Differential and Integral Equations Volume 22, Numbers 5–6 (2009), 561–574

ON THE LARGE-TIME BEHAVIOR OF ANISOTROPIC MAXWELL EQUATIONS

Cleverson R. da Luz
Institute of Mathematics, Federal University of Rio de Janeiro
P.O. Box 68530, Rio de Janeiro-RJ, Brazil

G. Perla Menzala
National Laboratory of Scientific Computation, LNCC/MCT
Av. Getúlio Vargas, 333, Quitandinha, Petrópolis
CEP 25651-070, Rio de Janeiro-RJ, Brazil
and
Institute of Mathematics, Federal University of Rio de Janeiro
P.O. Box 68530, Rio de Janeiro-RJ, Brazil

(Submitted by: J.A. Goldstein)

Abstract. Anisotropic Maxwell equations with electric conductivity are considered. Electromagnetic waves propagate in the exterior of a bounded connected obstacle with Lipschitz boundary. Our main result says that we can obtain uniform rates of decay of the total energy as $t \to +\infty$. No special requirements on the geometry of the obstacle are required. Previous results of this type were only given in the isotropic case. We use multipliers and properties of an associated evolution coupled system of first order.

1. Introduction

In many situations of practical importance, engineers, physicists and in general applied technicians must deal with reflections of electromagnetic waves from a rigid body. Maxwell equations provide the mathematical foundations for analyzing such kinds of problems. In most cases it is enough to consider the isotropic situation, that is, when the permittivity and permeability are positive constants or scalar-valued functions. In recent years, due to the enormous amount of research in applied technologies, say for instance "smart materials" ([1], [30]), crystal optics ([19]) or biomedical technologies for sensor applications and photodynamic therapy, Maxwell equations are the fundamental equations governing the interaction of the radiation with

Accepted for publication: December 2008.
AMS Subject Classifications: 35Q99, 35Q60, 35L99.

the tissue ([22], [27], [13]). In those situations the correct models are the anisotropic Maxwell equations. In this case, the permittivity $\epsilon(x)$ and permeability $\mu(x)$ are 3×3 symmetric matrices, uniformly positive definite. Analytical results in this case became harder to obtain because it is not possible to reduce Maxwell equations to a second-order vector wave equation for which large amounts of techniques and results are available.

This work is devoted to studying the anisotropic Maxwell equations in the exterior of a bounded connected obstacle with Lipschitz boundary. Our main result says that we can find uniform rates of decay of the total energy $\mathcal{L}(t)$ associated with the system. The rates of decay are of polynomial type.

The anisotropic Maxwell equations with electric conductivity read as follows:

$$\begin{cases} \epsilon(x)E_t - curl\, H + \sigma E = 0 \\ \mu(x)H_t + curl\, E = 0 \qquad \text{in } \Omega \times (0, +\infty) \\ div\,(\mu(x)H) = 0. \end{cases} \tag{1.1}$$

Here, $E = E(x,t)$ and $H = H(x,t)$ denote the electric and magnetic fields, respectively, $\epsilon(x)$ and $\mu(x)$ denote the electric permittivity and magnetic permeability respectively. They are 3×3 symmetric matrices, uniformly positive definite whose entries are real-valued functions and belong to $L^\infty(\Omega)$. The parameter $\sigma > 0$ is called the conductivity constant. Observe that the condition $div\,(\epsilon E) = 0$ does not appear in (1.1) because it is not in concordance when we take the divergence of the first equation in (1.1).

We complement (1.1) with boundary conditions

$$E \times \eta = 0 \qquad\qquad \text{on } \partial\Omega \times (0, +\infty), \tag{1.2}$$

where $\eta = \eta(x)$ denotes the exterior unit normal at $x \in \partial\Omega$ and $\times$ is the usual vector product in $\mathbb{R}^3$.

The initial conditions of (1.1) are

$$E(x, 0) = E_0(x), \qquad H(x, 0) = H_0(x), \qquad\qquad x \in \Omega, \tag{1.3}$$

where E_0 and H_0 will belong to standard functional spaces.

There are very few papers in the mathematical literature giving rigorous results in the anisotropic case for Maxwell equations. Let us mention some of them. M. M. Eller [6] established an observability inequality also known as an inverse inequality. By a duality argument this observability inequality implies exact controllability of an electromagnetic field in Ω by a current flux on the boundary $\partial\Omega$. V. Vogelsang [29] and T. Okaji [25] both proved strong unique continuation in the time-harmonic case, M. M. Eller and M. Yamamoto [8] established a Carleman estimate for the stationary anisotropic

Maxwell equations. The total energy of system (1.1)-(1.3) is given by

$$\mathcal{L}(t) = \frac{1}{2}\int_\Omega \Big\{ \epsilon(x)E(x,t)\,.\,E(x,t) + \mu(x)H(x,t)\,.\,H(x,t) \Big\}\; dx.$$

Here the dot . means the usual inner product in $\mathbb{R}^3$. Formally, an easy calculation shows that the derivative of $\mathcal{L}(t)$ is given by

$$\frac{d\mathcal{L}}{dt}(t) = -\sigma \int_\Omega |E(x,t)|^2 \; dx \leq 0, \tag{1.4}$$

because $\sigma > 0$. Thus, the total energy $\mathcal{L}(t)$ decreases along trajectories.

In this work we are interested in finding uniform rates of decay for the total energy $\mathcal{L}(t)$ associated with problem (1.1)-(1.3).

The decay rates obtained are of polynomial type. This result is in agreement with known results in the special case when $\Omega = \mathbb{R}^3$. Related results can be found in W. Dan and Y. Shibata [3] for the linear wave equation in an exterior domain, M. Nakao [23], R. Ikehata [15], [16] among many others and M. V. Ferreira and G. P. Menzala [10], R. C. Charão and R. Ikehata [2] in the case of elastic waves in exterior domains.

The behavior of $\mathcal{L}(t)$ as $t \to +\infty$ has been previously considered only in the isotropic bounded domain case with internal or boundary dissipation (see [7], [18], [20], [24] and the references therein). For isotropic Maxwell equations in exterior domains and Silver-Muller boundary conditions, B. V. Kapitonov [17] obtained the decay of the local energy as $t \to \infty$. M. V. Ferreira and G. P. Menzala [11] proved the uniform stabilization with polynomial rates for the second-order energy of the solutions of coupled systems of electromagnetic-elasticity in an exterior domain.

The paper is organized as follows. Well posedness of the problem is analyzed in Section 2 using semigroup theory. Uniform decay of the total energy is established in Section 3 using the multiplier method and properties of an evolution coupled system of first order.

2. Well posedness

Let Ω be an open set in $\mathbb{R}^3$ which is the exterior of an open bounded connected body in $\mathbb{R}^3$ with Lipschitz boundary. Let us describe the function spaces where we will consider the solution of problem (1.1)-(1.3). We consider the set $\mathcal{M}$ of matrices $\alpha(x)$ such that $\alpha(x) = \big[\alpha_{ij}(x)\big]_{3\times 3}$ is a symmetric and uniformly positive definite matrix; that is, there exist $\alpha_0 > 0$ such that

$$\xi^t\,\alpha(x)\,\xi \geq \alpha_0\,|\,\xi\,|^2, \quad \text{for every} \;\; \xi \in \mathbb{R}^3, \text{ a.e. on } \;\; \Omega;$$

here if $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}$ we denote by $\xi^t = \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 \end{pmatrix}$ and $|\xi|^2 = \sum_{j=1}^{3} \xi_j^2$. We assume that all entries α_{ij} of α belong to $L^\infty(\Omega)$. Clearly different such matrices $\alpha(x)$ could have different constants α_0.

If α belongs to $\mathcal{M}$ we consider the space

$$L^2(\Omega;\alpha) = \Big\{ v = (v_1, v_2, v_3) : \ v_i \text{ measurable}, \ i = 1,2,3 \text{ and } \sum_{i,j=1}^{3} \int_\Omega \alpha_{i,j}(x)\, v_i(x)\, v_j(x)\, dx < +\infty \Big\},$$

with inner product

$$(v,u)_{L^2(\Omega;\,\alpha)} = \sum_{i,j=1}^{3} \int_\Omega \alpha_{i,j}(x)\, v_i(x)\, u_j(x)\, dx = \int_\Omega [u(x)]^t\, \alpha(x)\, v(x)\, dx,$$

and norm

$$\|v\|^2_{L^2(\Omega;\,\alpha)} = \sum_{i,j=1}^{3} \int_\Omega \alpha_{i,j}(x)\, v_i(x)\, v_j(x)\, dx = \int_\Omega [v(x)]^t\, \alpha(x)\, v(x)\, dx.$$

Clearly, $L^2(\Omega;\alpha) = [L^2(\Omega)]^3$ and the norms $\|\,.\,\|_{L^2(\Omega;\,\alpha)}$, $\|\,.\,\|_{[L^2(\Omega)]^3}$ are equivalent in the space $[L^2(\Omega)]^3$. Throughout this work we shall denote by $\|\,.\,\|$ the norm in $[L^2(\Omega)]^3$.

Let $\epsilon(x)$ and $\mu(x)$ belong to $\mathcal{M}$ with constants ϵ_0 and μ_0 respectively.

In this section we briefly describe well posedness for problem (1.1)-(1.3) using semigroup theory. Let $X = L^2(\Omega;\epsilon) \times L^2(\Omega;\mu)$ be the Hilbert space equipped with the inner product

$$\langle w, v\rangle_X = (w_1, v_1)_{L^2(\Omega;\,\epsilon)} + (w_2, v_2)_{L^2(\Omega;\,\mu)},$$

for every $w = (w_1, w_2)$ and $v = (v_1, v_2)$ in X.

We also consider the Hilbert space

$$H(curl;\,\Omega) = \big\{ v \in [L^2(\Omega)]^3 : \ curl\, v \in [L^2(\Omega)]^3 \big\},$$

with inner product

$$\langle v, u\rangle_{H(curl;\,\Omega)} = (v,u)_{[L^2(\Omega)]^3} + (curl\, v, curl\, u)_{[L^2(\Omega)]^3}$$
$$= \int_\Omega v(x)\,.\,u(x)\, dx + \int_\Omega curl\, v(x)\,.\,curl\, u(x)\, dx.$$

It is well known (see for instance [5]) that the map $w \longrightarrow \eta \times w\big|_{\partial\Omega}$ from $[C_0^1(\bar\Omega)]^3$ into $[C^1(\partial\Omega)]^3$ extends by continuity to a continuous linear map from $H(curl;\,\Omega)$ into $[H^{-1/2}(\partial\Omega)]^3$. This result allows us to consider the subspace

$$H_0(curl;\,\Omega) = \{w \in H(curl;\,\Omega): \quad \eta \times w = 0 \quad \text{on} \quad \partial\Omega\}.$$

It follows that $H_0(curl;\,\Omega)$ is a closed subspace and $[C_0^\infty(\Omega)]^3$ is dense in $H_0(curl;\,\Omega)$ (see [5]). If $v \in H_0(curl;\,\Omega)$ and $u \in H(curl;\,\Omega)$, then the following equality is true:

$$\int_\Omega v(x)\,.\,curl\,u(x)\;dx = \int_\Omega curl\,v(x)\,.\,u(x)\;dx.$$

Now, let us consider the linear unbounded operator $A : D(A) \subset X \to X$ with domain

$$D(A) = H_0(curl;\,\Omega) \times H(curl;\,\Omega),$$

given by

$$Aw = (\epsilon^{-1}\,curl\,w_2, -\mu^{-1}\,curl\,w_1), \quad \text{for every } w = (w_1, w_2) \in D(A). \tag{2.1}$$

Here, $\epsilon^{-1}(x)$ and $\mu^{-1}(x)$ denote the inverses of $\epsilon(x)$ and $\mu(x)$ respectively. The matrices ϵ and μ are invertible almost everywhere in Ω because they belong to $\mathcal{M}$, therefore the eigenvalues of $\epsilon(x)$ and $\mu(x)$ are positive (see [28] page 243). Consequently the determinant of each of those matrices is positive. Hence, ϵ and μ are invertible almost everywhere in Ω. We can also prove that the entries of ϵ^{-1} and μ^{-1} belong to $L^\infty(\Omega)$.

Now, we consider the bounded linear operator $B : X \longrightarrow X$ given by

$$Bw = (-\sigma\epsilon^{-1}w_1, 0), \qquad \text{for every } w = (w_1, w_2) \in X.$$

With the above notation, (1.1)-(1.3) can be written as

$$\frac{dU}{dt}(t) = (A + B)U(t), \quad U(0) = U_0,$$

where $U(t) = (E(t), H(t))$ and $U_0 = (E_0, H_0)$. Clearly $D(A)$ is dense in X since $[\mathcal{D}(\Omega)]^3 \times [\mathcal{D}(\Omega)]^3 \subset D(A)$. We will show that A is skew adjoint.

Lemma 2.1. *Let A^* the adjoint operator of A. Then $D(A) \subset D(A^*)$ and, for any $v \in D(A)$, $A^*v = -Av$.*

Proof. Let $v = (v_1, v_2) \in D(A)$; then for every $w = (w_1, w_2) \in D(A)$ we have

$$\begin{aligned}\langle Aw, v\rangle_X &= \big\langle(\epsilon^{-1}\,curl\,w_2, -\mu^{-1}\,curl\,w_1), (v_1, v_2)\big\rangle_X \\ &= (\epsilon^{-1}\,curl\,w_2, v_1)_{L^2(\Omega;\,\epsilon)} - (\mu^{-1}\,curl\,w_1, v_2)_{L^2(\Omega;\,\mu)}\end{aligned}$$

$$= \int_\Omega [\epsilon^{-1}\, curl\, w_2]^t\, \epsilon\, v_1\ dx - \int_\Omega [\mu^{-1}\, curl\, w_1]^t\, \mu\, v_2\ dx$$
$$= \int_\Omega curl\, w_2 \,.\, v_1\ dx - \int_\Omega curl\, w_1 \,.\, v_2\ dx,$$

due to the symmetry of ϵ and μ. Since $v_1, w_1 \in H_0(curl;\Omega)$ and $v_2, w_2 \in H(curl;\Omega)$, from the above identity it follows that

$$\langle Aw, v\rangle_X = \int_\Omega w_2 \,.\, curl\, v_1\ dx - \int_\Omega w_1 \,.\, curl\, v_2\ dx$$
$$= \big\langle (w_1, w_2), (-\epsilon^{-1}\, curl\, v_2, \mu^{-1}\, curl\, v_1)\big\rangle_X .$$

Thus, if $v \in D(A)$ we can choose $g = (-\epsilon^{-1}\, curl\, v_2, \mu^{-1}\, curl\, v_1) \in X$ and

$$\langle Aw, v\rangle_X = \langle w, g\rangle_X , \qquad \text{for every } \ w \in D(A).$$

Therefore, $v \in D(A^*)$ and $A^* v = -Av$, for every $v \in D(A)$ which proves Lemma 2.1. □

Lemma 2.2. *$D(A^*) \subset D(A)$ and, for any $v \in D(A^*)$,*

$$A^* v = -Av.$$

Proof. Let $v \in D(A^*)$. Thus, there exists $g = (g_1, g_2) \in X$ such that

$$\langle Aw, v\rangle_X = \langle w, g\rangle_X , \qquad \text{for every } w \in D(A). \tag{2.2}$$

In particular, if $w = (0, w_2) \in D(A)$ with $w_2 \in [\mathcal{D}(\Omega)]^3$, from (2.2) we obtain

$$\int_\Omega curl\, w_2 \,.\, v_1\ dx = \int_\Omega w_2 \,.\, [\mu g_2]\ dx.$$

Hence, $curl\, v_1 \in [L^2(\Omega)]^3$. Thus, $v_1 \in H(curl;\Omega)$ and

$$g_2 = \mu^{-1}\, curl\, v_1 \qquad \text{in} \quad [L^2(\Omega)]^3.$$

Next, we consider $w = (w_1, 0) \in D(A)$ with $w_1 \in [\mathcal{D}(\Omega)]^3$. Using (2.2) we obtain that $v_2 \in H(curl;\Omega)$ and

$$g_1 = -\epsilon^{-1}\, curl\, v_2 \qquad \text{in} \ \ [L^2(\Omega)]^3.$$

In conclusion, we proved that $D(A^*) \subset H(curl;\Omega) \times H(curl;\Omega)$ and the element $g \in X$ which satisfies (2.2) is

$$g = (-\epsilon^{-1}\, curl\, v_2,\ \mu^{-1}\, curl\, v_1). \tag{2.3}$$

To conclude the proof it only remains to show that $\eta \times v_1 = 0$ on $\partial\Omega$. Again we use identity (2.2) together with (2.3) to obtain

$$\int_\Omega curl\, w_2 \,.\, v_1\ dx = \int_\Omega w_2 \,.\, curl\, v_1\ dx, \qquad \text{for every } w_2 \in H(curl;\Omega).$$

The above identity implies that $v_1 \in H_0(curl;\Omega)$ (see for instance Lemma 1, Chapter IX in [4]). Thus $v \in D(A)$ and $g = -Av$. $\square$

From Lemmas 2.1 and 2.2 it follows that $D(A^*) = D(A)$ and $A^* = -A$. Thus, using Stone's theorem (see [26]) we conclude that the operator A given by (2.1) is the infinitesimal generator of a group of unitary operators $\{T(t)\}_{t\in\mathbb{R}}$ of class $\mathcal{C}_0$ in X. Since B is linear and bounded, it follows that the operator $A+B$ with domain $D(A+B) = D(A)$ is the infinitesimal generator of a group of operators $\{S(t)\}_{t\in\mathbb{R}}$ of class $\mathcal{C}_0$ in X (see for instance Theorem 13.2.2 in [14]). It remains to fulfill the requirement that μH is divergence free. However, taking the divergence of the second equation in (1.1) we obtain

$$div\,(\mu H_t(t)) + div\,curl\,E(t) = 0,$$

in the sense of distributions. Consequently,

$$div\,(\mu H(t)) = div\,(\mu H_0),$$

in the sense of distributions. Thus, it is natural to consider the subspace

$$Y = \{(w, v) \in X \,:\, div\,(\mu\, v) = 0\}.$$

The above considerations imply the following.

Theorem 2.1. *Let the assumptions at the beginning of this section hold for Ω, σ, ϵ and μ. Let $(E_0, H_0) \in Y$. Then there is a unique mild solution (E, H) of problem (1.1)-(1.3) which belongs to $\mathcal{C}([0,\infty);Y)$. Furthermore, if $(E_0, H_0) \in D(A)\cap Y$, then the initial-boundary-value problem (1.1)-(1.3) has a unique strong solution $(E, H) \in \mathcal{C}([0,\infty); D(A)\cap Y) \cap \mathcal{C}^1([0,\infty);Y)$.*

Remark 2.1. In the standard way we could obtain more regular solutions if we assume that the initial data has more regularity. For example, if $(E_0, H_0) \in D(A^2)\cap Y$ we would obtain that the solution of (1.1)-(1.3) belongs to the space

$$\mathcal{C}([0,\infty); D(A^2)\cap Y) \cap \mathcal{C}^1([0,\infty); D(A)\cap Y) \cap \mathcal{C}^2([0,\infty);Y).$$

3. Stabilization

We consider $\Omega = \mathbb{R}^3\backslash\bar{\mathcal{O}}$, with $\mathcal{O}$ an open bounded connected subset of $\mathbb{R}^3$ having Lipschitz boundary $\partial\mathcal{O}$. The matrices ϵ and μ belong to $\mathcal{M}$ as in the previous section. In order to study the asymptotic behavior of solutions of problem (1.1)-(1.3), we consider three cases: If $(E_0, H_0) \in Y$ and $\mu H_0 = curl\,\psi_0$ for some $\psi_0 \in H_0(curl;\Omega)$; the case $(E_0, H_0) \in D(A)\cap Y$ without additional assumptions and finally when $(E_0, H_0) \in D(A)\cap Y$ and $\mu H_0 = curl\,\psi_0$, with $\psi_0 \in H_0(curl;\Omega)$.

Remark 3.1. *Note that if* $H_0 \in H(curl, \Omega)$ *is a function different from zero such that* $div\,(\mu H_0) = curl\, H_0 = 0$ *in* Ω*, then* $(E(t), H(t)) = (0, H_0)$ *is the solution of the problem* (1.1)-(1.3) *in* Ω*. Obviously, the total energy does not decay to zero when* $t \to +\infty$*.*

Theorem 3.1. *Let* $(E_0, H_0) \in Y$ *such that* $\mu H_0 = curl\, \psi_0$*, for some* $\psi_0 \in H_0(curl; \Omega)$*. Then the mild solution* (E, H) *of problem* (1.1)-(1.3) *satisfies*

$$\|E(t)\|^2 + \|H(t)\|^2 \le C I_0 (1+t)^{-1}, \quad \text{for every } t \ge 0,$$

with C *a positive constant which does not depend on the initial data and* $I_0 = \|E_0\|^2 + \|H_0\|^2 + \|\psi_0\|^2$*.*

The proof of Theorem 3.1 follows from the next two lemmas.

Lemma 3.1. *Let* (E, H) *be the mild solution of system* (1.1)-(1.3) *with initial data* $(E_0, H_0) \in Y$*. Then, the following identities hold:*

$$a) \quad \|E(t)\|^2_{L^2(\Omega;\epsilon)} + \|H(t)\|^2_{L^2(\Omega;\mu)} + 2\sigma \int_0^t \|E(s)\|^2 ds$$
$$= \|E_0\|^2_{L^2(\Omega;\epsilon)} + \|H_0\|^2_{L^2(\Omega;\,\mu)},$$

$$b) \quad (1+t)\|E(t)\|^2_{L^2(\Omega;\epsilon)} + (1+t)\|H(t)\|^2_{L^2(\Omega;\mu)} + 2\sigma \int_0^t (1+s)\|E(s)\|^2 ds$$
$$= \|E_0\|^2_{L^2(\Omega;\,\epsilon)} + \|H_0\|^2_{L^2(\Omega;\,\mu)} + \int_0^t \|E(s)\|^2_{L^2(\Omega;\epsilon)} ds + \int_0^t \|H(s)\|^2_{L^2(\Omega;\mu)} ds.$$

Proof. a) follows from identity (1.4). To prove b) we multiply (1.4) by $(1+t)$ and integrate by parts.

Lemma 3.2. *Assume the hypotheses of Theorem* 3.1*. Then the estimate*

$$\int_0^\infty \|H(s)\|^2_{L^2(\Omega;\,\mu)}\; ds \le C I_0$$

holds. Here the positive constant C *does not depends on the initial data.*

Proof. Let $(E_0, H_0) \in D(A)$. We define

$$W(t) = \int_0^t E(s)\; ds \;\; \text{and} \;\; F(t) = \int_0^t H(s)\; ds;$$

then $\{W, F\}$ is a solution of the following system:

$$\epsilon W_t - curl\, F + \sigma W = \epsilon E_0 \qquad \text{in} \;\; \Omega \times (0, \infty), \tag{3.1}$$

$$\mu F_t + curl\, W = \mu H_0 \qquad \text{in} \;\; \Omega \times (0, \infty), \tag{3.2}$$

$$W(x, 0) = 0, \quad F(x, 0) = 0 \qquad \text{in} \;\; \Omega, \tag{3.3}$$

$$W \times \eta = 0 \qquad \text{on} \quad \partial\Omega \times (0,\infty). \tag{3.4}$$

We find the derivative of equation (3.1) with respect to t and take the inner product in $L^2(\Omega)$ with $W(t)$. Afterwards, take the inner product of equation (3.2) with $F_t(t)$ in $L^2(\Omega)$. Adding the results we obtain

$$\frac{d}{dt}(W_t(t), W(t))_{L^2(\Omega;\,\epsilon)} - \|W_t(t)\|^2_{L^2(\Omega;\,\epsilon)} + \|F_t(t)\|^2_{L^2(\Omega;\,\mu)} + \frac{\sigma}{2}\frac{d}{dt}\|W(t)\|^2 = \frac{d}{dt}\int_\Omega [\mu H_0]\,.\,F(t)\;dx.$$

Since $W_t(t) = E(t)$ and $F_t(t) = H(t)$, integrating over the interval $[0,t]$ the previous identity we obtain

$$\frac{\sigma}{4}\|W(t)\|^2 + \int_0^t \|H(s)\|^2_{L^2(\Omega;\,\mu)}\;ds \le CI_0 + \int_\Omega [\mu H_0]\,.\,F(t)\;dx.$$

Since $D(A)$ is dense in X, the above estimate is valid whenever $(E_0, H_0) \in X$. By the hypotheses, there exists $\psi_0 \in H_0(curl;\,\Omega)$ such that $\mu H_0 = curl\,\psi_0$. Thus,

$$\frac{\sigma}{4}\|W(t)\|^2 + \int_0^t \|H(s)\|^2_{L^2(\Omega;\,\mu)}\;ds \le CI_0 + \int_\Omega \psi_0\,.\,curl\,F(t)\;dx.$$

Using (3.1), we conclude that

$$\frac{\sigma}{4}\|W(t)\|^2 + \int_0^t \|H(s)\|^2_{L^2(\Omega;\,\mu)}\;ds \le CI_0 + \frac{1}{2}\|E(t)\|^2_{L^2(\Omega;\,\epsilon)} + \frac{\sigma}{8}\|W(t)\|^2.$$

Thus, by Lemma 3.1 we deduce

$$\int_0^t \|H(s)\|^2_{L^2(\Omega;\,\mu)}\;ds \le CI_0 \quad \text{for any} \quad t \ge 0. \qquad \square$$

Clearly, the proof of Theorem 3.1 follows from Lemmas 3.1 and 3.2.

Theorem 3.2. *Let (E,H) be the strong solution of problem (1.1)-(1.3) for initial data $(E_0, H_0) \in D(A) \cap Y$. Then, there exists a constant $C > 0$, which does not depends on the initial data, such that*

$$\|E(t)\|^2 + \|curl\,H(t)\|^2 \le CI_1(1+t)^{-1}, \quad \text{for every} \quad t \ge 0,$$
$$\|E_t(t)\|^2 + \|H_t(t)\|^2 + \|curl\,E(t)\|^2 \le CI_1(1+t)^{-2}, \quad \text{for every} \quad t \ge 0,$$

where $I_1 = \|E_0\|^2 + \|curl\,E_0\|^2 + \|H_0\|^2 + \|curl\,H_0\|^2$.

Proof. Initially we assume that $(E_0, H_0) \in D(A^2)$. We differentiate system $(1.1) - (1.3)$ with respect to t to obtain

$$\epsilon E_{tt} - curl\,H_t + \sigma E_t = 0 \qquad \text{in} \quad \Omega \times (0,\infty), \tag{3.5}$$

$$\mu H_{tt} + curl\, E_t = 0 \quad \text{in} \quad \Omega \times (0,\infty), \tag{3.6}$$

$$E_t \times \eta = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty), \tag{3.7}$$

$$E_t(0) = E_1 = \epsilon^{-1} curl\, H_0 - \sigma\epsilon^{-1} E_0 \quad \text{in} \quad \Omega, \tag{3.8}$$

$$H_t(0) = H_1 = -\mu^{-1} curl\, E_0 \quad \text{in} \quad \Omega. \tag{3.9}$$

We take the inner product of (3.5) with $E_t(t)$ and of (3.6) with $H_t(t)$; integration over Ω and addition give us

$$\frac{d}{dt}\left\{\|E_t(t)\|^2_{L^2(\Omega;\,\epsilon)} + \|H_t(t)\|^2_{L^2(\Omega;\,\mu)}\right\} + 2\sigma\|E_t(t)\|^2 = 0. \tag{3.10}$$

Integrating (3.10) over the interval $[0,t]$ it follows that

$$\|E_t(t)\|^2_{L^2(\Omega;\,\epsilon)} + \|H_t(t)\|^2_{L^2(\Omega;\,\mu)} + 2\sigma\int_0^t \|E_s(s)\|^2\, ds \tag{3.11}$$
$$= \|E_1\|^2_{L^2(\Omega;\,\epsilon)} + \|H_1\|^2_{L^2(\Omega;\,\mu)}.$$

Multiplying identity (3.10) by $(1+t)$ and integrating over $[0,t]$ we obtain

$$(1+t)\|E_t(t)\|^2_{L^2(\Omega;\epsilon)} + (1+t)\|H_t(t)\|^2_{L^2(\Omega;\mu)} + 2\sigma\int_0^t (1+s)\|E_s(s)\|^2 ds \tag{3.12}$$
$$= \|E_1\|^2_{L^2(\Omega;\epsilon)} + \|H_1\|^2_{L^2(\Omega;\mu)} + \int_0^t \|E_s(s)\|^2_{L^2(\Omega;\epsilon)} ds + \int_0^t \|H_s(s)\|^2_{L^2(\Omega;\mu)} ds.$$

We take the inner product of (3.5) with $E(t)$ and of the second equation of (1.1) with $H_t(t)$; integration over Ω and addition give us

$$\frac{d}{dt}(E_t(t), E(t))_{L^2(\Omega;\epsilon)} - \|E_t(t)\|^2_{L^2(\Omega;\epsilon)} + \|H_t(t)\|^2_{L^2(\Omega;\mu)} + \frac{\sigma}{2}\frac{d}{dt}\|E(t)\|^2 = 0. \tag{3.13}$$

Let $\delta > 0$. Integrating over the interval $[0,t]$ we obtain

$$\int_0^t \|H_s(s)\|^2_{L^2(\Omega;\,\mu)}\, ds + \frac{\sigma}{2}\|E(t)\|^2 = \frac{\sigma}{2}\|E_0\|^2 + \int_0^t \|E_s(s)\|^2_{L^2(\Omega;\,\epsilon)}\, ds$$
$$- (E_t(t), E(t))_{L^2(\Omega;\,\epsilon)} + (E_1, E_0)_{L^2(\Omega;\,\epsilon)} \le C\|E_0\|^2 + \|E_1\|^2_{L^2(\Omega;\,\epsilon)}$$
$$+ C\int_0^t \|E_s(s)\|^2\, ds + \delta^{-1}\|E_t(t)\|^2_{L^2(\Omega;\,\epsilon)} + C\delta\|E(t)\|^2,$$

where we use the equivalence of the norms $\|\,.\,\|_{L^2(\Omega;\,\alpha)}$ and $\|\,.\,\|$. Choosing δ small enough and using (3.11) we have

$$\int_0^t \|H_s(s)\|^2_{L^2(\Omega;\,\mu)}\, ds \le CI_1.$$

Thus, combining the above estimate and identity (3.11) with (3.12) we conclude that

$$(1+t)\|E_t(t)\|^2_{L^2(\Omega;\,\epsilon)}+(1+t)\|H_t(t)\|^2_{L^2(\Omega;\,\mu)}+2\sigma\int_0^t(1+s)\|E_s(s)\|^2\ ds\le CI_1. \tag{3.14}$$

Now, we multiply (3.10) by $(1+t)^2$ and integrate on $[0,t]$ to have that

$$\begin{aligned}&(1+t)^2\|E_t(t)\|^2_{L^2(\Omega;\epsilon)}+(1+t)^2\|H_t(t)\|^2_{L^2(\Omega;\mu)}+2\sigma\int_0^t(1+s)^2\|E_s(s)\|^2ds\\&=\|E_1\|^2_{L^2(\Omega;\epsilon)}+\|H_1\|^2_{L^2(\Omega;\mu)}+2\int_0^t(1+s)\|E_s(s)\|^2_{L^2(\Omega;\epsilon)}ds\\&\quad+2\int_0^t(1+s)\|H_s(s)\|^2_{L^2(\Omega;\mu)}ds.\end{aligned} \tag{3.15}$$

Multiplying identity (3.13) by $(1+t)$ and integrating over $[0,t]$ we obtain

$$\begin{aligned}&\int_0^t(1+s)\|H_s(s)\|^2_{L^2(\Omega;\mu)}ds+\frac{\sigma}{2}(1+t)\|E(t)\|^2\\&=\frac{\sigma}{2}\|E_0\|^2+\frac{\sigma}{2}\int_0^t\|E(s)\|^2ds+\int_0^t(1+s)\|E_s(s)\|^2_{L^2(\Omega;\epsilon)}ds\\&\quad-(1+t)(E_t(t),E(t))_{L^2(\Omega;\epsilon)}+(E_1,E_0)_{L^2(\Omega;\epsilon)}+\int_0^t(E_s(s),E(s))_{L^2(\Omega;\epsilon)}ds\\&\le CI_1+C\int_0^t\|E(s)\|^2ds+C\int_0^t(1+s)\|E_s(s)\|^2ds\\&\quad+\delta^{-1}(1+t)\|E_t(t)\|^2_{L^2(\Omega;\epsilon)}+C\delta(1+t)\|E(t)\|^2.\end{aligned}$$

Choosing δ small enough, using (3.14) and Lemma 3.1, we have

$$\int_0^t(1+s)\|H_s(s)\|^2_{L^2(\Omega;\mu)}ds+\frac{\sigma}{4}(1+t)\|E(t)\|^2\le CI_1. \tag{3.16}$$

Placing information obtained in (3.14) and (3.16) into (3.15) we deduce

$$\begin{aligned}&(1+t)^2\|E_t(t)\|^2_{L^2(\Omega;\epsilon)}+(1+t)^2\|H_t(t)\|^2_{L^2(\Omega;\mu)}\\&\quad+2\sigma\int_0^t(1+s)^2\|E_s(s)\|^2ds+\sigma(1+t)\|E(t)\|^2\le CI_1.\end{aligned} \tag{3.17}$$

By density arguments the above estimate also holds for initial data $(E_0,H_0)\in D(A)$. The conclusions of Theorem 3.2 follow due to the equivalence of the norms $\|\,.\,\|_{L^2(\Omega;\alpha)}$ and $\|\,.\,\|$ and using equation (1.1). □

Theorem 3.3. *Let (E, H) be the strong solution of problem* (1.1)-(1.3) *with initial data $(E_0, H_0) \in D(A) \cap Y$. Suppose $\mu H_0 = curl\, \psi_0$, with $\psi_0 \in H_0(curl; \Omega)$. Then, there exists a constant $C > 0$, which does not depends on the initial data, such that*

$$\|H(t)\|^2 \leq CI_0(1+t)^{-1}, \qquad \text{for every} \quad t \geq 0,$$
$$\|E(t)\|^2 + \|curl\, H(t)\|^2 \leq CI_2(1+t)^{-2}, \qquad \text{for every} \quad t \geq 0,$$
$$\|E_t(t)\|^2 + \|H_t(t)\|^2 + \|curl\, E(t)\|^2 \leq CI_2(1+t)^{-3}, \qquad \text{for every} \quad t \geq 0,$$

where $I_2 = \|E_0\|^2 + \|curl\, E_0\|^2 + \|H_0\|^2 + \|curl\, H_0\|^2 + \|\psi_0\|^2$ and I_0 is as in Theorem 3.1.

Proof. Initially we assume that $(E_0, H_0) \in D(A^2)$. Multiplying (3.10) by $(1+t)^3$ and integrating over $[0, t]$ we obtain

$$\begin{aligned}
&(1+t)^3\|E_t(t)\|^2_{L^2(\Omega;\epsilon)} + (1+t)^3\|H_t(t)\|^2_{L^2(\Omega;\mu)} + 2\sigma\int_0^t (1+s)^3\|E_s(s)\|^2 ds \\
&= \|E_1\|^2_{L^2(\Omega;\epsilon)} + \|H_1\|^2_{L^2(\Omega;\mu)} + 3\int_0^t (1+s)^2\|E_s(s)\|^2_{L^2(\Omega;\epsilon)} ds \\
&+ 3\int_0^t (1+s)^2\|H_s(s)\|^2_{L^2(\Omega;\mu)} ds.
\end{aligned} \tag{3.18}$$

Multiplying (3.13) by $(1+t)^2$ and integrating over $[0, t]$ we deduce

$$\begin{aligned}
&\int_0^t (1+s)^2\|H_s(s)\|^2_{L^2(\Omega;\mu)} ds + \frac{\sigma}{2}(1+t)^2\|E(t)\|^2 \\
&= \frac{\sigma}{2}\|E_0\|^2 + \sigma\int_0^t (1+s)\|E(s)\|^2 ds \\
&+ \int_0^t (1+s)^2\|E_s(s)\|^2_{L^2(\Omega;\epsilon)} ds - (1+t)^2(E_t(t), E(t))_{L^2(\Omega;\epsilon)} \\
&+ (E_1, E_0)_{L^2(\Omega;\epsilon)} + 2\int_0^t (1+s)(E_s(s), E(s))_{L^2(\Omega;\epsilon)} ds \\
&\leq CI_1 + C\int_0^t (1+s)\|E(s)\|^2 ds + C\int_0^t (1+s)^2\|E_s(s)\|^2 ds \\
&+ \delta^{-1}(1+t)^2\|E_t(t)\|^2_{L^2(\Omega;\epsilon)} + C\delta(1+t)^2\|E(t)\|^2.
\end{aligned}$$

Choosing δ small enough and using (3.17) it follows that

$$\int_0^t (1+s)^2\|H_s(s)\|^2_{L^2(\Omega;\mu)} ds + \frac{\sigma}{4}(1+t)^2\|E(t)\|^2 \tag{3.19}$$

$$\leq CI_1 + C\int_0^t (1+s)\|E(s)\|^2 ds.$$

Placing information obtained in (3.17) and (3.19) into (3.18) we deduce

$$(1+t)^3\|E_t(t)\|^2_{L^2(\Omega;\epsilon)} + (1+t)^3\|H_t(t)\|^2_{L^2(\Omega;\mu)} + \sigma(1+t)^2\|E(t)\|^2$$
$$\leq CI_1 + C\int_0^t (1+s)\|E(s)\|^2 ds.$$

Using density arguments the above estimate also holds for initial data $(E_0, H_0) \in D(A)$. Thus, by Lemmas 3.1 and 3.2 we deduce

$$(1+t)^3\|E_t(t)\|^2_{L^2(\Omega;\epsilon)}+(1+t)^3\|H_t(t)\|^2_{L^2(\Omega;\mu)}+\sigma(1+t)^2\|E(t)\|^2 \leq CI_2. \quad (3.20)$$

Using system (1.1), it follows from (3.20) that

$$(1+t)^2\|curlH(t)\|^2 + (1+t)^3\|curlE(t)\|^2 \leq CI_2. \quad (3.21)$$

Theorem 3.3 follows from estimate (3.20), the equivalence of the norms $\|\,.\,\|_{L^2(\Omega;\alpha)}$, $\|\,.\,\|$, estimate (3.21) and Theorem 3.1. □

Acknowledgements: The second author (G. P. M.) was partially supported by a Research Grant of the Brazilian Government (MCT, Brasil), Proc. 305086/2006-5.

References

[1] H.T. Banks, R.C. Smith, and Y. Wang, "Smart Materials Structures, Modeling, Estimation and Control," RAM, Wiley, Chichester, Masson, Paris, 1996.

[2] R. C. Charão, and R. Ikehata, *Decay of solutions for a semilinear system of elastic waves in an exterior domain with damping near infinity*, Nonlinear Anal., 67 (2007), 398–429.

[3] W. Dan, and Y. Shibata, *On a local energy decay of solutions of a dissipative wave equation*, Funkcialaj Ekvacioj, 38 (1995), 545–568.

[4] R. Dautray, and J.L. Lions, "Mathematical Analysis and Numerical Methods for Science and Technology, Spectral Theory and Applications," Vol. 3, Springer-Verlag, Berlin, 1990.

[5] G. Duvaut and J.L. Lions, "Les Inéquations in Mécanique et en Physique," Dunod, 1972.

[6] M.M. Eller, *Continuous observability for the anisotropic Maxwell system*, Appl. Math. Optim., 55 (2007), 185–201.

[7] M. M. Eller, J. E. Lagnese, and S. Nicaise, *Decay rates for solutions of a Maxwell system with nonlinear boundary damping*, Comp. Appl. Math., 21 (2002), 135–165.

[8] M.M. Eller and M. Yamamoto, *A Carleman inequality for the stationary anisotropic Maxwell system*, J. Math. Pures Appl., 86 (2006), 449–462.

[9] M.V. Ferreira, "Ondas Elásticas e Electromagnéticas em Domínios Exteriores: Propriedades Assintóticas," Tese de Doutorado, Instituto de Matemática, Universidade Federal do Rio de Janeiro, Brasil, 2005 (in Portuguese).

[10] M.V. Ferreira and G.P. Menzala, *Energy decay for solutions to semilinear systems of elastic waves in exterior domains*, Electron. J. Diff. Eqns., 65 (2006), 1–13.
[11] M.V. Ferreira and G.P. Menzala, *Uniform stabilization of an electromagnetic-elasticity problem in exterior domains*, Discrete Contin. Dyn. Syst., 18 (2007), 719–746.
[12] E.P. Furlani, "Analysis of axial-field actuators," 6^{th} Joint MMM-Intermag Conference, Albuquerque, New Mexico, June-1994.
[13] E.P. Furlani, J.K. Lee, and D. Dowe, *Predicting the dynamic behavior of moving magnet actuators*, J. Appl. Phys., 73 (1993), 3555–3559.
[14] E. Hille and R. Phillips, "Functional Analysis and Semi-Groups," AMS Coll. Publ. Vol. 31, Providence, Rhode Island, 1957.
[15] R. Ikehata, *Energy decay of solutions for the semilinear dissipative wave equations in an exterior domain*, Funkcialaj Ekvacioj., 44 (2001), 487–499.
[16] R. Ikehata, *Small data global existence of solutions for dissipative wave equations in an exterior domain*, Funkcial. Ekvac., 45 (2002), 259–269.
[17] B.V. Kapitonov, *On exponential decay as $t \to \infty$ of solutions of an exterior boundary value problem for the Maxwell system*, Math. USSR Sbornik, 66 (1990), 475–497.
[18] B.V. Kapitonov and G.P. Menzala, *Uniform stabilization for Maxwell's equations with boundary conditions with memory*, Asymp. Anal., 26 (2001), 91–104.
[19] M. Kline and D. Kay, "Electromagnetic Theory and Geometric Optics, Pure and Applied Math." Interscience John Wiley & Sons Vol. XII, New York, 1965.
[20] V. Komornik, *Boundary stabilization, observation and control of Maxwell's equations*, PanAm. Math. J., 4 (1994), 47–61.
[21] L. D. Landau, and E. M. Lifshitz, "Electrodynamics of Continuous Media", Pergamon Press, New York, 1960.
[22] G.J. Müller et. al. (Editors), "Medical Optical Tomography: Functional Imaging and Monitoring," SPIE Optical Engineering Press Vol. IS11, Bellingham, Washington, 1993.
[23] M. Nakao, *Energy decay for the linear and semilinear wave equations in exterior domains with some localized dissipations*, Math. Z., 238 (2001), 781–797.
[24] S. Nicaise and C. Pignotti, *Partially delayed stabilizing feedbacks for Maxwell's system*, Adv. Differ. Equations, 12 (2007), 27–54.
[25] T. Okaji, *Strong unique continuation property for time harmonic Maxwell equations*, J. Math. Soc. Japan, 54 (2002), 89–122.
[26] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Applied Mathematical Sciences Vol. 44, Springer-Verlag, New York, 1983.
[27] S.A. Prahl, M. Keijzer, S.J. Jacques, and A.J. Welch, *A Monte Carlo model of light propagation in tissue, Dosimetry of Laser Radiation in Medicine and Biology*, SPIE Optical Engineering Press, 1-5 (1989), 102–111.
[28] G. Strang, "Introduction to Linear Algebra," Wellesley Cambridge, 1993.
[29] V. Vogelsang, *On the strong unique continuation principle for inequalities of Maxwell type*, Math. Ann., 289 (1991), 285–295.
[30] M.P. Wolorowicz and G.A. Sobolev, "Piezoelectric Method of Geophysical Prospecting of Quartz," Izd. Nauka, Moscou, 1969 (in Russian).