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# POSITIVE SOLUTIONS FOR A CLASS OF SINGULAR SEMIPOSITONE $p$–LAPLACIAN PROBLEMS

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## 1. Introduction

In this paper, we consider the existence of positive solutions to the boundary-value problems

$$\begin{cases} (q(t)\phi(u'))' + \lambda r(t)f(u) = 0, \quad t \in (a,b) \\ u(a) = u(b) = 0, \end{cases} \tag{1.1}$$

where $\phi(x) = |x|^{p-2}x, p > 1,\ q, r : [a,b] \to [0,\infty),\ f : (0,\infty) \to \mathbb{R}$ may have negative values and may become infinite at 0, and $\lambda$ is a positive parameter.

When $q(t) = t^{N-1}$, (1.1) occurs in the study of radial solutions for the $p$−Laplacian boundary-value problem

$$\begin{cases} \Delta_p u + \lambda r(|x|)f(u) \ \text{ in } \ \Omega \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Delta_p u \equiv div(|\nabla u|^{p-2}\nabla u)$ and $\Omega$ is the annulus $0 < a < |x| < b,\ x \in \mathbb{R}^N$. We refer to the survey article [3] for the literature on this problem when $f$ is nonsingular. The case when $p = 2$ and $f$ is possibly singular at 0 with negative values was considered in [1,5]. In [5], Liu obtained the existence of at least two positive solutions to the problem

$$\begin{cases} u'' + \lambda f(t,u) = 0, \quad t \in (0,1) \\ u(0) = u(1) = 0, \end{cases}$$

for $\lambda > 0$ small, where $f$ satisfies, among other assumptions, $f(t,u) + M \geq 0$ for all $(t,u) \in (0,1) \times (0,\infty)$, for some $M > 0$, and $\lim s_{u\to 0^+} f(t,u) = \infty$,

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$\lim_{u\to\infty} \frac{f(t,u)}{u} = \infty$ uniformly for $t$ in a compact subset of $(0,1)$. In particular, the results in [5] apply to the model case

$$\begin{cases} u'' + \lambda(u^{-\alpha} + u^{\beta} - C) = 0, \ t \in (0,1) \\ u(0) = u(1) = 0, \end{cases} \tag{1.3}$$

where $\alpha, C > 0$, $\beta > 1$. Note that the results in [1] when applied to (1.3) give the existence of one positive solution for $\lambda > 0$ small. In this paper, we shall establish existence and multiplicity results for positive solutions of (1.1) with $r \in L^1(a,b)$ when $f$ is $p-$superlinear at $\infty$ and $\lambda > 0$ is sufficiently small. Our results complement and extend corresponding ones in [1,5] even in the case when $p = 2$. Our approach uses degree theory arguments and is similar in spirit to the one in [2].

## 2. Statement of main results

We make the following assumptions:

(A.1) $q \in C[a,b]$ and there exist $q_0, q_1 > 0$ such that

$$q_0 \le q(t) \le q_1 \text{ for all } t \in [a,b].$$

(A.2) $r \in L^1(a,b)$, $r \ge 0$ and $\int_a^b r(t)dt > 0$.

(A.3) $f : (0,\infty) \to \mathbb{R}$ is continuous.

(A.4) There exists a positive number $K$ such that

$$f(u) \ge -K \text{ for all } u > 0.$$

(A.5) $\liminf\limits_{u\to 0^+} f(u) > 0$.

(A.6) $\lim\limits_{u\to\infty} \dfrac{f(u)}{\phi(u)} = \infty$.

By a solution of (1.1), we mean a function $u \in C[a,b] \cap C^1(a,b)$ that satisfies (1.1). Our main results are:

**Theorem 2.1.** *Let* (A.1)-(A.5) *hold and* $\delta > 0$. *Then there exists a positive number* $\lambda_\delta$ *such that, for* $\lambda < \lambda_\delta$, (1.1) *has a positive solution* $u$ *with* $|u|_\infty < \delta$.

**Theorem 2.2.** *Let* (A.1)-(A.4), (A.6) *hold and* $\delta > 0$. *Then there exists a positive number* $\tilde{\lambda}_\delta$ *such that, for* $\lambda < \tilde{\lambda}_\delta$, (1.1) *has a positive solution* $\tilde{u}$ *with* $|\tilde{u}|_\infty > \delta$.

Since $(A.4)$ follows from (A.5) and (A.6), the following result holds:

**Theorem 2.3.** *Let* (A.1)-(A.3), (A.5), (A.6) *hold and* $\delta > 0$. *Then there exists a positive number* $\bar{\lambda}_\delta$ *such that, for* $\lambda < \bar{\lambda}_\delta$, (1.1) *has at least two positive solutions* $u, \tilde{u}$ *with* $|u|_\infty < \delta < |\tilde{u}|_\infty$.

**Example.** Let $0 < a < b$, $r(t) = \frac{1}{\sqrt{(t-a)(b-t)}}$, $q(t) = t^{N-1}$, and

$$f(u) = Au^{-\alpha} + Bu^{\beta} - C,$$

where $\alpha, \beta > 0, A, B \geq 0, A + B > 0$, and $C \in \mathbb{R}$. Then using Theorems $2.1 - 2.3$, we obtain the following results:

*i*) If $A > 0$, then (1.2) has a positive radial solution for $\lambda > 0$ small. If, in addition, $B > 0$ and $\beta > p - 1$, then (1.2) has at least two positive radial solutions for $\lambda > 0$ small.

*ii*) If $B > 0$ and $\beta > p-1$, then (1.2) has a positive radial solution for $\lambda > 0$ small. It should be noted that there is no $M > 0$ so that $r(t)f(u) + M \geq 0$ for all $(t, u) \in (a, b) \times (0, \infty)$, and hence the results in [5] do not apply even when $p = 2$.

## 3. Preliminary results

We shall denote the usual norm in $L^k(a, b)$ by $|.|_k,\ 1 \leq k \leq \infty$.

**Lemma 3.1.** *Let* (A.1), (A.2) *hold. Then there exists a unique positive solution* $z \in C^1[a, b]$ *to the problem*

$$\begin{cases} -(q(t)\phi(z'))' = r(t), & t \in (a, b) \\ z(a) = z(b) = 0. \end{cases} \tag{3.1}$$

*Furthermore,* $z$ *satisfies*

$$|z'|_\infty \leq \phi^{-1}\Big(\frac{|r|_1}{q_0}\Big). \tag{3.2}$$

**Proof.** By integrating the equation in (3.1), we see that

$$z(t) = \int_a^t \phi^{-1}\Big(\frac{C - \int_a^s r(\tau)d\tau}{q(s)}\Big)ds$$

is the unique solution of (3.1), where the constant $C$ is such that $z(b) = 0$. By the mean value theorem, there exists $s_0 \in [a, b]$ such that $C = \int_a^{s_0} r(\tau)d\tau$. Hence

$$z'(t) = \phi^{-1}\Big(\frac{\int_t^{s_0} r(\tau)d\tau}{q(t)}\Big),$$

and (3.2) follows. Finally, $z > 0$ in $(a, b)$ by Remark 3.1 below.

**Lemma 3.2.** *Let* $u_1, u_2 \in C[a, b] \cap C^1(a, b)$ *satisfy*

$$\begin{cases} -(q(t)\phi(u_1'))' \geq -(q(t)\phi(u_2'))', & t \in (a, b) \\ u_1(a) \geq u_2(a),\ u_1(b) \geq u_2(b). \end{cases}$$

*Then* $u_1 \geq u_2$ *on* $[a, b]$.

**Proof.** Since $(q(t)(\phi(u_1') - \phi(u_2'))' \leq 0$ on $(a,b)$, $q(t)(\phi(u_1') - \phi(u_2'))$ is nonincreasing on $(a,b)$. If there exists $\tau \in (a,b)$ such that $u_1'(\tau) = u_2'(\tau)$, then

$$\phi(u_1') - \phi(u_2') \leq 0 \quad \text{on } [\tau, b),$$

which implies $u_1' \leq u_2'$ on $[\tau, b)$, and therefore

$$u_1(t) - u_2(t) \geq u_1(b) - u_2(b) \geq 0 \text{ for } t \in [\tau, b].$$

Similarly, $u_1 \geq u_2$ on $[a, \tau]$. If $u_1'(t) \neq u_2'(t)$ for all $t \in (a,b)$ then either $u_1' > u_2'$ or $u_1' < u_2'$ on $(a,b)$. In the former case,

$$u_1(t) - u_2(t) > u_1(a) - u_2(a) \geq 0,$$

while in the latter case

$$u_1(t) - u_2(t) > u_1(b) - u_2(b) \geq 0$$

for $t \in (a,b)$. This completes the proof of Lemma 3.2.

**Lemma 3.3.** $\phi^{-1}(x-y) \geq \phi^{-1}(x/2) - 2\phi^{-1}(y)$ for all $x, y \geq 0$.

**Proof.** For $y \leq x/2$,

$$\phi^{-1}(x-y) \geq \phi^{-1}(x/2) \geq \phi^{-1}(x/2) - 2\phi^{-1}(y),$$

while for $y > x/2$,

$$\phi^{-1}(x-y) \geq \phi^{-1}(-y) = -\phi^{-1}(y) \geq \phi^{-1}(x/2) - 2\phi^{-1}(y).$$

The next result generalizes and improves Lemma 2.3 in [4], where $q(t) = t^{N-1}$ and $r$ is a positive constant.

**Lemma 3.4.** *Let (A.1),(A.2) hold and $\lambda \geq 0, K > 0$. Suppose that $u \in C[a,b] \cap C^1(a,b)$ satisfies*

$$\begin{cases} (q(t)\phi(u'))' \leq \lambda K r(t), \ t \in (a,b) \\ u(a) \geq 0, \ u(b) \geq 0. \end{cases}$$

*Then*

$$u(t) \geq \left(c_0|u|_\infty - 2\phi^{-1}\left(\lambda c_1\right)\right) p(t) \tag{3.3}$$

*for $t \in [a,b]$, where $p(t) = \min(t-a, b-t)$, $c_0 = \phi^{-1}\left(\frac{q_0}{2q_1}\right)\frac{1}{b-a}$, and $c_1 = \frac{K}{q_0}|r|_1$.*

**Proof.** By Lemma 3.2, $u \geq z_\lambda$, where $z_\lambda$ is the solution of

$$\begin{cases} (q(t)\phi(z_\lambda'))' = \lambda K r(t), \ \ t \in (a,b) \\ z_\lambda(a) = z_\lambda(b) = 0. \end{cases}$$

Note that $z_\lambda = -\phi^{-1}(\lambda K)z$, where $z$ is given by Lemma 3.1. Let $|u|_\infty = |u(t_0)|$ for some $t_0 \in [a,b]$.

We shall distinguish two cases.

**Case 1**. $u(t_0) \leq 0$. By (3.2) and the mean value theorem,

$$|z(t)| \leq \phi^{-1}\Big(\frac{|r|_1}{q_0}\Big)p(t)$$

for $t \in [a,b]$. Hence

$$u(t) \geq z_\lambda(t) \geq -\phi^{-1}\Big(\frac{\lambda K|r|_1}{q_0}\Big)p(t) = -\phi^{-1}\left(\lambda c_1\right)p(t), \tag{3.4}$$

where $c_1 = \frac{K}{q_0}|r|_1$, and

$$|u|_\infty = -u(t_0) \leq -z_\lambda(t_0) \leq \phi^{-1}\left(\lambda c_1\right)(b-a). \tag{3.5}$$

Combining (3.4) and (3.5), we obtain

$$u(t) \geq \Big(\frac{|u|_\infty}{b-a} - 2\phi^{-1}(\lambda c_1)\Big)p(t) \geq \big(c_0|u|_\infty - 2\phi^{-1}(\lambda c_1)\big)\,p(t),$$

where $c_0 = \phi^{-1}(\frac{q_0}{2q_1})\frac{1}{b-a}$.

**Case 2.** $u(t_0) > 0$. Let $w$ be the solution of

$$\begin{cases} (q(t)\phi(w'))' = \lambda K r(t), \quad t \in (a,b) \\ w(a) = 0, \; w(t_0) = |u|_\infty. \end{cases}$$

Then $u \geq w$ on $[a,t_0]$, by Lemma 3.2. A calculation shows that

$$w(t) = \int_a^t \phi^{-1}\Big(\frac{C_1 + \lambda K\int_a^s r(\tau)d\tau}{q(s)}\Big)ds,$$

where the constant $C_1$ is such that $w(t_0) = |u|_0$.

By the mean value theorem, there exists $s_1 \in [a,b]$ such that

$$\frac{|u|_\infty}{t_0 - a} = \phi^{-1}\Big(\frac{C_1 + \lambda K\int_a^{s_1} r(\tau)d\tau}{q(s_1)}\Big),$$

which implies

$$C_1 = \phi\Big(\frac{|u|_\infty}{t_0-a}\Big)q(s_1) - \lambda K\int_a^{s_1} r(\tau)d\tau.$$

Using (A.1) and Lemma 3.3, we deduce that

$$u(t) \geq w(t) = \int_a^t \phi^{-1}\Big(\frac{\phi\big(\frac{|u|_\infty}{t_0-a}\big)q(s_1) + \lambda K\int_{s_1}^s r(\tau)d\tau}{q(s)}\Big)ds$$

$$\geq \int_a^t \phi^{-1}\Big(\frac{\phi\big(\frac{|u|_\infty}{t_0-a}\big)q_0 - \lambda K|r|_1}{q_1}\Big)ds \geq \big(\phi^{-1}(c_0|u|_\infty - 2\phi^{-1}(\lambda c_1)\big)\,(t-a)$$

for $t \in [a, t_0]$. Similarly,

$$u(t) \geq \left(\phi^{-1}(c_0|u|_\infty - 2\phi^{-1}(\lambda c_1)\right)(b-t)$$

for $t \in [t_0, b]$, which completes the proof of Lemma 3.4.

**Remark 3.1.** When $\lambda = 0$, (3.3) becomes $u(t) \geq c_0|u|_\infty p(t)$, which is a special case of Lemma 2.2 in [2]. In particular, the solution $z$ in Lemma 3.1 satisfies $z(t) \geq c_0|z|_\infty p(t)$ *for* $t \in [a, b]$.

In Lemmas 3.5-3.7 that follow, we assume that $\varepsilon \geq 0$.

**Lemma 3.5.** *Let* (A.1)-(A.4) *hold and* $r, \varepsilon_0 > 0$. *Suppose that* $f(T+r+\varepsilon_0) > 0$ *for some* $T \geq 0$. *Then there exists a positive number* $\bar{\lambda}_r$ *such that, for* $\varepsilon < \varepsilon_0$ *and* $\lambda < \bar{\lambda}_r$, *any solution* $u$ *of*

$$\begin{cases} -(q(t)\phi(u'))' = \lambda\phi(\theta)r(t)f(|u|+\varepsilon), & t \in (a, b) \\ u(a) = u(b) = 0 \end{cases} \tag{3.6}$$

*for some* $\theta \in [0, 1]$ *satisfies* $|u|_\infty \neq r$.

**Proof.** Let $u$ be a solution of (3.6) for some $\theta \in [0, 1]$, and suppose $|u|_\infty = r$. Since (A.4) holds, it follows from Lemma 3.4 that

$$u(t) \geq (c_0 r - 2\phi^{-1}(\lambda c_1))p(t).$$

Suppose

$$c_0 r - 2\phi^{-1}(\lambda c_1) > 0.$$

Then $u(t) > 0$ for $t \in (a, b)$.

Define

$$\tilde{f}(t) = \sup_{t \leq s \leq T_r} f(s), \quad h(t) = \frac{1}{t}\int_0^t \frac{1}{\tilde{f}(s)} ds$$

for $t \in (0, T_r)$, where $T_r = T + r + \varepsilon_0$. Then, on $(0, T_r)$, $\tilde{f}, h > 0$, $\tilde{f}$ is nonincreasing, and $f \leq \tilde{f}$. Hence,

$$h(t) \leq \frac{1}{\tilde{f}(t)}$$

and

$$h'(t) = -\frac{1}{t^2}\int_0^t \frac{1}{\tilde{f}(s)} ds + \frac{1}{t\tilde{f}(t)} \geq 0 \tag{3.7}$$

for $t \in (0, T_r)$. Consequently, $u$ satisfies

$$-(q(t)\phi(u'))' = \lambda\phi(\theta)r(t)f(u(t)+\varepsilon) \leq \lambda r(t)\tilde{f}(u(t)+\varepsilon) \leq \frac{\lambda r(t)}{h_\varepsilon(u(t))},$$

where $h_\varepsilon(u) = h(u+\varepsilon)$. Since $h_\varepsilon > 0$ on $(0, r]$, this implies

$$-(q(t)\phi(u'))' h_\varepsilon(u) \le \lambda r(t) \tag{3.8}$$

for $t \in (a, b)$. Let $v = \int_0^u \phi^{-1}(h_\varepsilon(s))ds$. A calculation shows that

$$-(q(t)\phi(v'))' = -(q(t)h_\varepsilon(u)\phi(u'))' = -(q(t)\phi(u'))'h_\varepsilon(u) - q(t)\phi(u')h'_\varepsilon(u)u',$$

which, together with (3.7) and (3.8), implies

$$-(q(t)\phi(v'))' \le \lambda r(t),\ t \in (a, b).$$

Applying Lemma 3.2 gives

$$\int_0^{u(t)} \phi^{-1}(h(s))ds \le v(t) \le \phi^{-1}(\lambda)z(t) \le \phi^{-1}\Big(\frac{\lambda |r|_1}{q_0}\Big)(b-a),$$

where $z$ is given by Lemma 3.1. In particular,

$$\int_0^{r} \phi^{-1}(h(s))ds \le \phi^{-1}\Big(\frac{\lambda |r|_1}{q_0}\Big)(b-a).$$

From this, we conclude that if $\lambda < \bar\lambda_r$, where $\bar\lambda_r > 0$ satisfies

$$\phi^{-1}\Big(\frac{\bar\lambda_r |r|_1}{q_0}\Big)(b-a) < \int_0^{r} \phi^{-1}(h(s))ds$$

and $c_0 r - 2\phi^{-1}(\bar\lambda_r c_1) > 0$, thus $|u|_\infty \ne r$. This completes the proof of Lemma 3.5.

From the proof of Lemma 3.5, we have:

**Lemma 3.6.** *Let (A.1)-(A.4) hold and $\varepsilon_0 > 0$. Suppose that $f(T+\varepsilon_0) > 0$ for some $T > 0$. Let $\varepsilon < \varepsilon_0$ and $u$ be a positive solution of*

$$\begin{cases} -(q(t)\phi(u'))' = \lambda r(t) f(u+\varepsilon), & t \in (a,b) \\ u(a) = u(b) = 0 \end{cases}$$

*with $|u|_\infty \le T$. Then*

$$\int_0^{u(t)} \phi^{-1}(h(s))ds \le \phi^{-1}(\lambda) z(t),$$

*where $h(t) = \frac{1}{t}\int_0^t \frac{1}{\tilde f(s)}ds,\ \tilde f(t) = \sup_{t \le s \le T+\varepsilon_0} f(s)$.*

**Lemma 3.7.** *Let* (A.1)-(A.4) *and* (A.6) *hold. Let $\alpha$ be a nonnegative number and $\lambda > 0$. Suppose that $u$ satisfies*

$$\begin{cases} -(q(t)\phi(u'))' = \lambda r(t) f(|u|+\varepsilon),\ t \in (a,b) \\ u(a) = u(b) = \alpha. \end{cases} \tag{3.9}$$

*Then there exists $R_\lambda > 0$ such that $|u|_\infty < R_\lambda$. In particular, (3.9) has no solution for $\alpha \geq R_\lambda$.*

**Proof.** Let $[c,d] \subset (a,b)$ be such that $\int_c^d r(t)dt > 0$ and let $\min\limits_{c \leq s \leq d} p(s) = 2\gamma > 0$, where $p$ is given by Lemma 3.4. Let $\tilde{z}$ be the solution of

$$\begin{cases} -(q(t)\phi(\tilde{z}'))' = r(t), \quad t \in (c,d) \\ \tilde{z}(c) = \tilde{z}(d) = 0, \end{cases}$$

and choose $M > 0$ so that

$$\phi^{-1}(\lambda M)c_0\gamma|\tilde{z}|_\infty > 1 \tag{3.10}$$

where $c_0$ is given by Lemma 3.4. By (A.6), there exists $B > 0$ such that

$$f(u) > M\phi(u) \text{ for } u > B.$$

Suppose that $|u|_\infty \geq R_\lambda >> 1$. Then, by Lemma 3.4,

$$u(t) \geq (c_0|u|_\infty - 2\phi^{-1}(\lambda c_1))p(t) > 0$$

for $t \in [a,b]$, which implies

$$u(t) \geq \frac{c_0|u|_\infty}{2}p(t) \geq c_0\gamma|u|_\infty > B$$

for $t \in [c,d]$. Thus, $u$ satisfies

$$\begin{cases} -(q(t)\phi(u'))' \geq \lambda M\phi(c_0\gamma|u|_\infty)r(t), \ t \in (c,d) \\ u(c) \geq 0, \ u(d) \geq 0. \end{cases}$$

Hence, using Lemma 3.2, we obtain

$$u(t) \geq \phi^{-1}(\lambda M)c_0\gamma|u|_\infty\tilde{z}(t),$$

for $t \in [c,d]$. Consequently,

$$|u|_\infty \geq \phi^{-1}(\lambda M)c_0\gamma|u|_\infty|\tilde{z}|_\infty,$$

and so

$$1 \geq \phi^{-1}(\lambda M)c_0\gamma|\tilde{z}|_\infty,$$

a contradiction with (3.10) and the lemma follows.

Next, for $\varepsilon > 0$ and $v \in C[a,b]$, define $A_\varepsilon v = u$, where $u$ is the solution of

$$\begin{cases} -(q(t)\phi(u'))' = \lambda r(t)f(|v| + \varepsilon), \ t \in (a,b) \\ u(a) = u(b) = 0. \end{cases} \tag{3.13}$$

Then $A_\varepsilon : C[a,b] \to C[a,b]$ is completely continuous and fixed points of $A_\varepsilon$ are solutions of (3.13).

**Proposition 3.1.** *Let* (A.1)-(A.5) *hold and $\delta > 0$. Then there exist positive numbers $\varepsilon_0$, $\lambda_\delta$ with $\varepsilon_0 < \delta$ such that (3.13) has a positive solution $u_\varepsilon$ with*

$|u_\varepsilon|_\infty < \varepsilon_0$ *for* $\lambda < \lambda_\delta$ *and* $\varepsilon < \varepsilon_0$. *Furthermore, there exists* $c_\lambda > 0$ *such that* $u_\varepsilon(t) \geq c_\lambda p(t)$ *for* $t \in [a, b]$.

**Proof.** By (A.5), there exist $c, \varepsilon_0 > 0$ with $\varepsilon_0 < \delta$ such that

$$f(u) \geq c \quad \text{for } u \in (0, 2\varepsilon_0]. \tag{3.14}$$

Let $\varepsilon \in (0, \varepsilon_0)$ and $\lambda \in (0, \lambda_\delta)$, where $\lambda_\delta = \bar{\lambda}_{\varepsilon_0}$ and $\bar{\lambda}_r$ is given by Lemma 3.5. Using Lemma 3.5 with $r = \varepsilon_0$ and $T = 0$, it follows that any solution of

$$u = \theta A_\varepsilon u \text{ for some } \theta \in [0, 1]$$

satisfies $|u|_\infty \neq \varepsilon_0$. Hence

$$\deg(I - A_\varepsilon, B_{\varepsilon_0}, 0) = 1,$$

where $B_{\varepsilon_0}$ denotes the open ball centered at 0 with radius $\varepsilon_0$ in $C[a, b]$, and therefore $A_\varepsilon$ has a fixed point $u_\varepsilon$ with $|u_\varepsilon|_\infty < \varepsilon_0$. Since

$$|u_\varepsilon|_\infty + \varepsilon < 2\varepsilon_0,$$

it follows from (3.13) and (3.14) that

$$-(q(t)\phi(u'_\varepsilon))' \geq \lambda c r(t) \text{ for } t \in (a, b).$$

This, together with Lemma 3.2, implies

$$u_\varepsilon(t) \geq \phi^{-1}(\lambda c) z(t) > 0$$

for $t \in (a, b)$. Hence,

$$|u_\varepsilon|_\infty \geq \phi^{-1}(\lambda c)|z|_\infty \equiv r_\lambda,$$

and, by Lemma 3.4,

$$u_\varepsilon(t) \geq c_0 r_\lambda p(t)$$

for $t \in [a, b]$, which completes the proof of Proposition 3.1.

**Proposition 3.2.** *Let* (A.1)-(A.4), (A.6) *hold and* $\delta > 0$. *Then there exists a positive numbers* $R_\lambda$ *such that (3.13) has a positive solution* $\tilde{u}_\varepsilon$ *with* $\delta < |\tilde{u}_\varepsilon|_\infty < R_\lambda$ *for* $\varepsilon < 1$ *and* $\lambda < \bar{\lambda}_\delta$, *where* $\bar{\lambda}_\delta$ *is given by Lemma 3.5. Furthermore, there exists* $\tilde{c}_\lambda > 0$ *such that* $\tilde{u}_\varepsilon(t) \geq \tilde{c}_\lambda p(t)$ *for* $t \in [a, b]$.

**Proof.** Since (A.6) holds, there exists $T > 0$ such that $f(u) > 0$ for $u > T$. Using Lemma 3.5 with $\varepsilon_0 = 1$ and $r = \delta$, it follows that for $\varepsilon < 1$ and $\lambda < \bar{\lambda}_\delta$,

$$\deg(I - A_\varepsilon, B_\delta, 0) = 1. \tag{3.15}$$

By Lemma 3.7, there exists $R_\lambda >> 1$ such that any solution $u$ of

$$u = A_\varepsilon u + \alpha, \tag{3.16}$$

where $\alpha \in [0, \infty)$, satisfies $|u|_\infty < R_\lambda$. Hence

$$\deg(I - (A_\varepsilon + \alpha), B_{R_\lambda}, 0) \text{ is a constant for all } \alpha \geq 0,$$

and since (3.16) has no solution for $\alpha \geq R_\lambda$,

$$\deg(I - A_\varepsilon, B_{R_\lambda}, 0) = 0. \tag{3.17}$$

Combining (3.15) and (3.17), we obtain

$$\deg(I - A_\varepsilon, B_{R_\lambda} \backslash \bar{B}_\delta, 0) = -1,$$

and therefore there is a solution $\tilde{u}_\varepsilon$ of (3.13) with $\delta < |\tilde{u}_\varepsilon|_\infty < R_\lambda$. By Lemma 3.4 and the choice of $\bar{\lambda}_\delta$,

$$\tilde{u}_\varepsilon(t) \geq (c_0\delta - 2\phi^{-1}(\bar{\lambda}_\delta c_1))p(t) > 0$$

for $t \in (a, b)$, which completes the proof of Proposition 3.2.

## 4. Proof of main results

**Proof of Theorem 2.1.** By Proposition 3.1, there exists $\lambda_\delta > 0$ and $\varepsilon_0 \in (0, \delta)$ such that, for $\lambda < \lambda_\delta$ and each $n \geq 1$, there exist positive solutions $u_n$ to the problem

$$\begin{cases} -(q(t)\phi(u'))' = \lambda r(t) f(u_n + \frac{\varepsilon_0}{2n}), & t \in (a, b) \\ u(a) = u(b) = 0 \end{cases}$$

with $|u_n|_\infty < \varepsilon_0$ and

$$u_n(t) \geq c_\lambda p(t) \tag{3.18}$$

for $t \in [a, b]$. Applying Lemma 3.6 with $T = \varepsilon_0$ gives

$$\int_0^{u_n(t)} \phi^{-1}(h(s))ds \leq \phi^{-1}(\lambda)z(t), \tag{3.19}$$

where $h(t) = \frac{1}{t}\int_0^t \frac{1}{\tilde{f}(s)}ds$ and $\tilde{f}(t) = \sup\limits_{t \leq s \leq 2\varepsilon_0} f(s)$. Suppose that $|u_n|_\infty = u(t_n)$ for some $t_n \in (a, b)$. From (3.18) and (3.19), we deduce that

$$\int_0^{r_\lambda} \phi^{-1}(h(s))ds \leq \int_0^{u_n(t_n)} \phi^{-1}(h(s))ds \leq \phi^{-1}(\lambda)z(t_n),$$

where $r_\lambda = c_\lambda |p|_\infty$. Hence, there exists $[a_1, b_1] \subset (a, b)$ such that $t_n \in [a_1, b_1]$ for all $n \geq 1$. By passing to a subsequence, we may assume that there exists $t_0 \in [a_1, b_1]$ such that $(t_n)$ converges to $t_0$.

Next, choose $a_k, b_k, k \geq 2$, so that $(a_k)$ is decreasing, $(b_k)$ is increasing, and $(a,b) = \cup_{k=1}^{\infty}(a_k, b_k)$. By (3.18), there exists $\delta_1 > 0$ such that $\delta_1 < u_n(t) < \delta$ for $t \in [a_1, b_1]$ and $n \geq 1$. Since $u_n'(t_n) = 0$, it follows by integrating that

$$u_n'(t) = -\phi^{-1}\Big(\frac{\lambda}{q(t)}\int_{t_n}^{t} r(s)f(u_n(s) + \frac{\varepsilon_0}{2n}\Big)ds.$$

Consequently, $|u_n'(t)| \leq K_1$ for $t \in [a_1, b_1]$, where $K_1 = \phi^{-1}(\frac{\lambda}{q_0}|r|_1 k_1),\ k_1 = \sup\limits_{\delta_1 \leq s \leq 2\varepsilon_0} |f(s)|$.

Hence, there exists $u_1 \in C[a_1, b_1]$ and a subsequence $(u_{1n})$ of $(u_n)$ such that $u_{1n} \to u_1$ in $C[a_1, b_1]$. Since

$$u_{1n}'(t) \to -\phi^{-1}\Big(\frac{\lambda}{q(t)}\int_{t_0}^{t} r(s)f(u_1(s)\Big)ds$$

for $t \in (a_1, b_1)$, we deduce that

$$u_1'(t) = -\phi^{-1}\Big(\frac{\lambda}{q(t)}\int_{t_0}^{t} r(s)f(u_1(s)\Big)ds$$

for $t \in (a_1, b_1)$; i.e., $u_1$ satisfies

$$-(q(t)\phi(u_1'))' = \lambda r(t)f(u_1(t)),\ \ t \in (a_1, b_1).$$

Similarly, there exists $u_2 \in C[a_2, b_2]$ and a subsequence $(u_{2n})$ of $(u_{1n})$ such that $u_{2n} \to u_2$ in $C[a_2, b_2]$. Clearly, $u_2 = u_1$ on $[a_1, b_1]$, and reasoning as above, we see that $u_2$ satisfies

$$-(q(t)\phi(u_2'))' = \lambda r(t)f(u_2(t)),\ \ t \in (a_2, b_2).$$

Continuing and using a diagonalization process, we obtain a subsequence $(u_{nn})$ of $(u_n)$ and $u \in C(a,b)$ such that $u_{nn} \to u$ in $C[a_k, b_k]$ for all $k$, and $u$ satisfies

$$-(q(t)\phi(u'))' = \lambda r(t)f(u(t)),\ \ t \in (a,b)$$

together with

$$c_\lambda p(t) \leq u(t) \leq \varepsilon_0$$

for $t \in [a,b]$. By Lemma 3.6,

$$\int_0^{u_{nn}(t)} \phi^{-1}(h(s))ds \leq \phi^{-1}(\lambda)z(t),$$

and by letting $n \to \infty$, we obtain

$$0 < \int_0^{u(t)} \phi^{-1}(h(s))ds \leq \phi^{-1}(\lambda)z(t)$$

for $t \in (a,b)$. This, in turn, implies

$$\lim_{t\to a} u(t) = \lim_{t\to b} u(t) = 0;$$

i.e., $u \in C[a,b]$ and is a positive solution of (1.1) with $|u|_\infty < \delta$.

**Proof of Theorem 2.2.** By Proposition 3.2, for $\lambda < \bar{\lambda}_\delta$ and each $n \geq 1$, there exists a positive solution $\tilde{u}_n$ to the problem

$$\begin{cases} -(q(t)\phi(u'))' = \lambda r(t) f(u_n + \frac{1}{2n}) \\ u(a) = u(b) = 0 \end{cases}$$

with $\delta < |\tilde{u}_n|_\infty < R_\lambda$ and

$$\tilde{u}_n(t) \geq \tilde{c}_\lambda p(t) \tag{3.20}$$

for $t \in [a,b]$. In view of (A.6), we can assume that $f(R_\lambda + 1) > 0$. Let $|\tilde{u}_n|_\infty = \tilde{u}_n(s_n)$ for some $s_n \in (a,b)$. Reasoning as in the proof of Theorem 3.1, we obtain a solution $\tilde{u} \in C[a,b]$ of (1.1) and a subsequence $(\tilde{u}_{nn})$ of $(\tilde{u}_n)$ such that

$$\tilde{u}_{nn} \to \tilde{u} \text{ in } C[\tilde{a}_k, \tilde{b}_k] \text{ for all } k, \tag{3.21}$$

where $\tilde{a}_k, \tilde{b}_k$ are such that $(a,b) = \cup_{k=1}^{\infty}(\tilde{a}_k, \tilde{b}_k)$, $\tilde{a}_k$ is decreasing, $\tilde{b}_k$ is increasing, and $(s_n) \subset [\tilde{a}_1, \tilde{b}_1]$.

From (3.20), we conclude that $\tilde{u} > 0$ in $(a,b)$. To complete the proof, we shall verify that $|\tilde{u}|_\infty > \delta$. Indeed, suppose $|\tilde{u}_{nn}|_\infty = \tilde{u}_{nn}(s_{nn})$ and without loss of generality assume that $(s_{nn})$ converges to some $s_0 \in [\tilde{a}_1, \tilde{b}_1]$. Then, by (3.21), $|\tilde{u}_{nn}|_\infty \to \tilde{u}(s_0)$, and since $|\tilde{u}_{nn}|_\infty > \delta$, it follows that $|\tilde{u}|_\infty \geq \delta$. By Lemma 3.5, $|\tilde{u}|_\infty \neq \delta$, and Theorem 2.2 follows.

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