ON THE REGULARITY CRITERIA FOR THE GENERALIZED NAVIER-STOKES EQUATIONS AND LAGRANGIAN AVERAGED EULER EQUATIONS

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Abstract. We obtain some regularity conditions for solutions of the 3D generalized Navier-Stokes equations with fractional powers of the Laplacian, in terms of the velocity, the vorticity, and the pressure in Besov space, Triebel-Lizorkin space, and Lorentz space, respectively. We also present a regularity condition for the 3D Lagrangian averaged Euler equations.

1. INTRODUCTION

We study the following generalized Navier-Stokes equations:

\[ \partial_t u + (u \cdot \nabla) u + \nabla \pi + (-\Delta)^{\ell/2} u = 0, \]
\[ \text{div } u = 0, \]
\[ u|_{t=0} = u_0(x), \quad \text{div } u_0 = 0, \quad x \in \mathbb{R}^3, \]

(1.1)-(1.3)

where \( u \) is the velocity field, \( \pi \) is the scalar pressure, and \( \ell \) is a positive constant. The case \( \ell = 2 \) corresponds to the usual Navier-Stokes equations.

The system (1.1)-(1.3) was first considered by J.L. Lions in [23], and the global regularity for \( \ell \geq 5/2 \) is shown there. When \( \ell > 2 \), existence and uniqueness of solutions for (1.1)-(1.3) has been studied by S. Tourville [31]. Y. Zhou [35] obtained the regularity condition:

\[ u \in L^r(0,T; L^p(\mathbb{R}^3)), \quad \text{with } \frac{\ell}{r} + \frac{3}{p} = \ell - 1, \quad \frac{3}{\ell - 1} < p \leq \infty, \]  

(1.4)

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or

\[ \Lambda^{\ell/2} u \in L^r(0, T; L^p(\mathbb{R}^3)), \quad \text{with} \quad \frac{\ell}{r} + \frac{3}{p} = \frac{3\ell}{2} - 1, \quad \frac{6}{3\ell - 2} < p < \frac{6}{\ell - 2}. \quad (1.5) \]

Here, \( \Lambda := (-\Delta)^{1/2} \). When \( 0 < \ell < 2 \), there were studies on the small data global well-posedness for (1.1)-(1.3) by D. Chae [4], M. Cannone and G. Karch [3], D. Chae and J. Lee [5], and J. Wu [33]. D. Chae [4] proved the regularity criterion:

\[ \omega := \text{curl} \ u \in L^r(0, T; L^p(\mathbb{R}^3)), \quad \text{with} \quad \frac{3}{p} + \frac{\ell}{r} \leq \ell, \quad (1.6) \]

where \( \frac{6}{\ell} < p \leq \infty \). This covered the case \( \ell = 2 \) due to H.B. da Veiga [1]. In this note, we are concerned with the regularity condition of the strong solutions to (1.1)-(1.3). The first result for Navier-Stokes equations in this direction is obtained independently by Serrin [26] and Ohyama [25] which states that if weak solution \( u \) satisfies

\[ u \in L^r(0, T; L^p(\mathbb{R}^3)), \quad \text{with} \quad \frac{2}{r} + \frac{3}{p} \leq 1, \quad 3 < p \leq \infty, \quad (1.7) \]

then \( u \) is smooth. After that there are further developments and refinements by Fabes, Jones, and Riviere [12], Giga [14], Sohr and Von Wahl [27], and Struwe [29]. In the important case \( p = 3, r = \infty \) in (1.7), a smallness condition was required at first and removed by Escauriaza, Sverak, and Seregin [11]. Kozono, Ogawa, and Taniuchi [21] proved the following regularity condition:

\[ \omega := \text{curl} u \in L^1(0, T; \dot{B}^0_{\infty, \infty}(\mathbb{R}^3)), \quad (1.8) \]

where \( \dot{B}^0_{\infty, \infty} \) denotes the homogeneous Besov spaces. Kozono and Shimada [22] refined (1.7) by the following condition:

\[ u \in L^{\frac{2}{r-s}}(0, T; \dot{F}^{-s}_{\infty, \infty}) \quad \text{for} \quad 0 < s < 1, \quad (1.9) \]

where \( \dot{F}^{-s}_{\infty, \infty} \) denotes the homogeneous Triebel-Lizorkin space. Y. Zhou [34] and M. Struwe [30] proved similar regularity conditions on the pressure:

\[ \pi \in L^r(0, T; L^p(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{r} + \frac{3}{p} = 2 \quad \text{for} \quad \frac{3}{2} < p \leq \infty, \quad (1.10) \]

or

\[ \nabla \pi \in L^r(0, T; L^p(\mathbb{R}^3)) \quad \text{with} \quad \frac{2}{r} + \frac{3}{p} = 3 \quad \text{for} \quad 1 < p \leq \infty. \quad (1.11) \]

See also Cai and Zhai [2]. The purpose of this paper is to generalize all those available results for (1.1) in terms of homogeneous Besov space \( \dot{B}^0_{p, \infty} \),
homogeneous Triebel-Lizorkin space $\dot{F}_{\infty, \infty}^{-s}$, and Lorentz space $L^{p, \infty}$. We now state our main results in this paper.

**Theorem 1.1.** Let $u_0$ be smooth and $\text{div} u_0 = 0$ in $\mathbb{R}^3$ and $\ell < \frac{5}{2}$. Assume that one of the following conditions is satisfied:

\begin{align*}
\ell > 1, & \quad u \in L^r(0, T; \dot{B}_{p, \infty}^0) \quad \text{with} \quad \frac{\ell}{r} + \frac{3}{p} = \ell - 1, \quad \frac{3}{\ell - 1} < p < \infty; \quad (1.12) \\
\ell > 0, & \quad \omega \in L^r(0, T; \dot{B}_{p, \infty}^0) \quad \text{with} \quad \frac{\ell}{r} + \frac{3}{p} = \ell, \quad \frac{3}{\ell} < p \leq \infty; \quad (1.13) \\
\ell > 1, & \quad u \in L^r(0, T; \dot{F}_{\infty, \infty}^{-s}) \quad \text{with} \quad \frac{\ell}{r} = \ell - 1 - s, \quad 0 < s < 1, \quad (1.14) \\
\ell > 0, & \quad \omega \in L^r(0, T; \dot{F}_{\infty, \infty}^{-s}) \quad \text{with} \quad \frac{\ell}{r} = \ell - s, \quad 0 < s < 1, \quad 0 < s < \frac{\ell}{2}; \quad (1.15) \\
\ell \geq 2, & \quad \pi \in L^r(0, T; L^{p, \infty}) \quad \text{with} \quad \frac{\ell}{r} + \frac{3}{p} = 2\ell - 2, \quad \frac{3}{2\ell - 2} < p \leq \frac{3}{\ell - 2}. \quad (1.16)
\end{align*}

Then there is no singularity up to $T$.

**Remark 1.1.** We observe that the system (1.1) is invariant under scaling transform $(u, \pi) \mapsto (u_\lambda, \pi_\lambda)$, where

$$u_\lambda(t, x) := \lambda^{\ell - 1} u(\lambda^\ell t, \lambda x), \quad \pi_\lambda(t, x) := \lambda^{2\ell - 2} \pi(\lambda^\ell t, \lambda x), \quad \lambda > 0,$$

which induces the scaling for the vorticity, $\omega \mapsto \omega_\lambda$, $\omega_\lambda(t, x) := \lambda^\ell \omega(\lambda^\ell t, \lambda x)$. Furthermore, we note that

$$\|u\|_{L^r(0, T; L^p(\mathbb{R}^3))} = \|u_\lambda\|_{L^r(0, \lambda^\ell T; L^p(\mathbb{R}^3))}, \quad \text{if} \quad \frac{\ell}{r} + \frac{3}{p} = \ell - 1;$$

$$\|\pi\|_{L^r(0, T; L^p(\mathbb{R}^3))} = \|\pi_\lambda\|_{L^r(0, \lambda^\ell T; L^p(\mathbb{R}^3))}, \quad \text{if} \quad \frac{\ell}{r} + \frac{3}{p} = 2\ell - 2;$$

$$\|\omega\|_{L^r(0, T; L^p(\mathbb{R}^3))} = \|\omega_\lambda\|_{L^r(0, \lambda^\ell T; L^p(\mathbb{R}^3))}, \quad \text{if} \quad \frac{\ell}{r} + \frac{3}{p} = \ell.$$

In this sense, our conditions (1.12)-(1.16) are optimal.

**Remark 1.2.** Since $L^p \subset \dot{B}_{p, \infty}^0$, $L^{3/p} \subset \dot{F}_{\infty, \infty}^{-s}$, and $L^\infty \subset BMO \subset \dot{B}_{\infty, \infty}^0$, our conditions (1.13) and (1.15) refines (1.6) in D.Chae [4]. Even when $\ell = 2$, our condition (1.15) is new.
Remark 1.3. It is a simple matter to generalize the conditions in Theorem 1.1 to the $n$-dimensional case. However, for simplicity we omit the details here.

2. Preliminaries

We first recall the definition of the homogeneous Littlewood-Paley decomposition which will be used to define function spaces. We follow [32]. Let $\mathcal{S}$ be the Schwartz class of rapidly decreasing smooth functions. Let $\hat{\varphi} \in \mathcal{S}$ satisfying $\text{supp } \hat{\varphi} \subset \{ \xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2 \}$, and $\hat{\varphi} > 0$ if $\frac{2}{3} < |\xi| < \frac{3}{2}$. Setting $\hat{\varphi}_j(\xi) = \hat{\varphi}(2^{-j} \xi)$ (in other words, $\varphi_j(x) = 2^{jn} \varphi(2^j x)$), we can adjust the normalization constant in front of $\hat{\varphi}$ so that

$$\sum_{j \in \mathbb{Z}} \hat{\varphi}_j(\xi) = 1, \quad \xi \in \mathbb{R}^n \setminus \{0\}.$$ 

Given $k \in \mathbb{Z}$, we define the function $S_k \in \mathcal{S}$ by its Fourier transform

$$\hat{S}_k(\xi) := 1 - \sum_{j \geq k+1} \hat{\varphi}_j(\xi).$$

We observe

$$\text{supp } \hat{\varphi}_j \cap \text{ supp } \hat{\varphi}_{j'} = \emptyset \quad \text{if} \quad |j - j'| \geq 2.$$ 

Let $s \in \mathbb{R}$, $(p, q) \in [1, \infty) \times [1, \infty]$. Given $f \in \mathcal{S}'$, we denote $\Delta_j f := \varphi_j \ast f$, and then the homogeneous Triebel-Lizorkin semi-norm $\|f\|_{\dot{F}^s_{p,q}}$ is defined by

$$\|f\|_{\dot{F}^s_{p,q}} := \begin{cases} \left\| \left( \sum_{j \in \mathbb{Z}} 2^{jsq} |\Delta_j f(\cdot)|^q \right)^{1/q} \right\|_{L^p} & \text{if } q \in [1, \infty), \\ \left\| \sup_{j \in \mathbb{Z}} \left( 2^{js} |\Delta_j f(\cdot)| \right) \right\|_{L^p} & \text{if } q = \infty. \end{cases}$$

The homogeneous Triebel-Lizorkin space $\dot{F}^s_{p,q}$ is a quasi-normed space with the quasi-norm given by $\| \cdot \|_{\dot{F}^s_{p,q}}$. For $s > 0$, $(p, q) \in [1, \infty) \times [1, \infty]$. We define the inhomogeneous Triebel-Lizorkin space $F^s_{p,q}$ of $f \in \mathcal{S}'$ as

$$\|f\|_{F^s_{p,q}} := \|f\|_{L^p} + \|f\|_{\dot{F}^s_{p,q}}.$$
The inhomogeneous Triebel-Lizorkin space is a Banach space equipped with the norm, \( \| \cdot \|_{F^{s,p,q}_{p,q}} \). Similarly, for \( s \in \mathbb{R}, (p, q) \in [1, \infty]^2 \), the homogeneous Besov norm \( \| f \|_{\dot{B}^{s,p,q}_{p,q}} \) is defined by

\[
\| f \|_{\dot{B}^{s,p,q}_{p,q}} := \begin{cases} 
( \sum_{j} 2^{js} \| \varphi_j * f \|_{L^p}^q )^{1/q} & \text{if } q \in [1, \infty), \\
\sup_{j} (2^{js} \| \varphi_j * f \|_{L^p}) & \text{if } q = \infty.
\end{cases}
\]

The homogeneous Besov space \( \dot{B}^{s,p,q}_{p,q} \) is a quasi-normed space with the quasi-norm given by \( \| \cdot \|_{\dot{B}^{s,p,q}_{p,q}} \). For \( s > 0 \) we define the inhomogeneous Besov space norm \( \| f \|_{B^{s,p,q}_{p,q}} \) of \( f \in \mathcal{S}' \) as

\[
\| f \|_{B^{s,p,q}_{p,q}} := \| f \|_{L^p} + \| f \|_{\dot{B}^{s,p,q}_{p,q}}.
\]

Below we recall lemmas that will be used in the proof of our results.

**Lemma 2.1.** (Bernstein’s lemma). Let \( 1 \leq p \leq \infty \). Then there exists a constant \( C_k \) such that for any \( f \in L^p \) with \( \text{supp} \hat{f} \subset \{ 2^{j-2} \leq |\xi| < 2^j \} \)

\[
C_k^{-1} 2^kJ^k \| f \|_{L^p} \leq \sum_{|\alpha| = k} \| \partial^\alpha f \|_{L^p} \leq C_k 2^kJ^k \| f \|_{L^p}. \tag{2.1}
\]

For the proof see J.Y. Chemin [6]. The following lemma is also well known (see, e.g., [32]).

**Lemma 2.2.** For any \( k \in \mathbb{Z}_+ \), there exists a constant \( C_k \) such that the following inequality holds:

\[
C_k^{-1} \sum_{|\alpha| = k} \| \partial^\alpha f \|_{\dot{F}^{s,p,q}_{p,q}} \leq \| f \|_{\dot{F}^{s+k,p,q}_{p,q}} \leq C_k \sum_{|\alpha| = k} \| \partial^\alpha f \|_{\dot{F}^{s,p,q}_{p,q}}. \tag{2.2}
\]

**Lemma 2.3.** Let \( \sigma > 0, 1 \leq p_i, p_2 \leq \infty, \sigma_1, \sigma_2 \in \mathbb{R} \) such that \( \frac{3}{p_i} - \sigma_i > 0 \) \( i = 1, 2 \), then the following inequality holds:

\[
\left( \sum_{j \in \mathbb{Z}} 2^{2j\sigma} \| [f, \Delta_j] \nabla g \|_{L^2}^2 \right)^{1/2} \leq C \left( \| \nabla f \|_{\dot{B}^{\sigma_1,1}_{p_1,\infty}} \| g \|_{\dot{B}^{\sigma_2,\sigma_1+\frac{1}{p_1}}_{p_2,2}} + \| \nabla g \|_{\dot{B}^{\sigma_2,\sigma_2+\frac{1}{p_2}}_{p_2,\infty}} \| f \|_{\dot{B}^{\sigma_1,\sigma_2+\frac{1}{p_2}}_{p_1,2}} \right). \tag{2.3}
\]

If \( \sigma_1 = 0, p_1 = \infty, \| \nabla f \|_{\dot{B}^{\sigma_1,1}_{p_1,\infty}} \) has to be replaced by \( \| \nabla f \|_{L^\infty} \), or if \( \sigma_2 = 0, p_2 = \infty, \| \nabla g \|_{\dot{B}^{\sigma_2,\infty}_{p_2,\infty}} \) has to be replaced by \( \| \nabla g \|_{L^\infty} \). Here \([f, \Delta_j] \nabla g := f \Delta_j (\nabla g) - \Delta_j (f \nabla g)\).

For the proof, see [8].
Lemma 2.4. (1) Let $1 < p < \infty, 1 < q < \infty$ and let $s > 0, \alpha > 0, \beta > 0$. Let $1 < p_1 < \infty, 1 < p_2 \leq \infty$ and $1 < r_1 \leq \infty, 1 < r_2 < \infty$ so that

$$1/p = 1/p_1 + 1/p_2 = 1/r_1 + 1/r_2.$$ Then for any $f \in \dot{F}^{s+\alpha}_{p_1,q} \cap \dot{F}^{-\beta}_{r_1,\infty}$ and $g \in \dot{F}^{-\alpha}_{p_2,\infty} \cap \dot{F}^{s+\beta}_{r_2,q}$ we have $fg \in \dot{F}^s_{p,q}$ with the estimate

$$\|fg\|_{\dot{F}^s_{p,q}} \leq C \left( \|f\|_{\dot{F}^{s+\alpha}_{p_1,q}} \|g\|_{\dot{F}^{-\beta}_{r_1,\infty}} + \|f\|_{\dot{F}^{-\alpha}_{p_2,\infty}} \|g\|_{\dot{F}^{s+\beta}_{r_2,q}} \right). \tag{2.4}$$

(2) Let $1 < p \leq \infty$ and let $s > 0, \alpha > 0, \beta > 0$. Let $1 < p_1, p_2, r_1, r_2 \leq \infty$ satisfy $1/p = 1/p_1 + 1/p_2 = 1/r_1 + 1/r_2$. Then for any $f \in \dot{F}^{s+\alpha}_{p_1,\infty} \cap \dot{F}^{-\beta}_{r_1,\infty}$ and $g \in \dot{F}^{-\alpha}_{p_2,\infty} \cap \dot{F}^{s+\beta}_{r_2,\infty}$ we have $fg \in \dot{F}^s_{p,\infty}$ with the estimate

$$\|fg\|_{\dot{F}^s_{p,\infty}} \leq C \left( \|f\|_{\dot{F}^{s+\alpha}_{p_1,\infty}} \|g\|_{\dot{F}^{-\alpha}_{p_2,\infty}} + \|f\|_{\dot{F}^{-\beta}_{r_1,\infty}} \|g\|_{\dot{F}^{s+\beta}_{r_2,\infty}} \right). \tag{2.5}$$

For the proof see [22].

3. Proof of Theorem 1.1

The proof is based on the establishment of a priori estimates for $u$ which can then be used to extend a smooth local solution globally in time.

We first show that Theorem 1.1 holds under the condition (1.12). Let $u$ be a smooth solution of (1.1)-(1.3). Applying the operator $\Delta_k$ to (1.1), then multiplying it by $\Delta_k u$ and integrating by parts and taking into account $\operatorname{div} u = 0$, we find that

$$\frac{1}{2} \frac{d}{dt} \|\Delta_k u\|_{L^2}^2 + \|\Lambda^{\ell/2} \Delta_k u\|_{L^2}^2 = \int_{\mathbb{R}^3} [u, \Delta_k] \nabla u \cdot \Delta_k u \, dx. \tag{3.1}$$

Integrating (3.1) over $(0,t)$ and multiplying by $2^{2km}(m \geq 3)$, then summing over $k \in \mathbb{Z}$, we obtain

$$\frac{1}{2} \|u\|_{H^m}^2 + \int_0^t \|u\|_{H^{m+\ell/2}}^2 \, d\tau$$

$$\leq \frac{1}{2} \|u_0\|_{H^m}^2 + \int_0^t \sum_{k \in \mathbb{Z}} 2^{2km} \|[u, \Delta_k] \nabla u\|_{L^2} \|\Delta_k u\|_{L^2} \, d\tau$$

$$= : \frac{1}{2} \|u_0\|_{H^m}^2 + \int_0^t I(\tau) \, d\tau. \tag{3.2}$$

$I(\tau)$ can be bounded as follows:

$$I \leq \left( \sum_{k \in \mathbb{Z}} 2^{2k(m-\ell/2)} \|[u, \Delta_k] \nabla u\|_{L^2}^2 \right)^{1/2} \left( \sum_{k \in \mathbb{Z}} 2^{2k(m+\ell/2)} \|\Delta_k u\|_{L^2}^2 \right)^{1/2}$$
\[ \leq C\|u\|_{\dot{B}_{p,\infty}^0} \|u\|_{\dot{B}_{2,2}^{m-\ell/2+1+\frac{3}{p}}} \|u\|_{\dot{H}^{m+\ell/2}} \]

\[ \leq C\|u\|_{\dot{B}_{p,\infty}^0} \|\theta\|_{\dot{B}_{p,\infty}^{\theta}} \|\nabla u\|_{L^2}^{2-\theta} \|u\|_{\dot{H}^{m+\ell/2}} \leq \frac{1}{2} \|u\|_{\dot{H}^{m+\ell/2}}^2 + C\|u\|_{\dot{B}_{p,\infty}^0} \|u\|_{\dot{H}^{m+\ell/2}}^2, \quad (3.3) \]

by (2.3) (taking \(\sigma_1 = \sigma_2 = -1, p_1 = p_2 = p, \sigma = m - \ell/2\)) and the interpolation inequality [32]:

\[ \|f\|_{\dot{B}_{p,q}^{\theta_1+(1-\theta)s_2}} \leq \|f\|_{\dot{B}_{p,q}^{\theta_1}} \|f\|_{L^2}^{1-\theta}, \quad (3.4) \]

with \(s_1 = m, s_2 = m + \ell/2, \theta = \frac{\ell-(1+3/p)}{\ell/2}\) and \(r = \frac{2}{\theta}\).

Inserting (3.3) into (3.2) and using the Gronwall’s inequality gives the result.

Assume that (1.13) holds with \(p < \infty\). Similarly, we have (3.2).

Using (2.3) with \(\sigma_1 = \sigma_2 = 0, p_1 = p_2 = p > \frac{3}{2}\) and \(\sigma = m - \frac{3}{2p}\) and the interpolation inequality (3.4) with \(s_1 = m, s_2 = m + \ell, \theta = \frac{p-3}{2p}\) and \(r = \frac{1}{p}\), then \(I(\tau)\) can be bounded as follows.

\[ I \leq \left( \sum_{k \in \mathbb{Z}} 2^{2k(m-\frac{3}{2p})} \|\nabla u\|_{L^2}^2 \left( \sum_{k \in \mathbb{Z}} 2^{2k(m+\frac{3}{2p})} \|\Delta_k u\|_{L^2}^2 \right)^{1/2} \right) \]

\[ \leq C\|\nabla u\|_{\dot{B}_{p,\infty}^0} \|u\|_{L^2}^{2(1-\theta)} \leq C\|\Delta u\|_{\dot{B}_{p,\infty}^0} \|\Delta u\|_{L^2}^{2(1-\theta)} \]

\[ \leq \frac{1}{2} \|u\|_{\dot{H}^{m+\ell/2}}^2 + C\|\nabla u\|_{\dot{B}_{p,\infty}^0} \|u\|_{\dot{H}^{m+\ell/2}}^2 \]

\[ \leq \frac{1}{2} \|u\|_{\dot{H}^{m+\ell/2}}^2 + C\|\omega\|_{\dot{B}_{p,\infty}^0} \|\omega\|_{H^{m+\ell/2}}^2. \quad (3.5) \]

Here, we have used the following inequality [28]:

\[ \|\nabla u\|_{\dot{B}_{p,\infty}^0} \leq C\|\omega\|_{\dot{B}_{p,\infty}^0}. \quad (3.6) \]

Inserting (3.5) into (3.2) and the Gronwall’s inequality yields the result.

Assume that (1.13) holds with \(p = \infty\). Similarly, we have (3.2). Using (2.3) with \(\sigma_1 = \sigma_2 = 0, p_1 = p_2 = p = \infty\) and \(\sigma = m\), then \(I(\tau)\) can be estimated as follows.

\[ I \leq \left( \sum_{k \in \mathbb{Z}} 2^{2km} \|\nabla u\|_{L^2}^2 \left( \sum_{k \in \mathbb{Z}} 2^{2km} \|\Delta_k u\|_{L^2}^2 \right)^{1/2} \right) \]

\[ \leq C\|\nabla u\|_{L^\infty} \|u\|_{H^m}^2 \leq C\|\nabla u\|_{\dot{B}_{\infty,c}^0} \|u\|_{H^m}^2 \log (3 + \|u\|_{H^m}^2) \]

\[ \leq C\|\omega\|_{\dot{B}_{\infty,c}^0} \|u\|_{H^m}^2 \log (3 + \|u\|_{H^m}^2), \quad (3.7) \]
where we have used the logarithmic Sobolev inequality [21]:
\[
\|\nabla u\|_{L^\infty} \leq C \|\nabla u\|_{\dot{B}_{0,\infty}^{1}} \log (3 + \|u\|_{H^m}^2),
\]
and the following inequality [28]:
\[
\|\nabla u\|_{\dot{B}_{0,\infty}^{3/2}} \leq C \|\omega\|_{\dot{B}_{0,\infty}^{3/2}}.
\]
Inserting (3.7) into (3.2) and the Gronwall’s inequality leads to the result.

Assume that (1.14) holds. Let \(\alpha = (\alpha_1, \alpha_2, \alpha_3)\) be a multi-index with \(|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 \geq 1\), and let \(v_\alpha := \partial^\alpha u = \frac{\partial^{\alpha_1} u}{\partial x_1} \frac{\partial^{\alpha_2} u}{\partial x_2} \frac{\partial^{\alpha_3} u}{\partial x_3}\).

Applying \(\partial^\alpha\) to (1.1), we have for \(v_\alpha\) the equation
\[
\partial_t v_\alpha + \nabla q_\alpha + (\Delta)^{\ell/2} v_\alpha = F_\alpha,
\]
where \(q_\alpha := \nabla^\alpha \pi\) and
\[
F_\alpha := -\partial^\alpha (u \cdot \nabla u) = -\partial^\alpha \nabla \cdot (u \otimes u).
\]
Multiplying (3.10) by \(v_\alpha\) and integrating by parts, we see that
\[
\frac{1}{2} \|v_\alpha(t)\|_{L^2}^2 + \int_0^t \|\Lambda^{\ell/2} v_\alpha(\tau)\|_{L^2}^2 d\tau \leq \frac{1}{2} \|v_\alpha(0)\|_{L^2}^2 + \int_0^t |(F_\alpha, v_\alpha)| d\tau.
\]
By the Schwarz inequality, there holds
\[
|(F_\alpha, v_\alpha)| = |(\Lambda^{-\ell/2} \partial^\alpha \nabla \cdot u \otimes u, \Lambda^{\ell/2} v_\alpha)| \leq \|u \otimes u\|_{H^{1+|\alpha| - \ell/2}} \|\Lambda^{\ell/2} v_\alpha\|_{L^2},
\]
it follows from Lemma 2.4 and the interpolation inequality that
\[
|(F_\alpha, v_\alpha)| \leq C \|u \otimes u\|_{F_{2,2}^{1+|\alpha| - \ell/2}} \|\Lambda^{\ell/2} v_\alpha\|_{L^2} \leq C \|u\|_{F_{2,2}^{\ell}} \|\dot{\omega}\|_{H^{1+|\alpha| - \ell/2}} \|\Lambda^{\ell/2} v_\alpha\|_{L^2} \leq C \|u\|_{\dot{F}_{\infty,\infty}^{1+|\alpha| - \ell/2}} \|\Lambda^{\ell/2} v_\alpha\|_{L^2}^2 \leq C \|u\|_{F_{\infty,\infty}^{-s}} \|v_\alpha\|_{L^2}^2 + C \|u\|_{F_{\infty,\infty}^{-s}} \|v_\alpha\|_{L^2}^2,
\]
where \(\theta = \frac{\ell-1-s}{\ell/2}\) and \(r = \frac{2}{\ell/2} + 1\).

Inserting (3.12) into (3.11) and the result follows from the Gronwall’s inequality.

Assume that (1.15) holds. Applying curl to (1.1), we easily get for \(\omega := \text{curl } u\) the equation
\[
\partial_t \omega + u \cdot \nabla \omega + (-\Delta)^{\ell/2} \omega = \omega \cdot \nabla u.
\]
The generalized NS equations 451

Multiplying (3.13) by \(-\Delta \omega\) and integrating by parts, we have

\[
\frac{1}{2} \frac{d}{dt} \|\nabla \omega\|^2_{L^2} + \|\Lambda^{1+\ell/2} \omega\|^2_{L^2} = \int_{\mathbb{R}^3} \omega \cdot \nabla u \cdot \Delta \omega dx + \int_{\mathbb{R}^3} u \cdot \nabla \omega \cdot \Delta \omega dx =: I_1 + I_2. \tag{3.14}
\]

Using Lemma 2.4 and the interpolation inequality, we bound \(I_1\) as follows.

\[
I_1 = \int_{\mathbb{R}^3} \omega \cdot \nabla u \cdot \Lambda^2 \omega dx = \int_{\mathbb{R}^3} \Lambda^{1-s}(\omega \cdot \nabla u) \cdot \Lambda^{1+s} \omega dx
\leq \|\Lambda^{1-s}(\omega \cdot \nabla u)\|_{L^2} \|\Lambda^{1+s} \omega\|_{L^2} \leq C \|\omega \cdot \nabla u\|_{F^1_{2,2}} \|\Lambda^{1+s} \omega\|_{L^2}
\leq C \|\omega\|_{F^s_{\infty,\infty}} \|\nabla \omega\|_{L^2} \|\Lambda^{1+s} \omega\|_{L^2}
\leq C \|\omega\|_{F^s_{\infty,\infty}} \|\nabla \omega\|_{L^2} \|\Lambda^{1+\ell/2} \omega\|_{L^2}
\leq \frac{1}{4} \|\Lambda^{1+\ell/2} \omega\|^2_{L^2} + C \|\omega\|_{F^s_{\infty,\infty}} \|\nabla \omega\|^2_{L^2}, \tag{3.15}
\]

with \(r = \frac{2}{1+\theta}\) and \(\theta = \frac{\ell-2s}{\ell}\). Here, we have used the following inequalities [28]:

\[
\|\nabla \omega\|_{F^s_{\infty,\infty}} \leq C \|\omega\|_{F^s_{\infty,\infty}}, \tag{3.16}
\]

and

\[
\|\nabla u\|_{F^1_{2,2}} \leq C \|\omega\|_{F^1_{2,2}}. \tag{3.17}
\]

By integration by parts, we rewrite \(I_2\) as

\[
I_2 = \sum_{i,k} \int_{\mathbb{R}^3} u_i \partial_i \omega \cdot \partial_k^2 \omega dx = - \sum_{i,k} \int_{\mathbb{R}^3} u_i \omega \partial_i \partial_k^2 \omega dx
= \sum_{i,k} \int_{\mathbb{R}^3} \partial_k u_i \cdot \omega \partial_i \partial_k \omega dx
= \sum_{i,k} \int_{\mathbb{R}^3} \Lambda^{1-s}(\partial_k u_i \cdot \omega) \cdot \partial_i \partial_k (-\Delta)^{-1} \cdot \Lambda^{1+s} \omega dx
\leq \sum_{i,k} \|\Lambda^{1-s}(\omega \partial_k u_i)\|_{L^2} \|\partial_i \partial_k (-\Delta)^{-1} \cdot \Lambda^{1+s} \omega\|_{L^2}
\leq C \|\omega \cdot \nabla u\|_{F^{1-s}_{2,2}} \|\Lambda^{1+s} \omega\|_{L^2},
\]
and we obtain, in the same as that of $I_1$,

$$
I_2 \leq \frac{1}{4} \|\Lambda^{1+\ell/2}\omega\|_{L^2}^2 + C \|\omega\|_{F^r_{\infty:\infty}} \|\nabla\omega\|_{L^2}^2.
$$

(3.18)

Inserting (3.15) and (3.18) into (3.14) and we apply the Gronwall’s inequality to get the result.

Finally, we assume that (1.16) holds. Multiplying (1.1) by $|u|^2 u$ and integrating by parts and using (1.2), we obtain

$$
\frac{1}{4} \frac{d}{dt} \int_{\mathbb{R}^3} |u|^4 dx + \int_{\mathbb{R}^3} \Lambda^\ell u \cdot |u|^2 u dx = - \int_{\mathbb{R}^3} |u|^2 u \nabla \pi dx =: J(t).
$$

(3.19)

The viscosity term on the left hand side is estimated by

$$
\int_{\mathbb{R}^3} \Lambda^\ell u \cdot |u|^2 u dx \geq \frac{1}{4} \int_{\mathbb{R}^3} |\Lambda^{\ell/2}(|u|^2)|^2 dx,
$$

(3.20)

where we used Lemma 2.4 of \cite{9} for the estimate of the fractional derivative.

Next, we estimate $J$ as

$$
J \leq 2 \int_{\mathbb{R}^3} |\pi| |u|^2 |\nabla| u| dx = \int_{\mathbb{R}^3} |\pi| \cdot |u||\nabla| u| | dx.
$$

(3.21)

For simplicity, denote $v = |u|^2$. Then we have by (3.19)-(3.21) that

$$
\frac{d}{dt} \int_{\mathbb{R}^3} v^2 dx + \int_{\mathbb{R}^3} |\Lambda^{\ell/2} v|^2 dx \\
\leq \frac{1}{4} \int_{\mathbb{R}^3} |\pi| \cdot |v|^{1/2} |\nabla v| dx = C \int_{\mathbb{R}^3} |\pi|^{1/2} \cdot |\nabla| v|^{1/2} \cdot |\nabla v| dx \\
\leq C \|\pi\|_{L^{2p,\infty}} \|\nabla v\|_{L^{2q,4}} \|\nabla v\|_{L^{6/(5-\ell),2}} \\
\leq C \|\pi\|_{L^{2p,\infty}} \|\nabla v\|_{L^{2q,2}} \|\nabla v\|_{L^{6/(5-\ell),2}},
$$

where we have used the generalized Hölder inequality \cite{32} with $1/2p + 1/q + (5 - \ell)/6 = 1$ and basic properties of the non-increasing rearrangement.

We use the representation formula of the pressure by means of the velocity and Sobolev and Gagliardo-Nirenberg inequalities involving Lorentz spaces to obtain

$$
\|\pi\|_{L^{q,2}} \leq C \|v\|_{L^{6,2}}, \\
\|\nabla v\|_{L^{6/(5-\ell),2}} \leq C \|\Lambda^{\ell/2} v\|_{L^2}, \\
\|v\|_{L^{q,2}} \leq C \|\Lambda^{\ell/2} v\|_{L^2} \|v\|^{1-\theta}_{L^2},
$$

where $\theta = 3/\ell - 6/\ell q$. 
Combining these estimates, we have
\[
\frac{d}{dt} \int_{\mathbb{R}^3} v^2 \, dx + \int_{\mathbb{R}^3} |\Lambda^{\ell/2} v|^2 \, dx \leq C \|\pi\|_{L^p, \infty}^{1/2} \|v\|_{L^2}^{1-\theta} \|\Lambda^{\ell/2} v\|_{L^2}^{1+\theta}.
\] (3.22)

For \( \theta \) with \( 0 \leq \theta < 1 \), which is equivalent to \( 3/(2\ell - 2) < p \leq 3/(\ell - 2) \), the right hand side on (3.22) is estimated as
\[
\frac{1}{2} \|\Lambda^{\ell/2} v\|_{L^2}^2 + C \|\pi\|_{L^p, \infty}^{1/2} \|v\|_{L^2}^2,
\]
where \( 1/r = 1 - \theta = 2 - 2/\ell - 3/\ell p \).

Substituting this into (3.19) and using the Gronwall’s inequality, we obtain
\[
u \in L^\infty(0, T; L^4(\mathbb{R}^3)),
\]
which implies the result from (1.12). This completes the proof. \( \square \)

4. The Lagrangian averaged Euler equations

In this section, we consider the 3D Lagrangian averaged Euler equations in the following form [15, 16, 24]:
\[
\begin{align*}
\partial_t u + (u_\alpha \cdot \nabla) u + (\nabla u_\alpha)^T \cdot u &= -\nabla \pi, \\
u = \text{curl} \psi, \\
u_\alpha &= (1 - \alpha^2 \Delta)^{-1} u, \quad \alpha > 0, \\
u|_{t=0} &= u_0(x), \quad x \in \mathbb{R}^3.
\end{align*}
\] (4.1-4.4)

One of the important properties of the averaged Euler equations is the following identity:
\[
\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^3} |u_\alpha|^2 + \alpha^2 |\nabla u_\alpha|^2 \, dx = 0.
\]

This conservation property gives an a priori bound on the \( H^1 \) norm of \( u_\alpha \):
\[
\|u_\alpha\|_{H^1} \leq C_\alpha.
\] (4.5)

The averaged Euler models have been used to study the average behavior of the 3D Euler and Navier-Stokes equations and used as a turbulent closure model [7]. However, the global existence of the 3D Lagrangian averaged Euler equations is still open, although the Lagrangian averaged Navier-Stokes equations have been shown to have global existence [24, 13]. Very recently, Hou-Li [17] obtain the following regularity condition:
\[
\psi \in L^1(0, T; BMO(\mathbb{R}^3)),
\] (4.6)

with the initial condition
\[
u_0 \in H^1(\mathbb{R}^3).
\] (4.7)
We will prove

**Theorem 4.1.** Let $\alpha > 0$ and let $u_0 \in H^s(\mathbb{R}^3)$ with $s > \frac{1}{2}$. Then the Lagrangian averaged 3D Euler equations (4.1)-(4.4) have a unique global solution $u \in L^\infty(0, T; H^s(\mathbb{R}^3))$ satisfying

$$
\|u(t)\|_{H^s} \leq C\|u_0\|_{H^s} \quad \text{for} \quad 0 \leq t \leq T,
$$

(4.8)

provided that

$$
\psi \in L^1(0, T; \dot{B}^0_{\infty, \infty}(\mathbb{R}^3)).
$$

(4.9)

**Remark 4.1.** Since $L^\infty \subset BMO \subset \dot{B}^0_{\infty, \infty}$, our result (4.9) generalizes (4.6) in [17]. It is interesting to note that our initial condition $u_0 \in H^s$ with $s > \frac{1}{2}$ is weaker than (4.7) in [17].

**Proof** Testing (4.1) by $u$, we see that

$$
\frac{1}{2} \frac{d}{dt} \|u\|_{L^2}^2 \leq \|\nabla u\|_{L^\infty} \|u\|_{L^2}^2.
$$

(4.10)

Applying $\Lambda^s$ to equation (4.1), we have

$$
\partial_t \Lambda^s u + \Lambda^s(u_\alpha \cdot \nabla u) + \Lambda^s((\nabla u_\alpha)^T \cdot u) = -\nabla \Lambda^s \pi.
$$

Testing this equation by $\Lambda^s u$ and using the divergence free property, we see that

$$
\frac{1}{2} \frac{d}{dt} \|\Lambda^s u\|_{L^2}^2 \leq \left| \int_{\mathbb{R}^3} (\Lambda^s(u_\alpha \cdot \nabla u) - u_\alpha \cdot \nabla \Lambda^s u) \cdot \Lambda^s u\, dx \right|
$$

$$
+ \left| \int_{\mathbb{R}^3} \Lambda^s((\nabla u_\alpha)^T \cdot u) \cdot \Lambda^s u\, dx \right| =: J_1 + J_2.
$$

(4.11)

We will use the following bilinear commutator and product estimates due to David-Journé [10], Kato-Ponce [18] and Kenig-Ponce-Vega [19]:

$$
\|\Lambda^r(fg) - f\Lambda^r g\|_{L^p} \leq C\|\nabla f\|_{L^{p_1}} \|\Lambda^{r-1} g\|_{L^{q_1}} + \|\Lambda^r f\|_{L^{p_2}} \|g\|_{L^{q_2}},
$$

(4.12)

$$
\|\Lambda^r(fg)\|_{L^p} \leq C\|f\|_{L^{p_1}} \|\Lambda^r g\|_{L^{q_1}} + \|\Lambda^r f\|_{L^{p_2}} \|g\|_{L^{q_2}},
$$

(4.13)

with $r > 0$ and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{q_2}$. We apply (4.12) to $f = (u_\alpha)_i$, $g = u$ and $p_1 = \infty, p_2 = q_1 = 2$ to bound $J_1$ as

$$
J_1 = \left| \sum_{i=1}^{3} \int_{\mathbb{R}^3} (\Lambda^s \partial_i((u_\alpha)_i u) - (u_\alpha)_i \cdot \Lambda^s \partial_i u) \cdot \Lambda^s u\, dx \right|
$$

$$
\leq \sum_{i=1}^{3} \|\Lambda^s \partial_i((u_\alpha)_i u) - (u_\alpha)_i \cdot \Lambda^s \partial_i u\|_{L^2} \|\Lambda^s u\|_{L^2}
$$
\[
\leq C(\|\nabla u_\alpha\|_{L^\infty} \|\Lambda^s u\|_{L^2} + \|\Lambda^{s+1} u_\alpha\|_{L^p}\|u\|_{L^{q^*}}) \|\Lambda^s u\|_{L^2},
\]
with \( \frac{1}{2} = \frac{1}{p_2} + \frac{1}{q_2} \). We use (4.13) to \( f = (\nabla u_\alpha)T, g = u \) and \( p_1 = \infty, p = q_1 = 2 \) to bound \( J_2 \) as follows
\[
J_2 \leq C(\|\nabla u_\alpha\|_{L^\infty} \|\Lambda^s u\|_{L^2} + \|\Lambda^{s+1} u_\alpha\|_{L^p}\|u\|_{L^{q^*}}) \|\Lambda^s u\|_{L^2},
\]
with \( \frac{1}{2} = \frac{1}{p_2} + \frac{1}{q_2} \). We will use the following Gagliardo-Nirenberg inequalities:
\[
\|\Lambda^{s+1} u_\alpha\|_{L^p} \leq C\|\nabla u_\alpha\|_{L^\infty}^\theta \|\Lambda^{s+2} u_\alpha\|_{L^2}^{1-\theta},
\]
\[
\|\Lambda^2 u_\alpha\|_{L^q} \leq C\|\nabla u_\alpha\|_{L^\infty}^{1-\theta} \|\Lambda^{s+2} u_\alpha\|_{L^2}^\theta,
\]
where \( \theta = \frac{\frac{3}{q_2} - \frac{1}{2}}{\frac{3}{p_2} - \frac{1}{2}} \in (0, 1), 3 < p_2 < 6, \) and \( 2 < q_2 < 6 \).

Using (4.16), (4.17), (4.3) and (4.5), we see that
\[
\|\Lambda^{s+1} u_\alpha\|_{L^p} \|u\|_{L^q} \leq C\|\Lambda^{s+1} u_\alpha\|_{L^p}(\|\Lambda^2 u_\alpha\|_{L^q} + \|u_\alpha\|_{L^q})
\leq C\|\nabla u_\alpha\|_{L^\infty} \|\Lambda^{s+2} u_\alpha\|_{L^2} + C\|\nabla u_\alpha\|_{L^\infty} \|\Lambda^{s+2} u_\alpha\|_{L^2}^{1-\theta}
\leq C(\|\nabla u_\alpha\|_{L^\infty} + 1)\|u\|_{H^s}. \tag{4.18}
\]

Inserting (4.18) into (4.14) and (4.15) and using the logarithmic Sobolev inequality (3.8), we find that
\[
\frac{d}{dt}\|u\|_{H^s}^2 \leq C(\|\nabla u_\alpha\|_{L^\infty} + 1)\|u\|_{H^s}^2
\leq C(\|\nabla u_\alpha\|_{\dot{H}^{s}_0} + 1)\log(e + \|u_\alpha\|_{H^{s+2}})\|u\|_{H^s}^2
\leq C(\|\psi\|_{\dot{H}^{s}_0} + 1)\log(e + \|u\|_{H^s})\|u\|_{H^s}^2, \tag{4.19}
\]
where we have used the following inequality [28]:
\[
\|\nabla u_\alpha\|_{\dot{H}^{s}_0} \leq C\|\psi\|_{\dot{H}^{s}_0}.
\]

Thus, (4.8) follows from (4.19). The proof is complete. \( \square \)

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**References**

