Differential and Integral Equations Volume 22, Numbers 5–6 (2009), 519–542

MAXIMAL REGULARITY FOR EVOLUTION PROBLEMS ON THE LINE

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Abstract. Let A be a hyperbolic bisectorial operator on a Banach space. In this paper we study the optimal regularity of the solutions of the abstract first-order evolution equation $u'(t) = Au(t) + f(t)$ on the whole line, depending on the regularity of the inhomogeneity f.

1. Introduction

Let X be a Banach space and let $A : D(A) \subset X \to X$ be a closed, linear operator. We say that A is bisectorial if

$$\sigma(A) \subseteq S_\eta \doteq \{z \in \mathbb{C} : |\arg(\pm z)| \le \eta\} \cup \{0\},$$
$$\|R(z, A)\| \le c/|z|, \qquad z \in \mathbb{C} \setminus S_\eta,$$

and it is hyperbolic bisectorial if, moreover, $0 \in \rho(A)$. We are interested in optimal regularity results for the evolution problem

$$u'(t) = Au(t) + f(t), \qquad t \in \mathbb{R}, \tag{1.1}$$

when f belongs to various functional spaces. We say that a Banach space E of real variable, X-valued functions, enjoys the optimal regularity property (or the maximal regularity property) if, for every $f \in E$ problem (1.1) has a unique solution (classical or strong) u, and both u' and Au belong to E. The study of problem (1.1) has been done when $f \in C^\alpha_b(\mathbb{R}; X)$ and the optimal regularity is proved by Arendt and Bu in [2] as an application of the general theory on the sum of two commuting operators in Banach spaces, and also by Arendt, Batty and Bu in [1] with the technique of Fourier multipliers. In [2] Arendt and Bu also proved the optimal regularity in $L^p(\mathbb{R}; (X, D(A))_{\alpha,p})$, still applying the theory of the sum of two operators. Here we want to find again these results as well as new ones by using a direct approach based on a suitable characterization of the interpolation spaces $(X, D(A))_{\alpha,p}$ for each $\alpha \in (0, 1)$ and each $p \in [1, \infty]$.

Accepted for publication: December 2008.
AMS Subject Classifications: 34G10, 47D06.

Using this approach and the theory of bisectorial operators developed in [3] we are able to prove maximal regularity in $C(\mathbb{R};X)\cap B(\mathbb{R};D_A(\alpha,\infty))$, not previously considered, and where $B(\mathbb{R};E)$ denotes the space of bounded functions in $\mathbb{R}$ with values in E.

The plan of the paper is as follows. In section 2 we give the definitions and the main properties of bisectorial operators and their associated semigroups $(T^{\pm}(t))_{t>0}$, which are needed in our treatment. In section 3 we characterize the interpolation spaces $(X,D(A))_{\alpha,p}$ with $\alpha\in(0,1)$ and $p\in[1,\infty]$ in terms of the semigroups $(T^{\pm}(t))_{t>0}$. In Section 4 we introduce various kinds of solutions for problem (1.1) and we recall the representation of the solutions in terms of the norms of the semigroups $(T^{\pm}(t))_{t>0}$, which has been introduced in [7] and [3], studying some of its properties. Finally, in sections 5-6 we prove the optimal regularity for the evolution problem (1.1) in various Banach spaces by the direct approach.

The contents of this paper are part of the Ph.D thesis of the author.

2. Notation and preliminary results

Let X be a Banach space and let A be a closed linear operator on X with domain $D(A)\subset X$. We said that A is bisectorial if there exist $\eta\in(0,\pi/2)$ and $c>0$ such that

(i) $\sigma(A)\subseteq S_\eta \doteq \{z\in\mathbb{C} : |\arg(\pm z)|\le\eta\}\cup\{0\}$ and
(ii) $\|zR(z,A)\|\le c, \qquad z\in\mathbb{C}\setminus S_\eta$.

The angle $\theta_A := \inf\{\eta\in[0,\frac{\pi}{2}) : (i) \text{ and } (ii) \text{ hold}\}$ is called the spectral angle of A. Moreover we say that A is hyperbolic if $0\in\rho(A)$. In this case the spectrum of A can be seen as $\sigma(A)=\sigma^-(A)\cup\sigma^+(A)$ where $\sigma^-(A)$ and $\sigma^+(A)$ are the intersection of the spectrum of A with $\{z\in\mathbb{C} : \mathrm{Re}(z)<0\}$ and $\{z\in\mathbb{C} : \mathrm{Re}(z)>0\}$ respectively.

Moreover, let us define the positive and the negative spectral bound of A by $\omega_+ = \inf\{\mathrm{Re}(\lambda) : \lambda\in\sigma^+(A)\}$, $\omega_- = -\sup\{\mathrm{Re}(\lambda) : \lambda\in\sigma^-(A)\}$, and let us define the spectral bound of A by $\overline{\omega}_A=\min(\omega_-,\omega_+)$. Using the curves $\Gamma^{\pm}_{\theta,r}$ given by

$$\Gamma^-_{\theta,r}\doteq\begin{cases} tre^{i\theta} & t\le -1,\\ -re^{-i\theta t} & -1<t<1,\\ -tre^{-i\theta} & t\ge 1,\end{cases}\qquad \Gamma^+_{\theta,r}\doteq\begin{cases} -tre^{i\theta} & t\le -1,\\ re^{-i\theta t} & -1<t<1,\\ tre^{-i\theta} & t\ge 1,\end{cases}$$

where $\theta > \overline{\theta}_A$ and $r > 0$ is such that $B(0,r) = \{\lambda \in \mathbb{C} : |\lambda| \leq r\} \subseteq \rho(A)$, we define the following families of operators:

$$T^-(t) \doteq \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,r}} e^{\lambda t}R(\lambda,A)d\lambda, \quad T^+(t) \doteq \frac{1}{2\pi i}\int_{\Gamma^+_{\theta,r}} e^{-\lambda t}R(\lambda,A)d\lambda, \quad (2.1)$$

for $t > 0$, which are analytic semigroups on X.

In the sequel we shall use some properties of the semigroups $(T^\pm(t))_{t>0}$ which are proved in [3] and [7], and which we recall in the following proposition.

Proposition 2.1. *Let $A : D(A) \subset X \to X$ be a hyperbolic bisectorial operator and let $(T^\pm(t))_{t>0}$ be the associated semigroups defined by* (2.1). *Then the following properties hold.*

(i) $T^\pm(t)T^\mp(s) = 0, \qquad s,t > 0.$

(ii) *There exist $M^\pm_{0,0} > 0$ such that, for each $t > 0$ and for each $x \in X$, we have*

$$\|T^\pm(t)x\| \leq M^\pm_{0,0}(1 + |\log(t)|)\,\|x\|. \quad (2.2)$$

(iii) *The functions $t \mapsto T^\pm(t)$ belong to $C^\infty((0,+\infty);\mathcal{L}(X))$ and we have for $t > 0$ and for each $k \in \mathbb{N}$*

$$\frac{d^k}{dt^k}T^\pm(t) = (\mp)^k AT^\pm(t). \quad (2.3)$$

(iv) *Let $\omega \in (0,\overline{\omega})$. For every $k \in \mathbb{N}\cup\{0\}$ there exist $M^\pm_{k,\infty} = M^\pm_{k,\infty}(\omega) > 0$ such that, for $t \geq 1$ and for each $x \in X$, we have*

$$\|A^kT^\pm(t)x\| \leq M^\pm_{k,\infty}e^{-\omega t}\|x\|. \quad (2.4)$$

(v) *There exists $M_0 > 0$ such that, for each $x \in X$ and for each $t > 0$,*

$$\|(T^+ + T^-)(t)x\| \leq M_0\|x\|. \quad (2.5)$$

Moreover, the following estimates hold.

Proposition 2.2. *Under the assumptions of Proposition* 2.1, *for every $k \in \mathbb{N}$ there exist $M^\pm_{k,0} > 0$ such that, for each $t > 0$ and for each $x \in X$ we have*

$$t^k\|A^kT^\pm(t)x\| \leq M^\pm_{k,0}\|x\|. \quad (2.6)$$

Proof. We have

$$\begin{aligned} AT^-(t) &= \frac{1}{2\pi i}\int_{\Gamma^-_{\theta,r}} e^{\lambda t}AR(\lambda,A)d\lambda \\ &= \frac{1}{2\pi i}\Big(\int_r^{+\infty} e^{-\rho(\cos(\theta)+i\sin(\theta))t}e^{i\theta}AR(-\rho e^{i\theta},A)d\rho \end{aligned}$$

$$- \int_r^{+\infty} e^{-\rho(\cos(\theta)-i\sin(\theta))t} e^{i\theta} AR(\rho e^{-i\theta}, A) d\rho$$
$$+ \int_{-\theta}^{\theta} ire^{-r(\cos(\alpha)-i\sin(\alpha))t} e^{-i\alpha} AR(-re^{-i\alpha}, A) d\alpha\Big).$$

Then

$$\|AT^-(t)\|_{\mathcal{L}(X)} \leq \frac{1+c}{\pi} \int_r^{+\infty} e^{-\rho t\cos(\theta)} d\rho + \frac{(1+c)r}{\pi} \int_{-\theta}^{\theta} e^{-r\cos(\alpha)} d\alpha.$$

Therefore, letting $r \to 0$ we get

$$\|AT^-(t)\|_{\mathcal{L}(X)} \leq \frac{1+c}{\pi t\cos(\theta)} \doteq \frac{M_{1,0}^-}{t}.$$

The proof for $AT^+(t)$ is similar. Let now $k \geq 2$. Using the semigroup property and (i) of Proposition 2.1 we have

$$A^k T^{\pm}(t) = A^k \underbrace{T^{\pm}(\frac{t}{k}) \cdot T^{\pm}(\frac{t}{k}) \cdots T^{\pm}(\frac{t}{k})}_{\text{k times}} = \underbrace{AT^{\pm}(\frac{t}{k}) \cdot AT^{\pm}(\frac{t}{k}) \cdots AT^{\pm}(\frac{t}{k})}_{\text{k times}}, \tag{2.7}$$

so that

$$\|A^k T^{\pm}(t)\|_{\mathcal{L}(X)} \leq \frac{(M_{1,0}^{\pm})^k}{(t/k)^k} = \frac{(kM_{1,0}^{\pm})^k}{t^k} \doteq \frac{M_{k,0}^{\pm}}{t^k}. \qquad \square$$

Remark 2.3. Let $\omega \in (0, \overline{\omega}_A)$. Using (2.2), (2.4) and (2.6), for each $k \in \mathbb{N}$, there exist $M_k^{\pm}$ such that, for each $t > 0$ and for each $x \in X$, we have

$$\|A^k T^{\pm}(t)x\| \leq M_k^{\pm} \frac{e^{-\omega t}}{t^k} \|x\|.$$

Moreover, there exist $M_0^{\pm} > 0$ such that, for each $t > 0$ and for each $x \in X$ we have

$$\|T^{\pm}(t)x\| \leq M_0^{\pm}(1 + |\log(t)|)e^{-\omega t}\|x\|.$$

3. Intermediate spaces between X and $D(A)$

The characterization of the real interpolation spaces between X and $D(A)$ using the semigroups generated by A is crucial in order to give a direct proof of the optimal regularity for the problem (1.1). The same results, when A is a sectorial operator, can be found, for example, in [8, Section 1.14.5]. In this section we do the same for bisectorial operators. We use a representation formula given by the following proposition.

Proposition 3.1. *Let $A : D(A) \subset X \to X$ be a hyperbolic bisectorial operator. For every $x \in X$,*

$$\int_0^{+\infty} (T^+(\sigma) - T^-(\sigma))x d\sigma \in D(A)$$

and

$$A\Big(\int_0^{+\infty} \Big(T^+(\sigma) - T^-(\sigma)\Big) x d\sigma\Big) = x.$$

Proof. Fix $\varepsilon > 0$ and $\lambda \in \rho(A)$. Then

$$\begin{aligned} &\int_\varepsilon^{+\infty} T^-(\sigma)xd\sigma - \int_\varepsilon^{+\infty} T^+(\sigma)xd\sigma \\ &= \int_\varepsilon^{+\infty} (\lambda I - A)R(\lambda, A)T^-(\sigma)xd\sigma - \int_\varepsilon^{+\infty} (\lambda I - A)R(\lambda, A)T^+(\sigma)xd\sigma \\ &= \lambda \int_\varepsilon^{+\infty} R(\lambda, A)T^-(\sigma)xd\sigma + T^-(\varepsilon)R(\lambda, A)x \\ &- \lambda \int_\varepsilon^{+\infty} R(\lambda, A)T^+(\sigma)xd\sigma + T^+(\varepsilon)R(\lambda, A)x. \end{aligned}$$

Let $P^\pm$ be the projections defined in [3, Section 3]. We recall that

$$P^\pm x = \frac{1}{2\pi i} A \int_{\Gamma^\pm_{\theta,r}} R(\lambda, A)x \frac{d\lambda}{\lambda}, \qquad x \in D(P^\pm),$$

where $\Gamma^\pm_{\theta,r}$ are defined as in the previous section and

$$D(P^\pm) := \Big\{x \in X : \int_{\Gamma^\pm_{\theta,r}} R(\lambda, A)\frac{d\lambda}{\lambda} \in D(A)\Big\}.$$

Since $R(\lambda, A)x \in D(A)$ it follows that $\lim_{\varepsilon\to 0} T^\mp(\varepsilon)R(\lambda, A)x = P^\mp R(\lambda, A)x$ (see [3, Proposition 4.3(a)]), and letting $\varepsilon \to 0$ we get

$$\begin{aligned} \lambda \int_0^{+\infty} T^-(\sigma)xd\sigma - \int_0^{+\infty} T^+(\sigma)xd\sigma &= \lambda \int_0^{+\infty} R(\lambda, A)T^-(\sigma)xd\sigma \\ - \lambda \int_0^{+\infty} R(\lambda, A)T^+(\sigma)xd\sigma &+ (P^+ + P^-)(R(\lambda, A)x). \end{aligned}$$

Then we have

$$\begin{aligned} &\int_0^{+\infty} T^-(\sigma)xd\sigma - \int_0^{+\infty} T^+(\sigma)xd\sigma \\ = \quad &\lambda R(\lambda, A) \int_0^{+\infty} T^-(\sigma)xd\sigma - \lambda R(\lambda, A) \int_0^{+\infty} T^+(\sigma)xd\sigma + R(\lambda, A)x. \end{aligned}$$

Therefore,

$$\int_0^{+\infty} T^-(\sigma)xd\sigma - \int_0^{+\infty} T^+(\sigma)d\sigma \in D(A)$$

and we have

$$\begin{aligned}&(\lambda I - A)\Big(\int_0^{\infty} T^-(\sigma)xd\sigma - \int_0^{+\infty} T^+(\sigma)xd\sigma\Big)\\ &= (\lambda I - A)\lambda R(\lambda, A)\int_0^{+\infty} T^-(\sigma)xd\sigma\\ &- (\lambda I - A)\lambda R(\lambda, A)\int_0^{+\infty} T^+(\sigma)xd\sigma + (\lambda I - A)R(\lambda, A)x\end{aligned}$$

which implies

$$A\Big(\int_0^{+\infty} T^+(\sigma)xd\sigma - \int_0^{+\infty} T^-(\sigma)xd\sigma\Big) = x. \qquad \square$$

We shall use the following characterization of the real interpolation spaces $(X, D(A))_{\alpha,p}$ for each $\alpha \in (0,1)$ and for each $p \in [1,\infty]$ through the resolvent. This is a consequence of the fact that the halfline $\{\rho e^{i\theta} : \rho \in (0,+\infty)\}$, where $\theta \in (\overline{\theta}_A, \pi - \overline{\theta}_A)$, is a ray of minimal growth for A; i.e., it is contained in $\rho(A)$ and $\sup\{\|\rho R(\rho e^{i\theta}, A)\| : \rho \in (0,+\infty)\} < \infty$.

Proposition 3.2. *Let $A : D(A) \subset X \to X$ be a hyperbolic bisectorial operator, let $\theta \in (\overline{\theta}_A, \pi - \overline{\theta}_A)$, $\alpha \in (0,1)$ and $p \in [1,+\infty]$. Then*

$$(X, D(A))_{\alpha,p} = \{x \in X : \rho \mapsto \varphi(\rho) = \rho^{\alpha-1/p}\|AR(\rho e^{i\theta})x\| \in L^p(0,+\infty)\}$$

and the norms $\|x\|_{\alpha.p}$ and $\|x\|^*_{\alpha,p} = \|x\| + \|\varphi\|_{L^p(0,+\infty)}$ are equivalent.

Let us introduce the following spaces.

Definition 3.3. *Let $A : D(A) \subset X \to X$ be a hyperbolic bisectorial operator. For every $\alpha \in (0,1)$ and for every $p \in [1,\infty]$ we define the space*

$$D_A(\alpha,p) \doteq \{x \in X : t \mapsto \varphi^\pm(t) \doteq t^{1-\alpha}\|AT^\pm(t)x\| \in L^p_\star(0,+\infty)\},$$

endowed with the norm

$$\|x\| + \|\varphi^+\|_{L^p_\star(0,+\infty)} + \|\varphi^-\|_{L^p_\star(0,+\infty)}.$$

In the following proposition we prove a characterization of the real interpolation spaces $(X, D(A))_{\alpha,p}$, for each $\alpha \in (0,1)$ and each $p \in [1,\infty]$, in terms of the norms of the semigroups $(T^\pm)_{t>0}$ associated with A.

Proposition 3.4. *Let $A : D(A) \subset X \to X$ be a hyperbolic bisectorial operator. Then, for every $\alpha \in (0,1)$ and for every $p \in [1,\infty]$, we have*

$$\begin{aligned}(X,D(A))_{\alpha,p} &= \{x \in X : t \mapsto \varphi^{\pm}(t) \doteq t^{1-\alpha}\|AT^{\pm}(t)x\| \in L^p_*(0,+\infty)\}\\ &= D_A(\alpha,p)\end{aligned}$$

and the norm $x \mapsto \|x\| + \|\varphi^+\|_{L^p_*(0,+\infty)} + \|\varphi^-\|_{L^p_*(0,+\infty)}$ is equivalent to the norm of $(X,D(A))_{\alpha,p}$.

Proof. Let $p=\infty$ and let $x \in (X,D(A))_{\alpha,\infty}$. We have

$$\sup_{t\in(0,+\infty)} t^{1-\alpha}\|AT^{\pm}(t)x\| \le \sup_{t\in(0,1)} t^{1-\alpha}\|AT^{\pm}(t)x\| + \sup_{t\in[1,+\infty)} t^{1-\alpha}\|AT^{\pm}(t)x\|$$

and, since $\sup_{t\in[1,+\infty)} t^{1-\alpha}\|AT^{\pm}(t)x\| \le M^{\pm}_{1,\infty}e^{-\omega}\|x\|$ for a suitable $\omega \in (0,\overline{\omega}_A)$, the second term is finite. Let us show that the first term is also finite. By Proposition 3.2, there exists $C>0$ such that $|\lambda|^{\alpha}\|AR(\lambda,A)x\| \le C\|x\|_{\alpha,\infty}$ for every $\lambda \in \Gamma^{\pm}_{\theta,r}$. Then we have

$$\begin{aligned}&\sup_{t\in(0,1)} t^{1-\alpha}\|AT^{\pm}(t)x\| = \sup_{t\in(0,1)} t^{1-\alpha}\left\|\frac{1}{2\pi i}\int_{\Gamma^{\pm}_{\theta,r}} e^{\mp\lambda t}AR(\lambda,A)xd\lambda\right\|\\
&\le C\|x\|_{\alpha,\infty}\sup_{t\in(0,1)}\Big(2\int_1^{+\infty}\frac{rte^{-\rho rt\cos(\theta)}}{(\rho rt)^{\alpha}}d\rho + (rt)^{1-\alpha}\int_{-1}^{1} e^{rt\cos(\rho\theta)}d\rho\Big)\\
&= C\|x\|_{\alpha,\infty}\sup_{t\in(0,1)}\Big(2\int_{rt}^{+\infty}\frac{e^{-u\cos(\theta)}}{u^{\alpha}}du + (rt)^{1-\alpha}\int_{-1}^{1} e^{rt\cos(\rho\theta)}d\rho\Big)\\
&\le C\|x\|_{\alpha,\infty}\Big(2\int_0^{+\infty}\frac{e^{-u\cos(\theta)}}{u^{\alpha}}du + r^{1-\alpha}\int_{-1}^{1} e^{-r\cos(\rho\theta)d\rho}\Big),\end{aligned}$$

and thus $x \in D_A(\alpha,\infty)$. Now let $x \in D_A(\alpha,\infty)$ and let $t>0$ be fixed. Then

$$\begin{aligned}x &= A\int_0^{+\infty}(T^+(s)-T^-(s))xds\\ &= A\int_0^{t}(T^+(s)-T^-(s))xds + A\int_t^{+\infty}(T^+(s)-T^-(s))xds\end{aligned}$$

and let us define

$$a_t \doteq A\int_0^{t}(T^+(s)-T^-(s))xds, \qquad b_t \doteq A\int_t^{+\infty}(T^+(s)-T^-(s))xds.$$

Then we have

$$\|a_t\| = \left\|A\int_0^{t}(T^+(s)-T^-(s))xds\right\| \le \int_0^t \|AT^+(s)x\|ds + \int_0^t \|AT^-(s)x\|ds$$

$$\leq \int_0^t \frac{1}{s^{1-\alpha}} \sup_{t\in(0,+\infty)} \|s^{1-\alpha}AT^+(s)x\|ds$$
$$+ \int_0^t \frac{1}{s^{1-\alpha}} \sup_{t\in(0,+\infty)} \|s^{1-\alpha}AT^-(s)x\|ds$$
$$\leq \|\varphi^+\|_\infty \int_0^t \frac{ds}{s^{1-\alpha}} + \|\varphi^-\|_\infty \int_0^t \frac{ds}{s^{1-\alpha}} = \frac{\|\varphi^+\|_\infty}{\alpha}t^\alpha + \frac{\|\varphi^-\|_\infty}{\alpha}t^\alpha$$

so that

$$\sup_{t\in(0,+\infty)} t^{-\alpha}\|a_t\| \leq \frac{\|\varphi^+\|_\infty + \|\varphi^-\|_\infty}{\alpha}.$$

Moreover, we have

$$\sup_{t\in(0,+\infty)} t^{1-\alpha}\|b_t\| \leq \sup_{t\in(0,+\infty)} \Big(\int_t^{+\infty} \|AT^+(s)x\|ds + \int_t^{+\infty} \|AT^-(s)x\|ds\Big).$$

Using the fact that, in the integrals above, $s \geq t$, so that $t^{1-\alpha} \leq s^{1-\alpha}$, we have

$$\sup_{t\in(0,+\infty)} t^{1-\alpha}\Big(\int_t^{+\infty} \|AT^+(s)x\|ds + \int_t^{+\infty} \|AT^-(s)x\|ds\Big)$$
$$\leq \int_0^{+\infty} s^{1-\alpha}\|AT^+(s)x\|ds + \int_0^{+\infty} s^{1-\alpha}\|AT^-(s)x\|ds$$
$$\leq M_1^+\|x\| \int_0^{+\infty} s^{-\alpha}e^{-\omega s}ds + M_1^-\|x\| \int_0^{+\infty} s^{-\alpha}e^{-\omega s}ds \leq C\|x\|,$$

where

$$C \doteq (M_1^+ + M_1^-) \int_0^{+\infty} s^{-\alpha}e^{-\omega s}ds.$$

Using the fact that $A^2T^\pm(s)x = AT^\pm(s/2)AT^\pm(s/2)x$ for each $s > 0$ one can deduce that

$$\|A^2T^\pm(s)x\| \leq 2^{2-\alpha}\|\varphi^\pm\|_\infty s^{\alpha-2}$$

and that $b_t \in D(A)$. Moreover, we have

$$\sup_{t\in(0,+\infty)} t^{1-\alpha} \int_t^{+\infty} \|A^2T^\pm(s)x\|ds$$
$$\leq 2^{2-\alpha}\|\varphi^\pm\|_\infty \sup_{t\in(0,+\infty)} t^{1-\alpha} \int_t^{+\infty} s^{\alpha-2}ds = \frac{2^{2-\alpha}\|\varphi^\pm\|_\infty}{1-\alpha}.$$

This means that

$$\sup_{t\in(0,+\infty)} t^{1-\alpha}\|Ab_t\| \le \frac{2^{2-\alpha}}{1-\alpha}(\|\varphi^+\|_\infty + \|\varphi^-\|_\infty).$$

So we have proved that

$$\sup_{t\in(0,+\infty)} t^{-\alpha}(\|a_t\| + t\|b_t\|_{D(A)}) \le C\|x\| + \Big(\frac{1}{\alpha} + \frac{2^{2-\alpha}}{1-\alpha}\Big)(\|\varphi^+\|_\infty + \|\varphi^-\|_\infty);$$

that is, $x \in (X, D(A))_{\alpha,\infty}$ and the norms are equivalent.

Now let $p < \infty$ and let $x \in (X, D(A))_{\alpha,p}$. Then, by Proposition 3.2, for every $\theta \in [\overline{\theta}_A, \pi/2)$, we have that

$$\begin{aligned} \rho &\mapsto \rho^\alpha\|AR(\rho e^{-i\theta}, A)x\| \in L^p_*(0,+\infty) \quad \text{and} \\ \rho &\mapsto \rho^\alpha\|AR(\rho e^{i\theta}, A)x\| \in L^p_*(0,+\infty). \end{aligned}$$

Then we have

$$\begin{aligned} &t^{1-\alpha}\|AT^+(t)x\| = t^{1-\alpha}\Big\|\frac{1}{2\pi i}\int_{\Gamma^+_{\theta,r}} e^{-\lambda t}AR(\lambda, A)x\,d\lambda\Big\| \\ &\le \frac{t^{1-\alpha}}{2\pi}\int_r^{+\infty} e^{-\rho t}\|AR(\rho e^{i\theta}, A)x\|d\rho + \frac{t^{1-\alpha}}{2\pi}\int_r^{+\infty} e^{-\rho t}\|AR(\rho e^{-i\theta}, A)x\|d\rho \\ &+ \frac{t^{1-\alpha}}{2\pi}\int_{-\theta}^{\theta} re^{-rt\cos(\rho)}\|AR(re^{-i\rho}, A)x\|d\rho \\ &\le \frac{1}{2\pi}\int_0^{+\infty}(\rho t)^{1-\alpha}e^{-\rho t}\rho^\alpha\|AR(\rho e^{i\theta}, A)x\|\rho^{-1}d\rho \\ &+ \frac{1}{2\pi}\int_0^{+\infty}(\rho t)^{1-\alpha}e^{-\rho t}\rho^\alpha\|AR(\rho e^{i\theta}, A)x\|\rho^{-1}d\rho \\ &+ \frac{1}{2\pi}\int_{-\theta}^{\theta}(rt)^{1-\alpha}e^{-rt\cos(\rho)}r^\alpha\|AR(re^{-i\rho}, A)x\|d\rho. \end{aligned} \tag{3.1}$$

Defining $f, g : \mathbb{R}^+ \to \mathbb{R}$ by

$$f(y) \doteq y^{1-\alpha}e^{-y} \quad \text{and} \quad g(y) \doteq y^\alpha\|AR(ye^{i\theta}, A)x\|$$

the first term of (3.1) is equal to

$$\frac{1}{2\pi}\int_0^{+\infty} f(t\rho)g(\rho)\frac{d\rho}{\rho},$$

where $f \in L^1_*(0,+\infty)$ and $g \in L^p_*(0,+\infty)$. Moreover,

$$\frac{1}{2\pi}f(t\rho)g(\rho)\frac{d\rho}{\rho} = \frac{1}{2\pi}\int_0^{+\infty} f\Big(\frac{t}{\tau}\Big)\tilde{g}(\tau)\frac{d\tau}{\tau} = \frac{1}{2\pi}(f \star \tilde{g})(t),$$

where $\star$ denotes the multiplicative convolution, $\tilde{g} : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$\tilde{g}(\tau) = \tau^{-\alpha}\|AR(\tau^{-1}e^{i\theta}, A)x\|$$

and belongs to $L^p_*(0,+\infty)$ with the same norm of g. Thus $(f\star\tilde{g}) \in L^p_*(0,+\infty)$ and its norm does not exceed $\|f\|_{L^1_*(0,+\infty)}\|g\|_{L^p_*(0,+\infty)}$. The same is true for the second term. In order to show that also the third term of (3.1) belongs to $L^p_*(0,+\infty)$ let us notice that, using Proposition 3.2, we have that for every $x \in (X, D(A))_{\alpha,p}$ there exists $C > 0$ such that

$$|\lambda|^\alpha \|AR(\lambda, A)x\| \le C\|x\|_{\alpha,p}, \qquad \lambda \in \Gamma^+_{\theta,r}.$$

Thus the third term is less than or equal to

$$\frac{C\|x\|_{\alpha,p}}{2\pi}\int_{-\theta}^{\theta}(rt)^{1-\alpha}e^{-rt\cos(\rho)}d\rho.$$

Then we have

$$\begin{aligned}
&\frac{C\|x\|_{\alpha,p}}{2\pi}\int_0^{+\infty}\Big(\int_{-\theta}^{\theta}(rt)^{1-\alpha}e^{-rt\cos(\rho)}d\rho\Big)^p\frac{dt}{t}\\
&\le \frac{\theta C\|x\|_{\alpha,p}}{2\pi}\int_0^{+\infty}(rt)^{p(1-\alpha)}e^{-rtp\cos(\theta)}\frac{dt}{t}\\
&= \frac{\theta C\|x\|_{\alpha,p}}{2\pi}\int_0^{+\infty}u^{p(1-\alpha)}e^{-up\cos(\theta)}du \le K\|x\|_{\alpha,p}
\end{aligned}$$

where

$$K \doteq \frac{\theta C}{2\pi}\int_0^{+\infty}u^{p(1-\alpha)}e^{-up\cos(\theta)}du.$$

In the same way, one can prove that $t^{1-\alpha}\|AT^-(t)x\| \in L^p_*(0,+\infty)$, and thus $x \in D_A(\alpha,p)$.

Conversely, let $x \in D_A(\alpha,p)$ and let $t > 0$ be fixed. Let us define, as before,

$$\begin{aligned}
a_t &\doteq A\int_0^t (T^+(s) - T^-(s))x\,ds, \qquad \text{and}\\
b_t &\doteq A\int_t^{+\infty}(T^+(s) - T^-(s))x\,ds.
\end{aligned}$$

Then we have

$$\begin{aligned}
\|a_t\| &= \Big\|A\int_0^t (T^+(s) - T^-(s))x\,ds\Big\|\\
&\le \int_0^t \|AT^+(s)x\|ds + \int_0^t \|AT^-(s)x\|ds.
\end{aligned}$$

By the assumptions, the functions $\psi^{\pm}(t) \doteq \|AT^{\pm}(t)x\|$ are such that $t \mapsto \psi_\alpha(t) \doteq t^{1-\alpha-1/p}\psi^{\pm}(t)$ belongs to $L^p(0,+\infty)$ and then the same property holds for the mean values $t \mapsto \zeta^{\pm} \doteq t^{-1}\int_0^t \psi^{\pm}(s)ds$ and, setting $\zeta^{\pm}_\alpha \doteq t^{\alpha-1/p}\zeta^{\pm}(t)$, we have

$$\|\zeta^{\pm}_\alpha\|_{L^p(0,+\infty)} \leq \frac{1}{1-\alpha}\|\psi^{\pm}_\alpha\|_{L^p(0,+\infty)} = \frac{1}{1-\alpha}\|\varphi^{\pm}\|_{L^p_*(0,+\infty)}$$

(this is an easy consequence of the Hardy-Young inequalities [5, pages 245-246]). Therefore,

$$\begin{aligned}\int_0^{+\infty}(t^{\alpha-1/p}\|a_t\|)^p dt \quad &\leq \quad 2^{p-1}\Big(\int_0^{+\infty}(t^{1-\alpha-1/p}\frac{1}{t}\int_0^t \|AT^{+}(s)x\|ds)^p dt \\ &\qquad + \int_0^{+\infty}(t^{1-\alpha-1/p}\frac{1}{t}\int_0^t \|AT^{-}(s)x\|ds)^p dt\Big) \\ &\leq \quad \frac{2^{p-1}}{1-\alpha}(\|\varphi^{+}\|_{L^p_*(0,+\infty)} + \|\varphi^{-}\|_{L^p_*(0,+\infty)}).\end{aligned}$$

From the definition of b_t we have

$$t^{1-\alpha-1/p}\|b_t\| \leq t^{1-\alpha-1/p}\Big(\int_t^{+\infty}\|AT^{+}(s)x\|ds + \int_t^{+\infty}\|AT^{-}(s)x\|ds\Big).$$

Using one of the Hardy-Young inequalities we have that

$$\begin{aligned}&\int_0^{+\infty} t^{p(1-\alpha)}\Big(\int_t^{+\infty} s\|AT^{\pm}(s)x\|\frac{ds}{s}\Big)^p\frac{dt}{t} \\ \leq \quad &\frac{1}{1-\alpha}\int_0^{+\infty} t^{p(1-\alpha)}t^p\|AT^{\pm}(t)x\|^p\frac{dt}{t} \\ \leq \quad &\frac{M_1^{\pm}\|x\|}{1-\alpha}\int_0^{+\infty} t^{p(1-\alpha)}e^{-\omega pt}\frac{dt}{t} \leq C\|x\|,\end{aligned}$$

where

$$C \doteq \frac{M_1^{\pm}}{1-\alpha}\int_0^{+\infty} t^{p(1-\alpha)}e^{-\omega tp}\frac{dt}{t}.$$

Finally, using the fact that, for every $s > 0$,

$$A^2T^{\pm}(s)x = AT^{\pm}(s/2)AT^{\pm}(s/2)x,$$

the Hölder inequality and the estimates (2.4), we have, for $1 < p < \infty$,

$$\begin{aligned}&\int_t^{+\infty}\|A^2T^{\pm}(s)x\|ds \leq \int_t^{+\infty}\|AT^{\pm}(s/2)\|_{\mathcal{L}(X)}\|AT^{\pm}(s/2)x\|ds \\ &\leq M^{\pm}_{1,\infty}\int_t^{+\infty} e^{-\omega s/2}\|AT^{\pm}(s/2)x\|ds\end{aligned}$$

$$\leq M^{\pm}_{1,\infty} \|\varphi^{\pm}\|_{L^p_*(0,+\infty)} \Big(\int_t^{+\infty} e^{-\omega s/2} \Big(\frac{s}{2}\Big)^{\frac{1+\alpha p - p}{p-1}} ds \Big)^{\frac{p-1}{p}}.$$

For $p = 1$ we have

$$\begin{aligned} \int_t^{+\infty} \|AT^2 T^{\pm}(s)x\| ds &\leq \int_t^{+\infty} \|AT^{\pm}(s/2)\|_{\mathcal{L}(X)} \|AT^{\pm}(s/2)x\| ds \\ &\leq M^{\pm}_{1,\infty} \|\varphi^{\pm}\|_{L^1_*(0,+\infty)} \sup_{s\in(t,+\infty)} (s^{\alpha} e^{-\omega s/2}). \end{aligned}$$

Then $b_t \in D(A)$ and, using one of the Hardy-Young inequalities and estimates (2.6) we have

$$\begin{aligned} & \int_0^{+\infty} t^{p(1-\alpha)} \Big(\int_t^{+\infty} \|A^2 T^{\pm}(s)x\| ds \Big)^p \frac{dt}{t} \\ \leq\ & 2^p \int_0^{+\infty} t^{p(1-\alpha)} \Big(\int_t^{+\infty} \|AT^{\pm}(s/2)x\| \frac{ds}{s} \Big)^p \frac{dt}{t} \\ \leq\ & \frac{2^p}{(1-\alpha)^p} \int_0^{+\infty} s^{p(1-\alpha)} \|AT^{\pm}(s/2)x\| \frac{ds}{s} \leq \frac{2^{p(2-\alpha)-1}}{(1-\alpha)^p} \|x\|_{\alpha,p}. \end{aligned}$$

Thus $x \in (X, D(A))_{\alpha,p}$ and this concludes the proof. □

From the above characterization we can now give some estimates for the functions $t \mapsto A^k T^{\pm}(t)$, for each $k \in \mathbb{N}$, which will be used later.

Proposition 3.5. *Let $k \in \mathbb{N}$, $0 < \alpha < 1$ and $\omega \in (0, \overline{\omega})$. Then there exist $M^{\pm}_{k,0,\alpha} > 0$ and $M^{\pm}_{k,\infty,\alpha} > 0$ such that*

$$\begin{cases} (i) & \|t^{k-\alpha} A^k T^{\pm}(t)\|_{\mathcal{L}(D_A(\alpha,\infty);X)} \leq M^{\pm}_{k,0,\alpha}, \quad 0 < t \leq 1; \\ (ii) & \|t^k A^k T^{\pm}(t)\|_{\mathcal{L}(D_A(\alpha,\infty);X)} \leq M^{\pm}_{k,\infty,\alpha} e^{-\omega t}, \quad t \geq 1. \end{cases} \tag{3.2}$$

Proof. (i) If $k = 1$ the statement follows from the definition of $D_A(\alpha,\infty)$. Now let $k \geq 2$ and let $x \in D_A(\alpha,\infty)$. Then

$$\begin{aligned} \|t^k A^k T^{\pm}(t)x\| &\leq 2^{k-\alpha} t^{\alpha} \|(t/2)^{k-1} A^{k-1} T^{\pm}(t/2)\|_{\mathcal{L}(X)} \|(t/2)^{1-\alpha} AT^{\pm}(t/2)x\| \\ &\leq 2^{k-\alpha} t^{\alpha} M^{\pm}_{k,0} \|x\|_{\alpha,\infty}, \end{aligned}$$

where the constants $M^{\pm}_{k,0}$ are finite by Proposition (2.1)(iv).

Estimates (3.2)(ii) follow easily from (3.2)(i) and Proposition 2.1(iv): it is sufficient to recall that for $t \geq 1$

$$\begin{aligned} & \|t^k A^k T^{\pm}(t)\|_{\mathcal{L}(D_A(\alpha,\infty);X)} \leq \|t^k A^k T^{\pm}(t)\|_{\mathcal{L}(X)} \\ & \leq \|T^{\pm}(1/2)\|_{\mathcal{L}(X)} M^{\pm}_{k,\infty} \Big(\frac{t}{t-1/2}\Big)^k e^{-\omega(t-1/2)} \end{aligned}$$

$$\leq \|T^{\pm}(1/2)\|_{\mathcal{L}(X)} 2^k e^{\omega/2} M^{\pm}_{k,\infty} e^{-\omega t}. \qquad \square$$

Proposition 3.6. *Suppose $A : D(A) \subset X \to X$ is a hyperbolic bisectorial operator. Then $\sup_{t\in(0,+\infty)} \|T^{\pm}(t)\|_{\mathcal{L}(D_A(\alpha,p))} < \infty$ for each $\alpha \in (0,1)$ and each $p \in [1,\infty]$.*

Proof. Let us assume first that $p = \infty$. As a consequence of Proposition 3.2 we have that, if $x \in D_A(\alpha,\infty)$, then there exists $C > 0$ such that

$$\|AR(\lambda, A)x\| \leq C|\lambda|^{-\alpha}\|x\|_{\alpha,\infty}, \qquad \lambda \in \Gamma^-_{\theta,r} \cup \Gamma^+_{\theta,r}$$

for every choice of $\theta \in [\bar{\theta}_A, \pi/2)$ and $r > 0$.

Since $D_A(\alpha,\infty) = D_{iA}(\alpha,\infty)$, in order to prove the statement one has to estimate $\tau^\alpha AR(-\tau, iA)T^+(t)x$.

By the resolvent identity and Cauchy's theorem we have

$$\tau^\alpha AR(-\tau, iA)T^+(t)x = \frac{1}{2\pi i}\int_{\Gamma^+_{\theta,r}} \frac{e^{-\lambda t}\tau^\alpha}{\tau + i\lambda} AR(\lambda, A)x d\lambda.$$

Then it follows that

$$\|\tau^\alpha AR(-\tau, iA)T^+(t)x\| \leq \frac{C\|x\|_{\alpha,\infty}}{2\pi}\int_{\Gamma^+_{\theta,r}} \frac{e^{-\lambda t}\tau^\alpha}{|\tau + i\lambda|}\frac{d\lambda}{|\lambda|^\alpha},$$

and the above integral is uniformly bounded in the pair (τ, t) and this shows that $\sup_{t\in(0,+\infty)} \|T^+(t)\|_{\mathcal{L}(D_A(\alpha,\infty))}$ is finite. In the same way one can show that $\sup_{t\in(0,+\infty)} \|T^-(t)\|_{\mathcal{L}(D_A(\alpha,\infty))}$ is also finite.

Now let $p < \infty$. By the reiteration theorem (see [8, Theorem 1.10.2]) we know that, for each $\varepsilon \in (0,\alpha)$, we have that

$$D_A(\alpha, p) = (D_A(\alpha - \varepsilon, \infty), D_A(\alpha + \varepsilon, \infty))_{1/2,p}.$$

Then the statement follows also for $p \in [1,\infty)$ by interpolation. $\square$

4. Solutions of first-order abstract equations

Let $A : D(A) \subset X \to X$ be a hyperbolic bisectorial operator and let $(T^{\pm}(t))_{t>0}$ be the associated semigroups defined by (2.1). Given $f \in E$ we shall be concerned with the evolution problem

$$u'(t) = Au(t) + f(t), \qquad t \in \mathbb{R}, \tag{4.1}$$

where $E = C_b(\mathbb{R}; X)$ or $E = L^p(\mathbb{R}; X)$ for $1 \leq p < +\infty$.

If $E = C_b(\mathbb{R}, X)$, a function u is called a classical solution of (1.1) if $u \in C(\mathbb{R}; D(A)) \cap C^1(\mathbb{R}; X)$ and we have $u'(t) = Au(t) + f(t)$ for each $t \in \mathbb{R}$.

If $E = L^p(\mathbb{R}; X)$ for some $1 \le p < +\infty$, a function u is called a strong solution of (4.1) if $u \in L^p(\mathbb{R}; D(A)) \cap W^{1,p}(\mathbb{R}; X)$ and $u'(t) = Au(t) + f(t)$ for almost all $t \in \mathbb{R}$.

Moreover, we say that a continuous function $u : \mathbb{R} \to X$ is a mild solution of (4.1) if $\int_0^t u(s)ds \in D(A)$ and

$$u(t) = u(0) + A \int_0^t u(s)ds + \int_0^t f(s)ds, \tag{4.2}$$

for all $t \in \mathbb{R}$.

In [3, Section 6] one finds the following representation formula.

Theorem 4.1. *The function $u : \mathbb{R} \to X$ defined by*

$$u(t) = \int_{-\infty}^t T^-(t-s)f(s)ds - \int_t^{+\infty} T^+(s-t)f(s)ds, \qquad t \in \mathbb{R}, \tag{4.3}$$

is the unique mild solution of (4.1).

Now we show that if u is a classical solution or a strong solution of (4.1) then u is given by the representation formula (4.3).

Proposition 4.2.

(i) *Let $f \in C_b(\mathbb{R}; X)$. If u is a classical solution of* (4.1), *then u is given by* (4.3).

(ii) *Let $f \in L^p(\mathbb{R}; X)$ for some $1 < p < +\infty$. If u is a strong solution of* (4.1) *then u is given by* (4.3) *almost everywhere on $\mathbb{R}$.*

Proof. (i) Let $f \in C_b(\mathbb{R}; X)$ and let u be a classical solution of (4.1) in $\mathbb{R}$ and let $t \in \mathbb{R}$. Since $u \in C^1(\mathbb{R}; X) \cap C(\mathbb{R}; D(A))$, $u(t)$ belongs to $\overline{D(A)}$ for every $t \in \mathbb{R}$, so that the functions

$$\begin{aligned} v_-(s) &= T^-(t-s)u(s), \qquad s \le t, \\ v_+(s) &= T^+(s-t)u(s), \qquad s \ge t, \end{aligned}$$

belong to the spaces $C((-\infty, t); X) \cap C^1((-\infty, t); X)$ and $C((t, +\infty); X) \cap C^1((t, +\infty); X)$, respectively, and we have

$$\begin{aligned} v'_-(s) &= -AT^-(t-s)u(s) + T^-(t-s)u'(s) = T^-(t-s)f(s), \\ v'_+(s) &= -AT^+(s-t)u(s) + T^+(s-t)u'(s) = T^+(s-t)f(s). \end{aligned}$$

Let $\varepsilon > 0$. Then we have

$$\int_{-\infty}^{t-\varepsilon} v'_-(s)ds - \int_{t+\varepsilon}^{+\infty} v'_+(s)ds = \int_{-\infty}^{t-\varepsilon} T^-(t-s)f(s)ds - \int_{t+\varepsilon}^{+\infty} T^+(s-t)f(s)ds$$

while, on the other hand, we also have

$$\int_{-\infty}^{t-\varepsilon} v_-'(s)ds - \int_{t+\varepsilon}^{+\infty} v_+'(s)ds = T^-(\varepsilon)u(t-\varepsilon) + T^+(\varepsilon)u(t+\varepsilon).$$

Since we have

$$\lim_{\varepsilon\to 0} T^-(\varepsilon)u(t-\varepsilon) + T^+(\varepsilon)u(t+\varepsilon) = (P^- + P^+)u(t) = u(t),$$

and

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_{-\infty}^{t-\varepsilon} T^-(t-s)f(s)ds - \int_{t+\varepsilon}^{+\infty} T^+(s-t)f(s)ds \\ = &\int_{-\infty}^{t} T^-(t-s)f(s)ds - \int_{t}^{+\infty} T^+(s-t)f(s)ds, \end{aligned}$$

the statement follows.

(ii) Let $f \in L^p(\mathbb{R}; X)$ and let u be a strong solution of (4.1) in $\mathbb{R}$, and let $\varepsilon > 0$. Since $u \in C(\mathbb{R}; X) \cap L^p(\mathbb{R}; D(A)) \cap W^{1,p}(\mathbb{R}; X)$, the functions

$$\begin{aligned} v_-(s) &= T^-(t-s)u(s), \qquad s \le t, \\ v_+(s) &= T^+(s-t)u(s), \qquad s \ge t, \end{aligned}$$

belong to $C((-\infty,t); X) \cap L^p((-\infty, t-\varepsilon); D(A)) \cap W^{1,p}((-\infty, t-\varepsilon); X)$ and to $C((t,+\infty); X) \cap L^p((t+\varepsilon,+\infty); D(A)) \cap W^{1,p}((t+\varepsilon,+\infty); X)$ respectively, and

$$\begin{aligned} v_-'(s) &= T^-(t-s)f(s), \qquad s \le t \quad a.e. \\ v_+'(s) &= T^+(s-t)f(s), \qquad s \ge t \quad a.e. \end{aligned}$$

Then we have

$$\int_{-\infty}^{t-\varepsilon} v_-'(s)ds - \int_{t+\varepsilon}^{+\infty} v_+'(s)ds = \int_{-\infty}^{t-\varepsilon} T^-(t-s)f(s)ds - \int_{t+\varepsilon}^{+\infty} T^+(s-t)f(s)ds$$

as well as

$$\int_{-\infty}^{t-\varepsilon} v_-'(s)ds - \int_{t+\varepsilon}^{+\infty} v_+'(s)ds = T^-(\varepsilon)u(t-\varepsilon) + T^+(\varepsilon)u(t+\varepsilon).$$

Since $u \in C(\mathbb{R}; X) \cap L^p(\mathbb{R}; D(A))$ we have that

$$\lim_{\varepsilon\to 0} T^-(\varepsilon)u(t-\varepsilon) + T^+(\varepsilon)u(t+\varepsilon) = (P^+ + P^-)u(t) = u(t)$$

almost everywhere on $\mathbb{R}$. Moreover,

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{t-\varepsilon} T^-(t-s)f(s)ds - \int_{t+\varepsilon}^{+\infty} T^+(s-t)f(s)ds$$

$$= \int_{-\infty}^{t} T^{-}(t-s)f(s)ds - \int_{t}^{+\infty} T^{+}(s-t)f(s)ds,$$

and the statement follows. □

As in the sectorial case, the result of Theorem 4.1 is used to give sufficient conditions in order that the function (4.3) be classical or strong.

Lemma 4.3. *Let $f \in C_b(\mathbb{R}; X)$ and let u be the representation formula* (4.3). *The following conditions are equivalent.*

(a) $u \in C(\mathbb{R}; D(A))$,
(b) $u \in C^1(\mathbb{R}; X)$,
(c) *u is a classical solution of*(4.1).

Proof. Of course, (c) is stronger than (a) and (b). Let us show that if either (a) or (b) holds, then u is a classical solution. From estimates (2.2) and (2.4) it follows that

$$t \to \int_{-\infty}^{t} T^{-}(t-s)f(s)ds$$

and

$$t \to \int_{t}^{+\infty} T^{+}(s-t)f(s)ds$$

belong to $C(\mathbb{R}; X)$ so that u belongs to $C(\mathbb{R}; X)$. Moreover, by Theorem 4.1, u satisfies (4.2). Therefore, for every $t, h \in \mathbb{R}$ we have

$$\frac{u(t+h)-u(t)}{h} = \frac{1}{h} A \int_{t}^{t+h} u(s)ds + \frac{1}{h}\int_{t}^{t+h} f(s)ds. \tag{4.4}$$

Since f is continuous at t,

$$\lim_{h\to 0} \frac{1}{h}\int_{t}^{t+h} f(s)ds = f(t). \tag{4.5}$$

Let (a) hold. Then Au is continuous at t, so that

$$\lim_{h\to 0} \frac{1}{h} A \int_{t}^{t+h} u(s)ds = \lim_{h\to 0} \frac{1}{h}\int_{t}^{t+h} Au(s)ds = Au(t).$$

By (4.4) and (4.5) we get now that u is differentiable at the point t, with $u'(t) = Au(t) + f(t)$. Since both Au and f are continuous in $\mathbb{R}$ it folows that u' is continuous too, and u is a classical solution.

Now let (b) hold. Since u is continuous at t,

$$\lim_{h\to 0} \frac{1}{h}\int_{t}^{t+h} u(s)ds = u(t).$$

On the other hand, by (4.4) and (4.5), there exists the limit

$$\lim_{h\to 0} A\Big(\frac{1}{h}\int_t^{t+h} u(s)ds\Big) = u'(t) - f(t).$$

Since A is a closed operator, $u(t)$ belongs to $D(A)$, and $Au(t) = u'(t) - f(t)$. Since both u' and f are continuous in $\mathbb{R}$, Au is also continuous in $\mathbb{R}$, so that u is a classical solution of (4.1). □

Lemma 4.4. *Let $f \in L^p(\mathbb{R}; X)$ and let u be the representation formula* (4.3). *Then the following conditions are equivalent.*

(a) $u \in L^p(\mathbb{R}; D(A))$,
(b) $u \in W^{1,p}(\mathbb{R}; X)$,
(c) *u is a strong solution of* (4.1).

Proof. It is obvious that (c) is stronger than (a) and (b). Now let u be the function (4.3). Then $u \in C(\mathbb{R}; X)$ and, by Theorem 4.1, $\int_0^t u(s)ds \in D(A)$ for each $t \in \mathbb{R}$, and

$$u(t) = u(0) + A\int_0^t u(s)ds + \int_0^t f(s)ds, \qquad t \in \mathbb{R}.$$

$(b) \Rightarrow (c)$. If $u \in W^{1,p}(\mathbb{R}; X)$, then for $t \in \mathbb{R}$ and $h > 0$, we have

$$\lim_{h\to 0}\frac{1}{h}\int_t^{t+h} u(s)ds = u(t),$$

and, by definition of the integral solution,

$$\frac{1}{h}\int_t^{t+h} u(s)ds \in D(A).$$

From Theorem (4.1) and from the fact that $u \in W^{1,p}(\mathbb{R}; X)$ we deduce also the existence of

$$\lim_{h\to 0} A\frac{1}{h}\int_t^{t+h} u(s)ds = u'(t) - f(t), \qquad t \in \mathbb{R} \quad \text{a.e.}$$

Since A is closed we obtain $u(t) \in D(A)$ and

$$Au(t) = u'(t) - f(t), \qquad t \in \mathbb{R} \quad \text{a.e.}$$

so u is a strong solution of (4.1).

$(a) \Rightarrow (c)$. If $u \in L^p(\mathbb{R}; D(A))$ we get from Theorem (4.1) that

$$u(t) = u(0) + \int_0^t Au(s)ds + \int_0^t f(s)ds, \qquad t \in \mathbb{R}.$$

This implies again that u is a strong solution of (4.1). □

5. Time Regularity

Let u be the mild solution of (4.1) and set $u = u_1 + u_2$, where

$$u_1(t) = \int_{-\infty}^{t} T^-(t-s)(f(s)-f(t))ds - \int_t^{+\infty} T^+(s-t)(f(s)-f(t))ds,$$
$$u_2(t) = \int_{-\infty}^{t} T^-(t-s)f(t)ds - \int_t^{+\infty} T^+(s-t)f(t)ds.$$

Theorem 5.1. *Let $0 < \alpha < 1$ and let $f \in C_b^\alpha(\mathbb{R}; X)$. Then the function*

$$u(t) = \int_{-\infty}^{t} T^-(t-s)f(s)ds - \int_t^{+\infty} T^+(s-t)f(s)ds, \qquad t \in \mathbb{R}$$

has values in $D(A)$, it is differentiable with values in X, and it is the classical solution of

$$u'(t) = Au(t) + f(t), \qquad t \in \mathbb{R}.$$

Moreover, u' and Au belong to $C_b^\alpha(\mathbb{R}; X)$ and there exists $C > 0$ such that

$$\|u'\|_{C_b^\alpha(\mathbb{R};X)} + \|Au\|_{C_b^\alpha(\mathbb{R};X)} \le C\|f\|_{C_b^\alpha(\mathbb{R};X)}.$$

Proof. Let u_1 and u_2 be defined as above. Then $u_1(t), u_2(t) \in D(A)$ for each t and, by Proposition 3.1, $Au_2(t) = -f(t)$, so that $Au_2(t)$ is Hölder-continuous in $\mathbb{R}$. Then we need to prove that Au_1 is also Hölder-continuous in $\mathbb{R}$. We have

$$Au_1(t) = \int_{-\infty}^{t} AT^-(t-s)(f(s)-f(t))ds - \int_t^{+\infty} AT^+(s-t)(f(s)-f(t))ds.$$

Let $s \le t$. Then we get

$$\begin{aligned}
&Au_1(t) - Au_1(s)\\
&= \int_{-\infty}^{t} AT^-(t-\sigma)(f(\sigma)-f(t))d\sigma - \int_t^{+\infty} AT^+(\sigma-t)(f(\sigma)-f(t))d\sigma\\
&- \int_{-\infty}^{s} AT^-(s-\sigma)(f(\sigma)-f(s))d\sigma + \int_s^{+\infty} AT^+(\sigma-s)(f(\sigma)-f(s))d\sigma\\
&= \int_{-\infty}^{s} A(T^-(t-\sigma)-T^-(s-\sigma))(f(\sigma)-f(s))d\sigma\\
&- \int_t^{+\infty} A(T^+(\sigma-t)-T^+(\sigma-s))(f(\sigma)-f(t))d\sigma\\
&+ \int_s^{t} AT^-(t-\sigma)(f(\sigma)-f(t))d\sigma + \int_s^{t} AT^+(\sigma-s)(f(\sigma)-f(s))d\sigma
\end{aligned}$$

$$+\int_{t-s}^{+\infty} A(T^-(\sigma)-T^+(\sigma))(f(s)-f(t))d\sigma,$$

and, recalling that $\pm AT^{\mp}(\tau)=\frac{d}{d\tau}T^{\mp}(\tau)$ for each $\tau>0$, the last term is equal to

$$\int_{t-s}^{+\infty}\frac{d}{d\sigma}(T^-(\sigma)+T^+(\sigma))(f(s)-f(t))d\sigma=(T^-+T^+)(t-s)(f(s)-f(t)).$$

Therefore, we have

$$\begin{aligned}
&\|Au_1(t)-Au_1(s)\|\le M_{2,0}^-\int_{-\infty}^{s}(s-\sigma)^\alpha\int_{s-\sigma}^{t-\sigma}\tau^{-2}d\tau d\sigma[f]_{C^\alpha}\\
&+M_{1,0}^-\int_s^t(t-\sigma)^{\alpha-1}d\sigma[f]_{C^\alpha}+M_{2,0}^+\int_t^{+\infty}(\sigma-t)^\alpha\int_{\sigma-t}^{\sigma-s}\tau^{-2}d\tau d\sigma[f]_{C^\alpha}\\
&+M_{1,0}^+\int_s^t(\sigma-s)^{\alpha-1}d\sigma[f]_{C^\alpha}+M_0(t-s)^\alpha[f]_{C^\alpha}\\
&\le M_{2,0}^-\int_{-\infty}^{s}d\sigma\int_{s-\sigma}^{t-\sigma}\tau^{\alpha-2}d\tau[f]_{C^\alpha}+M_{2,0}^+\int_t^{+\infty}d\sigma\int_{\sigma-t}^{\sigma-s}\tau^{\alpha-2}d\tau[f]_{C^\alpha}\\
&+\Big(\frac{M_{1,0}^+}{\alpha}(t-s)^\alpha+\frac{M_{1,0}^-}{\alpha}(t-s)^\alpha\Big)[f]_{C^\alpha}+M_0(t-s)^\alpha[f]_{C^\alpha}\\
&\le\Big(\frac{M_{2,0}^-}{\alpha(1-\alpha)}+\frac{M_{2,0}^+}{\alpha(1-\alpha)}+\frac{M_{1,0}^+}{\alpha}+\frac{M_{1,0}^-}{\alpha}+M_0\Big)(t-s)^\alpha[f]_{C^\alpha},
\end{aligned}$$

where the constants $M_{k,0}^{\pm}$, with $k=1,2$, follow from Proposition 2.2 and the constant M_0 follows from Proposition 2.1(v). It follows that Au_1 is also Hölder-continuous in $\mathbb{R}$. Therefore, by Lemma 4.3 u is a classical solution of (4.1). □

6. Space Regularity

Theorem 6.1. *Let $0<\alpha<1$ and let $f\in C(\mathbb{R};X)\cap B(\mathbb{R};D_A(\alpha,\infty))$. Then the function*

$$u(t)=\int_{-\infty}^{t}T^-(t-s)f(s)ds-\int_t^{+\infty}T^+(s-t)f(s)ds\qquad t\in\mathbb{R}$$

has values in $D(A)$, it is differentiable with values in X, and it is the classical solution of

$$u'(t)=Au(t)+f(t)\qquad(t\in\mathbb{R}).$$

Moreover, u' and Au belong to $C(\mathbb{R};X)\cap B(\mathbb{R},D_A(\alpha,\infty))$, Au belongs to $C^\alpha(\mathbb{R};X)$ and there exists $C>0$ such that

$$\|u'\|_{B(\mathbb{R};D_A(\alpha,\infty))}+\|Au\|_{B(\mathbb{R};D_A(\alpha,\infty))}+\|Au\|_{C^\alpha(\mathbb{R};X)}\le C\|f\|_{B(\mathbb{R};D_A(\alpha,\infty))}. \tag{6.1}$$

Proof. For $t\in\mathbb{R}$, $u(t)$ belongs to $D(A)$ and we have

$$\begin{aligned}
&\Big\|\int_{-\infty}^t AT^-(t-s)f(s)ds\Big\|\le\int_{-\infty}^t\|AT^-(t-s)f(s)\|ds\\
\le\;&\|f\|_{B(\mathbb{R};D_A(\alpha,\infty))}\Big(\int_{-\infty}^{t-1}\|AT^-(t-s)\|_{\mathcal{L}(D_A(\alpha,\infty);X)}ds\\
&\qquad+\int_{t-1}^t\|AT^-(t-s)\|_{\mathcal{L}(D_A(\alpha,\infty);X)}ds\Big)\\
\le\;&\|f\|_{B(\mathbb{R};D_A(\alpha,\infty))}\Big(M^-_{1,\infty,\alpha}\int_{-\infty}^{t-1}\frac{e^{-\omega(t-s)}}{t-s}ds+M^-_{1,0,\alpha}\int_{t-1}^t\frac{ds}{(t-s)^{1-\alpha}}\Big)\\
\le\;&C^-\|f\|_{B(\mathbb{R};D_A(\alpha,\infty))},
\end{aligned}$$

where the constants $M^\pm_{1,\infty,\alpha}$ and $M^\pm_{1,0,\alpha}$ follow from Proposition 3.5. Analogously, one can show that

$$\Big\|\int_t^{+\infty}AT^+(t-s)f(s)ds\Big\|\le C^+\|f\|_{B(\mathbb{R};D_A(\alpha,\infty))}.$$

Therefore, we have

$$\begin{aligned}
\|Au(t)\|&=\Big\|\int_{-\infty}^t AT^-(t-s)f(s)ds-\int_t^{+\infty}AT^+(s-t)f(s)ds\Big\|\\
&\le(C^-+C^+)\|f\|_{B(\mathbb{R};D_A(\alpha,\infty))}.
\end{aligned}\tag{6.2}$$

Moreover, for $\tau\in\mathbb{R}$ we have

$$\begin{aligned}
&\|\tau^{1-\alpha}AT^-(\tau)Au(t)\|\\
=\;&\tau^{1-\alpha}\Big\|AT^-(\tau)\Big(\int_{-\infty}^t AT^-(t-s)f(s)ds-\int_t^{+\infty}AT^+(s-t)f(s)ds\Big)\Big\|\\
=\;&\tau^{1-\alpha}\Big\|\int_{-\infty}^t A^2T^-(\tau+t-s)f(s)ds\Big\|\\
\le\;&M^-_{2,0,\alpha}\tau^{1-\alpha}\int_{-\infty}^t(\tau+t-s)^{\alpha-2}ds\|f\|_{B(\mathbb{R},D_A(\alpha,\infty))}\\
=\;&\frac{M^-_{2,0,\alpha}}{1-\alpha}\|f\|_{B(\mathbb{R};D_A(\alpha,\infty))},
\end{aligned}\tag{6.3}$$

where the constant $M^-_{2,0,\alpha}$ follows from Proposition 3.5. Similarly one can prove that

$$\|\tau^{1-\alpha}AT^+(\tau)Au(t)\| \le \frac{M^+_{2,0,\alpha}}{1-\alpha}\|f\|_{B(\mathbb{R};D_A(\alpha,\infty))},$$

and the constant $M^+_{2,0,\alpha}$ still follows from Proposition 3.5. Thus Au is bounded with values in $D_A(\alpha,\infty)$.

Let us show that Au is also Hölder-continuous with values in X. Let $s \le t$. Then we have

$$\begin{aligned}
& \|Au(t)-Au(s)\| \\
= \ & \Big\|A\Big(\int_{-\infty}^{t} T^-(t-\sigma)f(\sigma)d\sigma - \int_{t}^{+\infty} T^+(\sigma-t)f(\sigma)d\sigma\Big) \\
& -A\Big(\int_{-\infty}^{s} T^-(s-\sigma)f(\sigma)d\sigma - \int_{s}^{+\infty} T^+(\sigma-s)f(\sigma)d\sigma\Big)\Big\| \\
= \ & \Big\|\int_{-\infty}^{s} A(T^-(t-\sigma)-T^-(s-\sigma))f(\sigma)d\sigma + \int_{s}^{t} AT^-(t-\sigma)f(\sigma)d\sigma \\
& -\int_{t}^{+\infty} A(T^+(\sigma-t)-T^+(\sigma-s))f(\sigma)d\sigma + \int_{s}^{t} T^+(\sigma-s)f(\sigma)d\sigma\Big\| \\
\le \ & \Big\|\int_{-\infty}^{s} A(T^-(t-\sigma)-T^-(s-\sigma))f(\sigma)d\sigma\Big\| \\
& +\Big\|\int_{t}^{+\infty} A(T^+(\sigma-t)-T^+(\sigma-s))f(\sigma)d\sigma\Big\| \\
& +\Big\|\int_{s}^{t} AT^-(t-\sigma)f(\sigma)d\sigma\Big\| + \Big\|\int_{s}^{t} AT^+(\sigma-s)f(\sigma)d\sigma\Big\| \\
\le \ & M^-_{2,\alpha}\|f\|_{B(\mathbb{R};D_A(\alpha,\infty))}\int_{-\infty}^{t}\int_{s-\sigma}^{t-\sigma}\tau^{\alpha-2}d\tau d\sigma \\
& +M^+_{2,\alpha}\|f\|_{B(\mathbb{R};D_A(\alpha,\infty))}\int_{t}^{+\infty}\int_{s-\sigma}^{t-\sigma}\tau^{\alpha-2}d\tau d\sigma \\
& +M^-_{1,\alpha}\|f\|_{B(\mathbb{R};D_A(\alpha,\infty))}\int_{s}^{t}(t-s)^{\alpha-1}d\sigma \\
& +M^+_{1,\alpha}\|f\|_{B(\mathbb{R};D_A(\alpha,\infty))}\int_{s}^{t}(\sigma-s)^{\alpha-1}d\sigma \\
\le \ & \Big(\frac{M^-_{2,\alpha}}{\alpha(1-\alpha)} + \frac{M^+_{2,\alpha}}{\alpha(1-\alpha)} + \frac{M^-_1}{\alpha} + \frac{M^+_1}{\alpha}\Big)(t-s)^{\alpha}\|f\|_{B(\mathbb{R};D_A(\alpha,\infty))}. \quad (6.4)
\end{aligned}$$

Then Au is Hölder-continuous in $\mathbb{R}$ and (6.1) follows from (6.2), (6.3) and (6.4). $\square$

Theorem 6.2. *Let $0 < \alpha < 1$, $1 \leq p < \infty$ and let $f \in L^p(\mathbb{R}; D_A(\alpha, p))$. Then the function*

$$u(t) = \int_{-\infty}^{t} T^-(t-s)f(s)ds - \int_{t}^{+\infty} T^+(s-t)f(s)ds, \qquad t \in \mathbb{R},$$

has values in $D(A)$, it is differentiable in the L^p-sense for almost all $t \in \mathbb{R}$, and it is a strong solution of (4.1). Moreover, u' and Au belong to $L^p(\mathbb{R}; D_A(\alpha, p))$ and there exists $C > 0$ such that

$$\|u'\|_{L^p(\mathbb{R};D_A(\alpha,p))} + \|Au\|_{L^p(\mathbb{R};D_A(\alpha,p))} \leq C\|f\|_{L^p(\mathbb{R};D_A(\alpha,p))}.$$

Proof. As we have seen in Proposition 2.2, the norms $\|T^{\pm}(t)\|$ are not bounded, in general, near $t = 0$. However, to prove the statement, we prefer to work with spaces where these norms are bounded up to $t = 0$. To this aim, let us consider the subspace $Y \doteq D_A(\alpha/2, p)$ and let us denote by A_0 the part of A in Y. By Proposition 3.6 we know that $\sup_{t\in(0,+\infty)} \|T^{\pm}(t)\|_{\mathcal{L}(Y)}$ is finite. Moreover, by the reiteration theorem, it follows that $D_A(\alpha, p) = (Y, D(A_0))_{\alpha/2,p}$, so that $L^p(\mathbb{R}; D_A(\alpha, p)) = L^p(\mathbb{R}; D_{A_0}(\alpha/2, p))$ where A_0 is the part of A in Y. This implies that in order to prove maximal regularity in $L^p(\mathbb{R}; D_A(\alpha, p)$ it suffices to prove maximal regularity in $L^p(\mathbb{R}; D_{A_0}(\alpha/2, p))$. We have

$$\begin{aligned}
&\int_{\mathbb{R}} \|A_0 u(t)\|^p dt \\
&= \int_{\mathbb{R}} \Big\|A_0\Big(\int_{-\infty}^{t} T^-(t-s)f(s)ds - \int_{t}^{+\infty} T^+(s-t)f(s)ds\Big)\Big\|^p dt \\
&\leq \int_{\mathbb{R}} \Big\|A_0 \int_{-\infty}^{t} T^-(t-s)f(s)ds\Big\|^p dt + \int_{\mathbb{R}} \Big\|A_0 \int_{t}^{+\infty} T^+(s-t)f(s)ds\Big\|^p dt \\
&\leq \int_{\mathbb{R}} \Big(\int_{-\infty}^{t} \|A_0 T^-(t-s)\|_{\mathcal{L}(D_{A_0}(\alpha/2,p);X)} \|f(s)\|_{D_{A_0}(\alpha/2,p)} ds\Big)^p dt \\
&+ \int_{\mathbb{R}} \Big(\int_{t}^{+\infty} \|A_0 T^+(s-t)\|_{\mathcal{L}(D_{A_0}(\alpha/2,p);X)} \|f(s)\|_{D_{A_0}(\alpha/2,p)} ds\Big)^p dt \\
&\leq \int_{\mathbb{R}} \Big(\int_{-\infty}^{t} \frac{M^-_{1,\alpha/2} e^{-\omega(t-s)}}{(t-s)^{1-\alpha/2}} \|f(s)\|_{D_{A_0}(\alpha/2,p)} ds\Big)^p dt \\
&+ \int_{\mathbb{R}} \Big(\int_{t}^{+\infty} \frac{M^+_{1,\alpha/2} e^{-\omega(s-t)}}{(s-t)^{1-\alpha/2}} \|f(s)\|_{D_{A_0}(\alpha/2,p)} ds\Big)^p dt
\end{aligned}$$

$$= \Big(M^-_{\alpha/2} + M^+_{\alpha/2}\Big) \int_{\mathbb{R}} \Big((g_1 * g_2)(t)\Big)^p dt,$$

where

$$g_1(\tau) = \frac{e^{-\omega\tau}}{\tau^{1-\alpha/2}} \chi_{[0,+\infty)}(\tau) \in L^1(\mathbb{R}), \quad g_2(\tau) = \|f(\tau)\|_{D_{A_0}(\alpha/2,p)} \in L^p(\mathbb{R}).$$

Thus $(g_1 * g_2) \in L^p(\mathbb{R})$ and this proves that $t \mapsto \|A_0 u(t)\|$ is in $L^p(\mathbb{R})$. Now we have to prove that the function

$$\begin{aligned} t \mapsto [A_0 u(t)]_{D_{A_0}(\alpha/2,)} \quad &\doteq \quad \Big(\int_0^{+\infty} \tau^{p(1-\alpha/2)-1} \|A_0 T^-(\tau) A_0 u(t)\|^p d\tau\Big)^{1/p} \\ &+ \Big(\int_0^{+\infty} \tau^{p(1-\alpha/2)-1} \|A_0 T^+(\tau) A_0 u(t)\|^p d\tau\Big)^{1/p} \end{aligned}$$

is also in $L^p(\mathbb{R})$.

Using the Hardy-Young inequalities and recalling that $T^{\pm}(t)T^{\mp}(s) = 0$ for every $t, s > 0$ and that $\sup_{t\in(0,+\infty)} \|T^{\pm}(t)\|_{\mathcal{L}(D_{A_0}(\alpha/2,p))} < \infty$, we have the following estimates (it is understood that the constants M_0^- and M_1^- will be referring to the norm $\|T^{\pm}(t)\|_{\mathcal{L}(D_{A_0}(\alpha,p))}$)

$$\begin{aligned} & \int_{\mathbb{R}} \int_0^{+\infty} \tau^{p(1-\frac{\alpha}{2})} \|A_0 T^-(\tau) A_0 u(t)\|^p \frac{d\tau}{\tau} dt \\ \le\ & \int_{\mathbb{R}} \int_0^{+\infty} \tau^{p(1-\frac{\alpha}{2})} \Big(\int_0^{+\infty} \|A_0^2 T^-(\tau+s) f(t-s)\| ds\Big)^p \frac{d\tau}{\tau} dt \\ \le\ & C \int_{\mathbb{R}} \int_0^{+\infty} \tau^{p(1-\frac{\alpha}{2})} \Big(\int_0^{+\infty} \frac{1}{\tau+s} \Big\|A_0 T^-\Big(\frac{s}{2}\Big) f(t-s)\Big\| ds\Big)^p \frac{d\tau}{\tau} dt \\ \le\ & C \int_{\mathbb{R}} \int_0^{+\infty} \Big(\tau^{-\frac{\alpha}{2}} \int_0^{\tau} \Big\|A_0 T^-\Big(\frac{s}{2}\Big) f(t-s)\Big\| ds\Big)^p \frac{d\tau}{\tau} dt + \\ & + C \int_{\mathbb{R}} \int_0^{+\infty} \Big(\tau^{1-\frac{\alpha}{2}} \int_{\tau}^{+\infty} \Big\|A_0 T^-\Big(\frac{s}{2}\Big) f(t-s)\Big\| \frac{ds}{s}\Big)^p \frac{d\tau}{\tau} dt \\ \le\ & \frac{2^p C}{\alpha^p (2-\alpha)^p} \int_{\mathbb{R}} \int_0^{+\infty} s^{p(1-\frac{\alpha}{2})} \Big\|A_0 T^-\Big(\frac{s}{2}\Big) f(t-s)\Big\|^p \frac{ds}{s} dt \\ =\ & \frac{2^p C}{\alpha^p (2-\alpha)^p} \int_0^{+\infty} s^{p(1-\frac{\alpha}{2})} \int_{\mathbb{R}} \Big\|A_0 T^-\Big(\frac{s}{2}\Big) f(t)\Big\|^p dt \frac{ds}{s} \\ =\ & \frac{2^p C}{\alpha^p (2-\alpha)^p} \int_{\mathbb{R}} \int_0^{+\infty} s^{p(1-\frac{\alpha}{2})} \Big\|A_0 T^-\Big(\frac{s}{2}\Big) f(t)\Big\|^p \frac{ds}{s} dt \\ \le\ & 2^{2p-\frac{\alpha p}{2}-1} \frac{C}{\alpha^p (2-\alpha)^p} \|f\|^p_{L^p(\mathbb{R};D_{A_0}(\alpha/2,p))}, \end{aligned}$$

where $C := (2^p M_0^1 M_1^1)^p$. Thus $Au \in L^p(\mathbb{R}; D_{A_0}(\alpha/2, p))$ and this proves maximal regularity in the space $L^p(\mathbb{R}; D_{A_0}(\alpha/2, p))$, which implies maximal regularity in the space $L^p(\mathbb{R}; D_A(\alpha, p))$. □

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