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STRUCTURE OF THE SET OF BOUNDED SOLUTIONS AND EXISTENCE OF PSEUDO ALMOST-PERIODIC SOLUTIONS OF A LIÉNARD EQUATION

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Abstract. We study some of the properties of bounded, asymptotically almost-periodic or pseudo almost-periodic solutions of the Liénard equation,

$$x'' + f(x)x' + g(x) = p(t),$$

where $p : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous, bounded, asymptotically almost-periodic or pseudo almost-periodic function, f and $g : (a, b) \longrightarrow \mathbb{R}$ are continuous and g is strictly decreasing. Notably, we describe the set of initial conditions of the bounded solutions on $(0, +\infty)$ and we state some results of existence of pseudo almost-periodic solutions.

1. Introduction

In this paper, we study some of the properties of bounded, asymptotically almost-periodic or pseudo almost-periodic solutions of the following Liénard equation:

$$x'' + f(x)x' + g(x) = p(t), \tag{1.1}$$

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where $p : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function and $f, g : (a, b) \longrightarrow \mathbb{R}$ $(-\infty \leq a < b \leq +\infty)$ are locally Lipschitz-continuous functions. Throughout the paper, we assume the following hypotheses:

(H1) *g is strictly decreasing.*

(H2) *$f(x) \geq 0$ for all $x \in (a, b)$.*

(H3) $\sup_{t \in \mathbb{R}} |p(t)| < +\infty$.

The model of equation (1.1) is

$$x'' + cx' + \frac{1}{x^\alpha} = p(t), \tag{1.2}$$

where $c \geq 0$, $\alpha > 0$ and $p : \mathbb{R} \longrightarrow \mathbb{R}$ is an almost-periodic function, that appears when the restoring force is a singular nonlinearity which becomes infinite at zero. Martínez-Amores and Torres in [12], then Campos and Torres in [5] describe the dynamics of equation (1.1) in the periodic case, namely the forcing term p is periodic. Recall that the existence of periodic solutions of equation (1.1) without the friction term ($f = 0$) is proved by Lazer and Solimini in [11] and by Habets and Sanchez in [9] for some Liénard equations with singularities, more general than equation (1.1). In [5], Campos and Torres prove that the existence of a bounded solution on $(0, +\infty)$ implies the existence of a unique periodic solution that attracts all bounded solutions on $(0, +\infty)$. In addition, they state that the set of initial conditions of bounded solutions on $(0, +\infty)$ is the graph of a continuous nondecreasing function. Then Cieutat [6] extends results of the paper [5] to the almost-periodic case. In [5], the authors use topological tools, such as free homeomorphisms (cf. [4]), together with truncation arguments. The homeomorphisms used in [5] are the Poincaré operators of equation (1.1); therefore, these topological tools are not adapted to the almost-periodic case. In [6], the methods used are essentially the recurrence property of the almost-periodic functions.

A well known extension of almost-periodicity is the notion of asymptotical almost-periodicity historically due to Fréchet (cf. [8] for détails). In [16], Zhang introduced another extension of the almost-periodic functions, the so-called pseudo almost-periodic functions. In [17], Zhang investigated the existence of a pseudo almost-periodic solution for a pseudo almost-periodic nonlinear perturbation of a linear autonomous ordinary differential equation. Ait Dads and Arino [1] introduced the generalized pseudo almost-periodic functions and extended Zhang's results. More details on the concept of pseudo almost-periodicity can be found in [1, 2, 17]. The question of pseudo almost-periodic solutions to some differential equations has been studied by various authors; see [3, 7, 10, 15, 18] and references therein.

Our aim is to extend the main results of the paper [6] to the bounded or pseudo almost-periodic case. Notably, we describe the set of initial conditions of the bounded solutions on $(0, +\infty)$ and we state some results of existence of pseudo almost-periodic solutions.

Concerning notation and definitions, $v \in C(\mathbb{R})$ (continuous) is said to be *almost periodic* (in the sense of Bohr), if for each $\varepsilon > 0$, there exists $\ell > 0$, for each $\alpha \in \mathbb{R}$, there exists $\tau \in [\alpha, \alpha + \ell]$ such that

$$\sup_{t \in \mathbb{R}} |v(t + \tau) - v(t)| < \varepsilon.$$

Denote by $AP(\mathbb{R})$ the set of all such functions. If v is almost periodic, the *mean value*

$$\mathcal{M}\{v(t)\}_t := \lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} v(t)dt$$

exists; furthermore,

$$\lim_{r \to +\infty} \frac{1}{2r} \int_{-r+a}^{r+a} v(t)dt = \mathcal{M}\{v(t)\}_t,$$

uniformly with respect to $a \in \mathbb{R}$. For each $\omega \in \mathbb{R}$, we denote by $a(v, \omega) := \mathcal{M}\{v(t)e^{-i\omega t}\}_t$ the *Fourier-Bohr coefficient* of v associated with ω and by $\Lambda(v) := \{\omega \in \mathbb{R}; a(v, \omega) \neq 0\}$ the set of *exponents* of v. The *module* of v, denoted by $mod(v)$, is the additive group generated by $\Lambda(v)$ (cf. [8, Chapters 1 and 4]). An important property of almost-periodic functions is the *recurrence property*, which says that once a value is taken by $v(t)$ at some point $t \in \mathbb{R}$, it will be "almost" taken arbitrarily far in the future and in the past. More precisely, we have that for an almost-periodic function v, there exists a real sequence $(\tau_n)_n$ such that

$$\lim_{n \to +\infty} \tau_n = +\infty \quad (\text{respectively} - \infty)$$

and

$$\lim_{n \to +\infty} \sup_{t \in \mathbb{R}} |v(t + \tau_n) - v(t)| = 0.$$

A function $u : (c, +\infty) \longrightarrow \mathbb{R}$ (with $-\infty \leq c < +\infty$) is called *asymptotically almost periodic*, if $u = v + \phi$, where $v \in AP(\mathbb{R})$ and $\phi \in C(c, +\infty)$ with $\lim_{t \to +\infty} \phi(t) = 0$. The function v is called the *almost-periodic part* of the function u (cf. [8, Chapter 9]).

We denote by $PAP_0(\mathbb{R})$ the space of continuous functions $\phi : \mathbb{R} \longrightarrow \mathbb{R}$, such that $\sup_{t \in \mathbb{R}} |\phi(t)| < +\infty$ and

$$\lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} |\phi(t)|dt = 0. \tag{1.3}$$

A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is called *pseudo almost periodic*, if $f = g + \phi$, where $g \in AP(\mathbb{R})$ and $\phi \in PAP_0(\mathbb{R})$. Note that g and ϕ are uniquely determined. The functions g and ϕ are called the *almost-periodic component* and the *ergodic perturbation*, respectively, of the function f. We denote by $PAP(\mathbb{R})$ the set of pseudo almost-periodic functions. If $f \in PAP(\mathbb{R})$, the function $[t \to f(t)e^{-i\omega t}] \in PAP(\mathbb{R})$ and the mean value exists,

$$\lim_{r \to +\infty} \frac{1}{2r} \int_{-r}^{r} f(t)dt,$$

and is equal to the mean value of its almost-periodic component. So, similarly as in the case of almost-periodic functions, we can define the mean value, the Fourier coefficients and the module, for any pseudo almost-periodic functions. These are the mean value, the Fourier coefficients and the module of its almost-periodic component. For some preliminary results on such functions, we refer to [1, 2, 17].

We say that a function $x : (c, +\infty) \longrightarrow \mathbb{R}$ (with $-\infty \leq c < +\infty$) is *bounded in the future* if there exist $r, s \in \mathbb{R}$ and $t_0 > c$ such that $a < r \leq x(t) \leq s < b$ for all $t > t_0$ and a function $x : \mathbb{R} \longrightarrow \mathbb{R}$ is *bounded on* $\mathbb{R}$ if there exist r and $s \in \mathbb{R}$ such that $a < r \leq x(t) \leq s < b$ for all $t \in \mathbb{R}$.

Remark that if x is a periodic solution of equation (1.1), then x is *bounded on* $\mathbb{R}$ (in the sense of the above definition), but an almost-periodic solution, therefore a pseudo almost-periodic solution, is not necessarily *bounded on* $\mathbb{R}$ (of course $\sup_{t \in \mathbb{R}} |x(t)| < +\infty$), because there exists an almost-periodic solution x such that $\inf_{t \in \mathbb{R}} x(t) = a$ (if $a \in \mathbb{R}$). For example, we consider $x(t) := \cos(t) - \cos(2\pi t) + 2$. Since $x(t) > 0$ for all $t \in \mathbb{R}$, then x is an almost-periodic solution of equation (1.1), where $a := 0$, $b := +\infty$, $f(x) := 0$, $g(x) := -x$ and $p(t) := ((2\pi)^2 + 1)\cos(2\pi t) - 2\cos(t) - 2$. Moreover, there exists a sequence $(a_n)_n$ of integers such that $\lim_{n \to +\infty} \cos(a_n) = -1$; therefore, $\lim_{n \to +\infty} x(a_n) = 0$, so x is not *bounded on* $\mathbb{R}$.

The paper is organized as follows: in Section 2, we study the structure of solutions that are bounded in the future, when the second member p is bounded. The main result of this section (Theorem 2.5) extends a result in the almost-periodic case [6, Theorem 2.2] to the bounded case. To establish this last result, we state some results of comparison of bounded solutions. In Section 3, we establish that all solutions that are bounded in the future are asymptotically almost periodic, when the second member p is asymptotically almost periodic. In fact, we state this result for a larger class of second member p. Then in the Section 4, we study the existence of pseudo almost-periodic solutions.

2. Structure of bounded solutions

In this section, first we establish some results of comparison of bounded solutions, then we study the structure of solutions that are bounded on the future when the second member p is bounded. We describe the set of initial conditions of the solutions that are bounded in the future and we state that this set is the graph of a continuous strictly decreasing function.

We denote by $x(t;t_0,x_0,v_0)$ the unique solution of the Cauchy problem $x(t_0;t_0,x_0,v_0)=x_0$ and $x'(t_0;t_0,x_0,v_0)=v_0$ of equation (1.1) with $(T_{min}(t_0,x_0,v_0),T_{max}(t_0,x_0,v_0))$ its maximal interval of definition. We start by recalling two results of Campos and Torres. These results are the monotonicity properties of equation (1.1) when the forcing term p is only continuous.

Theorem 2.1. (Campos and Torres [5]). *Let $x_1(t)=x(t;t_0,x_1,v_1)$ and $x_2(t)=x(t;t_0,x_2,v_2)$ be different solutions of* (1.1) *such that $a<x_1\leq x_2<b$ and $v_1\leq v_2$. Then $x_1(t)<x_2(t)$ for all $t\in(t_0,\min_{i=1,2}T_{max}(t_0,x_i,v_i))$.*

Corollary 2.2. (Campos and Torres [5]). *Any couple of different solutions of* (1.1) *has at most one point in common; i.e., there exists at most t_* such that $x_1(t_*)=x_2(t_*)$ where x_1 and x_2 are solutions of* (1.1).

Proposition 2.3. *i) Any couple of different solutions x_1 and x_2 bounded in the future of equation* (1.1) *satisfy*

$$(x_1(t)-x_2(t))(x_1'(t)-x_2'(t))<0 \tag{2.1}$$

for every t where both solutions are defined and

$$\lim_{t\to+\infty}|x_1(t)-x_2(t)|+|x_1'(t)-x_2'(t)|=0; \textit{and} \tag{2.2}$$

ii) equation (1.1) *has at most one bounded solution on $\mathbb{R}$.*

Remarks. 1) Relation (2.1) implies that the function $t\longrightarrow|x_1(t)-x_2(t)|$ is strictly decreasing and that any couple of different solutions bounded on the future has no point in common.

2) In the almost-periodic case, namely when the forcing term p is almost periodic, Proposition 2.3 is established in [6, Proposition 3.1] by using the notion of the recurrence property of an almost-periodic function p. For this reason, we cannot extend this proof to the bounded case we give another proof.

For the proof of Proposition 2.3, we need of the following lemma.

Lemma 2.4. (Cieutat [6]). *Let $I=(t_0,+\infty)$ with $t_0=-\infty$ or $t_0\in\mathbb{R}$. If x is a solution of equation* (1.1) *that is bounded in the future* (*respectively* *bounded on* $\mathbb{R}$*), i.e.,* $a < r \leq x(t) \leq s < b$ *for all* $t > t_0$ *(respectively* $t \in \mathbb{R}$*), then the derivatives* x' *and* x'' *are bounded in the future (respectively bounded on* $\mathbb{R}$*); i.e.,* $\sup_{t\in I} |x'(t)| \leq c_1 < +\infty$ *and* $\sup_{t\in I} |x''(t)| \leq c_2 < +\infty$*, where*

$$c_0 := \max(|r|, |s|), \tag{2.3}$$

$$c_1 := \frac{1}{2}\sup_{t\in\mathbb{R}} |p(t)| + \frac{1}{2}\sup_{r\leq z\leq s} |g(z)| + 2c_0 + 4c_0 \sup_{r\leq z\leq s} |f(z)| < +\infty \tag{2.4}$$

and

$$c_2 := \sup_{t\in I} |p(t)| + \sup_{r\leq z\leq s} |g(z)| + c_1 \sup_{r\leq z\leq s} |f(z)| < +\infty. \tag{2.5}$$

Proof of Proposition 2.3. i) Let there be the open interval $(c, +\infty)$ (with $-\infty \leq c < +\infty$) where both solutions are defined. By Corollary 2.2, there exists $\tau > c$ such that $x_1(t) \neq x_2(t)$ for each $t \geq \tau$. We can assume that

$$\forall t \geq \tau, \qquad x_2(t) > x_1(t). \tag{2.6}$$

Since x_1 and x_2 are bounded in the future, there exist r and $s \in \mathbb{R}$ such that $a < r < s < b$ and

$$\forall t \geq \tau, \qquad x_1(t) \text{ and } x_2(t) \in [r, s]. \tag{2.7}$$

By Lemma 2.4, we obtain

$$\sup_{t\geq\tau} |x_1'(t)| \leq c_1, \qquad \sup_{t\geq\tau} |x_2'(t)| \leq c_1 \tag{2.8}$$

and

$$\sup_{t\geq\tau} |x_1''(t)| \leq c_2, \qquad \sup_{t\geq\tau} |x_2''(t)| \leq c_2, \tag{2.9}$$

where c_1 and c_2 are defined by (2.3)–(2.5). We denote by α the numerical function defined by

$$\forall t \geq \tau, \qquad \alpha(t) := x_2'(t) - x_1'(t) + \int_{x_1(t)}^{x_2(t)} f(z)\, dz. \tag{2.10}$$

By using (2.7), (2.8) and the following inequality,

$$|\alpha(t)| \leq |x_2'(t) - x_1'(t)| + |x_2(t) - x_1(t)| \sup_{r\leq z\leq s} |f(z)|,$$

we deduce that the function α is bounded on $[\tau, +\infty)$:

$$\forall t \geq \tau, \qquad \alpha(t) \leq 2c_1 + 2\sup(|r|, |s|) \sup_{r\leq z\leq s} |f(z)| < +\infty. \tag{2.11}$$

The function α is of class C^1 with

$$\alpha'(t) = x_2''(t) - x_1''(t) + f(x_2(t))x_2'(t) - f(x_1(t))x_1'(t)$$

and by subtracting the respective equations and by (H1), we obtain for each $t \geq \tau$

$$\alpha'(t) = g(x_1(t)) - g(x_2(t)) > 0. \tag{2.12}$$

Then the function α is strictly increasing on $[\tau, +\infty)$ and by (2.11), we deduce that α admits a finite limit as t tends to ∞:

$$\lim_{t \to +\infty} \alpha(t) = \sup_{t \geq \tau} \alpha(t) \in \mathbb{R}. \tag{2.13}$$

By (2.8), x_1 and x_2 are uniformly continuous on $[\tau, +\infty)$, g is uniformly continuous on $[r, s]$, then the function $t \to g(x_1(t)) - g(x_2(t))$ is uniformly continuous on $[\tau, +\infty)$. Since the numerical function α admits a finite limit as t tends to ∞ and its derivative α' is uniformly continuous on $[\tau, +\infty)$, we deduce that $\lim_{t \to +\infty} \alpha'(t) = 0$; therefore, by (2.12), we obtain

$$\lim_{t \to +\infty} g(x_1(t)) - g(x_2(t)) = 0. \tag{2.14}$$

Since g is continuous and strictly decreasing on $[r, s]$, we deduce that

$$\lim_{t \to +\infty} x_1(t) - x_2(t) = 0. \tag{2.15}$$

By the following inequality,

$$\left| \int_{x_1(t)}^{x_2(t)} f(z)\ dz \right| \leq |x_2(t) - x_1(t)| \sup_{r \leq z \leq s} |f(z)|,$$

we obtain

$$\lim_{t \to +\infty} \int_{x_1(t)}^{x_2(t)} f(z)\ dz = 0;$$

therefore, by (2.10) and (2.13), one has

$$\lim_{t \to +\infty} x_2'(t) - x_1'(t) = \sup_{t \geq \tau} \alpha(t). \tag{2.16}$$

By (2.9), x_1' and x_2' are uniformly continuous on $[\tau, +\infty)$, and by (2.15) and (2.16), we deduce that

$$\lim_{t \to +\infty} x_2'(t) - x_1'(t) = 0; \tag{2.17}$$

therefore, by (2.16) and (2.17), we obtain

$$\sup_{t \geq \tau} \alpha(t) = 0. \tag{2.18}$$

First by (2.15) and (2.17), relation (2.2) is established. Then by (2.18), we deduce that $\alpha(t) < 0$ for each $t \geq \tau$ because α is strictly increasing; therefore by using (H2) and (2.10), we obtain

$$\forall t \geq \tau, \qquad x_2'(t) - x_1'(t) < 0. \tag{2.19}$$

Let $\tau_* := \inf\{\tau > c; x_2(t) > x_1(t) \text{ on } [\tau, +\infty)\}$. We have established

$$\forall t > \tau_*, \quad x_2(t) > x_1(t) \quad \text{and} \quad x_2'(t) < x_1'(t). \tag{2.20}$$

It remains to prove that $\tau_* = c$. Assume the contrary: $\tau_* > c$. Then by continuity of x_1 and x_2, one has $x_1(\tau_*) = x_2(\tau_*)$, which is a contradiction with (2.20).

ii) To prove the uniqueness, we consider x_1 and x_2, two bounded solutions on $\mathbb{R}$ of equation (1.1). If we assume that these two bounded solutions are different, then the proof of

$$\lim_{t \longrightarrow -\infty} |x_1(t) - x_2(t)| = 0 \tag{2.21}$$

is similar to the proof of (2.2). By (2.1) the function $t \longrightarrow |x_1(t) - x_2(t)|$ is strictly decreasing, which is a contradiction with (2.21); therefore, the bounded solution on $\mathbb{R}$ is unique. □

Now we describe the set of initial conditions of the solutions that are bounded on the future and we state that this set is the graph of a continuous strictly decreasing function. For $t_0 \in \mathbb{R}$, we denote by $W_s^{t_0}$ the following set

$$W_s^{t_0} := \{(x_0, v_0) \in (a,b) \times \mathbb{R} \ ; \ x(t; t_0, x_0, v_0) \text{ is bounded in the future}\}.$$

Theorem 2.5. *If equation* (1.1) *has at least one solution that is bounded in the future, then there exist a non-empty open interval $I \subset (a,b)$ and a continuous strictly decreasing function $\xi : I \longrightarrow \mathbb{R}$ such that*

$$W_s^{t_0} = \{(x, \xi(x)) \ ; \ x \in I\}.$$

Remark. Theorem 2.5 was established, first in the periodic case by Campos and Torres in [5, Theorem 4.6], then in the almost-periodic case by Cieutat in [6, Theorem 2.2]. In [6], Theorem 2.2 is proved by using two results of Campos and Torres (Theorem 2.1 and Corollary 2.2) and a result of Cieutat [6, Proposition 3.1]. These two results of Campos and Torres are monotonicity properties of equation (1.1) when the forcing term p is only continuous, and [6, Proposition 3.1] is a result of comparison of bounded solutions on the future, in the almost-periodic case. Proposition 2.3 extends to the bounded case the result [6, Proposition 3.1]; therefore, the proof given in [6, Theorem 2.2] is valid for Theorem 2.5.

3. Asymptotically almost-periodic solutions

In this section, when the second member p of equation (1.1) is asymptotically almost periodic, we establish that each solution that is bounded in the future is asymptotically almost periodic. In fact, we state this result for a larger class of second members. For that we introduce the following sets of functions. Let $c \in \mathbb{R}$.

$$C_b(c,+\infty) := \{x \in C(c,+\infty);\ \sup_{t>c} |x(t)| < +\infty\},$$

$$C_0^+(c,+\infty) := \{x \in C_b(c,+\infty);\ \lim_{t\to+\infty} |x(t)| = 0\},$$

$$\mathcal{E}^+(c,+\infty) := \{x \in C_b(c,+\infty);\ \lim_{n\to+\infty} \int_n^{n+1} |x(t)|\ dt = 0\ (n \in \mathbb{N})\},$$

$$PAP_0^+(c,+\infty) := \{x \in C_b(c,+\infty);\ \lim_{r\to+\infty} \frac{1}{r} \int_c^r |x(t)|\ dt = 0\}.$$

Proposition 3.1. *The following assertions hold.*

i) $x \in \mathcal{E}^+(c,+\infty)$ if and only if $x \in C_b(c,+\infty)$ and for each $l > 0$, one has

$$\lim_{r\to+\infty} \int_r^{r+l} |x(t)|\ dt = 0.$$

ii) $x \in PAP_0^+(c,+\infty)$ if and only if $x \in C_b(c,+\infty)$ and for each $a > c$, one has

$$\lim_{r\to+\infty} \frac{1}{r} \int_a^{a+r} |x(t)| dt = 0.$$

iii) $C_0^+(c,+\infty) \subset \mathcal{E}^+(c,+\infty) \subset PAP_0^+(c,+\infty)$.

iv) $L^p(c,+\infty) \cap C_b(c,+\infty) \subset \mathcal{E}^+(c,+\infty)$ $(p \geq 1)$.

Inclusions of iii) and iv) are strict.

Proof. i) Let $x \in \mathcal{E}^+(c,+\infty)$. By the following equality,

$$\int_n^{n+p} |x(t)|\ dt = \sum_{k=n}^{n+p-1} \int_k^{k+1} |x(t)|\ dt,$$

we deduce that

$$\forall p \in \mathbb{N}^*,\ \lim_{n\to+\infty} \int_n^{n+p} |x(t)|\ dt = 0. \tag{3.1}$$

For $x \in \mathbb{R}$, we denote by $E(x)$ the integer part of x. By the following inequality,

$$0 \leq \int_r^{r+l} |x(t)|\ dt \leq \int_{E(r)}^{E(r)+E(l)+2} |x(t)|\ dt,$$

and by (3.1), we deduce that

$$\forall l > 0, \lim_{r \to +\infty} \int_r^{r+l} |x(t)| \, dt = 0; \tag{3.2}$$

therefore,

$$\mathcal{E}^+(c, +\infty) \subset \{x \in C_b(c, +\infty) : \forall l > 0, \lim_{r \to +\infty} \int_r^{r+l} |x(t)| \, dt = 0\}.$$

The reciprocal inclusion is trivial.

ii) Let $x \in PAP_0^+(c, +\infty)$ and let $a > c$. By the following inequality,

$$\frac{1}{r} \int_a^{a+r} |x(t)| \, dt \leq \frac{a+r}{r} \frac{1}{a+r} \int_c^{a+r} |x(t)| \, dt,$$

we deduce that

$$\lim_{r \to +\infty} \frac{1}{r} \int_a^{a+r} |x(t)| = 0;$$

therefore,

$$PAP_0^+(c, +\infty) \subset \bigcap_{a > c} \{x \in C_b(c, +\infty); \lim_{r \to +\infty} \frac{1}{r} \int_a^{a+r} |x(t)| \, dt = 0\}.$$

For the reciprocal inclusion, we use the following equality:

$$\frac{1}{r} \int_c^r |x(t)| \, dt$$

$$= \frac{1}{r} \int_c^{c+1} |x(t)| \, dt + \frac{r-c-1}{r} \frac{1}{r-c-1} \int_{c+1}^{c+1+(r-c-1)} |x(t)| \, dt.$$

iii) The inclusion $C_0^+(c, +\infty) \subset \mathcal{E}^+(c, +\infty)$ is evident. Now consider the following function in $C_b(0, +\infty)$ defined by $x(t) = n(t-n)$ for $n \leq t < n + \frac{1}{n}$, $x(t) = -n(t-n) + 2$ for $n + \frac{1}{n} \leq t \leq n + \frac{2}{n}$, for each $n \in \mathbb{N}$ such that $n \geq 2$ and $x(t) = 0$ otherwise. Since $x(n + \frac{1}{n}) = 1$ and $\int_n^{n+1} |x(t)| \, dt = \frac{1}{n}$, one has $C_0^+(0, +\infty) \neq \mathcal{E}^+(0, +\infty)$.

Let $x \in \mathcal{E}^+(c, +\infty)$. Let $a > c$. For $r \geq 1$, one has

$$\frac{1}{r} \int_a^{a+r} |x(t)| \, dt \leq \frac{1}{E(r)} \int_a^{a+E(r)+1} |x(t)| \, dt = \frac{1}{E(r)} \sum_{n=0}^{E(r)} \int_{a+n}^{a+n+1} |x(t)| \, dt. \tag{3.3}$$

By i), one has

$$\lim_{n \to +\infty} \int_{a+n}^{a+n+1} |x(t)| \, dt = 0 \tag{3.4}$$

and by (3.3), (3.4) and Cesaro's theorem, we deduce that

$$\lim_{r\to+\infty}\frac{1}{r}\int_a^{a+r}|x(t)|\,dt=0;$$

therefore, by ii), one has $\mathcal{E}^+(c,+\infty)\subset PAP_0^+(c,+\infty)$. Consider the following function in $C_b(1,+\infty)$ defined by $x(t)=t-n^2$ for $n^2\le t<n^2+\frac{1}{2}$, $x(t)=n^2-t+1$ for $n^2+\frac{1}{2}\le t\le n^2+1$, for each $n\in\mathbb{N}^*$, and $x(t)=0$ otherwise. Since

$$\int_{n^2}^{n^2+1}|x(t)|\,dt=\frac{1}{4}, \tag{3.5}$$

the function x is not in $\mathcal{E}^+(1,+\infty)$. On other hand, one has

$$\int_1^{N^2+1}|x(t)|\,dt=\frac{N}{4}$$

for each $N\in\mathbb{N}$. Let $r>0$. There exists $N\in\mathbb{N}$ such that $N^2\le r<(N+1)^2$; therefore,

$$\int_1^r|x(t)|\,dt\le\int_1^{(N+1)^2}|x(t)|\,dt=\int_1^{N^2+1}|x(t)|\,dt=\frac{N}{4}\le\frac{\sqrt{r}}{4}.$$

Consequently,

$$\lim_{r\to+\infty}\frac{1}{r}\int_1^r|x(t)|\,dt=0; \tag{3.6}$$

therefore, x is in $PAP_0^+(1,+\infty)$. By (3.5) and (3.6), we deduce that

$$\mathcal{E}^+(1,+\infty)\neq PAP_0^+(1,+\infty).$$

iv) In the particular case where p=1, the inclusion results from the relation

$$\sum_{n=n_0}^{+\infty}\int_n^{n+1}|x(t)|\,dt=\int_{n_0}^{+\infty}|x(t)|\,dt,$$

where $n_0\in\mathbb{N}$ such that $n_0>c$. For the general case, $p\ge1$, we use the previous result and the Holder's inequality

$$\int_n^{n+1}|x(t)|\,dt\le\left(\int_n^{n+1}|x(t)|^p\,dt\right)^{\frac{1}{p}}.$$

Let $p\ge1$. The following function $x(t)=t^{-\frac{1}{p}}$ is in $\mathcal{E}^+(1,+\infty)$ but not in $L^p(1,+\infty)$; therefore, $L^p(1,+\infty)\cap C_b(1,+\infty)\neq\mathcal{E}^+(1,+\infty)$. □

Theorem 3.2. *Let $c\in\mathbb{R}$. In addition, suppose that $p=p^{ap}+p^e$ where $p^{ap}\in AP(\mathbb{R})$ and $p^e\in\mathcal{E}^+(c,+\infty)$.*

i) If equation (1.1) *has at least one solution that is bounded in the future, then*

$$x'' + f(x)x' + g(x) = p^{ap}(t) \tag{3.7}$$

has exactly one solution ϕ *that is bounded on* $\mathbb{R}$*. Moreover, this solution* ϕ *and its derivatives* ϕ' *and* ϕ'' *are almost periodic and* $mod(\phi) \subset mod(p^{ap})$*.*

ii) Every solution x *bounded in the future of equation* (1.1) *is asymptotically almost periodic:*

$$\lim_{t\to+\infty} \big(|x(t) - \phi(t)| + |x'(t) - \phi'(t)|\big) = 0. \tag{3.8}$$

Remark. In the particular case where the second member is asymptotically almost periodic, then every solution x bounded in the future of equation (1.1) is asymptotically almost periodic (cf. Proposition 3.1).

For the proof of Theorem 3.2, we use the following lemma.

Lemma 3.3. *Let* $c \in \mathbb{R}$*. In addition, suppose that* $p = p^{ap} + p^e$*, where* $p^{ap} \in AP(\mathbb{R})$ *and* $p^e \in \mathcal{E}^+(c, +\infty)$*. We assume that* x *is a solution of equation* (1.1) *that is bounded in the future. If there exist a numerical sequence* $(t_n)_n$ *and* $p_*^{ap} \in AP(\mathbb{R})$ *such that*

$$\lim_{n\to+\infty} t_n = +\infty \tag{3.9}$$

and

$$\lim_{n\to+\infty} \sup_{t\in\mathbb{R}} |p^{ap}(t + t_n) - p_*^{ap}(t)| = 0, \tag{3.10}$$

then for a subsequence, one has

$$\forall t \in \mathbb{R}, \qquad \lim_{n\to+\infty} x(t + t_n) = x_*(t), \tag{3.11}$$

where x_* *is a bounded solution on* $\mathbb{R}$ *of*

$$x_*'' + f(x_*)x_*' + g(x_*) = p_*^{ap}(t). \tag{3.12}$$

Proof of Lemma 3.3. Since x is bounded in the future, there exist r, s and $t_0 \in \mathbb{R}$ such that

$$a < r \le x(t) \le s < b \quad \text{for all} \quad t > t_0. \tag{3.13}$$

By Lemma 2.4, we obtain

$$\sup_{t>t_0} |x'(t)| \le c_1 < +\infty \tag{3.14}$$

and

$$\sup_{t>t_0} |x''(t)| \le c_2 < +\infty, \tag{3.15}$$

where c_1 and c_2 are defined by (2.3)–(2.5). Given any interval $(\tau, +\infty)$, for $n \in \mathbb{N}$ sufficiently large ($\tau + t_n \geq t_0$), $t \longrightarrow x(\cdot + t_n)$ is defined on $(\tau, +\infty)$, and by (3.13), one has

$$a < r \leq x(t + t_n) \leq s < b \qquad \text{for all } t \in (\tau, +\infty). \tag{3.16}$$

By (3.14) and (3.15), we obtain

$$|x'(t + t_n)| \leq c_1 \qquad \text{for all } t \in (\tau, +\infty)$$
$$|x''(t + t_n)| \leq c_2 \qquad \text{for all } t \in (\tau, +\infty).$$

By taking τ as a sequence going to $-\infty$, by Ascoli's theorem and a diagonalization argument, we can assert that there exist $x_* \in C^1(\mathbb{R})$ and a subsequence of $(t_n)_n$ such that

$$x(t + t_n) \to x_*(t) \qquad \text{as} \quad n \to +\infty \tag{3.17}$$
$$x'(t + t_n) \to x_*'(t) \qquad \text{as} \quad n \to +\infty \tag{3.18}$$

uniformly on each compact subinterval of $\mathbb{R}$. By (3.16) and (3.17), we deduce that x_* is bounded on $\mathbb{R}$: $a < r \leq x_*(t) \leq s < b$ for all $t \in \mathbb{R}$. It remains to prove that x_* is a solution of equation (3.12). Consider (t_1, t_2). Since the function $t \longrightarrow x(t + t_n)$ is a solution of the equation

$$x'' + f(x)x' + g(x) = p^{ap}(t + t_n) + p^e(t + t_n)$$

on (t_1, t_2) for $n \in \mathbb{N}$ sufficiently large ($t_1 + t_n \geq t_0$), by integrating on (t_1, t_2), we obtain that x is a solution on (t_1, t_2) of

$$\begin{aligned} x'(t + t_n) &+ \int_{t_1}^{t} f(x(s + t_n))x'(s + t_n) + g(x(s + t_n))ds \\ &= x'(t_1 + t_n) + \int_{t_1}^{t} p^{ap}(s + t_n)ds + \int_{t_1}^{t} p^e(s + t_n)ds. \end{aligned} \tag{3.19}$$

Moreover, one has

$$\int_{t_1}^{t} p^e(s + t_n)ds = \int_{t_1+t_n}^{t+t_n} p^e(s)ds$$

and by i) of Proposition 3.1, we obtain that

$$\int_{t_1+t_n}^{t+t_n} p^e(s)ds \to 0 \qquad \text{as} \quad n \to +\infty. \tag{3.20}$$

By (3.10) and (3.17)–(3.20), we deduce that x_* is solution on (t_1, t_2) of

$$x_*'(t) + \int_{t_1}^{t} f(x_*(s))x_*'(s) + g(x_*(s))ds = x_*'(t_1) + \int_{t_1}^{t} p_*^{ap}(s)ds;$$

therefore, x_* is a solution on (t_1, t_2) of equation (3.12). Since this is fulfilled for all interval (t_1, t_2), then x_* is a solution on $\mathbb{R}$ of equation (3.12). □

Proof of Theorem 3.2. i) By using [6, Theorem 2.1], it suffices to prove the existence of a solution of equation (3.7) that is a bounded solution on the future. Let x be a solution of equation (1.1) that is a bounded solution on the future. Since p^{ap} is almost periodic, there exists a numerical sequence $(t_n)_n$ such that

$$\lim_{n\to+\infty} t_n = +\infty$$

and

$$\lim_{n\to+\infty} \sup_{t\in\mathbb{R}} |p^{ap}(t+t_n) - p^{ap}(t)| = 0.$$

By Lemma 3.3, there exists a solution ϕ of equation (3.7) that is bounded on $\mathbb{R}$. By [6, Theorem 2.1], we deduce the uniqueness of the bounded solution ϕ on $\mathbb{R}$ and that ϕ, ϕ' and ϕ'' are almost periodic and the formula of the module.

ii) We claim that

$$\lim_{t\to+\infty} |x(t) - \phi(t)| = 0. \tag{3.21}$$

Assume the contrary: there exist $\varepsilon > 0$ and a numerical sequence $(t_n)_n$ satisfying (3.9) and

$$\inf_{n\in\mathbb{N}} |x(t_n) - \phi(t_n)| \geq \varepsilon. \tag{3.22}$$

Since p^{ap} is an almost-periodic function, there exist a subsequence of $(t_n)_n$ and $p_*^{ap} \in AP(\mathbb{R})$ satisfying (3.10). By Lemma 3.3, there exists a subsequence of $(t_n)_n$ and a bounded solution x_* on $\mathbb{R}$ of equation (3.12) satisfying (3.11). Since ϕ is an almost-periodic solution of equation (3.7), then ϕ is a bounded solution on $\mathbb{R}$ of equation (1.1) with the perturbation null: $p^e = 0$. By Lemma 3.3 and by (3.10), there exists a subsequence of $(t_n)_n$ such that

$$\forall t \in \mathbb{R}, \quad \lim_{n\to+\infty} \phi(t+t_n) = \phi_*(t), \tag{3.23}$$

where ϕ_* is a bounded solution on $\mathbb{R}$ of equation (3.12). By (3.11), (3.22) and (3.23), one has

$$|x_*(0) - \phi_*(0)| \geq \varepsilon. \tag{3.24}$$

Moreover, x_* and ϕ_* are two bounded solutions on $\mathbb{R}$ of equation (3.12). By assertion i) of this theorem, we obtain $x_*(t) = \phi_*(t)$ for each $t \in \mathbb{R}$, which is a contradiction with (3.24); therefore, (3.21) is satisfied. Since x and ϕ are bounded in the future, by Lemma 2.4, there exists $t_0 \in \mathbb{R}$ such that $\sup_{t>t_0} |x''(t) - \phi''(t)| < +\infty$; therefore, the function $t \to |x'(t) - \phi'(t)|$ is uniformly continuous on $(t_0, +\infty)$. Since the function $t \to |x(t)-\phi(t)|$ admits a finite limit as t tends to infinity and its derivative is uniformly continuous on $(t_0, +\infty)$, we deduce that $\lim_{t\to+\infty} |x'(t) - \phi'(t)| = 0$ and obviously ii) is proved. □

4. PSEUDO ALMOST-PERIODIC SOLUTIONS

In this section, we study the existence and uniqueness of the pseudo almost-periodic solution. First, when the second member of equation (1.1) belongs to a particular class of pseudo almost-periodic functions, we state that the bounded solution on $\mathbb{R}$ is asymptotically almost periodic in $-\infty$ and $+\infty$, and therefore it is pseudo almost periodic. Then we state a result of existence of pseudo almost-periodic solutions. We introduce the following sets of functions:

$$C_b(\mathbb{R}) := \{x \in C(\mathbb{R}) : \sup_{t\in\mathbb{R}} |x(t)| < +\infty\},$$

$$C_0(\mathbb{R}) := \{x \in C_b(\mathbb{R}) : \lim_{|t|\to+\infty} |x(t)| = 0\},$$

$$x \in \mathcal{E}(\mathbb{R}) \Longleftrightarrow x \in C_b(\mathbb{R}), \lim_{n\to+\infty} \int_n^{n+1} |x(t)|dt = \lim_{n\to-\infty} \int_n^{n+1} |x(t)|dt = 0.$$

One has the analog of Proposition 3.1 on the whole real line.

Proposition 4.1. *The following assertions hold.*

i) $x \in \mathcal{E}(\mathbb{R})$ if and only if $x \in C_b(\mathbb{R})$, and for each $l > 0$, one has

$$\lim_{r\to+\infty} \int_r^{r+l} |x(t)|\ dt = \lim_{r\to-\infty} \int_r^{r+l} |x(t)|\ dt = 0.$$

ii) $C_0(\mathbb{R}) \subset \mathcal{E}(\mathbb{R}) \subset PAP_0(\mathbb{R})$.

iii) $L^p(\mathbb{R}) \cap C_b(\mathbb{R}) \subset \mathcal{E}(\mathbb{R})$ $(p \geq 1)$.

Inclusions of ii) and iii) are strict.

Proof. The proof is similar to the proof of Proposition 3.1. □

Proposition 4.2. *In addition, suppose that $p = p^{ap} + p^e$, where $p^{ap} \in AP(\mathbb{R})$ and $p^e \in \mathcal{E}(\mathbb{R})$. If equation (1.1) has at least one solution x that is bounded on $\mathbb{R}$, then this solution is unique and it is asymptotically almost periodic in $-\infty$ and $+\infty$:*

$$\lim_{|t|\to+\infty} \big(|x(t) - \phi(t)| + |x'(t) - \phi'(t)|\big) = 0, \tag{4.1}$$

where ϕ denotes the almost-periodic solution of equation (3.7).

Proof. Uniqueness of the bounded solution results from Proposition 2.3. By using Theorem 3.2, we obtain the existence of the almost-periodic solution of equation (3.7) satisfying (3.8). The proof of

$$\lim_{t\to-\infty}\big(|x(t)-\phi(t)|+|x'(t)-\phi'(t)|\big)=0$$

is similar to the proof of (3.8). □

When the second member p of equation (1.1) is pseudo almost periodic, the pseudo almost-periodic solution (if it exists) is not necessarily asymptotically almost periodic; in fact, it does not belong in the particular class $\mathcal{E}(\mathbb{R})$. For example, consider the function $\phi\in C^\infty(\mathbb{R})$ which is defined by $\phi(t)=\exp(\frac{1}{t(t-1)})$ on $(0,1)$ and $\phi(t)=0$ otherwise. Let $x\in C_b(\mathbb{R})$ be the function defined by $x(t)=\phi(t-n^2)$ for $n^2\le t\le n^2+1$ for each $n\in\mathbb{N}$, and $\phi(t)=0$ otherwise. The function x is a solution of the particular equation (1.1), $x''-x=p$, where p is defined by $p(t)=\phi''(t-n^2)-\phi(t-n^2)$ for $n^2\le t\le n^2+1$ and $p(t)=0$ otherwise. Moreover, the functions x and p are pseudo almost periodic, but x and p are not in $\mathcal{E}(\mathbb{R})$.

Now, we state a result of existence and uniqueness of the pseudo almost-periodic solution.

Theorem 4.3. *In addition, suppose that* $p\in PAP(\mathbb{R})$ (*pseudo almost periodic*). *If* $\inf_{t\in\mathbb{R}}p(t)$ *and* $\sup_{t\in\mathbb{R}}p(t)$ *are in the range of* g: $g(a,b)$, *then equation* (1.1) *has a unique bounded solution* x *on* $\mathbb{R}$ *which is pseudo almost periodic and satisfies*

$$mod(x)\subset mod(p). \tag{4.2}$$

Furthermore, if we denote by p^{ap} (*respectively* y) *the almost-periodic component of* p (*respectively* x), *then* y *is the almost-periodic solution of equation* (3.7).

Remarks. 1) In the particular case of equation (1.2), one has the existence and uniqueness of a pseudo almost-periodic solution, when the second member p satisfies $0<\inf_{t\in\mathbb{R}}p(t)\le\sup_{t\in\mathbb{R}}p(t)<+\infty$ and p is pseudo almost periodic.

2) In the almost-periodic case: $p\in AP(\mathbb{R})$, the unique bounded solution is almost periodic and satisfies the formula of modules.

For the proof of Theorem 4.3, we need the following lemma.

Lemma 4.4 (Slyusarchuk [14]). *Let* r *and* s *be such that* $r<s$. *Let* j *be a continuous and strictly increasing function on* $[r,s]$. *Then, for every* $\varepsilon\in(0,s-r)$, *there exists* $k>0$ *such that* $j(u)-j(v)\ge k(u-v)$ *for all* u *and* $v\in[r,s]$ *such that* $u-v\ge\varepsilon$.

Proof of Theorem 4.3. First we state the existence and uniqueness of the solution of equation (1.1) that is bounded on $\mathbb{R}$. The uniqueness of a solution that is bounded on $\mathbb{R}$ results from Proposition 2.3. In the particular case where $p(t) = p_0$ for each $t \in \mathbb{R}$, i.e., $\inf_{t\in\mathbb{R}} p(t) = \sup_{t\in\mathbb{R}} p(t)$, there exists $x_0 \in (a, b)$ such that $g(x_0) = p_0$; therefore, $x(t) = x_0$ for each $t \in \mathbb{R}$, is a solution that is bounded on $\mathbb{R}$. Now, we assume that $\inf_{t\in\mathbb{R}} p(t) < \sup_{t\in\mathbb{R}} p(t)$. By our hypothesis on the range of g and by (H1), there exist r and $s \in \mathbb{R}$ such that $g(r) = \sup_{t\in\mathbb{R}} p(t)$, $g(s) = \inf_{t\in\mathbb{R}} p(t)$ and $a < r < s < b$. Let $\tilde{f}$ and $\tilde{g}$ be extensions of $f_{/[r,s]}$ and $g_{/[r,s]}$. The extension $\tilde{f}$ is defined by $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{R}$ with

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } r \le x \le s \\ f(r) & \text{if } x < r \\ f(s) & \text{if } x > s. \end{cases}$$

The extension $\tilde{f}$ is continuous. We define in the same way the extension of $\tilde{g}$. We set

$$F(t, x, y) := p(t) - \tilde{f}(x)y - \tilde{g}(x), \qquad V(y) := 2 + |y|,$$
$$T(t) := \max\big(|p(t)|, \sup_{r\le x\le s} f(x), \sup_{r\le x\le s} |g(x)|\big),$$

for each t, x and $y \in \mathbb{R}$. Then

i) $F \in C(\mathbb{R}^3, \mathbb{R})$ and $F(t, r, 0) \le 0 \le F(t, s, 0)$ for each $t \in \mathbb{R}$,

ii) V and T are nonnegative and continuous functions on $\mathbb{R}$ such that V satisfies $\int_0^{+\infty} \frac{y}{V(y)} dy = +\infty$, $V(-y) = V(y)$ and $V(y) \ge 1$ for each $y \in \mathbb{R}$,

iii) $|F(t, x, y)| \le T(t)V(y)$ for each t, $y \in \mathbb{R}$ and $x \in [r, s]$.

By using [13, Théorème 2], we can assert that the equation

$$x'' = F(t, x, x')$$

admits at least a solution x satisfying $r \le x(t) \le s$ for each $t \in \mathbb{R}$; therefore, x is a solution of equation (1.1) that is bounded on $\mathbb{R}$.

Then we state the existence and uniqueness of the almost-periodic solution of equation (3.7) and the formula of the module. By using [1, Proposition 2.4], one has $p^{ap}(\mathbb{R}) \subset \overline{p(\mathbb{R})}$; therefore, $\inf_{t\in\mathbb{R}} p^{ap}(t)$ and $\sup_{t\in\mathbb{R}} p^{ap}(t) \in g(a, b)$. By using the preceding result, equation (3.7) admits a unique solution y that is bounded on $\mathbb{R}$. By using [6, Theorem 2.1], the bounded solution y is almost periodic and $mod(y) \subset mod(p^{ap})$.

If we denote by x the bounded solution of equation (1.1), y the almost-periodic solution of equation (3.7) and

$$h := x - y, \tag{4.3}$$

then $h \in C(\mathbb{R})$ and satisfies $\|h\|_\infty := \sup_{t\in\mathbb{R}} |h(t)| < +\infty$. Our results will be established if we prove that $h \in PAP_0(\mathbb{R})$, or equivalently

$$\forall \varepsilon > 0, \lim_{r\to+\infty} \frac{1}{2r} meas\{t \in [-r,r]; |h(t)| \geq \varepsilon\} = 0. \tag{4.4}$$

Since x and y are bounded on $\mathbb{R}$, there exist r and $s \in \mathbb{R}$ such that $a < r < s < b$ and

$$\forall t \in \mathbb{R}, \quad x(t) \text{ and } y(t) \in [r,s]. \tag{4.5}$$

Let $\varepsilon > 0$ and let $p^e(t) := p(t) - p^{ap}(t)$ be the ergodic perturbation of p. By subtracting equation (1.1) from equation (3.7), we obtain

$$h''(t) + f(x(t))x'(t) - f(y(t))y'(t) + g(x(t)) - g(y(t)) = p^e(t)$$

therefore,

$$\begin{aligned} &\frac{1}{2r}\int_{-r}^{r} \big(h''(t) + f(x(t))x'(t) - f(y(t))y'(t)\big)\, 1_{[h>0]}(t)dt \\ &= \frac{1}{2r}\int_{-r}^{r} p^e(t) 1_{[h>0]}(t)dt + \frac{1}{2r}\int_{-r}^{r} \left(g(y(t)) - g(x(t))\right) 1_{[h>0]}(t)dt, \end{aligned} \tag{4.6}$$

where $[h > 0]$ is defined by $[h > 0] := \{t \in \mathbb{R} : h(t) > 0\}$. First, one has

$$\left|\frac{1}{2r}\int_{-r}^{r} p^e(t) 1_{[h>0]}(t)dt\right| \leq \frac{1}{2r}\int_{-r}^{r} |p^e(t)|dt, \tag{4.7}$$

and by (H1) and (4.3), we obtain

$$\left(g(y(t)) - g(x(t))\right) 1_{[h>0]}(t) \geq 0;$$

therefore,

$$\frac{1}{2r}\int_{-r}^{r} \left(g(y(t)) - g(x(t))\right) 1_{[h\geq\varepsilon]}(t)dt \leq \frac{1}{2r}\int_{-r}^{r} \left(g(y(t)) - g(x(t))\right) 1_{[h>0]}(t)dt, \tag{4.8}$$

where $[h \geq \varepsilon]$ is defined by $[h \geq \varepsilon] := \{t \in \mathbb{R} : h(t) \geq \varepsilon\}$. By (H1), (4.5) and Lemma 4.4, there exists $k > 0$ such that

$$h(t) \geq \varepsilon \Longrightarrow g(y(t)) - g(x(t)) \geq kh(t);$$

therefore,

$$\frac{k\varepsilon}{2r} meas\{t \in [-r,r] : h(t) \geq \varepsilon\} \leq \frac{1}{2r}\int_{-r}^{r} \left(g(y(t)) - g(x(t))\right) 1_{[h\geq\varepsilon]}(t)dt. \tag{4.9}$$

By (4.6)–(4.9), we deduce that

$$\frac{k\varepsilon}{2r} meas\{t \in [-r,r]; h(t) \geq \varepsilon\} \tag{4.10}$$

$$\leq \frac{1}{2r}\int_{-r}^{r}|p^e(t)|dt + \frac{1}{2r}\int_{-r}^{r}\left(h''(t)+f(x(t))x'(t)-f(y(t))y'(t)\right)1_{[h>0]}(t)dt.$$

By using (4.5), one has

$$|h'(t)+\int_{y(t)}^{x(t)}f(z)dz| \leq |x'(t)-y'(t)| + |x(t)-y(t)|\sup_{r\leq z\leq t}|f(z)|,$$

and by Lemma 2.4, we deduce that for each $t\in\mathbb{R}$

$$|h'(t)+\int_{y(t)}^{x(t)}f(z)dz| \leq c_3 \tag{4.11}$$

with

$$c_3 := 2c_1 + 2c_0\sup_{r\leq z\leq t}|f(z)|, \tag{4.12}$$

where c_0 and c_1 are defined by (2.3) and (2.4). Now, we establish that

$$\frac{1}{2r}\int_{-r}^{r}\left(h''(t)+f(x(t))x'(t)-f(y(t))y'(t)\right)1_{[h>0]}(t)dt \leq \frac{c_3}{r}. \tag{4.13}$$

We denote by O_r the open subset of $(-r,r)$ defined by $O_r := \{t\in(-r,r) : h(t)>0\}$. The components of O_r are open intervals ω_i $(i\in I)$ included in $(-r,r)$, where the set I is countable. Let $m_i := \inf_{t\in\omega_i} t$ and $M_i := \sup_{t\in\omega_i} t$. In the particular case where there is one component, $O_r = (-r,r)$, i.e., $h(t)>0$ for each $t\in(-r,r)$, one has

$$\begin{aligned}&\frac{1}{2r}\int_{-r}^{r}\left(h''(t)+f(x(t))x'(t)-f(y(t))y'(t)\right)1_{[h>0]}(t)dt\\&=\frac{1}{2r}\int_{-r}^{r}(h''(t)+f(x(t))x'(t)-f(y(t))y'(t))dt=\frac{1}{2r}\left[h'(t)+\int_{y(t)}^{x(t)}f(z)dz\right]_{-r}^{r},\end{aligned}$$

and by (4.11) and (4.12), we deduce (4.13). In the general case where there are several components, one has

$$\forall t\in\omega_i,\quad h(t)>0, \tag{4.14}$$

$$\frac{1}{2r}\int_{\omega_i}\left(h''(t)+f(x(t))x'(t)-f(y(t))y'(t)\right)dt = \frac{1}{2r}\left[h'(t)+\int_{y(t)}^{x(t)}f(z)dz\right]_{m_i}^{M_i}. \tag{4.15}$$

If the component ω_i satisfies $-r<m_i<M_i<r$, then $h(m_i)=h(M_i)=0$. By (4.3) and (4.14), we deduce that $x(m_i)=y(m_i)$, $x(M_i)=y(M_i)$,

$h'(m_i) \geq 0$ and $h'(M_i) \leq 0$, and by (4.15) we obtain

$$\frac{1}{2r} \int_{\omega_i} \big(h''(t) + f(x(t))x'(t) - f(y(t))y'(t)\big)\, dt \leq 0. \tag{4.16}$$

If the component ω_i satisfies $-r = m_i < M_i < r$ (there is at most one such component), then $h(M_i) = 0$. By (4.3) and (4.14), we deduce that $x(M_i) = y(M_i)$ and $h'(M_i) \leq 0$, and by (4.15) we obtain

$$\frac{1}{2r}\int_{\omega_i}(h''(t)+f(x(t))x'(t)-f(y(t))y'(t))dt \leq -\frac{1}{2r}\Big(h'(m_i)+\int_{y(m_i)}^{x(m_i)} f(z)dz\Big).$$

By using (4.11) and (4.12), one has

$$\frac{1}{2r} \int_{\omega_i} \big(h''(t) + f(x(t))x'(t) - f(y(t))y'(t)\big)\, dt \leq \frac{c_3}{2r}. \tag{4.17}$$

For the same reason, if the component ω_i satisfies $-r < m_i < M_i = r$ (there is at most one such component), then the relation (4.17) is satisfied. With the following equality,

$$\frac{1}{2r} \int_{-r}^{r} \big(h''(t) + f(x(t))x'(t) - f(y(t))y'(t)\big)\, 1_{[h>0]}(t)dt$$
$$= \sum_{i\in I} \frac{1}{2r} \int_{\omega_i} \big(h''(t) + f(x(t))x'(t) - f(y(t))y'(t)\big)\, dt,$$

and by (4.16) and (4.17), by using the fact that there exists at most a component satisfying $-r = m_i < M_i < r$ (respectively $-r < m_i < M_i = r$), we deduce that the relation (4.13) is fulfilled. By (4.10) and (4.13), we deduce

$$\frac{k\varepsilon}{2r} meas\{t \in [-r,r] : h(t) \geq \varepsilon\} \leq \frac{1}{2r}\int_{-r}^{r} |p^e(t)|dt + \frac{c_3}{r};$$

therefore,

$$\lim_{r\to+\infty} \frac{1}{2r} meas\{t \in [-r,r] : h(t) \geq \varepsilon\} = 0. \tag{4.18}$$

The proof of

$$\lim_{r\to+\infty} \frac{1}{2r} meas\{t \in [-r,r] : h(t) \leq -\varepsilon\} = 0 \tag{4.19}$$

is similar to the proof of (4.18). By (4.18) and (4.19), we obtain the conclusion (4.4). This concludes the proof of Theorem 4.3. □

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