longer vanishes for $r > 0$ and $(\alpha, c) \in \mathcal{D}$.

As an immediate corollary, one gets a description of normal solutions which are limits of nontrivial symmetric vortices, i.e., which lie in the boundary of $\mathcal{N}$:

Corollary 4.2. $\partial\mathcal{N} \cap \big(\mathbb{R}_+ \times \{0\}\big) = [0, \alpha_4] \times \{0\}$.

This means that normal solutions which can be reached as limits of symmetric vortices with Neumann conditions correspond to values of the parameters $c = 0$ and $0 \leq \alpha \leq \alpha_4$. As $b'_{\alpha,0}(\bar{r}) = -2\alpha\bar{r}$, the corresponding value of the parameter $h = -\frac{b'(\bar{r})}{\kappa\bar{r}}$ lies in $[0, \frac{2\alpha_4}{\kappa}]$.

The proof of this proposition relies on the study of an auxiliary differential system.

An auxiliary family of functions. Notation: For $\beta \geq 0$, φ_β will denote the (only) solution on $\mathbb{R}_+$ of the differential equation of Fuchsian type

$$\varphi''(r) + \frac{1}{r}\varphi'(r) = \varphi(r)\Big(\frac{d^2}{r^2} - 1 + \beta^2 r^2\Big)$$

such that $\varphi(r) \sim r^d$ at the neighbourhood of 0.

(The existence and uniqueness of such φ_β is classical. Note that the functions φ_β and φ'_β depend continuously on β for the topology of uniform convergence on compact subsets of $\mathbb{R}_+$.)

Proposition 4.3. *There exist β_2 and β_4, with $0 < \beta_2 < \beta_4 < \frac{1}{2d}$, such that*

- for $\beta \in [0, \beta_2)$, φ_β vanishes on $\mathbb{R}_+^*$: there exists some $r > 0$ such that $\varphi_\beta(r) = 0$;

- *for* $\beta \in [\beta_2, +\infty)$, φ_β *no longer vanishes on* $\mathbb{R}_+^*$*: for any* $r > 0$*, one has* $\varphi_\beta(r) > 0$ *;*

- *for* $\beta \in [0, \beta_4]$, φ'_β *vanishes on* $\mathbb{R}_+^*$*: there exists some* $r > 0$ *such that* $\varphi'_\beta(r) = 0$ *;*

- *for* $\beta \in (\beta_4, +\infty)$, φ'_β *no longer vanishes on* $\mathbb{R}_+^*$*: for any* $r > 0$*, one has* $\varphi'_\beta(r) > 0$.

Proof. State first the following property:

Let β *and* $\widetilde{\beta}$ *be given, with* $\beta < \widetilde{\beta}$*, and* $R > 0$*. Then one has*

$\varphi_\beta < \varphi_{\widetilde{\beta}}$ *on* $(0, R]$ *if* $\varphi_\beta \geq 0$ *on* $[0, R]$

$\varphi'_\beta < \varphi'_{\widetilde{\beta}}$ *on* $(0, R]$ *if* $\varphi'_\beta \geq 0$ *on* $[0, R]$.

To prove it, it suffices to write the equation

$$r\big(\varphi'_{\widetilde{\beta}}(r)\varphi_\beta(r) - \varphi'_\beta(r)\varphi_{\widetilde{\beta}}(r)\big) = \Big(\widetilde{\beta}^2 - \beta^2\Big) \int_0^r s^3 \varphi_{\widetilde{\beta}}(s)\varphi_\beta(s) ds,$$

and notice that the left-hand side should be nonpositive at the first $r > 0$ such that $\varphi_{\widetilde{\beta}}(r) = \varphi_\beta(r)$ or $\varphi'_{\widetilde{\beta}}(r) = \varphi'_\beta(r)$.

Define then the following sets:

$\mathcal{M}_0 = \{\beta \geq 0 : \exists r > 0 : \varphi_\beta(r) = 0\}$, $\mathcal{M}_1 = \{\beta \geq 0 : \exists r > 0 : \varphi'_\beta(r) = 0\}$,

$\mathcal{M}_2 = \{\beta \geq 0 : \forall r > 0, \varphi'_\beta(r) > 0\}$.

One has $\mathcal{M}_0 \subset \mathcal{M}_1$, $\mathcal{M}_2 = \mathbb{R}_+ \backslash \mathcal{M}_1$, and the property above implies that $\mathcal{M}_0$ and $\mathcal{M}_1$ are intervals with left end 0, while $\mathcal{M}_2$ is a half line open at $+\infty$. What we show is that $\mathcal{M}_0$ is open and nonempty, $\mathcal{M}_1$ is closed or equivalently $\mathcal{M}_2$ open, and that $\mathcal{M}_2$ contains $\frac{1}{2d}$.

$\mathcal{M}_0$ *is open in* $\mathbb{R}_+$*:* let $\beta \in \mathcal{M}_0$ and R be the smallest strictly positive zero of φ_β. One has $\varphi'_\beta(R) \leq 0$ and, by the Cauchy-Lipschitz uniqueness property $\varphi'_\beta(R) \neq 0$; hence one has $\varphi_\beta(r) < 0$ for $r > R$ close to R, and by continuity $\varphi_{\widetilde{\beta}}$ takes strictly negative values for $\widetilde{\beta}$ close to β.

$\mathcal{M}_0$ *is not empty, and more precisely* $0 \in \mathcal{M}_0$*:* we suppose $\varphi = \varphi_0 \geq 0$ on $\mathbb{R}_+$, hence > 0 on $\mathbb{R}_+^*$, and we seek a contradiction. For $r > d$ one has then $\big(r\varphi'(r)\big)' < 0$; $r\varphi'(r)$ decreases and has a limit $\ell \in [-\infty, +\infty)$ as r tends to infinity.

Suppose $\ell < 0$: if ℓ is finite, then at infinity $\varphi'(r) \sim \frac{\ell}{r}$ and $\varphi(r) \sim \ell \log(r)$, which contradicts the positivity of φ. If $\ell = -\infty$, then for large r one has $\varphi'(r) \leq \frac{m}{r}$ for some $m < 0$, and the same argument leads to the same contradiction.

Suppose $\ell \geq 0$: then φ is increasing, and there exists $\delta > 0$ such that $\varphi(r) \geq \delta$ for any $r > 2d$, hence $\big(r\varphi'(r)\big)' \leq r\delta(\frac{d^2}{r^2} - 1) < -\delta r/2$ which implies $r\varphi'(r) < 2d\varphi'(2d) - \delta r^2/4$ and contradicts $\ell \geq 0$. In both cases, we get a contradiction, thus φ_0 vanishes on $\mathbb{R}_+^*$.

(Note that the same proof shows that φ_0 cannot have a constant sign in a neighbourhood of $+\infty$, and hence has infinitely many zeroes.)

At this point, we have proved that $0 \in \mathcal{M}_0$ and that

there exists $\beta_2 > 0$ *such that* $\mathcal{M}_0 = [0, \beta_2)$.

The next step is to show that $[\frac{1}{2d}, +\infty) \subset \mathcal{M}_2$*:* Notice that for $\beta \geq \frac{1}{2d}$, one has $(\frac{d^2}{r^2} - 1 + \beta^2 r^2) \geq 0$ for all $r > 0$, and apply the maximum principle.

We show then that the set $\mathcal{M}_1$ *is closed:* Let β_n a sequence in $\mathcal{M}_1$ with limit β in $\mathbb{R}_+$. If $\beta < \beta_2$, one has $\beta \in \mathcal{M}_0 \subset \mathcal{M}_1$ and there is nothing to prove. So we suppose $\beta \geq \beta_2$.

Note that, for any n, the first strictly positive zero $\widetilde{R}_n$ of φ'_{β_n} lies between the two positive zeroes of $\frac{d^2}{r^2} - 1 + \beta^2 r^2$, which will be denoted $R_-(\beta_n)$ and $R_+(\beta_n)$, with $R_-(\beta_n) < R_+(\beta_n)$. One has obviously $R_-(\beta_n) > d$. Moreover, $R_+(\beta)$ is a decreasing function of β, which implies $\limsup R_+(\beta_n) \leq R_+(\beta_2)$. Hence the $\widetilde{R}_n$ have an accumulation point $\widetilde{R}$ in $\mathbb{R}_+^*$, which will be such that $\varphi'_\beta(\widetilde{R}) = 0$; this proves $\beta \in \mathcal{M}_1$.

At this point, we have proved:

There exists β_4 *with* $\beta_2 \leq \beta_4 < \frac{1}{2d}$ *such that* $\mathcal{M}_1 = [0, \beta_4]$ *and* $\mathcal{M}_2 = (\beta_4, +\infty)$. The last property to prove is $\beta_2 < \beta_4$: for $\beta < \beta_2$, let $\widetilde{R}_\beta$ be the first positive zero of φ'_β and $z_\beta > \widetilde{R}_\beta$ the first positive zero of φ_β. The family z_β is increasing (by the property stated at the beginning of this proof), and must tend to $+\infty$ as $\beta \uparrow \beta_2$ (if not, it would have a limit $z_0 > 0$ such that $\varphi_{\beta_2}(z_0) = 0$, which would contradict $\beta_2 \notin \mathcal{M}_0$).

As in the proof of Theorem 1, for $\beta < \beta_2$, φ_β is decreasing on $[\widetilde{R}_\beta, z_\beta]$, so that one has $\varphi'_{\beta_2} \leq 0$ on $[\widetilde{R}_{\beta_2}, +\infty)$. There will exist some $r_0 > 0$ such that $\varphi'_{\beta_2}(r_0) < 0$. This implies $\varphi'_\beta(r_0) < 0$ and $\beta \in \mathcal{M}_1$ for $\beta > \beta_2$ close to β_2. □

Remark 4.4. One can repeat the arguments in the proof of Theorem 1 to obtain a more accurate description of the φ_β:

- for $\beta \in [0, \beta_2)$, φ'_β vanishes exactly once at $\widetilde{R}_\beta$ on $(0, z_\beta)$ (with z_β its first strictly positive zero) ; one has $\varphi''(\widetilde{R}_\beta) < 0$;

- for $\beta = \beta_2$, φ'_β has exactly one zero $\widetilde{R}_\beta$ in $\mathbb{R}_+^*$; φ_β is increasing on $[0, \widetilde{R}_\beta]$, decreasing on $[\widetilde{R}_\beta, +\infty)$, and tends to zero at infinity ; moreover, one has $\varphi''(\widetilde{R}_\beta) < 0$;

- for $\beta \in (\beta_2, \beta_4)$, φ'_β vanishes exactly twice on $\mathbb{R}_+^*$, at $\widetilde{R}_\beta$ and $\widetilde{\widetilde{R}}_\beta$; φ'_β is > 0 on $(0, \widetilde{R}_\beta)$ and $(\widetilde{\widetilde{R}}_\beta, +\infty)$, < 0 on $(\widetilde{R}_\beta, \widetilde{\widetilde{R}}_\beta)$; one has $\varphi''(\widetilde{R}_\beta) < 0$ and $\varphi''(\widetilde{\widetilde{R}}_\beta) > 0$;

- for $\beta = \beta_4$, one has $\varphi'_\beta \geq 0$ on $\mathbb{R}_+$; φ'_β vanishes exactly once on $\mathbb{R}_+^*$, at $\widetilde{R}_\beta$, and one has $\varphi''_\beta(\widetilde{R}_\beta) = 0$, $\varphi'''_\beta(\widetilde{R}_\beta) \neq 0$.

Proposition 4.1 will be an immediate consequence of the following lemmas, which provide a slightly more accurate picture:

Lemma 4.5. *Define a family* $\{F_{\alpha,c}\}_{\alpha \geq 0, c \geq 0}$ *of functions by the formula*

$$F_{\alpha,c} = \begin{cases} \frac{1}{c} f_{\alpha,c} & \text{if } c > 0 \\ \frac{\partial f_{\alpha,c}}{\partial c}_{|c=0} & \text{if } c = 0 \,. \end{cases} \tag{4.1}$$

Then

1. *The function* $(\alpha, c, r) \to F_{\alpha,c}(r)$ *is analytic on its domain*

$$\{(\alpha, c, r) \in \mathbb{R}_+^3 : r < \rho(\alpha, c)\}$$

with the convention $\rho(\alpha, c) = +\infty$ *if* $c = 0$*.*

2. *For* $\alpha \in \mathbb{R}_+$*, the function* $F_{\alpha,0}(\frac{r}{\sqrt{2\alpha d + \kappa^2}})$ *is proportional to the function* $\varphi_\beta(r)$ *of Proposition* 4.3*, with* $\beta = \frac{\alpha}{2\alpha d + \kappa^2}$*.*

Proof. The analyticity statement 1. comes from the analyticity of the map $(\alpha, c, r) \to f_{\alpha,c}(r)$ on $\alpha \geq 0$, $c \geq 0$ and $r < \rho(\alpha, c)$, together with $f_{\alpha,c} \equiv 0$ whenever $c = 0$.

The function $F = F_{\alpha,0}$ satisfies the linear differential equation $F''(r) + \frac{1}{r}F'(r) = F(r)(\frac{(d-\alpha r^2)^2}{r^2} - \kappa^2)$ with initial condition $F(r) \sim_0 r^d$, so that the function $G_\alpha(r) = F_{\alpha,0}(\frac{r}{\sqrt{\kappa^2 + 2\alpha d}})$ satisfies the equation $G''(r) + \frac{1}{r}G'(r) = G(r)(\frac{d^2}{r^2} - 1 + \beta^2 r^2)$ with $\beta = \frac{\alpha}{2\alpha d + \kappa^2}$. The space of solutions of this equation on $\mathbb{R}_+$ is a one-dimensional space, so that G_α is proportional to the corresponding φ_β of the previous section. □

Lemma 4.6. *Let* α_2 (*respectively* α_4) *be such that* $\beta_2 = \frac{\alpha_2}{2\alpha_2 d + \kappa^2}$ (*respectively* $\beta_4 = \frac{\alpha_4}{2\alpha_4 d + \kappa^2}$)*, with* β_2 *and* β_4 *provided by Proposition* 4.3*. They have the following properties:*

(a) *Let* $\alpha_0 \in (0, \alpha_4)$*. Then* $F'_{\alpha_0,0}$ *vanishes on* $\mathbb{R}_+^*$*. For* $(\alpha, c) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ *close to* $(\alpha_0, 0)$*, one has* $(\alpha, c) \in \mathcal{N}$*.*

(b) *Let* $\alpha_0 \in (0, \alpha_2)$. *Then* $F_{\alpha_0,0}$ *vanishes on* $\mathbb{R}_+^*$. *For* $(\alpha, c) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ *close to* $(\alpha_0, 0)$, *one has* $(\alpha, c) \in \mathcal{N}_1$. *For* $(\alpha, c) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ *close to* $(0,0)$, *one has* $(\alpha, c) \in \widetilde{\mathcal{N}}_1$.

(c) *Let* $\alpha_0 \in (\alpha_2, \alpha_4)$. *Then* $F_{\alpha_0,0}$ *is positive on* $\mathbb{R}_+^*$ *and* $F'_{\alpha_0,0}$ *vanishes exactly twice on* $\mathbb{R}_+^*$. *For* $(\alpha, c) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ *close to* $(\alpha_0, 0)$, *one has* $(\alpha, c) \in \mathcal{N}_3$.

(d) *Let* $\alpha_0 \in (\alpha_4, +\infty)$. *Then* $F'_{\alpha_0,0}$ *no longer vanishes on* $\mathbb{R}_+^*$. *For* $(\alpha, c) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ *close to* $(\alpha_0, 0)$, *one has* $(\alpha, c) \in \mathcal{D}$.

Proof. $\frac{\alpha}{\kappa^2+2\alpha d}$ increases from 0 to $\frac{1}{2d}$ when α increases from 0 to $+\infty$. So there exists α_2 (respectively α_4) such that $\frac{\alpha_2}{2\alpha_2 d+\kappa^2} = \beta_2$ (respectively $\frac{\alpha_4}{2\alpha_4 d+\kappa^2} = \beta_4$). Lemma 4.5 and Proposition 4.3 provide a description of the behaviour of $F_{\alpha,0}$ according to the value of α.

From that description, the arguments of Proposition 3.4 can be easily adapted to get all the conclusions of the lemma, except for the case $\alpha > \alpha_4$ which requires a more specific argument.

Fix $\alpha_0 > \alpha_4$, so that the corresponding β_0 is strictly greater than β_4 and $F'_{\alpha_0,0}$ no longer vanishes on $\mathbb{R}_+^*$.

One has $b_{\alpha,0}(r) = d - \alpha r^2$, so that there exists $r_1 > \sqrt{d/\alpha_0}$ such that, for α close to α_0 and small values of c, one has $b_{\alpha,c}(r_1) \le 0$ and $b'_{\alpha,c}(r_1) \le -2(\alpha_0 - \varepsilon) r_1$. For $r > r_1$, one has $b_{\alpha,c}(r) < 0$, $(\frac{b'_{\alpha,c}(r)}{r})' = \frac{b_{\alpha,c}(r) f_{\alpha,c}(r)^2}{r} < 0$ and $b'_{\alpha,c}(r) < -2(\alpha_0 - \varepsilon) r$. From this we deduce $b_{\alpha,c}(r) < -(\alpha_0 - \varepsilon)(r^2 - r_1^2)$ and the existence of $r_2 > r_1$ such that $b_{\alpha,c}(r) < -\kappa r$ for any $r \ge r_2$ and any $c > 0$ close to 0.

By the compact convergence of $\frac{1}{c} f_{\alpha,c}$ to $F_{\alpha_0,0}$ and $\frac{1}{c} f'_{\alpha,c}$ to $F'_{\alpha_0,0}$, one will have, for (α, c) close enough to $(\alpha_0, 0)$, $f_{\alpha,c}(r) > 0$ and $f'_{\alpha,c}(r) > 0$ for $r \in (0, r_2]$. For $r \ge r_2$, one has $\frac{b(r)^2}{r^2} - \kappa^2(1 - f(r)^2) > 0$, and the maximum principle implies $f_{\alpha,c}(r) > 0$ and $f'_{\alpha,c}(r) > 0$ for $r \ge r_2$.

According to Theorem 1, (α, c) coresponds either to a $\mathcal{D}$-vortex, or to a BC-vortex. But the BC-vortex case is not possible, since $b_{\alpha,c}$ takes strictly negative values, so that we have proved $(\alpha, c) \in \mathcal{D}$. □

Lemma 4.7. *Define* $\widetilde{R}(\alpha, 0)$ (*respectively* $\widetilde{\widetilde{R}}(\alpha, 0)$) *as the first* (*respectively the second*) *strictly positive zero of* $F'_{\alpha,0}$ *for* $\alpha \in [0, \alpha_4]$ (*respectively* $\alpha \in (\alpha_2, \alpha_4)$).

Then the map $(\alpha, c) \to \widetilde{R}(\alpha, c)$ *is continuous on* $\widetilde{\mathcal{N}} \cup ([0, \alpha_4] \times \{0\})$, *and the map* $(\alpha, c) \to \widetilde{\widetilde{R}}(\alpha, c)$ *is continuous on* $\mathcal{N}_3 \cup \mathcal{N}_4 \cup ((\alpha_2, \alpha_4] \times \{0\})$, *with the convention* $\widetilde{\widetilde{R}}(\alpha, c) = \widetilde{R}(\alpha, c)$ *if* $(\alpha, c) \in \mathcal{N}_4 \cup \{(\alpha_4, 0)\}$.

The proof is analogous to the proof of items 4 and 5 in Proposition 3.4.

Remark 4.8. As a consequence of the three previous lemmas, for any κ and d, the seven possible behaviours of f listed in Theorem 1 do actually appear. It is obvious for normal states, and proved in [6] for BC -vortices. Lemma 4.6 states the existence of $\mathcal{D}$-vortices, as well as $\mathcal{N}_1$-vortices and $\mathcal{N}_3$-vortices. Then one shows easily the existence of $\mathcal{N}_2$-vortices corresponding to some (α, c) close to $(\alpha_2, 0)$, as an intermediate type between $\mathcal{N}_1$-vortices and $\mathcal{N}_3$-vortices; and the existence of $\mathcal{N}_4$-vortices for some (α, c) close to $(\alpha_4, 0)$, as an intermediate type between $\mathcal{N}_3$-vortices and $\mathcal{D}$-vortices.

Remark 4.9. (Bauman-Phillips-Tang bifurcation branches). For fixed $\bar{r}$ (except perhaps for countably many ones) and suitable values h_0 of h, P. Bauman, D. Phillips and Q. Tang have proved the existence of branches of symmetric vortices of degree d (d depending on $\bar{r}$ and h_0) which are bifurcation branches emanating at $h = h_0$ from the branch of normal states defined for fixed $\bar{r}$ and h running in $\mathbb{R}_+^*$ (cf. [11]).

Those branches appear in our picture as a mere application of the implicit function theorem. For fixed d and κ, consider any h_0 such that the corresponding $\alpha_0 = \frac{\kappa h}{2}$ lies in $(0, \alpha_4]$. Let $\bar{r} > 0$ be such that $F'_{\alpha,0}(\bar{r}) = 0$, i.e., $\bar{r} = \widetilde{R}_{\alpha_0,0}$ or $\bar{r} = \widetilde{\widetilde{R}}_{\alpha_0,0}$: then a branch in $\mathcal{N}$ at $\bar{r} =$ constant will have the implicit equation $f'_{\alpha,c}(\bar{r}) = 0$ or, equivalently, $F'_{\alpha,c}(\bar{r}) = 0$.

It is not difficult to check that $\frac{\partial F'_{\alpha,c}(r)}{\partial c} = 0$, for all r, whenever $c = 0$, while $\frac{\partial F'_{\alpha,c}(\bar{r})}{\partial \alpha} \neq 0$ at $\alpha = \alpha_0$ and $c = 0$, except perhaps for finitely many values of α_0. Thus the implicit function theorem provides a curve defined for small values of c

$$c \to \alpha(c) \text{ with } \alpha(0) = \alpha_0 \,, \alpha'(0) = 0$$

such that

$$F'_{\alpha(c),c}(\bar{r}) = 0 \text{ hence } f'_{\alpha(c),c}(\bar{r}) = 0 \text{ for all } c \text{ small enough.}$$

This provides an alternative approach to the existence of phase transitions from normal states to superconducting symmetric vortices.

5. BOUNDEDNESS OF $\widetilde{\mathcal{N}}$ AND $\mathcal{N}$

The main result of this section is that the set $\mathcal{D}$ of (α, c)'s corresponding to monotone blowing-up vortices ($\mathcal{D}$-vortices) contains a neighbourhood of infinity in the quadrant $\alpha > 0$, $c > 0$:

Proposition 5.1. *There exists $(\alpha_\mu, c_\mu) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ such that*

$$\Big((0, \alpha_\mu) \times [c_\mu, +\infty)\Big) \bigcup \Big([\alpha_\mu, +\infty) \times (0, +\infty)\Big) \subset \mathcal{D}\,. \tag{5.1}$$

Before proving the proposition, let us state and prove some corollaries. The first one is an immediate consequence of the proposition:

Corollary 5.2. *$\widetilde{\mathcal{N}}$ and $\mathcal{N}$ are bounded subsets of $\mathbb{R}_+ \times \mathbb{R}_+$.*

As a consequence, any sequence (α_n, c_n) in $\mathcal{N}$ has an accumulation point and any sequence (ψ_n, A_n) of symmetric vortices with Neumann conditions has an accumulation point for the topology of compact convergence. As $\mathcal{D}$ is open (Proposition 3.4), such an accumulation point can be either a vortex with Neumann conditions (with possibly $h = 0$), or a BC-vortex, or a normal state.

The next corollary states that, for $\bar{r}$ small enough and independently of h, there is no $(\bar{r}, h)$-symmetric vortex; the normal solution is the only solution of the Ginzburg-Landau equation [GL] with boundary conditions (2.2), which is of radial form (1.3):

Corollary 5.3. *For given d and κ, there exists $R_\mu > 0$ such that $\widetilde{R}(\alpha, c) \geq R_\mu$ for all (α, c) in $\widetilde{\mathcal{N}}$.*

Proof. Set $\widetilde{R}(\alpha, c) = +\infty$ if (α, c) corresponds to a BC-vortex. The map $(\alpha, c) \to \widetilde{R}(\alpha, c)$ defined on $\widetilde{\mathcal{N}} \cup \big([0, \alpha_4] \times \{0\}\big)$ in Lemma 4.5 extends as a continuous map defined on the closure of $\widetilde{\mathcal{N}} \cup \big([0, \alpha_4] \times \{0\}\big)$, taking values in $(0, +\infty]$. A nonvanishing nonnegative continuous map on a compact set has a strictly positive lower bound. □

The next corollary states that a simple $(\bar{r}, h)$-symmetric vortex for a large value of the radius $\bar{r}$ is close either to a BC-vortex, or to a $\mathcal{N}_2$-vortex, or to the normal solution corresponding to a specific value $h_2 = 2\alpha_2/\kappa$ of the applied magnetic field:

Corollary 5.4. 1. *Let (α_n, c_n) be a sequence in $\mathcal{N}$ such that $\lim_n \widetilde{R}(\alpha_n, c_n) = +\infty$. This sequence has an accumulation point (α, c) in $\mathbb{R}_+ \times \mathbb{R}_+$ and the corresponding vortex is a BC-vortex.*

2. *Let* (α_n, c_n) *be a sequence in* $\mathcal{N}_3$ *such that* $\lim_n \widetilde{\widetilde{R}}(\alpha_n, c_n) = +\infty$. *This sequence has an accumulation point in* $\mathbb{R}_+ \times \mathbb{R}_+$ *which corresponds either to a BC-vortex, or to an* $\mathcal{N}_2$*-vortex, or to the normal solution* $(\alpha_2, 0)$ *with* α_2 *as defined in Proposition* 4.1. *In the last case, one has* $\lim_n \widetilde{R}(\alpha_n, c_n) = \widetilde{R}(\alpha_2, 0)$ (*defined in Lemma* 4.7).

Proof. By Corollary 5.2, one can suppose $(\alpha_n, c_n) \to (\alpha, c) \in \mathbb{R}_+ \times \mathbb{R}_+$.

Suppose first $\lim_n \widetilde{R}(\alpha_n, c_n) = +\infty$. We claim $c > 0$. If not, then by Corollary 4.2, α belongs to $[0, \alpha_4]$ and, by Lemma 4.7, one would have $\lim_n \widetilde{R}(\alpha_n, c_n) = \widetilde{R}(\alpha, 0) < +\infty$.

For any n, f_{α_n, c_n} is increasing and takes values in $[0, 1)$ on $[0, \widetilde{R}(\alpha_n, c_n)]$, so that $f_{\alpha,c}$ is defined on $\mathbb{R}_+$, is nondecreasing and takes values in $[0, 1]$. According to Theorem 1, since $c \neq 0$, (α, c) corresponds to a BC -vortex.

Suppose now $(\alpha_n, c_n) \in \mathcal{N}_3$ and $\lim_n \widetilde{\widetilde{R}}(\alpha_n, c_n) = +\infty$. If $\widetilde{R}(\alpha_n, c_n)$ tends to $+\infty$, the previous case provides (α, c) as corresponding to a BC-vortex. If not, one can suppose that $\widetilde{R} = \lim_n \widetilde{R}(\alpha_n, c_n)$ exists, and is > 0 by Corollary 5.3. If $c > 0$, we get $f_{\alpha,c}$ not identically zero, nonnegative, defined on the whole half line $\mathbb{R}_+$, increasing from 0 to $\widetilde{R}$ and decreasing from $\widetilde{R}$ to $+\infty$, which corresponds to $(\alpha, c) \in \mathcal{N}_2$. If $c = 0$, we get the same picture for $F_{\alpha,,0}$ defined in Lemma 4.5. This picture corresponds to $\alpha = \alpha_2$. □

Before proving the proposition, let us state a result which is an immediate corollary of [21], Lemma 2.4:

Comparison Lemma 5.5. *Let* R, α, $\widetilde{\alpha}$, c, $\widetilde{c}$ *be given in* $\mathbb{R}_+^*$, *such that* $\alpha \leq \widetilde{\alpha}$, $c \leq \widetilde{c}$ *and* $(\alpha, c) \neq (\widetilde{\alpha}, \widetilde{c})$.

Suppose $f_{\alpha,c}(r) \geq 0$ *and* $b_{\alpha,c}(r) \geq 0$ *for* r *in* $(0, R]$. *Then one has*

$$f_{\alpha,c}(r) < f_{\widetilde{\alpha},\widetilde{c}}(r)\ ,\ b_{\alpha,c}(r) < b_{\widetilde{\alpha},\widetilde{c}}(r)\ ,\ b'_{\alpha,c}(r) < b'_{\widetilde{\alpha},\widetilde{c}}(r)\ ,\ \forall r \in (0, R].$$

Moreover, if one supposes also $f'_{\alpha,c}(r) \geq 0$ *for* r *in* $[0, R]$, *one has*

$$f'_{\alpha,c}(r) < f'_{\widetilde{\alpha},\widetilde{c}}(r)\ ,\ \forall r \in (0, R].$$

Proof of Proposition 5.1. Apply the results of [21], Sections 7 and 8. There exists $L > 0$ such that, for any $\alpha > L$, there will exist $c(\alpha) > 0$ and $\widehat{R}(\alpha) > 0$ such that

$$f_{\alpha,c(\alpha)}(r) > 0 \text{ and } f'_{\alpha,c(\alpha)}(r) > 0 \text{ for } 0 < r < \rho(\alpha, c(\alpha))\ ; \tag{5.2}$$

$$f_{\alpha,c(\alpha)}(\widehat{R}(\alpha)) = 1\ ;\ b_{\alpha,c(\alpha)}(\widehat{R}(\alpha)) = 0\ ; \tag{5.3}$$

$$\widehat{R}(\alpha)=\sqrt{\frac{d}{\alpha}}+O(\frac{1}{\alpha}) \text{ and } \frac{b'_{\alpha,c(\alpha)}(\widehat{R}(\alpha))}{\widehat{R}(\alpha))}=-2\alpha+O(\alpha^{1/2}) \tag{5.4}$$

as $\alpha\to+\infty$. For such an α, we define $\widetilde{\alpha}=\alpha+O(\alpha^{1/2})$ by the formula

$$\frac{b'_{\alpha,c(\alpha)}(\widehat{R}(\alpha))}{\widehat{R}(\alpha))}=-2\widetilde{\alpha} \tag{5.5}$$

and we have, for $r\in[0,\widehat{R}(\alpha)]$,

$$-2\alpha r\leq b'_{\alpha,c(\alpha)}(r)\leq-2\widetilde{\alpha}r \text{ and } d-\alpha r^2\leq b_{\alpha,c(\alpha)}(r)\leq d-\widetilde{\alpha}r^2\ . \tag{5.6}$$

(For proving the first inequality, write $(\frac{b'(r)}{r})'=\frac{b(r)f(r)^2}{r}\geq 0$ on $[0,\widehat{R}(\alpha)]$, and $\frac{b'(r)}{r}\to-2\alpha$ as $r\to 0$.)

(5.6) implies in particular the inequalities

$$\sqrt{\frac{d}{\alpha}}\leq\widehat{R}(\alpha)\leq\sqrt{\frac{d}{\widetilde{\alpha}}}\ . \tag{5.7}$$

The main claim, from which the proposition will be deduced easily, is the following:

$$\exists\alpha_\mu>L:\forall\alpha\geq\alpha_\mu,\forall c\in(0,c(\alpha)]\ ,\forall r>0,\textit{ one has } f'_{\alpha,c}(r)>0\ . \tag{5.8}$$

The proof of this claim is divided into several steps. From now on, we fix $\alpha>L$ and $c\in(0,c(\alpha)]$ such that $f'_{\alpha,c}$ vanishes for some $r>0$, and we denote by $\widetilde{R}(\alpha,c)$ the first strictly positive zero of $f'_{\alpha,c}$. We seek a contradiction, not depending on the choice of c, when α is large enough.

Define a function $Z_{\alpha,c}$ by the formula

$$b_{\alpha,c}(r)=d-\alpha r^2+r^2Z_{\alpha,c}(r)\ .$$

Our intermediate purpose is to prove that $|Z_{\alpha,c}|=O(\alpha^{1/2})$ on $[0,\widetilde{R}(\alpha,c)]$. (In this proof, the notation $O(\alpha^p)$ will always denote a Landau estimate when $\alpha\to+\infty$, not depending on the choice of $c\in(0,c(\alpha)]$.)

Step 1. Let R be equal, either to $\widetilde{R}(\alpha,c)$ if $b_{\alpha,c}(\widetilde{R}(\alpha,c))\geq 0$, or to $z_{\alpha,c}$ such that $b_{\alpha,c}(z_{\alpha,c})=0$ if $b_{\alpha,c}(\widetilde{R}(\alpha,c))\leq 0$ (with $z_{\alpha,c}\leq\widetilde{R}(\alpha,c)$).

We claim that, for large α,

$$\begin{aligned} R\leq\widehat{R}(\alpha)&=\sqrt{\frac{d}{\alpha}}+O(\frac{1}{\alpha})\\ 0\leq Z_{\alpha,c}(r)\leq(\alpha-\widetilde{\alpha})&=O(\alpha^{1/2}) \text{ for } r\in[0,R]\ . \end{aligned} \tag{5.9}$$

To prove it, note that, on $[0, R]$, $f_{\alpha,c}$ and $b_{\alpha,c}$ are nonnegative. This implies first $b_{\alpha,c}(r) \geq d - \alpha r^2$ (cf. [S] Lemma 4.2), then $b_{\alpha,c}(r) \leq b_{\alpha,c(\alpha)}(r)$, for $r \in [0, R]$. Applying (5.6) and (5.7), one gets easily the result.

Step 2. We suppose now that $b_{\alpha,c}$ vanishes at $z_{\alpha,c} \leq \widetilde{R}(\alpha, c)$. Note that the results in step 1. hold with $R = z_{\alpha,c}$, so that one has

$$\sqrt{\frac{d}{\alpha}} \leq z_{\alpha,c} \leq \widehat{R}(\alpha) = \sqrt{\frac{d}{\alpha}} + O(\frac{1}{\alpha}) \,. \tag{5.10}$$

Moreover, the Comparison Lemma 5.5 implies $b'_{\alpha,c} \leq b'_{\alpha,c(\alpha)}$ on $[0, R]$. Applying (5.5), (5.10) and the fact that $b'_{\alpha,c}(r)/r$ and $b'_{\alpha,c(\alpha)}(r)/r$ are increasing functions on $[0, z_{\alpha,c}]$, taking the value -2α at $r = 0$, one gets

$$-2\alpha \leq \frac{b'_{\alpha,c}(z_{\alpha,c})}{z_{\alpha,c}} \leq \frac{b'_{\alpha,c(\alpha)}(z_{\alpha,c})}{z_{\alpha,c}} \leq -2\widetilde{\alpha} = -2\alpha + O(\alpha^{1/2}) \,. \tag{5.11}$$

For $r \geq z_{\alpha,c}$, $b'_{\alpha,c}(r)/r$ becomes decreasing and one has $b'_{\alpha,c}(r) \leq -2\widetilde{\alpha} r$, which provides

$$b_{\alpha,c}(r) \leq -\widetilde{\alpha}(r^2 - z_{\alpha,c}^2) \ \text{ for } \ r \geq z_{\alpha,c} \,. \tag{5.12}$$

Define now $R_+(\alpha, c)$ as the positive root of the equation

$$\widetilde{\alpha}(r^2 - z_{\alpha,c}^2) = \kappa\, r;$$

On one side, one checks easily

$$R_+(\alpha, c) = \sqrt{\frac{d}{\alpha}} + O(\frac{1}{\alpha}) \,. \tag{5.13}$$

On the other side, for $r \geq R_+(\alpha, c)$, one has $b(r) \leq -\kappa r$ and consequently $\frac{b(r)^2}{r^2} - \kappa^2(1 - f(r)^2) \geq 0$. If $f'_{\alpha,c}$ did not vanish on $[0, R_+(\alpha, c)]$, applying the maximum principle to $f_{\alpha,c}$ in the first equation of (2.1), one sees that $f'_{\alpha,c}$ would not vanish either for $r > R_+(\alpha, c)$, and this would contradict the existence of $\widetilde{R}(\alpha, c)$. We have shown

$$\widetilde{R}(\alpha, c) \leq R_+(\alpha, c) = \sqrt{\frac{d}{\alpha}} + O(\frac{1}{\alpha}) \,. \tag{5.14}$$

Step 3. We still suppose the existence of $z_{\alpha,c} \leq \widetilde{R}(\alpha, c)$. $Z_{\alpha,c}$ satisfies the differential equation

$$Z'' + \frac{3}{r} Z'(r) = \frac{b(r) f(r)^2}{r^2}$$

and one has

$$(r^3 Z'_{\alpha,c}(r))' = rb(r)f(r)^2 \; . \tag{5.15}$$

Integrating between 0 and $z_{\alpha,c}$, one gets

$$z_{\alpha,c}^3 Z'_{\alpha,c}(z_{\alpha,c}) \leq d \int_0^{z_{\alpha,c}} r dr = \frac{d}{2} z_{\alpha,c}^2$$

and

$$Z'_{\alpha,c}(z_{\alpha,c}) = O(\alpha^{1/2}) \; . \tag{5.16}$$

One has also

$$0 = b_{\alpha,c}(z_{\alpha,c}) = d - \alpha z_{\alpha,c}^2 + z_{\alpha,c}^2 Z_{\alpha,c}(z_{\alpha,c})$$

which implies $Z_{\alpha,c}(z_{\alpha,c}) = O(\alpha^{1/2})$.

Integrating (5.15) between $z_{\alpha,c}$ and any r in $[z_{\alpha,c}, R_+(\alpha,c)]$, one gets $Z'_{\alpha,c}(r) = O(\alpha)$ and

$$\begin{aligned} &\forall r \in [z_{\alpha,c}, R_+(\alpha,c)] \\ &Z_{\alpha,c}(r) \leq Z_{\alpha,c}(z_{\alpha,c}) + (R_+(\alpha,c) - z_{\alpha,c}) Z'_{\alpha,c}(z_{\alpha,c}) = O(\alpha^{1/2}) \; . \end{aligned} \tag{5.17}$$

(5.15) provides also, since $0 \geq b(r) \geq -\kappa r$ between $z_{\alpha,c}$ and $R_+(\alpha,c)$:

$$r^3 Z'_{\alpha,c}(r) \geq z_{\alpha,c}^3 Z'_{\alpha,c}(z_{\alpha,c}) - \frac{\kappa}{3}(r^3 - z_{\alpha,c}^3) \geq -\frac{\kappa}{3}(r^3 - z_{\alpha,c}^3) = O(\alpha^{-2})$$

for $r \in [z_{\alpha,c}, R_+(\alpha,c)]$, from which one deduces

$$Z_{\alpha,c}(r) \geq -\lambda \alpha^{-1/2} \text{ for some } \lambda > 0 \text{ and all } r \in [z_{\alpha,c}, R_+(\alpha,c)] \; . \tag{5.18}$$

(5.9), (5.17) and (5.18), with (5.14), lead to the intermediate result

$$\exists \nu_1 > 0 \; , \; |Z_{\alpha,c}(r)| \leq \nu_1 \alpha^{1/2} \quad , \quad \forall r \in (0, \widetilde{R}(\alpha,c)] \tag{5.19}$$

independently on $c \in (0, c(\alpha))$ such that $\widetilde{R}(\alpha,c)$ exists.

Step 4. From (5.19), one has on $[0, \widetilde{R}(\alpha,c)]$ that $|b_{\alpha,c}(r) - d + \alpha r^2| \leq \nu_1 \alpha^{1/2} r^2$ and consequently $|b_{\alpha,c}(r) + d - \alpha r^2| \leq 2|d - \alpha r^2| + \nu_1 \alpha^{1/2} r^2 \leq K$, where K is some constant independent from α, r and c. From this we deduce $b_{\alpha,c}(r)^2 - (d - \alpha r^2)^2 \geq -\nu \alpha^{1/2} r^2$ for some constant $\nu > 0$. We get then, for $r \in [0, \widetilde{R}(\alpha,c)]$:

$$\frac{b_{\alpha,c}(r)^2}{r^2} - \kappa^2(1 - f_{\alpha,c}(r)^2) \geq \frac{d^2}{r^2} - (2d\alpha + \nu\alpha^{1/2} + \kappa^2) + \alpha^2 r^2 \; . \tag{5.20}$$

Let Γ be the solution of the differential equation

$$\Gamma''(r) + \frac{1}{r}\Gamma'(r) = \Gamma(r)\left(\frac{d^2}{r^2} - (2d\alpha + \nu\alpha^{1/2} + \kappa^2) + \alpha^2 r^2\right) \; ,$$

with initial condition $\Gamma(r) \sim_0 r^d$.

Setting $\mu = \frac{1}{\sqrt{2d\alpha+\nu\alpha^{1/2}+\kappa^2}}$ and $\beta = \frac{\alpha}{2d\alpha+\nu\alpha^{1/2}+\kappa^2}$ one checks easily that $\Gamma(\mu r)$ is proportional to the function φ_β considered in Proposition 4.3. As $\lim_{\alpha\to\infty}\beta = 1/2d$, according to this proposition, there will exist some α_μ such that, for $\alpha \geq \alpha_\mu$, Γ and Γ' are strictly positive on $\mathbb{R}_+^*$.

On $[0, \widetilde{R}(\alpha, c)]$, one has $\big(r(f'_{\alpha,c}(r)\Gamma(r) - \Gamma'(r)f_{\alpha,c}(r)\big)' \geq 0$ and, integrating, $-\Gamma'(\widetilde{R}(\alpha,c))f_{\alpha,c}(\widetilde{R}(\alpha,c)) > 0$, which contradicts $\Gamma'(\widetilde{R}(\alpha,c)) > 0$ and $f_{\alpha,c}(\widetilde{R}(\alpha,c)) > 0$. The claim (5.8) is proved.

The proposition will appear now as an easy consequence of this claim. Note first that (5.8) implies:

$$\forall \alpha \geq \alpha_\mu, \quad \forall c \in (0, c(\alpha)] : \quad (\alpha, c) \in \mathcal{D}.$$

Note then that (5.2) and the Comparison Lemma 5.5 imply

$$\forall\, \alpha_1 > L\ ,\ \forall\, (\alpha, c) : \alpha \leq \alpha_1 \ \text{ and } \ c \geq c(\alpha_1)\ :\ (\alpha, c) \in \mathcal{D}\ . \tag{5.21}$$

Setting $c_\mu = c(\alpha_\mu)$, we get:

$$\begin{cases} \text{for } \alpha \leq \alpha_\mu & \text{and } c \geq c_\mu, \quad (\alpha,c) \in \mathcal{D} \text{ by (5.21)}, \\ \text{for } \alpha \geq \alpha_\mu & \text{and } c \leq c(\alpha),\ (\alpha,c) \in \mathcal{D} \text{ by (5.8)}, \\ \text{for } \alpha \geq \alpha_\mu & \text{and } c \geq c(\alpha),\ (\alpha,c) \in \mathcal{D} \text{ by (5.21)}, \end{cases}$$

and finally $\big((0, \alpha_\mu] \times [c_\mu, +\infty)\big) \bigcup \big((\alpha_\mu, +\infty) \times (0, +\infty)\big) \subset \mathcal{D}$. □

6. THE BOUNDARY CURVE AT $h = 0$.

Among the elements of $\widetilde{\mathcal{N}}$ belonging to the boundary $\partial\mathcal{N}$ of $\mathcal{N}$ are those corresponding to the limiting case $h = 0$, i.e., symmetric vortices which satisfy the boundary conditions (1.2) for some $\bar{r} > 0$ and for $h = 0$. They are those for which, at some $\bar{r} > 0$, one has $f'_{\alpha,c}(\bar{r}) = b'_{\alpha,c}(\bar{r}) = 0$.

The next lemma states that $\bar{r}$ should then be the first positive zero of $f'_{\alpha,c}$:

Lemma 6.1. *Let $(\alpha, c) \in \mathbb{R}_+^* \times \mathbb{R}_+^*$ and $\bar{r} > 0$ such that $f'_{\alpha,c}(\bar{r}) = 0$, $b'_{\alpha,c}(\bar{r}) = 0$ and $f_{\alpha,c}(r) > 0$ on $(0, \bar{r}]$. Then one has $\bar{r} = \widetilde{R}(\alpha, c)$ (i.e., $\bar{r}$ is the first strictly positive zero of $f'_{\alpha,c}$) and $f''_{\alpha,c}(\bar{r}) \neq 0$, i.e., $(\alpha, c) \notin \widetilde{\mathcal{N}}_4$.*

Definition 6.2. The *boundary curve at $h = 0$* will be the set

$$\Gamma_{h=0} = \{(\alpha, c) \in \widetilde{\mathcal{N}} : b'_{\alpha,c}(\widetilde{R}(\alpha, c)) = 0\}$$

and $\mathcal{A}_{h=0}$ will be its first projection, i.e.

$$\mathcal{A}_{h=0} = \{\alpha > 0 : \exists c > 0 : (\alpha, c) \in \Gamma_{h=0}\}\ .$$

Notice that, as a consequence of the Comparison Lemma 5.5, for a given α in $\mathcal{A}_{h=0}$ there exists only one $c > 0$ such that $(\alpha, c) \in \Gamma_{h=0}$. This c will be denoted $\gamma_0(\alpha)$.

Proposition 6.3. 1. *$\mathcal{A}_{h=0}$ is an open bounded subset of $\mathbb{R}_+^*$. The map γ_0 is analytic and strictly increasing on $\mathcal{A}_{h=0}$, with $\gamma_0'(\alpha) > 0$, for all $\alpha \in \mathcal{A}_{h=0}$.*

2. *$\mathcal{A}_{h=0}$ has a connected component $(0, A_0)$, with $\lim_{\alpha\to 0} \gamma_0(\alpha) = 0$, $\lim_{\alpha\to 0} \gamma_0'(\alpha) = +\infty$.*

3. *For each connected component (α_0, α_1) of $\mathcal{A}_{h=0}$:*

(i) *$\lim_{\alpha\downarrow\alpha_0} \gamma_0(\alpha) = c_0$ and $\lim_{\alpha\uparrow\alpha_1} \gamma_0(\alpha) = c_1$ exist.*

(ii) *If $\alpha_0 \neq 0$, one has $\lim_{\alpha\downarrow\alpha_0} \widetilde{R}(\alpha, \gamma_0(\alpha)) = +\infty$ and the vortex corresponding to (α_0, c_0) is a BC-vortex.*

(iii) *$\lim_{\alpha\uparrow\alpha_1} \widetilde{R}(\alpha, \gamma_0(\alpha)) = +\infty$ and the vortex corresponding to (α_1, c_1) is a BC-vortex.*

4. *If the BC-vortex is unique, and in particular if $\kappa^2 \geq 2d^2$ (cf. [1]), the set $\Gamma_{h=0}$ is exactly the smooth connected curve $\{(\alpha, \gamma_0(\alpha)); \alpha \in (0, A_0)\ \}$.*

Proof of Lemma 6.1. According to Lemma 2.1, for $0 < r \leq \bar{r}$ one has $b(r) > 0$, $b'(r) < 0$ and $(\frac{b(r)^2}{r^2})' < 0$, so that, if one supposes $\bar{r} = \widetilde{\widetilde{R}}(\alpha, c)$, one has that $\frac{b(r)^2}{r^2} - \kappa^2(1 - f(r)^2)$ would be decreasing between $\widetilde{R}(\alpha, c)$ and $\widetilde{\widetilde{R}}(\alpha, c)$, nonpositive at $r = \widetilde{R}(\alpha, c)$. Hence, $rf'(r)$ would decrease on $[\widetilde{R}(\alpha, c), \widetilde{\widetilde{R}}(\alpha, c)]$, which would contradict $f'_{\alpha,c}(\widetilde{R}(\alpha, c)) = f'_{\alpha,c}(\widetilde{\widetilde{R}}(\alpha, c)) = 0$.

Suppose $(\alpha, c) \in \Gamma_{h=0} \cap \widetilde{\mathcal{N}}_4$. One has $\frac{b(r)^2}{r^2} - \kappa^2(1 - f(r)^2) = 0$ and $(\frac{b(r)^2}{r^2} - \kappa^2(1 - f(r)^2))' = -\frac{b^2(r)}{r^3} < 0$ at $r = \widetilde{R}(\alpha, c)$. So, $rf'(r)$ increases for $r < \widetilde{R}(\alpha, c)$ close to $\widetilde{R}(\alpha, c)$, which leads to a contradiction. □

Proof of Proposition 6.3. 1. Let $(\alpha, c) \in \Gamma_{h=0}$. As in Section 8 of [21], one checks that the partial derivatives $\frac{\partial f'_{\alpha,c}}{\partial c}$ and $\frac{\partial b'_{\alpha,c}}{\partial c}$ remain strictly positive on $(0, \widetilde{R}(\alpha, c)]$ (since $f_{\alpha,c}$ and $b_{\alpha,c}$ are positive on this interval), while the partial derivatives $\frac{\partial f'_{\alpha,c}}{\partial \alpha}$ and $\frac{\partial b'_{\alpha,c}}{\partial \alpha}$ remain strictly negative. Moreover, one has $f''_{\alpha,c}(\widetilde{R}(\alpha, c)) < 0$ and $b''_{\alpha,c}(\widetilde{R}(\alpha, c)) > 0$.

$\Gamma_{h=0}$ is the projection of an analytic set

$$\big\{(\alpha, c, r) \in \mathbb{R}_+^* : f'_{\alpha,c}(r) = b'_{\alpha,c}(r) = 0\big\}.$$

At $r = \widetilde{R}(\alpha, c)$ the Jacobian $\frac{\partial f'_{\alpha,c}}{\partial c}\frac{\partial b'_{\alpha,c}}{\partial r} - \frac{\partial f'_{\alpha,c}}{\partial r}\frac{\partial b'_{\alpha,c}}{\partial c}$ is strictly positive, while the Jacobian $\frac{\partial f'_{\alpha,c}}{\partial \alpha}\frac{\partial b'_{\alpha,c}}{\partial r} - \frac{\partial f'_{\alpha,c}}{\partial r}\frac{\partial b'_{\alpha,c}}{\partial \alpha}$ is strictly negative. This proves that

$\mathcal{A}_{h=0}$ is open, and that $\alpha \to \gamma_0(\alpha)$ is an analytic map with strictly positive derivative.

The fact that γ_0 is strictly increasing is a consequence of the Comparison Lemma 5.5.

2. The map $F_{0,0}$ of Lemma 4.5 is positive increasing for small $r > 0$, then vanishes. It follows that, for (α, c) close to $(0, 0)$, $F_{\alpha,c}$ and $f_{\alpha,c}$ vanish on $\mathbb{R}_+^*$.

Fix $c_0 > 0$ such that f_{0,c_0} vanishes at some $r > 0$. As b_{0,c_0} is positive increasing for small $r > 0$, the maximum principle implies $b'_{0,c_0}(r) > 0$ for all $r > 0$. So, if $\widetilde{R}(0, c_0)$ is the first positive zero of f'_{0,c_0}, one has $b'_{0,c_0}(\widetilde{R}(0, c_0)) > 0$. The consequence is that one has $b'_{\alpha,c_0}(\widetilde{R}(\alpha, c_0)) > 0$ for $\alpha > 0$ small enough.

On the other hand, one has $b'_{\alpha,0} = -2\alpha r < 0$ for any $r > 0$ and $\alpha > 0$, and in particular $b'_{\alpha,0}(\widetilde{R}(\alpha, 0)) < 0$ for any α in $(0, \alpha_4)$. The intermediate value theorem implies, for any $\alpha > 0$ small enough, the existence of $c = \gamma_0(\alpha)$ such that $b'_{\alpha,c}(\widetilde{R}(\alpha, c)) = 0$. This proves that $\mathcal{A}_{h=0}$ has a connected component $(0, A_0)$, with $\lim_{\alpha\to 0} \gamma_0(\alpha) = 0$.

For computing $\gamma'_0(\alpha)$ at $\alpha = 0$, write it as $-J_\alpha/J_c$, where J_α is the value at $\alpha = c = 0$ of $\big((\partial F'_{\alpha,c}/\partial\alpha) b''_{\alpha,c} - (\partial b'_{\alpha,c}/\partial\alpha) F''_{\alpha,c}\big)(\widetilde{R}(\alpha, c))$ and J_c is the value at $\alpha = c = 0$ of $\big((\partial F'_{\alpha,c}/\partial c) b''_{\alpha,c} - (\partial b'_{\alpha,c}/\partial c) F''_{\alpha,c}\big)(\widetilde{R}(\alpha, c))$.

$b_{\alpha,c}$ has limited expansion at 0:

$$b_{\alpha,c}(r) = d - \alpha r^2 + \frac{c^2}{4d(d+1)} r^{2d+2} + O(r^{2d+4}) \,.$$

One has $b_{\alpha,0}(r) = d - \alpha r^2$ and $\frac{\partial}{\partial\alpha} b'_{\alpha,0}(r) = -2r < 0$ at $\alpha = 0$ and $r = \widetilde{R}(0, 0)$. At $c = 0$, since $f_{\alpha,0} \equiv 0$, $\frac{\partial}{\partial c} b_{\alpha,c}$ at $c = 0$ is a solution of the differential equation

$$B'' - \frac{1}{r} B' = 0$$

with initial conditions $B(r) = O(r^{2d+4})$, hence $\frac{\partial}{\partial c} b_{0,c_{|c=0}} \equiv 0$.

Moreover, one has $b''_{\alpha,0}(r) = -2\alpha = 0$ for $\alpha = 0$ and any r. One has also $F''_{0,0}(\widetilde{R}(0, 0)) < 0$. So, at $\alpha = c = 0$ and $r = \widetilde{R}(0, 0)$, $J_c = 0$ while $J_\alpha = -(\partial b'_{\alpha,c}/\partial\alpha) F''_{0,0}$ is strictly negative. We have proved $\gamma'(0) = +\infty$.

3. The existence of limits is a consequence of the fact that γ_0 is increasing and bounded.

If α_0 is any nonzero limit point of $A_{h=0}$, one has $\lim_{\alpha\to\alpha_0} \widetilde{R}(\alpha, \gamma_0(\alpha)) = +\infty$: for if this was not true, there would be a sequence (α_n) in $A_{h=0}$ with limit α_0, such that $\widetilde{R}(\alpha_n, \gamma_0(\alpha_n))$ is bounded. One can suppose that this sequence has a limit $\widetilde{R}$ in $\mathbb{R}_+$. By Corollary 5.3, one has $\widetilde{R} > 0$. One has also $f'_{\alpha_0,\gamma_0(\alpha_0)}(\widetilde{R}) = b'_{\alpha_0,\gamma_0(\alpha_0)}(\widetilde{R}) = 0$, which means $\alpha_0 \in A_{h=0}$ and contradicts the fact that $A_{h=0}$ is open.

4. is straightforward. □

7. THE ANALYTIC SET $\mathcal{N}_4$ AND THE BOUNDARY OF $\mathcal{N}$

The Comparison Lemma 5.5 implies that any (α, c), in the analytic set $\Gamma_{h=0}$ of the previous section, is a boundary point of $\mathcal{N}$. The pair (α_0, c_0) corresponding to a BC-vortex in Proposition 6.3 is also a boundary point of $\mathcal{N}$. It has been seen also (Corollary 4.2) that normal solutions $(\alpha, 0)$ in the boundary of $\mathcal{N}$ are those such that $\alpha \in [0, \alpha_4]$. By Proposition 3.4, as (α, c)'s in $\mathcal{N}_1$, $\mathcal{N}_2$ of $\mathcal{N}_3$ are interior points of $\mathcal{N}$, other points in the bondary of $\mathcal{N}$ must be, either some other BC -vortices, if they exist, or (α, c)'s belonging to $\mathcal{N}_4$.

In order to complete the map of the set $\mathcal{N}$, one should provide a description of the set

$$\begin{aligned}\mathcal{N}_4 &= \{(\alpha, c) \in (\mathbb{R}_+^*)^2 : \exists \bar{r} > 0 \quad f'_{\alpha,c}(\bar{r}) = f''_{\alpha,c}(\bar{r}) = 0\} \\ &= \{(\alpha, c) \in \widetilde{\mathcal{N}} : f''_{\alpha,c}(\widetilde{R}(\alpha, c)) = 0\} .\end{aligned} \tag{7.1}$$

$\mathcal{N}_4$ is an analytic set, as the projection on $(\mathbb{R}_+^*)^2$ of the analytic set $\{(\alpha, c, r) \in (\mathbb{R}_+^*)^3 : f'_{\alpha,c}(r) = f''_{\alpha,c}(r) = 0\}$.

There are some open questions to which only partial answers will be given:

- Is $\mathcal{N}_4$ connected ?
- Is any point of $\mathcal{N}_4$ a boundary point of $\mathcal{N}$?
- Is $\mathcal{N}_4$ a smooth curve ?

The results can be summarized in the following proposition:

Proposition 7.1. 1. $\mathcal{N}_4$ *is a one-dimensional analytic set.*

2. *The point* $(\alpha_4, 0)$ *is the only point of the horizontal axis belonging to the closure of* $\mathcal{N}_4$*. In a neighbourhood of* $(\alpha_4, 0)$*,* $\mathcal{N}_4$ *is a smooth analytic curve contained in the boundary* $\partial\mathcal{N}$*, with infinite slope at* $(\alpha_4, 0)$*.*

3. *If either* $\frac{\partial f'_{\alpha,c}}{\partial \alpha}$ *or* $\frac{\partial f'_{\alpha,c}}{\partial c}$ *does not vanish at* $\widetilde{R}(\alpha, c)$ *for an* (α, c) *in* $\mathcal{N}_4$*, then* (α, c) *belongs to the boundary* $\partial\mathcal{N}$ *and* $\partial\mathcal{N} \cap \mathcal{N}_4$ *is a smooth analytic curve in a neighbourhood of* (α, c)*.*

In order to prove the proposition, we state a preliminary lemma allowing us to write $\mathcal{N}_4$ as an analytic set with equation $D(\alpha, c) = 0$, where $D(\alpha, c)$ is an analytic extension of the reduced discriminant $\frac{1}{4}(\widetilde{\widetilde{R}}(\alpha, c) - \widetilde{R}(\alpha, c))^2$.

Lemma 7.2. *There exist an open neighbourhood $\mathcal{W}$ of $\mathcal{N}_3 \cup \mathcal{N}_4$ in $(\mathbb{R}_+^*)^2$, two analytic functions $(\alpha, c) \to \mu_1(\alpha, c)$ and $(\alpha, c) \to \mu_2(\alpha, c)$ defined on $\mathcal{W}$, an analytic function $(\alpha, c, r) \to h_{\alpha,c}(r)$ defined for $(\alpha, c) \in \mathcal{W}$ and $0 \leq r < \rho(\alpha, c)$, such that*

i) *for $(\alpha, c) \in \mathcal{N}_3 \cup \mathcal{N}_4$, $\mu_1(\alpha, c) = \frac{\widetilde{R}(\alpha,c)+\widetilde{\widetilde{R}}(\alpha,c)}{2}$ and $\mu_2(\alpha, c) = \widetilde{R}(\alpha, c)\widetilde{\widetilde{R}}(\alpha, c)$, with the convention $\widetilde{\widetilde{R}}(\alpha, c) = \widetilde{R}(\alpha, c)$ if (α, c) belongs to $\mathcal{N}_4$;*

ii) *for $(\alpha, c) \in \mathcal{W}$ and $0 < r < \rho(\alpha, c)$, one has $h_{\alpha,c}(r) > 0$ and*
$$f'_{\alpha,c}(r) = \left(r^2 - 2\mu_1(\alpha, c)r + \mu_2(\alpha, c)\right) h_{\alpha,c}(r); \tag{7.2}$$

iii) *setting $D(\alpha, c) = \mu_1(\alpha, c)^2 - \mu_2(\alpha, c)$, the following properties hold true:*

iii.a) *one has $(\alpha, c) \in \mathcal{N}_4$ (respectively $(\alpha, c) \in \mathcal{N}_3$) if and only if $(\alpha, c) \in \mathcal{W}$ and $D(\alpha, c) = 0$ (respectively $D(\alpha, c) > 0$);*

iii.b) *for $(\alpha, c) \in \mathcal{N}_4$, one has $\frac{\partial D(\alpha,c)}{\partial \alpha} = 0$ if and only if $\frac{\partial f'_{\alpha,c}}{\partial \alpha} = 0$ at $r = \widetilde{R}(\alpha, c)$, and $\frac{\partial D(\alpha,c)}{\partial c} = 0$ if and only if $\frac{\partial f'_{\alpha,c}}{\partial c} = 0$ at $r = \widetilde{R}(\alpha, c)$;*

iii.c) *if $(\alpha, c) \in \mathcal{N}_4$ is an interior point of $\mathcal{N}$, one has $\frac{\partial f'_{\alpha,c}}{\partial \alpha} = \frac{\partial f'_{\alpha,c}}{\partial c} = 0$ at $r = \widetilde{R}(\alpha, c)$.*

Proof. On $\mathcal{N}_3$, the maps $(\alpha, c) \to \frac{\widetilde{R}(\alpha,c)+\widetilde{\widetilde{R}}(\alpha,c)}{2}$, $(\alpha, c) \to \widetilde{R}(\alpha, c)\widetilde{\widetilde{R}}(\alpha, c)$ and $(\alpha, c) \to D(\alpha, c) = (\widetilde{\widetilde{R}}(\alpha, c) - \widetilde{R}(\alpha, c))^2$ are analytic functions.

Let (α_0, c_0) belong to $\mathcal{N}_4$, with $R_0 = \widetilde{R}(\alpha_0, c_0)$. By Theorem 1, one has $f'''_{\alpha_0,c_0}(R_0) > 0$, and the Riesz Presentation theorem implies the existence of a neighbourhood $\mathcal{U}_0$ of (α_0, c_0) and a neighbourhood $\mathcal{V}_0$ of R_0, of analytic functions $\nu_1(\alpha, c)$ and $\nu_2(\alpha, c)$ on $\mathcal{U}_0$, of an analytic function $(\alpha, c, r) \to h_{\alpha,c}(r)$ defined for $(\alpha, c) \in \mathcal{U}_0$ and $r \in \mathcal{V}_0$, such that, for all $(\alpha, c, r) \in \mathcal{U}_0 \times \mathcal{V}_0$,
$$\begin{aligned} &h_{\alpha,c}(r) > 0 \\ f'_{\alpha,c}(r) &= \left((r - R_0)^2 - 2\nu_1(\alpha, c)(r - R_0) + \nu_2(\alpha, c)\right) h_{\alpha,c}(r)\ . \end{aligned} \tag{7.3}$$
One can rewrite (7.3) as
$$f'_{\alpha,c}(r) = \left(r^2 - 2\mu_1(\alpha, c)r + \mu_2(\alpha, c)\right) h(\alpha, c, r) \tag{7.4}$$

for some analytic functions $\mu_1(\alpha,c)$ and $\mu_2(\alpha,c)$ defined on $\mathcal{U}_0$.

For $(\alpha,c)\in\mathcal{N}_4\cap\mathcal{U}_0$, we have $\mu_1(\alpha,c)=\widetilde{R}(\alpha,c)$ and $\mu_2(\alpha,c)=\widetilde{R}(\alpha,c)^2$.

For $(\alpha,c)\in\mathcal{U}_0\cap\mathcal{N}_3$, supposing $\mathcal{U}_0$ small enough, $\widetilde{R}(\alpha,c)$ and $\widetilde{\widetilde{R}}(\alpha,c)$ belong to $\mathcal{V}_0$, so that one has $\mu_1(\alpha,c)=\frac{\widetilde{R}(\alpha,c)+\widetilde{\widetilde{R}}(\alpha,c)}{2}$ and $\mu_2(\alpha,c)=\widetilde{R}(\alpha,c)\widetilde{\widetilde{R}}(\alpha,c)$. Finally, we have proved that the analytic maps on $\mathcal{N}_3$, $(\alpha,c)\to\frac{\widetilde{R}(\alpha,c)+\widetilde{\widetilde{R}}(\alpha,c)}{2}$ and $(\alpha,c)\to\widetilde{R}(\alpha,c)\widetilde{\widetilde{R}}(\alpha,c)$, extend as analytic maps on a neighbourhood $\mathcal{W}$ of $\mathcal{N}_3\cup\mathcal{N}_4$. Item i) of the lemma is proved.

For fixed $(\alpha,c)\in\mathcal{N}_3$ or $(\alpha,c)\in\mathcal{N}_4$, the two maps $r\to f'_{\alpha,c}$ and $r\to r^2-2\mu_1(\alpha,c)r+\mu_2(\alpha,c)$ have the same zeroes with the same multiplicity, so that the quotient map $h_{\alpha,c}(r)=\frac{f'_{\alpha,c}(r)}{r^2-2\mu_1(\alpha,c)r+\mu_2(\alpha,c)}$ is well defined and does not vanish for $r>0$. As it is >0 for $r>0$ close to 0, one has $h_{\alpha,c}(r)>0$ for any $r>0$.

For $(\alpha,c)\in\mathcal{W}$ not belonging to $\mathcal{N}_3\cup\mathcal{N}_4$, the two maps $f'_{\alpha,c}$ and $r\to r^2-2\mu_1(\alpha,c)r+\mu_2(\alpha,c)$ do not vanish for $r>0$, so that the quotient map $h_{\alpha,c}$ again exists and does not vanish for $r>0$.

This proves item ii) of the lemma.

Items iii.a) is straightforward. iii.b) is proved by differentiating (7.2) with respect to α or c at $r=\widetilde{R}(\alpha,c)$: if $(\alpha,c)\in\mathcal{N}_4$, one has, at $r=\widetilde{R}(\alpha,c)=\mu_1(\alpha,c)$

$$
\begin{aligned}
\frac{\partial f'_{\alpha,c}(r)}{\partial\alpha} &= \Big(-2\frac{\partial\mu_1(\alpha,c)}{\partial\alpha}r+\frac{\partial\mu_2(\alpha,c)}{\partial\alpha}\Big)h_{\alpha,c}(r)\\
&= \Big(-2\frac{\partial\mu_1(\alpha,c)}{\partial\alpha}\mu_1(\alpha,c)+\frac{\partial\mu_2(\alpha,c)}{\partial\alpha}\Big)h_{\alpha,c}(r)=-\frac{\partial D(\alpha,c)}{\partial\alpha}h_{\alpha,c}(r)
\end{aligned}
$$

with $h_{\alpha,c}(r)\neq 0$, so that one has $\frac{\partial f'_{\alpha,c}(r)}{\partial\alpha}=0$ at $r=\widetilde{R}(\alpha,c)$ if and only if $\frac{\partial D(\alpha,c)}{\partial\alpha}=0$.

The same proof applies for $\frac{\partial f'_{\alpha,c}(r)}{\partial c}=0$ at $r=\widetilde{R}(\alpha,c)$.

For iii.c), notice that if (α,c) in $\mathcal{N}_4$ is an interior point, then $D(\alpha,c)=0$ and D is nonnegative in a neighbourhood of (α,c) so that its partial derivatives will vanish at (α,c). □

Proof of Proposition 7.1. 1. The fact that $\mathcal{N}_4$ is an analytic set is obvious, and item 2 implies that it has at least dimension one. In order to prove that it is not of dimension two, it suffices to show that it has an empty interior. So, we assume that it has a nonempty interior, and we seek a contradiction.

Suppose that the disk of center (α_0, c_0) and radius δ_0 is contained in $\mathcal{N}_4$. Without loss of generality, we can suppose $\alpha_0 \neq \alpha_4$ (defined in Proposition 4.1).

We note that, for given α_0, the Comparison Lemma implies that there is at most one c such that (α_0, c) corresponds to a BC -vortex. It follows that at least one of the two following assertions holds true:

either (i) $\forall\, c \geq c_0$, (α_0, c) does not correspond to a BC -vortex
or (ii) $\forall\, c \leq c_0$, (α_0, c) does not correspond to a BC -vortex.

Let us suppose that (i) holds, and define

$$\Gamma = \{c \geq c_0 : \forall\, \widetilde{c} \in [c_0, c]\ (\alpha_0, c) \in \mathcal{N}_4\}\ .$$

By assumption, Γ is not empty, and by Corollary 5.2, it is bounded above, so that $c_1 = \sup \Gamma$ exists. By assumption (i), $\widetilde{R}(\alpha_0, c)$ remains bounded as c tends increasingly to c_1 (if not, then by Corollary 5.4, (α_0, c_1) would correspond to a BC -vortex). Then, arguments similar to those used in many proofs here show that $(\alpha_0, c_1) \in \mathcal{N}_4$. The map $D(\alpha, c)$ defined in the previous lemma is defined in a neighbourhood of (α_0, c_1) and vanishes at (α_0, c) for $c \in [c_0, c_1]$, so that, by analyticity, it will vanish at (α_0, c) for $c > c_1$ close to c_1. This proves that, for $c > c_1$ close to c_1, one has $(\alpha_0, c) \in \mathcal{N}_4$ and contradicts the definition of c_1 as the upper bound of Γ.

Let us suppose now that (ii) holds, and define

$$\Gamma = \{c \leq c_0 : \forall\, \widetilde{c} \in [c_0, c]\ (\alpha_0, c) \in \mathcal{N}_4\}\ .$$

A similar argument shows that c_1 must be equal to 0. This means that, for arbitrarily small $c > 0$, (α_0, c) lies in $\mathcal{N}_4$. As we have supposed $\alpha_0 \neq \alpha_4$, this contradicts the topology of symmetric vortices for small c established in Proposition 4.1.

2. As in the proof of Proposition 6.3, one has $\frac{\partial F_{\alpha,c}}{\partial c}_{|c=0} \equiv 0$. From the proof of Lemma 4.5 one deduces

$$F_{\alpha,0}(r) = \frac{1}{\sqrt{2\alpha d + \kappa^2}} \varphi_\beta\big(r\sqrt{2\alpha d + \kappa^2}\big)$$

with $\beta = \frac{\alpha}{2\alpha d + \kappa^2}$. At $\alpha = \alpha_4$ and $\beta = \beta_4$, since $\varphi_\beta'(\widetilde{R}_\beta) = \varphi_\beta''(\widetilde{R}_\beta) = 0$, one has

$$\frac{\partial F_{\alpha,0}'}{\partial \alpha}\big(\widetilde{R}(\alpha, 0)\big) = \frac{\partial \beta}{\partial \alpha}\, \frac{\partial \varphi_\beta'}{\partial \beta}\big(\widetilde{R}_\beta\big)$$

hence $\frac{\partial F_{\alpha,0}'}{\partial \alpha}\big(\widetilde{R}(\alpha, 0)\big) \neq 0$ since one has $\frac{\partial \varphi_\beta'}{\partial \beta} > 0$ on $\mathbb{R}_+^*$.

One has also $\varphi'''_\beta(\widetilde{R}_\beta) \neq 0$ at $\beta = \beta_4$, and consequently, according to Remark 4.4, $F'''_{\alpha,0}\big(\widetilde{R}(\alpha,0)\big) \neq 0$ at $\alpha = \alpha_4$. The implicit function theorem provides then two smooth maps $c \to \alpha(c)$ and $c \to R(c)$ such that $f'_{\alpha(c),c}(R(c)) = f''_{\alpha(c),c}(R(c)) = 0$; this defines a smooth curve in $\mathcal{N}_4$ issued from $(\alpha_4, 0)$ with infinite slope.

For small c, one will have on this curve $\frac{\partial f'_{\alpha,c}}{\partial\alpha}\big(\widetilde{R}(\alpha,c)\big) \neq 0$, so that (α, c) will correspond to a boundary point of $\mathcal{N}$ according to item 3 of the proposition, which in turn is a mere consequence of iii.c. of the previous lemma. □

8. Applied magnetic field and equilibrium states.

For a given body, has been shown the existence of a *superheating field* $\bar{h}$ such that, for $h > \bar{h}$, the only critical state of the Ginzburh-Landau functional is the normal state. In the case of thin films, one can refer to [9] and references therein, mainly [8]. For bounded bodies in $\mathbb{R}^3$ of cylindrical bodies with bounded cross section, this result is due to T. Giorgi and D. Phillips [15]. It implies that, for a given $\bar{r}$, there cannot exist $(\bar{r}, h)$-Neumann symmetric vortices when h is too large.

On the other hand, if $\bar{r}$ is allowed to vary, $(\bar{r}, h)$-Neumann symmetric vortices can exist for arbitrarily large values of h. What is shown in this section is that, for type II materials, symmetric vortices cannot appear as local minimizers of the energy functional when the intensity h of the applied magnetic field is either too high or too low.

Let us call *equilibrium state* for $\mathcal{G}_{\bar{r},h}$ a critical state for this functional which is a local minimizer. For such an equilibrium state (ψ, A), the quadratic form $D^2\mathcal{G}_{\bar{r},h}(\psi, A)$ on $H^1(B_{\bar{r}}, \mathbb{C} \times \mathbb{R}^2)$, is nonnegative. A critical state (ψ, A) of $\mathcal{G}_{\bar{r},h}$ will be said to be *unstable* if the second derivative $D^2\mathcal{G}_{\bar{r},h}(\psi, A)$, as a quadratic form on $H^1(B_{\bar{r}}, \mathbb{C} \times \mathbb{R}^2)$, takes at least one stricly negative value.

Proposition 8.1. *Let $\kappa > 1/\sqrt{2}$ and $d \geq 2$ be given. Then there exists a minimal value $h_\mu > 0$ and a maximal value $h_M > h_\mu$ such that*

> *for $\bar{r}$ and h in $\mathbb{R}^*_+$, if (ψ, A) is a symmetric vortex of degree d which is an equilibrium state for $\mathcal{G}_{\bar{r},h}$, then one has $h_\mu \leq h \leq h_M$.*

The proof relies on some preliminary lemmas. We suppose $\kappa > 1/\sqrt{2}$ and $d \geq 2$.

Lemma 8.2. *Let (α_0, c_0) in $\mathbb{R}^*_+$ correspond to a BC -vortex. There exists a neighbourhood $\mathcal{U}$ of (α_0, c_0) such that:*

- *If* $(\alpha, c) \in \mathcal{N} \bigcap \mathcal{U}$*, then, setting* $\bar{r} = \widetilde{R}(\alpha, c)$ *and* $h = -\frac{b'_{\alpha,c}(\bar{r})}{\kappa \bar{r}}$ *, the symmetric vortex* (ψ, A) *corresponding to* (α, c)*, as a critical state of* $\mathcal{G}_{\bar{r},h}$*, is unstable.*
- *If* $(\alpha, c) \in \mathcal{N}_3 \bigcap \mathcal{U}$*, then, setting* $\bar{r} = \widetilde{\widetilde{R}}(\alpha, c)$ *and* $h = -\frac{b'_{\alpha,c}(\bar{r})}{\kappa \bar{r}}$ *, the symmetric vortex* (ψ, A) *corresponding to* (α, c)*, as a critical state of* $\mathcal{G}_{\bar{r},h}$*, is unstable.*

Proof. Let (ψ_0, A_0) be the BC-vortex corresponding to (α_0, c_0). By a result of [16], there exists $(\varphi, B) \in H^1(\mathbb{R}^2, \mathbb{C} \times \mathbb{R}^2)$ such that the quadratic form $D^2 G_{+\infty,0}(\psi_0, A_0)$ is strictly negative when evaluated at (φ, B) . By density, one can suppose that (φ, B) is smooth with support in a bounded disk B_R. By continuity, for (α, c) close enough to (α_0, c_0), one will have $\bar{r} \geq R$ and

$$D^2 G_{\bar{r},h}(\psi, A)[(\varphi, B), (\varphi, B)] < 0$$

(with $\bar{r}$ and h as defined in the setting of the lemma). □

Lemma 8.3. *Let* (α_0, c_0) *belong to the curve* $\Gamma_{h=0}$ *of Definition* 6.2*, and* (ψ_0, A_0) *be the corresponding symmetric vortex. Set* $r_0 = \widetilde{R}(\alpha_0, c_0)$*. Then:*

1. (ψ_0, A_0) *is an unstable critical state of the functional* $\mathcal{G}_{\bar{r}_0,0}$*.*
2. *For* (α, c) *in* $\mathcal{N}$ *close to* (α_0, c_0)*, the corresponding symmetric vortex* (ψ, A) *is an unstable critical state of* $\mathcal{G}_{\bar{r},h}$ *with* $\bar{r} = \widetilde{R}(\alpha, c)$ *and* $h = -\frac{b'_{\alpha,c}(\bar{r})}{\kappa \bar{r}}$*.*

Proof. For 1., set $\sigma(r) = f'(r)$, $\tau(r) = f(r)b(r)/r$, $\beta(r) = -b'(r)/r$ with $f = f_{\alpha_0,c_0}$ and $b = b_{\alpha_0,c_0}$. Set

$$\varphi(r, \theta) = (\sigma(r) - \tau(r))e^{i(d+1)\theta} + (\sigma(r) + \tau(r))e^{i(d-1)\theta},$$
$$B(r, \theta) = \beta(r)\sin(\theta)\frac{\partial}{\partial r} + \beta(r)\cos(\theta)\frac{1}{r}\frac{\partial}{\partial \theta}.$$

The computations in sections 5 and 6 of [13] lead to

$$D^2 G_{\bar{r}_0,0}(\psi_0, A_0)[(\varphi, B), (\varphi, B)] =$$
$$\bar{r}\sigma(\bar{r})\sigma'(\bar{r}) + \bar{r}\tau(\bar{r})\tau'(\bar{r}) + 2\bar{r}\beta(\bar{r})\beta'(\bar{r}) = -\frac{f^2(\bar{r})b^2(\bar{r})}{\bar{r}} < 0,$$

which provides the result.

For 2., identify any $H^1(B_{\bar{r}}, \mathbb{C} \times \mathbb{R}^2)$ to $H^1(B_1, \mathbb{C} \times \mathbb{R}^2)$ by making a dilation of ratio $1/\bar{r}$, and invoke continuity. □

Lemma 8.4. *Let* (α, c) *belong to* $\mathcal{N}_2$ *and suppose that* $b_{\alpha,c}$ *is decreasing on* $\mathbb{R}_+$*. Then one has* $b_{\alpha,c}(r) < 0$ *for* r *large enough, and there exists* $\eta = \eta(\alpha, c)$

in $\mathbb{R}_+^*$ *such that*

$$\lim_{r\to+\infty} \frac{b'_{\alpha,c}(r)}{r} = -\kappa\eta\ .$$

Proof. Suppose that $b_{\alpha,c}$ is nonnegative and decreasing, thus bounded. Then one has

$$\lim_{r\to+\infty} \frac{b^2_{\alpha,c}(r)}{r^2} - \kappa^2\big(1 - f_{\alpha,c}(r)^2\big) = -\kappa^2 < 0$$

(since $f_{\alpha,c}(r) \to 0$ at infinity) and consequently $(rf'_{\alpha,c}(r))' < 0$ for r large enough, so that $\lim_{r\to+\infty} rf'_{\alpha,c}(r)$ exists in $[-\infty,+\infty)$. A nonzero limit would imply $\lim_{r\to+\infty} f_{\alpha,c}(r) = \pm\infty$ and a contradiction. A zero limit, and a negative derivative, would imply $rf'_{\alpha,c}(r) > 0$ and f increasing for r large enough, which would contradict the fact that f is positive and vanishes at infinity. Hence $b_{\alpha,c} > 0$ on $\mathbb{R}_+^*$ leads to a contradiction.

We have proved that b vanishes at some $z > 0$. For $r > z$, one has $b_{\alpha,c}(r) < 0$ and $(\frac{b'_{\alpha,c}(r)}{r})' = \frac{b_{\alpha,c}(r) f^2_{\alpha,c}(r)}{r} < 0$. So $\beta = \lim_{r\to+\infty} \frac{b'_{\alpha,c}(r)}{r}$ exists in $[-\infty, 0)$.

Consequently, there exists $\lambda > 0$ such that, for r large enough, one has $b_{\alpha,c}(r) \leq -\lambda r^2$. This implies that

$$\big(rf'_{\alpha,c}(r)\big)' = rf_{\alpha,c}(r)\Big(\frac{b^2_{\alpha,c}(r)}{r^2} - \kappa^2(1 - f^2_{\alpha,c}(r))\Big) \sim f_{\alpha,c}(r)\frac{b^2_{\alpha,c}(r)}{r} > 0$$

from which we deduce the existence of $\lim_{r\to+\infty} rf'_{\alpha,c}(r)$ in $(-\infty,+\infty]$. As above, a nonzero limit would contradict the fact that $f_{\alpha,c}$ vanishes at infinity, so that one has $\lim_{r\to+\infty} rf'_{\alpha,c}(r) = 0$.

This result implies in turn the convergence of the integral

$$\int_0^{+\infty} f_{\alpha,c}(r)\frac{b^2_{\alpha,c}(r)}{r} dr$$

which will imply the convergence of the integral $\int_0^{+\infty} \frac{b_{\alpha,c}(r) f^2_{\alpha,c}(r)}{r} dr$ and $\beta > -\infty$. □

Lemma 8.5. *Let* $(\alpha, c) \in \mathcal{N}_2$, *and let* (α_n, c_n) *be a sequence in* $\mathcal{N}_3$ *converging to* (α, c). *Suppose* $b'_{\alpha_n,c_n}(\tilde{\tilde{R}}(\alpha_n, c_n) < 0$ *for all* n. *Then, setting* $\bar{r}_n = \tilde{\tilde{R}}(\alpha_n, c_n)$ *and* $h_n = -\frac{b'_{\alpha_n,c_n}(\bar{r}_n)}{\kappa\bar{r}_n}$, *one has* $lim_{n\to\infty}\bar{r}_n = +\infty$ *and* $lim_{n\to\infty}h_n = \eta(\alpha, c)$ *(as defined in the previous lemma).*

Proof. Set $f_n = f_{\alpha_n,c_n}$, $f = f_{\alpha,c}$, $b_n = b_{\alpha_n,c_n}$, $b = b_{\alpha,c}$ and $\eta = \eta(\alpha, c)$.

One has $\lim_n \widetilde{R}(\alpha_n, c_n) = \widetilde{R}(\alpha, c)$. If the sequence $\bar{r}_n$ had an accumulation point $\bar{r}$ in $\mathbb{R}_+^*$, one would have $\bar{r} \geq \widetilde{R}(\alpha, c)$ and (α, c) would correspond either to an $\mathcal{N}_3$-vortex (if $\bar{r} > \widetilde{R}(\alpha, c)$) or an $\mathcal{N}_4$-vortex (if $\bar{r} = \widetilde{R}(\alpha, c)$). Hence we have a contradiction.

One has $\lim_n \bar{r}_n = +\infty$. As b_n' is negative on $(0, \bar{r}_n]$, b' is nonpositive on $\mathbb{R}_+$ and the conclusions of the previous lemma hold. In particular, b vanishes at some $z > 0$ and is strictly negative for $r > z$. By continuity, each b_n vanishes at some $z_n > 0$ (with $\lim_n z_n = z$) and there is some N_0 and some $R_0 > 0$ such that, for $n \geq N_0$ and $r \geq R_0$, one has $b_n(r) < 0$. Note that $b_n'(r)/r$ decreases then on $[R_0, +\infty)$.

Fix $\delta > 0$ small enough. Fix $R_1 \geq R_0$ such that $|\frac{b'(r)}{r} + \kappa\eta| \leq \delta$, for any $r \geq R_1$. Fix $N_1 \geq N_0$ such that, for $n \geq N_1$, one has $\bar{r}_n \geq R_1$ and $|b_n'(R_1) - b'(R_1)| \leq R_1\delta$. One has then, for $n \geq N_1$ and $r \in [R_1, \bar{r}_n]$:

$$-\kappa h_n = \frac{b_n'(\bar{r}_n)}{\bar{r}_n} \leq \frac{b_n'(r)}{r} \leq \frac{b_n'(R_1)}{R_1} \leq -\kappa\eta + 2\delta \ . \tag{8.1}$$

From this sequence of inequalities, one deduces easily $\liminf_n h_n \geq \eta$. We shall prove the reverse inequality.

Integrating $\frac{b_n'(s)}{s} \leq -\kappa\eta + 2\delta$ (with $-\kappa\eta + 2\delta < 0$), between R_1 and r, one gets

$$b_n(r) \leq b_n(R_1) - \frac{\kappa\eta - 2\delta}{2}\left(r^2 - R_1^2\right)$$

which provides

$$\begin{gathered} \exists R_2 > R_1 \quad \forall\, r \geq R_2 \quad \forall\, n \geq N_1 \\ b_n(r) \leq -1 \text{ and } \frac{b_n^2(r)}{r^2} - \kappa^2(1 - f_n(r)^2) \geq \frac{1}{2}\frac{b_n^2(r)}{r^2} \ . \end{gathered} \tag{8.2}$$

Replacing, if necessary, R_2 by some larger value, we can suppose $|R_2 f'(R_2)| \leq \delta$ (since, as seen in the proof of the previous lemma, $\lim_{r\to+\infty} r f'(r) = 0$). Replacing, if necessary, N_1 by a larger value, one can suppose $r_n \geq R_2$ and $R_2|f_n'(R_2) - f'(R_2)| \leq \delta$ for $n \geq N_1$. One has then, for $n \geq N_1$:

$$\begin{aligned} 2\delta \geq -R_2 f_n'(R_2) &= \int_{R_2}^{\bar{r}_n} \left(r f_n'(r)\right)' dr \\ &= \int_{R_2}^{\bar{r}_n} r f_n(r)\Big(\frac{b_n^2(r)}{r^2} - \kappa^2(1 - f_n(r)^2)\Big) dr \geq \frac{1}{2}\int_{R_2}^{\bar{r}_n} r f_n(r)\frac{b_n^2(r)}{r^2} dr. \end{aligned}$$

As $f_n(r) < 1$ and $b_n(r) < -1$ on $[R_2, \bar{r}_n]$, we have proved

$$\int_{R_2}^{\bar{r}_n} \frac{b_n(r) f_n^2(r)}{r} dr \geq -4\delta$$

and consequently

$$0 < \frac{b_n'(R_2)}{R_2} + \kappa h_n \leq 4\delta.$$

This last inequality provides

$$\frac{b'(R_2)}{R_2} + \liminf_n \kappa h_n \leq 4\delta$$

and $-\kappa h + \liminf_n \kappa h_n \leq 5\delta$ for any δ, i.e., $\liminf_n h_n \leq h$. □

Lemma 8.6. *Define $\mathcal{N}_2^-$ as the subset of $\mathcal{N}_2$:*

$$\{(\alpha, c) \in \mathcal{N}_2 : b'_{\alpha,c}(r) \leq 0 \ \forall \ r \in \mathbb{R}^+\} \ .$$

Then, on $\mathcal{N}_2^-$, the map $(\alpha, c) \to \eta(\alpha, c)$ provided by Lemma 8.4 is continuous.

The proof is quite similar to the proof of the previous lemma, and will be omitted.

Proof of Proposition 8.1. One checks easily that the set of (α, c)'s corresponding to BC-vortices is closed. By Proposition 5.1, this set is bounded, thus compact. Lemma 8.2 provides then an open neighbourhood W_1 of this set in $\mathbb{R}_+^* \times \mathbb{R}_+^*$ such that, for $(\alpha, c) \in \mathcal{N} \bigcap W_1$, for any pair $(\bar{r}, h)$ such that the corresponding vortex is a critical state for $\mathcal{G}_{\bar{r},h}$, it is not an equilibrium state.

With the notation of Definition 6.2, the set $\big(\Gamma_{h=0} \cup \{(0,0)\}\big) \backslash W_1$ is compact, and Lemma 8.5 provides an open neighbourhood W_2 of this set such that, for $(\alpha, c) \in W_2$, the corresponding vortex is a critical state for $\mathcal{G}_{\bar{r},h}$ but not an equilibrium state, with $\bar{r} = \widetilde{R}(\alpha, c)$ and $h = -\frac{b'_{\alpha,c}(\bar{r})}{\kappa \bar{r}}$.

Consider the compact set $(\mathcal{N} \cup ()0, \alpha_4] \times \{0\})) \backslash (W_1 \cup W_2)$ and, on it, the map

$$(\alpha, c) \to -\frac{b'_{\alpha,c}(\widetilde{R}(\alpha, c))}{\kappa \widetilde{R}(\alpha, c)} \ .$$

This map is continous, taking values in $\mathbb{R}_+^*$, so that there exist $\widetilde{h}_\mu > 0$ and $\widetilde{h}_M > \widetilde{h}_\mu$ such that it is bounded below by h_μ and bounded above by h_M.

Consider the set $\mathcal{N}_3^- = \{(\alpha, c) \in \mathcal{N}_3 : b'_{\alpha,c}(\widetilde{\widetilde{R}}(\alpha, c)) < 0\}$. As a consequence of Lemma 6.1, it is a closed subset of $\mathcal{N}_3$. The set $\mathcal{N}_2^-$ defined in

Lemma 8.6 is also a closed subset of $\mathcal{N}_2$. The following set

$$\Big(\mathcal{N}_2^- \cup \mathcal{N}_3^- \cup \mathcal{N}_4 \cup \big(]\alpha_2, \alpha_4] \times \{0\}\big)\Big) \setminus W_1$$

is again compact, and the map defined on it by

$$(\alpha, c) \to \begin{cases} \eta(\alpha, c) & \text{if } (\alpha, c) \in \mathcal{N}_2^- \\ -\frac{b'_{\alpha,c}(\widetilde{\widetilde{R}}(\alpha,c))}{\kappa \widetilde{\widetilde{R}}(\alpha,c)} & \text{if } (\alpha, c) \in \mathcal{N}_3^- \\ -\frac{b'_{\alpha,c}(\widetilde{R}(\alpha,c))}{\kappa \widetilde{R}(\alpha,c)} & \text{if } (\alpha, c) \in \mathcal{N}_4 \end{cases}$$

is continuous, taking values in $\mathbb{R}_+^*$ [Lemmas 8.4 and 8.5 extend to the case $c = 0$ without much difficulty].

This map is thus bounded below by $\widetilde{\widetilde{h}}_\mu > 0$ and bounded above by $\widetilde{\widetilde{h}}_M > \widetilde{\widetilde{h}}_\mu$. Hence the proposition, with $h_\mu = \min(\widetilde{h}_\mu, \widetilde{\widetilde{h}}_\mu)$ and $h_M = \max(\widetilde{h}_M, \widetilde{\widetilde{h}}_M)$.

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