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WELL POSEDNESS FOR A HYPERBOLIC-PARABOLIC CAUCHY PROBLEM ARISING IN POPULATION DYNAMICS

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Abstract. In some previous work [1]-[3], the authors have considered the diffusion of a population in a multilayered habitat, taking into account both the demographic structure, due to the age distribution of the individuals, and the spatial distribution related to population spread and diffusion. The development of the mathematical framework for this kind of problems leads the attention to a linear problem which incorporates all the features that make these kinds of problems unusual. This model is represented by a system of PDEs with discontinuous coefficients and data and sources at the boundaries between layers with different structure. In this paper we provide well posedness to such a problem together with regularity conditions, using m-accretiveness and fixed-points techniques.

1. Introduction

The modelling of age-structured populations, spreading in a geographical region, leads to the analysis of non-linear P.D.E.s with non-local terms that are strictly related to hereditary effects such as those represented in the framework of integral equations of Volterra type. In particular, in some previous work [1]-[3], the authors have considered the diffusion of a population in a multilayered habitat, taking into account both the demographic structure, due to the age distribution of the individuals, and the spatial distribution related to population spread and diffusion.

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However, the same problems studied in [1]-[3] lead to the consideration of several technical problems mainly focused on well posedness of a certain linear system which incorporates all the features that make these kinds of problems unusual. Namely, the hyperbolic character with respect to the age variable (denoted by $a \in [0, a^+]$) interacts with the parabolic features due to the spatial one (denoted by $y \in [0, L]$). The model is represented by a system of PDEs with discontinuous coefficients and data and sources at the boundaries between layers with different structure. Thus, in the present paper we provide a systematic approach to the problem, stating some basic results in a framework that allows us to treat various problems in population dynamics such as the control problems approached in [3].

The same results may be of interest for other modelling problems with hyperbolic-parabolic behaviour and discontinuous coefficients and data.

We consider the domain $\Omega = (0, a^+) \times (y_0, y_n)$ composed of n parallel layers

$$\Omega_j = (0, a^+) \times (y_{j-1}, y_j),\ j = 1, 2, ...n,$$

and denote

$$\Gamma_{y_j} = \{(a, y_j) : a \in (0, a^+)\},\ j = 0, ..., n,$$

where Γ_{y_j} with $j = 0$ and $j = n$ are the exterior boundaries, while Γ_{y_j} with $j = 1, \dots, n-1$ represent the interior ones. The time t runs within the finite interval $(0, T)$.

The model we are going to analyze reads

$$\begin{aligned}
&\frac{\partial q_j}{\partial t} + \frac{\partial q_j}{\partial a} - \frac{\partial}{\partial y}\Big(K_j(a)\frac{\partial q_j}{\partial y}\Big) + T_j(t)q_j = f_j \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{in } (0,T)\times\Omega_j, j = 1, ..., n \\
&q_j(0, a, y) = q_j^0(a, y) \qquad\qquad \text{in } \Omega_j,\ j = 1, ..., n, \\
&q_j(t, 0, y) = F_j(t, y) \qquad\qquad \text{in } (0,T)\times(y_{j-1}, y_j),\ j = 1, ..., n, \\
&q_j = q_{j+1} \qquad\qquad\qquad\qquad \text{on } (0,T)\times\Gamma_{y_j},\ j = 1, ...n-1, \\
&K_j(a)\frac{\partial q_j}{\partial y} = K_{j+1}(a)\frac{\partial q_{j+1}}{\partial y} + k_j(t, a) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{on } (0,T)\times\Gamma_{y_j},\ j = 1, ...n-1, \\
&-K_1(a)\frac{\partial q_1}{\partial y} = k_0(t, a) \qquad\qquad \text{on } (0,T)\times\Gamma_{y_0}, \\
&K_n(a)\frac{\partial q_n}{\partial y} = k_n(t, a) \qquad\qquad \text{on } (0,T)\times\Gamma_{y_n},
\end{aligned} \tag{1.1}$$

assuming that

$$\begin{aligned} & q_j^0 \in L^2(\Omega_j), \quad f_j \in L^2(0,T;L^2(\Omega_j)), \quad F_j \in L^2(0,T;L^2(y_{j-1},y_j)), \\ & T_j \in C([0,T];\mathcal{L}(L^2(\Omega_j),L^2(\Omega_j)), \quad k_j \in L^2(0,T;L^2(0,a^+)), \\ & K_j \in L^\infty(0,a^+), K_j(a) \geq K_0 > 0 \text{ a.e. in } \quad (0,a^+). \end{aligned} \tag{1.2}$$

We introduce the generic notation defining the function $\Phi(t,a,y)$ in the set $(0,T)\times\Omega$, by

$$\Phi(t,a,y) = \Phi_j(t,a,y), \text{ for } \quad y \in (y_{j-1},y_j) \tag{1.3}$$

where Φ_j stands for any function defined on $(0,T)\times\Omega_j$ (see also the notation (25)-(32) from [1]). Conversely, if Φ is a function defined on $(0,T)\times\Omega$, we define Φ_j in $(0,T)\times\Omega_j$ by setting

$$\Phi_j(t,a,y) = \Phi(a,t,y), \quad y \in (y_{j-1},y_j). \tag{1.4}$$

Then, denoting

$$H_\Omega = L^2(\Omega), \quad H = L^2(0,L), \quad V = H^1(0,L),$$

with the standard norms, and using assumptions (1.2) we have

$$\begin{aligned} & q^0 \in H_\Omega, \quad f \in L^2(0,T;H_\Omega), \quad F \in L^2(0,T;H), \\ & k \in L^2(0,T;L^2(0,a^+)), \quad K \in L^\infty(\Omega). \end{aligned} \tag{1.5}$$

Moreover, we may define the operator $\mathcal{T}(t): H_\Omega \to H_\Omega$ by setting

$$(\mathcal{T}(t)\theta)_j = (T_j(t)\theta_j),$$

where $\theta \in H_\Omega$ and we have used both (1.3) and (1.4). Then $\mathcal{T}(t)$ is continuous on H_Ω, for each $t \in [0,T]$, and we have that

$$\|\mathcal{T}(t)\theta\|_{H_\Omega} \leq M\|\theta\|_{H_\Omega}, \forall\theta \in H_\Omega, \quad M > 0. \tag{1.6}$$

Concerning the previous problem we are led to adopt the following definition:

Definition 1.1. A *weak solution* to problem (1.1) is a function

$$q \in C([0,T];H_\Omega) \cap L^2(0,T;L^2(0,a^+;V)) \cap C([0,a^+];L^2(0,T;H)) \tag{1.7}$$

satisfying

$$\begin{aligned} & -\int_0^T \int_\Omega q\, \frac{\partial \psi}{\partial t} dydadt - \int_0^T \Big\langle \frac{\partial \psi}{\partial a}(t), q(t) \Big\rangle dt \\ & + \int_\Omega q(T,a,y)\psi(T,a,y)dady + \int_0^T \int_0^L q(t,a^+,y)\psi(t,a^+,y)dydt \\ & - \int_\Omega q^0(a,y)\psi(0,a,y)dyda - \int_0^T \int_0^L F(t,y)\psi(t,0,y)dydt \end{aligned} \tag{1.8}$$

$$+\int_0^T\int_\Omega\left(K(a,y)\frac{\partial q}{\partial y}\frac{\partial\psi}{\partial y}+(\mathcal{T}(t)q(t))\,(a,y)\psi(t,a,y)\right)dydadt$$
$$=\int_0^T\int_\Omega f\psi\ dydadt+\int_0^T\int_0^{a^+}\sum_{j=0}^n k_j(t,a)\psi(t,a,y_j)dadt,$$

for any function ψ such that

$$\psi\in L^2(0,T;L^2(0,a^+;V))\cap W^{1,2}(0,T;H_\Omega),\quad \psi_a\in L^2(0,T;L^2(0,a^+;V')). \tag{1.9}$$

We specify that $\langle\cdot,\cdot\rangle$ denotes the duality between $L^2(0,a^+;V)$ and $L^2(0,a^+;V')$ defined as

$$\langle h,g\rangle=\int_0^{a^+}\langle h(a),g(a)\rangle_{V',V}\,da,\quad \forall h\in L^2(0,a^+;V'),\quad g\in L^2(0,a^+;V),$$

where $\langle\cdot,\cdot\rangle_{V',V}$ is the pairing between V' and V.

The aim of this paper is to prove existence and uniqueness for (1.1). To this end we will proceed by steps, considering different particular cases of the full problem.

2. The basic problem

To approach our problem we first introduce the linear operator

$$B_0:D(B_0)\subset L^2(0,a^+;V)\to L^2(0,a^+;V')$$

on the domain

$$D(B_0)=\{v\in L^2(0,a^+;V)\,:\,v_a\in L^2(0,a^+;V'),\ v(0,y)=0\}, \tag{2.1}$$

where we note that the condition $v(0,y)=0$ is meaningful because $v\in D(B_0)$ implies that $v\in C([0,a^+];H)$.

We define B_0 by setting, for $v\in D(B_0)$,

$$\langle(B_0v)(a),\psi\rangle_{V',V}=\langle v_a(a,\cdot),\psi\rangle_{V',V}+\int_0^L K(a,y)v_y(a,y)\psi_y(y)dy, \tag{2.2}$$

for almost every $a\in[0,a^+]$ and any $\psi\in V$.

Then we define the operator $B:D(B)\subset H_\Omega\to H_\Omega$, by

$$Bv=B_0v,$$

on the domain $D(B)=\{v\in D(B_0)\ :\ Bv\in H_\Omega\}$. We note that B is a quasi m-accretive operator because it is the linear case of the operator A

discussed in [1] and we specify that by the results in [2] (see Proposition 1 at page 894),

$$\overline{D(B)} = H_\Omega. \tag{2.3}$$

Then we may consider the Cauchy problem

$$q'(t) + Bq(t) = h(t), \text{ a.e. } \quad t \in (0,T), \quad q(0) = q^0, \tag{2.4}$$

which corresponds to the main problem (1.1) with

$$\mathcal{T}_j(t) = 0, \quad F_j = 0, \quad k_j = 0, \text{ for all } \quad j, \quad f = h.$$

Actually this is a first step for which we have the following result

Theorem 2.1. *Let $q^0 \in D(B)$, $h \in W^{1,1}(0,T;H_\Omega)$. Then problem (2.4) has a strong solution*

$$q \in L^\infty(0,T;D(B)) \cap W^{1,\infty}(0,T;H_\Omega) \tag{2.5}$$

satisfying the estimate

$$\begin{aligned} \|q(t)\|^2_{H_\Omega} + \int_0^t \left\|q(s,a^+,\cdot)\right\|^2_H ds + K_0 \int_0^t \|q(s)\|^2_{L^2(0,a^+;V)} ds \\ \le e^{2K_0T}\Big(\left\|q^0\right\|^2_{H_\Omega} + \frac{1}{K_0}\int_0^T \|h(s)\|^2_{L^2(0,a^+;V')} ds\Big). \end{aligned} \tag{2.6}$$

Proof. The existence of the solution satisfying (2.5) follows from the quasi m-accretiveness of B. We note that, since q satisfies (2.5), then

$$q \in L^2(0,T;C([0,a^+] \,:\, H)) \subset C([0,a^+];L^2(0,T;H)). \tag{2.7}$$

We note that (2.4) may be specified as

$$q'(t)(a) + (B_0q(t))(a) = h(t)(a), \quad q(0)(a) = q^0(a),$$

for almost every $a \in (0,a^+)$ and $t \in (0,T)$. Since the solution $q(t) \in V$ for almost every $t \in (0,T)$ we can write the previous equation in the equivalent form

$$\begin{aligned} \int_0^a \int_0^L q_t(t,\sigma,y)q(t,\sigma,y)dyd\sigma + \int_0^a \langle q_\sigma(t,\sigma,\cdot), q(t,\sigma,\cdot)\rangle_{V',V} \, d\sigma \\ + \int_0^a \int_0^L K(\sigma,y)q_y^2(t,\sigma,y)dyd\sigma = \int_0^a \int_0^L h(t,\sigma,y)q(t,\sigma,y)dyd\sigma. \end{aligned}$$

Then, recalling that

$$\int_0^a \langle q_\sigma(t,\sigma,\cdot), q(t,\sigma,\cdot)\rangle_{V',V} \, d\sigma = \frac{1}{2}\int_0^a \frac{d}{d\sigma}\|q(t,\sigma,\cdot)\|^2_H \, d\sigma = \frac{1}{2}\|q(t,a,\cdot)\|^2_H ,$$

we have the estimate

$$\begin{aligned}
&\frac{d}{dt}\int_0^a \|q(t,\sigma,\cdot)\|_H^2\,d\sigma + \|q(t,a,\cdot)\|_H^2 + 2K_0\int_0^a \|q(t,\sigma,\cdot)\|_V^2\,d\sigma \\
&\le 2\int_0^a \|h(t,\sigma,\cdot)\|_{V'}\,\|q(t,\sigma,\cdot)\|_V\,d\sigma + 2K_0\int_0^a \|q(t,\sigma,\cdot)\|_H^2\,d\sigma \\
&\le \frac{1}{K_0}\int_0^a \|h(t,\sigma,\cdot)\|_{V'}^2 d\sigma + K_0\int_0^a \|q(t,\sigma,\cdot)\|_V^2 d\sigma + 2K_0\int_0^a \|q(t,\sigma,\cdot)\|_H^2 d\sigma,
\end{aligned}$$

so that

$$\begin{aligned}
&\int_0^a \|q(t,\sigma,\cdot)\|_H^2\,d\sigma + \int_0^t \|q(s,a,\cdot)\|_H^2\,ds + K_0\int_0^t\int_0^a \|q(s,\sigma,\cdot)\|_V^2\,d\sigma ds \\
&\le \left\|q^0\right\|_{H_\Omega}^2 + \frac{1}{K_0}\int_0^t \|h(s)\|_{L^2(0,a^+;V')}^2\,ds + 2K_0\int_0^t \|q(s)\|_{H_\Omega}^2\,ds. \qquad (2.8)
\end{aligned}$$

From this inequality, taking $a = a^+$, we first derive

$$\|q(t)\|_{H_\Omega}^2 \le \left\|q^0\right\|_{H_\Omega}^2 + \frac{1}{K_0}\int_0^T \|h(s)\|_{L^2(0,a^+;V')}^2\,ds + 2K_0\int_0^t \|q(s)\|_{H_\Omega}^2\,ds,$$

and, by the Gronwall inequality, we have

$$\|q(t)\|_{H_\Omega}^2 \le e^{2K_0 t}\Big(\left\|q^0\right\|_{H_\Omega}^2 + \frac{1}{K_0}\int_0^T \|h(s)\|_{L^2(0,a^+;V')}^2\,ds\Big).$$

Then, plugging it into (2.8), we get

$$\begin{aligned}
&\int_0^a \|q(t,\sigma,\cdot)\|_H^2\,d\sigma + \int_0^t \|q(s,a,\cdot)\|_H^2\,ds + K_0\int_0^t\int_0^a \|q(s,\sigma,\cdot)\|_V^2\,d\sigma ds \\
&\qquad \le e^{2K_0 T}\Big(\left\|q^0\right\|_{H_\Omega}^2 + \frac{1}{K_0}\int_0^T \|h(s)\|_{L^2(0,a^+;V')}^2\,ds\Big). \qquad (2.9)
\end{aligned}$$

As a consequence (setting again $a = a^+$) we have (2.6). □

We note that in the previous theorem we have proved the estimate (2.6) in which we consider the term $h(t)$ as belonging to the space $L^2(0,a^+;V')$ while in the original problem (1.1) the source term $f(t)$ belongs to H_Ω. In fact problem (2.4) occurs in the intermediate cases (see Proposition 3.2 below) that we are going to consider with a source term $h(t) \in L^2(0,a^+;V')$ though the final issue concerns only the space H_Ω.

3. Intermediate reduced problems

We first discuss a particular reduced problem; namely we start considering vanishing boundary conditions both with respect to a and y, by taking

$$F_j \equiv 0, \quad k_j \equiv 0, \text{ for all } \quad j. \tag{3.1}$$

We shall denote by C some constants depending on the problem parameters $(K_0,\ T,\ M)$. Using the previous result, we have the following.

Proposition 3.1. *Let the assumptions* (1.2) *be satisfied in the particular case* (3.1). *Then problem* (1.1) *has a unique weak solution*

$$q \in C([0,T] \,:\, H_\Omega) \cap L^2(0,T;L^2(0,a^+;V)) \cap C([0,a^+];L^2(0,T;H)), \tag{3.2}$$

satisfying

$$\begin{aligned}\|q(t)\|^2_{H_\Omega} + \int_0^t \left\|q(s,a^+,\cdot)\right\|^2_H ds + K_0 \int_0^t \|q(s)\|^2_{L^2(0,a^+;V)} ds \\ \le C_1\Big(\left\|q^0\right\|^2_{H_\Omega} + \int_0^T \|f(s)\|^2_{L^2(0,a^+,V')} ds\Big),\end{aligned} \tag{3.3}$$

where

$$C_1 = C_0 e^{C_0 M^2 T}, \quad C_0 = 2e^{2K_0 T} \max\Big\{1, \frac{1}{K_0}\Big\}. \tag{3.4}$$

If, in addition, we assume that

$$q^0 \in D(B), \quad f \in C^1([0,T];H_\Omega), \quad \mathcal{T} \in C^1([0,T];\mathcal{L}(H_\Omega,H_\Omega)), \tag{3.5}$$

then

$$q \in C^1([0,T];H_\Omega) \cap C([0,T];D(B)) \tag{3.6}$$

and

$$\begin{aligned}&\|q(t)\|^2_{H_\Omega} + \left\|q'(t)\right\|^2_{H_\Omega} + K_0 \int_0^t \|q(s)\|^2_{L^2(0,a^+;V)} ds + \int_0^t \left\|q(s,a^+,\cdot)\right\|^2_H ds \\ &\le C\Big(\left\|q^0\right\|^2_{H_\Omega} + \left\|Bq^0 + f(0)\right\|^2_{H_\Omega} \\ &\qquad + \int_0^T \Big[\|f(s)\|^2_{L^2(0,a^+;V')} + \left\|f'(s)\right\|^2_{L^2(0,a^+;V')}\Big] ds\Big),\end{aligned} \tag{3.7}$$

where C is another constant which depends on the parameters of the problem.

Proof. We shall use an approximating procedure and a fixed point argument. We fix $\eta \in C([0,T];H_\Omega)$, and consider the problem

$$q'(t) + Bq(t) = f(t) - \mathcal{T}(t)\eta(t) \text{ a.e. } t \in (0,T), \quad q(0) = q^0. \tag{3.8}$$

Then, we take the sequences

$$\{q_m^0\}_{m\geq 1} \subset D(B), \qquad \{h_m\}_{m\geq 1} \subset W^{1,1}(0,T;H_\Omega),$$

such that

$$q_m^0 \to q^0 \text{ in } \quad H_\Omega, \text{ as } \quad m \to \infty \tag{3.9}$$

and

$$h_m(t) \to f(t) - \mathcal{T}(t)\eta(t) \text{ in } \quad L^2(0,T;H_\Omega) \text{ as } \quad m \to \infty. \tag{3.10}$$

Thus, according to Theorem 2.1, the Cauchy problem

$$q_m'(t) + Bq_m(t) = h_m(t), \text{ a.e. } \quad t \in (0,T), \quad q_m(0) = q_m^0, \tag{3.11}$$

provides a sequence of solutions

$$\{q_m\}_{m\geq 1} \subset L^\infty(0,T;D(B)) \cap W^{1,\infty}(0,T;H_\Omega)$$

satisfying the weak form of (2.4)

$$\begin{aligned}
&-\int_0^T \int_\Omega q_m \frac{\partial \psi}{\partial t} dydadt - \int_0^T \Big\langle \frac{\partial \psi}{\partial a}(t), q_m(t) \Big\rangle dt \\
&-\int_0^{a^+} \int_0^L q_m^0(a,y)\psi(0,a,y)dyda \\
&+\int_\Omega q_m(T,a,y)\psi(T,a,y)dyda + \int_0^T \int_0^L q_m(t,a^+,y)\psi(t,a^+,y)dydt \\
&+\int_0^T \int_\Omega \Big(K(a,y)\frac{\partial q_m}{\partial y}\frac{\partial \psi}{\partial y}\Big)dydadt = \int_0^T \int_\Omega h_m\psi \; dydadt
\end{aligned} \tag{3.12}$$

for any function ψ satisfying (1.9). Moreover, q_m satisfies (2.6) and since the problem is linear we have

$$\begin{aligned}
&\|q_l(t) - q_m(t)\|_{H_\Omega}^2 + \int_0^t \left\|q_l(s,a^+,\cdot) - q_m(s,a^+,\cdot)\right\|_H^2 ds \\
&\qquad\qquad + K_0 \int_0^t \|q_l(s) - q_m(s)\|_{L^2(0,a^+;V)}^2 ds \\
&\leq e^{2K_0T}\Big(\left\|q_l^0 - q_m^0\right\|_{H_\Omega}^2 + \frac{1}{K_0}\int_0^T \|h_l(s) - h_m(s)\|_{L^2(0,a^+;V')}^2 ds\Big).
\end{aligned} \tag{3.13}$$

This latter estimate implies that the sequence $\{q_m\}_{m\geq 1}$ is a Cauchy sequence in the spaces $C([0,T] : H_\Omega) \cap L^2(0,T;L^2(0,a^+,V)) \cap C([0,a^+] : L^2(0,T;H))$ and we can conclude that $q_m \to q$ in these spaces. Then, passing to the limit in (3.12), we get (1.8) which shows that $q(t)$ is a solution to (3.8).

Moreover, thanks to (2.8), written for $a = a^+$, we also get that $q(t)$ satisfies

$$\begin{aligned}
&\|q(t)\|^2_{H_\Omega} + K_0 \int_0^t \|q(s)\|^2_{L^2(0,a^+;V)}\,ds + \int_0^t \left\|q(s,a^+,\cdot)\right\|^2_H ds \qquad (3.14)\\
&\le C_0\Big(\left\|q^0\right\|^2_{H_\Omega} + \int_0^T \|f(s)\|^2_{L^2(0,a^+;V')}\,ds + \int_0^T \|\mathcal{T}(s)\eta(s)\|^2_{H_\Omega}\,ds\Big)\\
&\le C_0\Big(\left\|q^0\right\|^2_{H_\Omega} + \int_0^T \|f(s)\|^2_{L^2(0,a^+;V')}\,ds + M^2\int_0^t \|\eta(s)\|^2_{H_\Omega}\,ds\Big),
\end{aligned}$$

where $C_0 = 2e^{2K_0T}\max\left\{1, \frac{1}{K_0}\right\}$. Now if $q(t)$ and $\overline{q}(t)$ are solutions respectively corresponding to $\eta(t)$ and $\overline{\eta}(t)$, but with the same q^0 and $f(t)$, we have also

$$\|q(t) - \overline{q}(t)\|^2_{H_\Omega} \le C_0M^2\int_0^t \|\eta(s) - \overline{\eta}(s)\|^2_{H_\Omega}\,ds. \qquad (3.15)$$

Then, the mapping $\Gamma : C([0,T];H_\Omega) \to C([0,T];H_\Omega)$, defined by setting

$$\Gamma(\eta)(t) = q(t), \quad \forall\eta \in C([0,T];H_\Omega)$$

where $q(t)$ is the solution to (3.8), is a contraction in the norm

$$\|q\|_b = \sup_{t\in[0,T]}\left(e^{-\delta t}\,\|q(t)\|_{H_\Omega}\right),$$

(which is a norm equivalent to the usual norm on $C([0,T];H_\Omega)$), with δ suitably chosen. Indeed, from (3.15), multiplying by $e^{-2\delta t}$, we get

$$\|q - \overline{q}\|^2_b \le \frac{C_0M^2}{2\delta}\,\|\eta - \overline{\eta}\|^2_b$$

and, taking $2\delta > C_0M^2$, we see that Γ is a contraction. We conclude that the fixed point of Γ actually belongs to $C([0,T];H_\Omega) \cap L^2(0,T;L^2(0,a^+;V)) \cap C([0,a^+];L^2(0,T;H))$ and it is the unique weak solution to (1.1).

Therefore we can set $\eta = q$ in (3.14) and applying the Gronwall lemma we obtain (3.3) as claimed.

If we assume also (3.5) we may use a similar argument to get regularity. In fact, if we fix $\eta \in C^1([0,T];H_\Omega)$, we have

$$h(t) = f(t) - \mathcal{T}(t)\eta(t) \in C^1([0,T];H_\Omega)$$

and consequently the Cauchy problem

$$q'(t) + Bq(t) = h(t), \quad q(0) = q^0$$

has a solution $q \in C^1([0,T];H_\Omega) \cap C([0,T];D(B))$. Moreover, $z(t) = q'(t)$ is a mild solution of

$$z'(t)+Bz(t) = f'(t)-\mathcal{T}'(t)\eta(t)-\mathcal{T}(t)\eta'(t), \quad z(0) = -Bq^0+f(0)-\mathcal{T}(0)\eta(0),$$

and satisfies the estimate

$$\begin{aligned}&\|q(t)\|^2_{H_\Omega} + \left\|q'(t)\right\|^2_{H_\Omega} \\ &\qquad \le C\Big(\|q^0\|^2_{H_\Omega} + \|z(0)\|^2_{H_\Omega} + \int_0^t \left[\|h(s)\|^2_{H_\Omega} + \left\|h'(s)\right\|^2_{H_\Omega}\right] ds\Big),\end{aligned}$$

where C is again a suitable constant depending on K_0, T, M and M_1. Here, M_1 arises in

$$\left\|\mathcal{T}'(t)v\right\|_{H_\Omega} \le M_1 \left\|v\right\|_{H_\Omega}, \forall v \in H_\Omega.$$

All this allows us to use the previous argument to prove existence of a fixed point η in $C^1([0,T];H_\Omega)$. □

The next step is concerned with non-trivial boundary conditions in the space variable. In fact we consider problem (1.1)-(1.2) when it is assumed that

$$F \equiv 0, \tag{3.16}$$

but with boundary sources $k_j \neq 0$. In this case we have the following.

Proposition 3.2. *Let the assumptions* (1.2) *be satisfied in the particular case* (3.16)*. Then problem* (1.1) *has a unique weak solution*

$$q \in C([0,T];H_\Omega) \cap L^2(0,T;L^2(0,a^+;V)) \cap C([0,a^+];L^2(0,T;H)), \tag{3.17}$$

satisfying

$$\begin{aligned}&\|q(t)\|^2_{H_\Omega} + K_0 \int_0^t \|q(s)\|^2_{L^2(0,a^+;V)} ds + \int_0^t \left\|q(s,a^+,\cdot)\right\|^2_H ds \\ &\qquad \le C_2\Big(\left\|q^0\right\|^2_{H_\Omega} + \int_0^T \|f(t)\|_{V'} + \sum_{j=0}^n \int_0^T \|k_j(s)\|^2_{L^2(0,a^+)} ds\Big),\end{aligned} \tag{3.18}$$

where C_2 is a new constant depending only on K_0, M, T, n and the domain Ω.

Proof. We start defining $L_0 \in L^2(0,T;L^2(0,a^+;V'))$ by setting

$$\langle L_0(t), v\rangle = \sum_{j=0}^n \int_0^{a^+} k_j(t,a)v(a,y_j)da, \tag{3.19}$$

for any $v \in L^2(0,a^+;V)$. We see that L_0 is well defined in this way, because

$$\|L_0(t)\|_{L^2(0,a^+;V')} \;=\; \sup\Big\{|\langle L_0(t),v\rangle| : \|v\|_{L^2(0,a^+;V)} \le 1\Big\} \tag{3.20}$$

$$\leq \quad C_{tr}\sum_{j=0}^{n}\|k_j(t)\|_{L^2(0,a^+)}\,,$$

where C_{tr} is the constant occurring in the trace theorem. Here we have used the estimate

$$\begin{aligned}\Big|\int_0^{a^+} k_j(s,a)v(a,y_j)da\Big| &\leq \|k_j(s)\|_{L^2(0,a^+)}\Big(\int_0^{a^+} v^2(a,y_j)da\Big)^{1/2} \\ &\leq C_{tr}\,\|k_j(s)\|_{L^2(0,a^+)}\,\|v\|_{L^2(0,a^+;V)}\,,\end{aligned} \tag{3.21}$$

holding for $v\in L^2(0,a^+;V)$.

Then we consider a sequence $\{L_0^m\}_{m\geq 1}\subset W^{1,1}(0,T;H_\Omega)$, such that

$$\lim_{m\to\infty} L_0^m = L_0, \text{ in } \quad L^2(0,T;L^2(0,a^+;V')), \tag{3.22}$$

so that, for any $v\in L^2(0,T;L^2(0,a^+;V))$, we have

$$\begin{aligned}\lim_{m\to\infty}\int_0^T\int_\Omega L_0^m(t)v(t)dydadt &= \lim_{m\to\infty}\int_0^T \langle L_0^m(t),v(t)\rangle\,dt \\ = \int_0^T \langle L_0(t),v(t)\rangle\,dt &= \sum_{j=0}^{n}\int_0^T\int_0^{a^+} k_j(t,a)v(t,a,y_j)dadt.\end{aligned}$$

Let now q^m be the solution of the weak problem

$$\begin{aligned}&-\int_0^T\int_\Omega q^m\,\frac{\partial\psi}{\partial t}dydadt-\int_0^T\Big\langle\frac{\partial\psi}{\partial a}(t),q^m(t)\Big\rangle dt \\ &-\int_0^{a^+}\int_0^L q^0(a,y)\psi(0,a,y)dyda \\ &+\int_\Omega q^m(T,a,y)dyda+\int_0^{a^+}\int_0^L q^m(t,a^+,y)dyda \\ &+\int_0^T\int_\Omega\Big(K(a,y)\frac{\partial q^m}{\partial y}\frac{\partial\psi}{\partial y}+\mathcal{T}(t)q^m(t,a,y)\psi(t,a,y)\Big)dydadt \\ &\qquad\qquad= \int_0^T\int_\Omega L_0^m\psi\;dydadt+\int_0^T\int_\Omega f\psi\;dydadt,\end{aligned} \tag{3.23}$$

for any function ψ satisfying (1.9). We know by the previous Proposition 3.1 that this solution exists and satisfies

$$\Big\|q^m(t)-q^l(t)\Big\|^2_{H_\Omega}+K_0\int_0^t\Big\|q^m(s)-q^l(s)\Big\|^2_{L^2(0,a^+;V)}ds$$

$$+\int_0^t \left\|q^m(s,a^+,\cdot)-q^l(s,a^+,\cdot)\right\|_H^2 ds$$
$$\leq C_1\int_0^T \left\|L_0^m(s)-L_0^l(s)\right\|^2_{L^2(0,a^+;V')} ds.$$

Thus, $\{q^m\}$ is a Cauchy sequence in

$$C([0,T];H_\Omega)\cap L^2(0,T;L^2(0,a^+;V))\cap C([0,a^+];L^2(0,T;H))$$

and converges to a q which, going to the limit in (3.23), turns out to be a weak solution to (1.1), with $F=0$.

Finally, (3.18) is a simple consequence of (3.20), (3.3) with

$$C_2 = 2C_1\max(1,nC_{tr}^2). \qquad \square \tag{3.24}$$

From the previous proof we also draw the following result:

Corollary 3.3. *Under the assumption of Proposition* 3.2, *suppose in addition that* $q^0\in D(B)$, $\mathcal{T}\in C^1([0,T];\mathcal{L}(H_\Omega,H_\Omega))$, $L_0\in C^1([0,T];L^2(0,a^+;V'))$, $f\in C^1([0,T];H_\Omega)$. *Then the solution to* (1.1) *satisfies* $q\in C^1([0,T];H_\Omega)\cap C([0,T];D(B))$ *and*

$$\begin{aligned}
&\|q(t)\|^2_{H_\Omega}+\left\|q'(t)\right\|^2_{H_\Omega}+K_0\int_0^t\|q(s)\|^2_{L^2(0,a^+;V)}ds+\int_0^t\left\|q(s,a^+,\cdot)\right\|_H^2 ds\\
&\leq C\Big(\left\|q^0\right\|^2_{H_\Omega}+\left\|Bq^0+f(0)\right\|^2_{H_\Omega}+\|L_0(0)\|^2_{L^2(0,a^+;V')}\\
&+\int_0^T\|f(s)\|^2_{H_\Omega}ds+\int_0^T\|L_0(s)\|^2_{L^2(0,a^+;V')}ds\\
&+\int_0^T\left\|f'(s)\right\|^2_{H_\Omega}ds+\int_0^T\left\|(L_0)'(s)\right\|^2_{L^2(0,a^+;V')}ds\Big).
\end{aligned}\tag{3.25}$$

Proof. As in the proof of Proposition 3.2 we approximate $L_0(t)$ by a sequence $\{L_0^m\}_{m\geq 1}\subset C^1([0,T];H_\Omega)$ such that

$$L_0^m\to L_0 \quad \text{in} \quad C^1([0,T];L^2(0,a^+;V')).$$

Accordingly, we have the approximating problem

$$q_m'(t)+Bq_m(t)+\mathcal{T}(t)q_m(t)=f(t)+L_0^m(t),\quad q_m(0)=q^0,$$

which, by the regularity result in Proposition 3.1, has a unique solution

$$q_m\in C^1([0,T];H_\Omega)\cap C([0,T];D(B).$$

Moreover, this solution satisfies (3.7) with f replaced by $f+L_0^m$ and

$$\|q_m(t)\|^2_{H_\Omega}+\|q_m'(t)\|^2_{H_\Omega}+K_0\int_0^t\|q_m(s)\|^2_{L^2(0,a^+;V)}ds+\int_0^t\|q_m(s,a^+,\cdot)\|^2_H ds$$

$$
\begin{aligned}
&\le C\Big(\big\|q^0\big\|_{H_\Omega}^2 + \big\|Bq^0 + f(0)\big\|_{H_\Omega}^2 + \|L_0^m(0)\|_{L^2(0,a^+;V')}^2 \\
&+ \int_0^T \|f(s)\|_{H_\Omega}^2 \, ds + \int_0^T \|L_0^m(s)\|_{L^2(0,a^+;V')}^2 \, ds \\
&+ \int_0^T \big\|f'(s)\big\|_{H_\Omega}^2 \, ds + \int_0^T \big\|(L_0^m)'(s)\big\|_{L^2(0,a^+;V')}^2 \, ds\Big). \qquad (3.26)
\end{aligned}
$$

Considering two solutions q_m and q_l, with the same data, we have

$$
\begin{aligned}
&\|q_m(t) - q_l(t)\|_{H_\Omega}^2 + \|q_m'(t) - q_l'(t)\|_{H_\Omega}^2 + K_0 \int_0^t \|q_m(s) - q_l(t)\|_{L^2(0,a^+;V)}^2 ds \\
&+ \int_0^t \big\|q_m(s,a^+,\cdot) - q_l(s,a^+,\cdot)\big\|_H^2 \, ds \\
&\le C\Big(\Big\|L_0^m(0) - L_0^l(0)\Big\|_{L^2(0,a^+;V')}^2 + \int_0^T \Big\|L_0^m(t) - L_0^l(t)\Big\|_{L^2(0,a^+;V')}^2 \, ds \\
&\qquad + \int_0^T \Big\|(L_0^m)'(s) - (L_0^l)'(s)\Big\|_{L^2(0,a^+;V')}^2 \, ds\Big). \qquad (3.27)
\end{aligned}
$$

Hence, using (3.22) we deduce that q_m is a Cauchy sequence in

$$C^1([0,T]; H_\Omega) \cap L^2(0,T; L^2(0,a^+;V)) \cap C([0,a^+]; L^2(0,T;H))$$

and converges to q in these spaces. Finally, passing to the limit in (3.26), we obtain (3.25). □

4. The complete problem

Next we are concerned with the last step which covers the general case. We have the following.

Theorem 4.1. *Let the assumptions* (1.2) *be satisfied. Then problem* (1.1) *has a unique weak solution*

$$q \in C([0,T]; H_\Omega) \cap C([0,a^+]; L^2(0,T;H)) \cap L^2(0,T; L^2(0,a^+;V)) \qquad (4.1)$$

such that

$$
\begin{aligned}
&\|q(t)\|_{H_\Omega}^2 + \int_0^t \big\|q(s,a^+,\cdot)\big\|_H^2 \, ds + K_0 \int_0^t \|q(s)\|_{L^2(0,a^+;V)}^2 \, ds \qquad (4.2) \\
&\le C_2\Big(\big\|q^0\big\|_{H_\Omega}^2 + \int_0^T \|f(s)\|_{H_\Omega}^2 \, ds \\
&\qquad + \|F\|_{L^2(0,T;H)}^2 + \sum_{j=0}^n \int_0^T \|k(s)\|_{L^2(0,a^+)}^2 \, ds\Big),
\end{aligned}
$$

for any $t \in [0,T]$.

Proof. We consider an approximation of this problem by approximating q^0, f, k, F and $\mathcal{T}(t)$ by the sequences

$$q^{0m} \in D(B), \quad f^m \in C^1([0,T];H_\Omega), \quad k^m \in C^1([0,T];L^2(0,a^+)), \tag{4.3}$$
$$F^m \in C^2([0,T]\times[0,L]) \text{ such that } \quad F^m(0,y)=0 \text{ for } \quad y\in[0,L],$$
$$\mathcal{T}^m(t) \in C^1([0,T];\mathcal{L}(L^2(\Omega),L^2(\Omega)),$$

such that

$$q^{0m} \to q^0 \text{ in } \quad H_\Omega, \quad f^m \to f \text{ in } \quad L^2(0,T;L^2(\Omega)), \tag{4.4}$$
$$k^m \to k \text{ in } \quad L^2(0,T;L^2(0,a^+)), \quad F^m \to F \text{ in } \quad L^2(0,T;L^2(0,L)),$$
$$\mathcal{T}^m(t) \to \mathcal{T}(t) \text{ in } \quad C([0,T];\mathcal{L}(L^2(\Omega),L^2(\Omega)).$$

Then, we focus on the approximated problem

$$\frac{\partial q_j^m}{\partial t} + \frac{\partial q_j^m}{\partial a} - \frac{\partial}{\partial y}\Big(K_j(a)\frac{\partial q_j^m}{\partial y}\Big) + T_j^m(t)q_j^m = f_j^m \tag{4.5}$$
$$\text{in } (0,T)\times\Omega_j,\ j=1,\dots,n,$$
$$q_j^m(0,a,y) = q_j^{0m} \qquad \text{in } \Omega_j,\ j=1,\dots,n,$$
$$q_j^m(t,0,y) = F_j^m(t,y) \qquad \text{in } (0,T)\times(y_{j-1},y_j),\ j=1,\dots,n,$$
$$q_j^m = q_{j+1}^m \qquad \text{on } (0,T)\times\Gamma_{y_j},\ j=1,\dots,n-1,$$
$$K_j(a)\frac{\partial q_j^m}{\partial y} = K_{j+1}(a)\frac{\partial q_{j+1}^m}{\partial y} + k_j^m \qquad \text{on } (0,T)\times\Gamma_{y_j},\ j=1,\dots,n-1,$$
$$-K_1(a)\frac{\partial q_1^m}{\partial y} = k_0^m \qquad \text{on } (0,T)\times\Gamma_{y_0},$$
$$K_n(a)\frac{\partial q_n^m}{\partial y} = k_n^m \qquad \text{on } (0,T)\times\Gamma_{y_n}.$$

We note that we can transform this problem by performing the change

$$w^m = q^m - F^m,$$

getting the following system for w^m :

$$\frac{\partial w_j^m}{\partial t} + \frac{\partial w_j^m}{\partial a} - \frac{\partial}{\partial y}\Big(K_j(a)\frac{\partial w_j^m}{\partial y}\Big) + T_j^m(t)w_j^m = \widetilde{f}_j^m \tag{4.6}$$
$$\text{in } (0,T)\times\Omega_j,\ j=1,\dots,n,$$
$$w_j^m(0,a,y) = w_j^{0m} \qquad \text{in } \Omega_j,\ j=1,\dots,n,$$
$$w_j^m(t,0,y) = 0 \qquad \text{in } (0,T)\times(y_{j-1},y_j),\ j=1,\dots,n,$$

$$
\begin{aligned}
&w_j^m = w_{j+1}^m && \text{on } (0,T)\times\Gamma_{y_j},\ j=1,\dots,n-1,\\
&K_j(a)\frac{\partial w_j^m}{\partial y} = K_{j+1}(a)\frac{\partial w_{j+1}^m}{\partial y} + \widetilde{k}_j^m(t,a) && \\
& && \text{on } (0,T)\times\Gamma_{y_j},\ j=1,\dots,n-1,\\
&-K_1(a)\frac{\partial w_1^m}{\partial y} = \widetilde{k}_0^m(t,a) && \text{on } (0,T)\times\Gamma_{y_0},\\
&K_n(a)\frac{\partial w_n^m}{\partial y} = \widetilde{k}_n^m(t,a) && \text{on } (0,T)\times\Gamma_{y_n},
\end{aligned}
$$

where

$$
\begin{aligned}
&w_j^{0m}(a,y) = q_j^{0m}(a,y)\\
&\widetilde{k}_0^m(t,a) = K_1(a)\frac{\partial F_1^m}{\partial y}(t,0) + k_0^m(t,a)\\
&\widetilde{k}_n^m(t,a) = K_n(a)\frac{\partial F_n^m}{\partial y}(t,L) + k_n^m(t,a)\\
&\widetilde{k}_j^m(t,a) = K_{j+1}(a)\frac{\partial F_{j+1}^m}{\partial y}(t,y_j) - K_j(a)\frac{\partial F_j^m}{\partial y}(t,y_j) + k_j^m(t,a)\\
&\widetilde{f}^m(t,a,y) = f^m(t,a,y) + \frac{\partial}{\partial y}\Big(K(a,y)\frac{\partial F^m}{\partial y}(t,y)\Big)\\
&\qquad\qquad - \left(\mathcal{T}^m(t)F^m(t,\cdot)\right)(a,y) - \frac{\partial F^m}{\partial t}(t,y).
\end{aligned}
\tag{4.7}
$$

Hence, Corollary 3.3 provides the existence of a strong solution to (4.6)

$$
w^m \in C^1([0,T];H_\Omega)\cap C([0,T];D(B)),
$$

which satisfies

$$
\begin{aligned}
&-\int_0^T\int_\Omega w^m\frac{\partial\psi}{\partial t}dydadt - \int_0^T\Big\langle\frac{\partial\psi}{\partial a}(t), w^m(t)\Big\rangle dt\\
&+\int_\Omega w^m(T,a,y)\psi(T,a,y)dyda + \int_0^T\int_0^L w^m(t,a^+y)\psi(t,a^+,y)dydt\\
&-\int_\Omega w^{0m}(a,y)\psi(0,a,y)dyda\\
&+\int_0^T\int_\Omega\Big(K(a,y)\frac{\partial w^m}{\partial y}\frac{\partial\psi}{\partial y} + \left(\mathcal{T}(t)w^m(t)\right)(a,y)\psi(t,a,y)\Big)dydadt\\
&=\int_0^T\int_\Omega\widetilde{f}^m\psi dydadt + \sum_{j=0}^n\int_0^T\int_0^{a^+}\widetilde{k}_j^m(t,a)\psi(t,a,y)dadt,
\end{aligned}
\tag{4.8}
$$

for any ψ satisfying (1.9). Consequently, replacing

$$q^m = w^m + F^m \tag{4.9}$$

it follows that q^m satisfies the weak form of (4.5)

$$\begin{aligned}
&-\int_0^T \int_\Omega q^m \frac{\partial \psi}{\partial t} dydadt - \int_0^T \Big\langle \frac{\partial \psi}{\partial a}(t), q^m(t) \Big\rangle dt \\
&+ \int_\Omega q^m(T,a,y)\psi(T,a,y)dyda + \int_0^T \int_0^L q^m(t,a^+y)\psi(t,a^+,y)dydt \\
&- \int_\Omega q^{0m}(a,y)\psi(0,a,y)dyda - \int_0^T \int_0^L F^m(t,y)\psi(t,0,y)dydt \\
&+ \int_0^T \int_\Omega \Big(K(a,y)\frac{\partial q^m}{\partial y}\frac{\partial \psi}{\partial y} + (\mathcal{T}(t)q^m(t))\,(a,y)\psi(t,a,y)\Big) dydadt \\
&= \int_0^T \int_\Omega f^m \psi dydadt + \sum_{j=0}^n \int_0^T \int_0^{a^+} k_j^m(t,a)\psi(t,a,y)dadt,
\end{aligned} \tag{4.10}$$

for any ψ satisfying (1.9). Moreover, we have that

$$\begin{aligned}
&q^m \in C^1([0,T];H_\Omega) \cap L^2(0,T;L^2(0,a^+;V)) \cap C([0,a^+];L^2(0,T;H)), \\
&q_a^m \in L^2(0,T;L^2(0,a^+;V')).
\end{aligned} \tag{4.11}$$

Thus, q^m is regular enough so that we can replace $\psi = q^m$ in (4.10) to obtain

$$\begin{aligned}
&\|q^m(t)\|_{H_\Omega}^2 + \int_0^t \left\|q^m(s,a^+,\cdot)\right\|_H^2 ds + K_0 \int_0^t \|q^m(s)\|_{L^2(0,a^+;V)}^2 ds \\
&\le C_2\Big(\left\|q^0\right\|_{H_\Omega}^2 + \int_0^T \|f^m(s)\|_{H_\Omega}^2 ds \\
&\qquad + \|F^m\|_{L^2(0,T;H)}^2 + \sum_{j=0}^n \int_0^T \|k^m(s)\|_{L^2(0,a^+)}^2 ds\Big),
\end{aligned} \tag{4.12}$$

for any $t \in [0,T]$.

Now, since the system is linear, the estimate (4.12) can be written for the difference of two solutions as

$$\begin{aligned}
&\left\|q^m(t) - q^{m'}(t)\right\|_{H_\Omega}^2 + \int_0^t \left\|q^m(s,a^+,\cdot) - q^{m'}(s,a^+,\cdot)\right\|_H^2 ds \\
&\qquad + \int_0^t \left\|q^m(s) - q^{m'}(s)\right\|_{L^2(0,a^+;V)}^2 ds
\end{aligned} \tag{4.13}$$

$$\le C_2\Big(\int_0^T \Big\|f^m(s)-f^{m'}(s)\Big\|^2_{H_\Omega}\,ds+\Big\|F^m-F^{m'}\Big\|^2_{L^2(0,T;H)} \\ +\sum_{j=0}^n\int_0^T\Big\|k^m(s)-k^{m'}(s)\Big\|^2_{L^2(0,a^+)}\,ds\Big),$$

for any $t\in[0,T]$.

Since we have (4.4), the sequence $(q^m)_{m\geq 1}$ is Cauchy in

$$C([0,T];H_\Omega)\cap C([0,a^+];L^2(0,T;H))\cap L^2(0,T;L^2(0,a^+;V))$$

and its limit q exists in these spaces and satisfies (1.8), obtained from (4.10) by passing to the limit as $m\to\infty$.

Finally (4.2) follows from (4.12) by passing to the limit as $m\to\infty$.

Concerning the uniqueness of the solution we obtain it writing for example (4.2) for the difference of two solutions $q(t)$ and $\bar q(t)$ corresponding to the same data. □

Corollary 4.2. *In Theorem* 4.1, *if, in addition,*

$$\begin{array}{c} q^0\in D(B),\ \ f\in C^1([0,T];H_\Omega),\ \ F\in C^1([0,T];L^2(0,L)),\\ k\in C^1([0,T];L^2(0,a^+)),\quad \mathcal{T}\in C^1([0,T];\mathcal{L}(L^2(\Omega),L^2(\Omega)))\end{array}\tag{4.14}$$

then

$$\begin{array}{l} q\in C^1([0,T];H_\Omega)\cap L^2(0,T;L^2(0,a^+;V))\cap C([0,a^+];L^2(0,T;H)),\\ q_a\in L^2(0,T;L^2(0,a^+;V')).\end{array}$$

and

$$\|q(t)\|^2_{H_\Omega}+\big\|q'(t)\big\|^2_{H_\Omega}+\int_0^t\big\|q(s,a^+,\cdot)\big\|^2_H\,ds+K_0\int_0^t\|q(s)\|^2_{L^2(0,a^+;V)}\,ds \\ \le C\Big(\big\|q^0\big\|^2_{H_\Omega}+\big\|Bq^0+f(0)\big\|^2_{H_\Omega}+\|L_0(0)\|^2_{L^2(0,a^+;V')} \\ +\int_0^T\|f(s)\|^2_{H_\Omega}\,ds+\int_0^T\big\|f'(s)\big\|^2_{H_\Omega}\,ds+\int_0^T\|F(s)\|^2_H\,ds+\int_0^T\big\|F'(s)\big\|^2_H\,ds \\ +\sum_{j=0}^n\int_0^T\|k(s)\|^2_{L^2(0,a^+)}\,ds+\sum_{j=0}^n\int_0^T\big\|k'(s)\big\|^2_{L^2(0,a^+)}\,ds\Big),\tag{4.15}$$

with C a constant depending on the problem parameters.

Proof. We resume the proof of Theorem 4.1 and due to the hypotheses we proceed as in Corollary 3.3. □

Corollary 4.3. *In* (1.1) *let us assume* (1.5) *with*

$$f(t,a,y)\quad\geq\quad 0\ \ a.e.\ (t,a,y)\in(0,T)\times\Omega,\tag{4.16}$$

$$
\begin{aligned}
q^0(a,y) &\geq 0 \ a.e.\ (a,y)\in\Omega,\\
F(t,y) &\geq 0 \ a.e.\ (t,y)\in(0,T)\times(0,L)
\end{aligned}
$$

and let us consider that $\mathcal{T}(t)\equiv 0$ *for any* $t\in[0,T]$ *and* $k(t,a)\equiv 0$ *a.e.* $(t,a)\in(0,T)\times(0,a^+)$.

Then the weak solution to (1.1) *is non-negative; i.e.,*

$$
q(t,a,y)\geq 0 \text{ for any } t\in[0,T],\ \ a.e.\ (a,y)\in\Omega. \tag{4.17}
$$

Proof. Under (1.5) problem (1.1) has a unique weak solution satisfying (1.8). We shall resume the proof of Theorem 4.1, with non-negative q^{0m}, f^m, F^m, in view of (4.16). We recall in fact that the system (4.5), where $\mathcal{T}(t)=0$, $k(t,a)=0$, has a strong solution q^m regular enough (see (4.11).

Therefore, we can multiply (4.5) by $(q^m)^-$ and integrate with respect to $a\in(0,a^+)$, $y\in(0,L)$ and t. We get

$$
\begin{aligned}
&\left\|(q^m)^-(t)\right\|^2_{H_\Omega}+\int_0^t\left\|(q^m)^-(s,a^+,\cdot)\right\|^2_H ds+2K_0\int_0^t\left\|(q^m)^-_y(s)\right\|^2_{L^2(0,a^+;H)}ds\\
&\leq\left\|(q^{0m})^-\right\|^2_{H_\Omega}+\int_0^t\left\|(q^m)^-(s,0,\cdot)\right\|^2_H ds \qquad (4.18)\\
&\quad -2\int_0^t\int_\Omega f^m(s,a,y)(q^m)^-(s,a,y)dydads.
\end{aligned}
$$

Using (4.16), we obtain $\|(q^m)^-(t)\|^2_{H_\Omega}=0$ for any $t\in[0,T]$, whence we deduce that $q^m(t,a,y)\geq 0$ for any $t\in[0,T]$, for almost every $(a,y)\in\Omega$.

Finally, we recall that q^m is strongly convergent to q in $C([0,T];H_\Omega)$ and the limit preserves non-negativeness. □

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