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WELL-POSEDNESS OF THE CAUCHY PROBLEM FOR THE KORTEWEG-DE VRIES EQUATION AT THE CRITICAL REGULARITY

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Abstract. The Cauchy problem for the nonperiodic KdV equation is shown by the iteration method to be locally well-posed in $H^{-3/4}(\mathbb{R})$. In particular, solutions are unique in the whole Banach space for the iteration. This extends the previous well-posedness result in H^s, $s > -3/4$ obtained by Kenig, Ponce and Vega (1996) to the limiting case, and improves the existence result in $H^{-3/4}$ given by Christ, Colliander and Tao (2003). Our result immediately yields global well-posedness for the KdV equation in $H^{-3/4}(\mathbb{R})$ and for the modified KdV equation in $H^{1/4}(\mathbb{R})$, combined with the argument of Colliander, Keel, Staffilani, Takaoka and Tao (2003).

1. Introduction

We study well-posedness for the Cauchy problem of the nonperiodic Korteweg-de Vries equation

$$\begin{cases} \partial_t u + \partial_x^3 u = \partial_x(u^2), & u : \mathbb{R}\times\mathbb{R} \to \mathbb{R}, \\ u(0,x) = u_0(x). \end{cases} \tag{1.1}$$

The initial data u_0 are given in a Sobolev space $H^s(\mathbb{R})$ defined by the norm

$$\|\varphi\|_{H^s} := \|\langle\xi\rangle^s \hat{\varphi}\|_{L^2},$$

where $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$ and $\hat{\varphi}$ denotes the Fourier transform of φ; i.e.,

$$\hat{\varphi}(\xi) := \int_{\mathbb{R}} e^{-ix\xi}\varphi(x)\,dx.$$

The KdV equation, which was originally derived by Korteweg and de Vries [14] as a model equation for the propagation of shallow water waves along a canal, appears in various phases of mathematical physics, and it is also

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well-known as one of the simplest partial differential equations that have complete integrability.

The Cauchy problem (1.1) has been extensively studied. The viscosity method was first applied to establish the local well-posedness (LWP for short) in H^s with $s > \frac{3}{2}$ ([2]), which is improved to $s > \frac{3}{4}$ ([9]) by the iterative approach exploiting the local smoothing estimate for the Airy operator $e^{-t\partial_x^3}$. Well-posedness theory has made substantial progress since Bourgain introduced the Fourier restriction method ([3]), namely the iteration in the function space $X^{s,b}$ defined by

$$X^{s,b} := \left\{ u \in \mathcal{S}'(\mathbb{R}^2) \, : \, \big\| u \big\|_{X^{s,b}} := \big\| \langle \xi \rangle^s \langle \tau - \xi^3 \rangle^b \hat{u} \big\|_{L^2_{\tau,\xi}(\mathbb{R}^2)} < \infty \right\}.$$

By this method Kenig, Ponce and Vega obtained the lowest regularity for LWP, $s > -\frac{3}{4}$ ([10]). They also showed ([11]; see also [4]) that when $s < -\frac{3}{4}$ the data-to-solution map fails to be uniformly continuous as a map from H^s to $C_t^0(H^s)$, which means that $s = -\frac{3}{4}$ is the best regularity that can be achieved by iteration.

The above local results form the basis for the global well-posedness (GWP for short) of (1.1). GWP in L^2 immediately follows from the local result and the L^2-conservation law of the equation, while in the negative regularity the "I-method," introduced by Colliander, Keel, Staffilani, Takaoka and Tao, plays a great role in constructing global solutions. They obtained GWP first for $s > -\frac{3}{10}$ ([5]), and then introduced some correction terms to improve the result to the same regularity as the above LWP, $s > -\frac{3}{4}$ ([6]). Recently the I-method has been applied to GWP for a variety of nonlinear evolution equations and other topics (see [17] and references therein).

When we apply the Fourier restriction method to (1.1), LWP is reduced to the following bilinear estimate corresponding to the quadratic nonlinearity,

$$\big\| \partial_x(uv) \big\|_{X^{s,b-1}} \le C \big\| u \big\|_{X^{s,b}} \big\| v \big\|_{X^{s,b}}. \tag{1.2}$$

It was proved, however, that this estimate does not hold for any $b \in \mathbb{R}$ at the critical regularity $s = -\frac{3}{4}$ ([16]). Therefore, to obtain the remaining LWP in $H^{-\frac{3}{4}}$ we have to iterate in a space different from $X^{s,b}$, or abandon the direct iteration method.

Christ, Colliander and Tao [4] obtained the existence result for $s = -\frac{3}{4}$ by combining the (slightly modified) Miura transform with the corresponding LWP result for the modified Korteweg-de Vries equation in $H^{\frac{1}{4}}$ obtained before ([9]). The modified KdV equation,

$$\partial_t v + \partial_x^3 v = \pm \partial_x(v^3), \quad v(0,x) = v_0(x), \quad v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \tag{1.3}$$

is also well studied and linked with the KdV equation through the Miura transform $v \mapsto u := \partial_x v + cv^2$, which maps a solution of (1.3) to that of (1.1) (up to a constant). Since the Miura transform acts roughly as a derivative, many results for KdV have counterparts for modified KdV with one higher regularity; in fact, GWP of KdV for $s > -\frac{3}{4}$ was translated into GWP of modified KdV for $s > \frac{1}{4}$ ([6]), and for modified KdV the same ill-posedness assertion as that for KdV was obtained for $s < \frac{1}{4}$ ([11], [4]).

Although these results suggest that the LWP theory for the KdV and the modified KdV equations has been completed in some sense, we point out that the above result for KdV in $H^{-\frac{3}{4}}$ is relatively weak on at least two points. Firstly, as remarked in their paper, the uniqueness of solutions was obtained only in the Miura image, not in $C_t^0(H^{-\frac{3}{4}})$ or some Banach space contained in $C_t^0(H^{-\frac{3}{4}})$. Recall that in the case $s > -\frac{3}{4}$ solutions were shown to be unique in the Banach space of the Fourier restriction norm $X^{s,b}$. We find it difficult to verify whether a given function is in the Miura image or not. Secondly, while they constructed a local-in-time solution in $C_t^0([-T,T]; H^{-\frac{3}{4}})$, we do not have the bound of solutions in a stronger space well adapted to the I-method, such as $X^{s,b}$. This is why GWP for KdV in $H^{-\frac{3}{4}}$ (and GWP for modified KdV in $H^{\frac{1}{4}}$) is not easy to obtain.

In this article we shall establish LWP for KdV (1.1) by the direct iteration method. The function space X in which we seek solutions will be some Besov-like generalization of Bourgain's space $X^{-\frac{3}{4},\frac{1}{2}}$ with a slight modification in low frequency (see (2.1)–(2.4) in Section 2). The same Besov-type modified $X^{s,b}$ spaces have been used recently in the context of the nonlinear Schrödinger equation ([1], [12], [13]; also see [15], [18]), especially in order to avoid some "logarithmic" divergences of the crucial estimate occurring at the critical regularity, and we find them also useful in our KdV context. However the new space X fails to be embedded into $C_t^0(H^{-\frac{3}{4}})$, so we need to introduce an auxiliary space Y defined by the norm

$$\|u\|_Y := \|\langle\xi\rangle^{-\frac{3}{4}}\hat{u}\|_{L^2_\xi L^1_\tau}.$$

This space Y has also appeared in previous works (originally in [7]). For these spaces we have the following bilinear estimate similar to (1.2).

Proposition 1.1. *There exists $C > 0$ such that for any u, v we have*

$$\|\mathcal{F}^{-1}[\langle\tau-\xi^3\rangle^{-1}\widehat{\partial_x(uv)}]\|_X + \|\mathcal{F}^{-1}[\langle\tau-\xi^3\rangle^{-1}\widehat{\partial_x(uv)}]\|_Y \le C\|u\|_X\|v\|_X,$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

A standard iteration argument then implies our main theorem.

Theorem 1.2. *The Cauchy problem* (1.1) *is locally well-posed in* $H^{-\frac{3}{4}}(\mathbb{R})$*. More precisely, for any* $r > 0$ *there exists a Lipschitz continuous data-to-solution map from a ball* $\{ u_0 \in H^{-\frac{3}{4}}(\mathbb{R}) : \|u_0\|_{H^{-\frac{3}{4}}} \leq r \}$ *into* $X_T \cap C_t^0([-T,T]; H^{-\frac{3}{4}}(\mathbb{R}))$ *for some* $T = T(r) > 0$ *(*X_T *is the time restricted space of* X *defined by* (2.1)–(2.5) *in Section 2). Moreover, the solutions of* (1.1) *on the time interval* $[-T,T]$ *are unique in the class* X_T*.*

Remark 1.3. (i) If a solution u on $[-T,T]$ lies in X_T, we can easily bound u in $C_t^0([-T,T]; H^{-\frac{3}{4}})$ using the integral representation of (1.1). Therefore, for the uniqueness in X_T we just have to see it in the class $X_T \cap C_t^0([-T,T]; H^{-\frac{3}{4}})$, which we shall prove following the argument given by Muramatu and Taoka [15]. See Section 4 for details.

(ii) We can of course show the persistence of the regularity property, so the data-to-solution map given in the theorem agrees with the unique classical solution on smooth initial data. The same is true for the previous result in [4], and we conclude that the solution for (1.1) with $u_0 \in H^{-\frac{3}{4}}$ obtained in this theorem coincides with the weak solution given in [4].

This theorem combined with the I-method yields the global results:

Corollary 1.4. *The Cauchy problem* (1.1) *is globally well-posed in* $H^{-\frac{3}{4}}(\mathbb{R})$*.*

Corollary 1.5. *The Cauchy problem* (1.3) *is globally well-posed in* $H^{\frac{1}{4}}(\mathbb{R})$*.*

Since our function space X is very close to the usual Bourgain space $X^{s,b}$ (in fact satisfies the embedding $X^{-\frac{3}{4},b} \hookrightarrow X \hookrightarrow X^{-\frac{3}{4},\frac{1}{2}}$ for any $b > \frac{1}{2}$), proofs for these corollaries are almost identical with the case of $X^{s,b}$ for $s > -\frac{3}{4}$. We will omit them and refer to [6]. Note that these global results do not hold for the complex-valued case, while LWP in Theorem 1.2 also holds for the complex nonperiodic KdV equation with the identical proof.

Remark 1.6. After the manuscript had been completed the author heard that Guo [8] obtained similar results independently. The function spaces of Guo and the author slightly differ from each other in low frequency, and there is no inclusion relation between them. We will again compare these spaces after the definition of our space; see Remark 2.3. On the other hand, uniqueness of solutions in the whole Banach space of iteration was not mentioned in [8]; solutions were obtained just as the unique limit of smooth solutions, as in [4].

We are planning the rest of this article as follows. In Section 2 we shall see some typical nonlinear interactions and construct the function space where we carry out the iteration. Section 3 will be devoted to proving the essential bilinear estimate, Proposition 1.1. We shall sketch the proof of Theorem 1.2 in Section 4.

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2. Construction of the function space

Hereafter we use the notation "$A \lesssim B$" for the estimate $A \le C\,B$ with some constant $C > 0$, and "$A \sim B$" for $A \lesssim B \lesssim A$. We write "$A \ll B$" if the estimate $A \le C^{-1}B$ holds for some large positive constant C.

For the proof of LWP we have to find the appropriate function space which obeys the bilinear estimate. We recall that the bilinear estimate in $X^{s,b}$ (1.2) fails for any b at our regularity $s = -\frac{3}{4}$. Let us first see some counterexamples to get a hint about the creation of a new space.

Example 2.1. (High-High interaction) Let N be a sufficiently large positive number and $\mathcal{R}$ be the region in $\mathbb{R}^2_{\tau,\xi}$ inside the parallelogram with vertices

$$\begin{aligned}(\tau,\xi) &= (N^3, N), \quad (N^3 + N^{\frac{3}{2}}, N + \tfrac{1}{3}N^{-\frac{1}{2}}),\\ &\quad ((N + \tfrac{1}{3}N^{-\frac{1}{2}})^3, N + \tfrac{1}{3}N^{-\frac{1}{2}}), \quad (N^3 + \tfrac{1}{3} + \tfrac{1}{27}N^{-\frac{3}{2}}, N).\end{aligned}$$

Note that $\mathcal{R}$ is included in the region $\{\,|\tau - \xi^3| < 1\,\}$, has the longest side pointing at the direction $(3N^2, 1)$ and $|\mathcal{R}| \sim N^{-\frac{1}{2}}$. We put $\mathcal{R}_0$ equal to the translation of $\mathcal{R}$ centered at the origin.

We define u and v as $\hat{u} := \mathbf{1}_{\mathcal{R}}$, $\hat{v}(\tau,\xi) := \hat{u}(-\tau,-\xi)$, where $\mathbf{1}_\Omega$ denotes the characteristic function of a set Ω. A simple calculation shows

$$\|u\|_{X^{-\frac{3}{4},b}} = \|v\|_{X^{-\frac{3}{4},b}} \sim N^{-1}, \quad \widehat{uv} \gtrsim N^{-\frac{1}{2}}\mathbf{1}_{\mathcal{R}_0}, \quad \|\partial_x(uv)\|_{X^{-\frac{3}{4},b-1}} \gtrsim N^{\frac{6b-11}{4}},$$

and we see that the parameter b needs to satisfy $\frac{6b-11}{4} \le -2 \;\Leftrightarrow\; b \le \frac{1}{2}$ for (1.2) to hold with $s = -\frac{3}{4}$ and an arbitrary N. This is a counterexample in the case $b > \frac{1}{2}$.

Example 2.2. (High-Low interaction) We define $\hat{u} := \mathbf{1}_{\mathcal{R}}$, $\hat{v} := \mathbf{1}_{\mathcal{R}_0}$. In a similar manner we have

$$\|u\|_{X^{-\frac{3}{4},b}} \sim N^{-1}, \; \|v\|_{X^{-\frac{3}{4},b}} \sim N^{\frac{6b-1}{4}}, \; \widehat{uv} \gtrsim N^{-\frac{1}{2}}\mathbf{1}_{\mathcal{R}}, \; \|\partial_x(uv)\|_{X^{-\frac{3}{4},b-1}} \gtrsim N^{-\frac{1}{2}},$$

and the bilinear estimate in $X^{-\frac{3}{4},b}$ requires $-\frac{1}{2} \le \frac{6b-5}{4} \iff b \ge \frac{1}{2}$. This is a counterexample in the case $b < \frac{1}{2}$.

Although each of these examples is controlled well in $X^{-\frac{3}{4},\frac{1}{2}}$, it is merely an easy example of awkward nonlinear interactions; both High-High and High-Low interactions are in fact hard to completely control in $X^{-\frac{3}{4},\frac{1}{2}}$. In particular, a sophisticated counterexample to the bilinear estimate in $X^{-\frac{3}{4},\frac{1}{2}}$ was given in [16]. We recall this example, which is of High-High type, supported in frequency also in the region distant from the curve $\{\, \tau = \xi^3 \,\}$ (in contrast to the function in Example 2.1), producing the divergence of $O(\log N)$ in the bilinear estimate. We often encounter such logarithmic divergence when treating the critical regularity.

To avoid the logarithmic divergence, inspired by previous papers on the nonlinear Schrödinger equation ([1], [12], [13]), we replace the $X^{-\frac{3}{4},\frac{1}{2}}$ norm in high frequency (i.e., in $\{\, |\xi| \ge 1 \,\}$) with a Besov-like $X^{-\frac{3}{4},\frac{1}{2},1}$ norm. We first define two dyadic decompositions of the frequency space,

$$\begin{aligned} A_j &:= \{\, (\tau,\xi) \,:\, 2^j \le \langle \xi \rangle < 2^{j+1} \,\}, \\ B_k &:= \{\, (\tau,\xi) \,:\, 2^k \le \langle \tau - \xi^3 \rangle < 2^{k+1} \,\} \end{aligned} \tag{2.1}$$

for nonnegative integers j, k. The space $X^{s,b,1}$ is then defined by the norm

$$\begin{aligned} \|u\|_{X^{s,b,1}} &:= \Big\| \Big\{ \|\langle \xi \rangle^s \langle \tau - \xi^3 \rangle^b \hat{u}\|_{L^2_{\tau,\xi}(A_j \cap B_k)} \Big\}_{j,k \ge 0} \Big\|_{\ell^2_j \ell^1_k} \\ &\sim \Big(\textstyle\sum_j \Big(2^{sj} \sum_k 2^{bk} \|\hat{u}\|_{L^2_{\tau,\xi}(A_j \cap B_k)} \Big)^2 \Big)^{1/2}. \end{aligned} \tag{2.2}$$

The $X^{s,b,1}$ norm is slightly stronger than the $X^{s,b}$ norm. This type of space has many good properties such as the embedding $X^{s,\frac{1}{2},1} \hookrightarrow C^0_t(\mathbb{R}; H^s)$ (note that $X^{s,b}$, $b > \frac{1}{2}$ has the same embedding property but $X^{s,\frac{1}{2}}$ does not).

The only remaining problem is how we should design the final space X in low frequency (i.e., in $\{\, |\xi| \le 1 \,\}$). This issue is actually quite sensitive. If we set $X = X^{-\frac{3}{4},\frac{1}{2},1}$ in the whole frequency range, then we find this space so strong that we cannot control the High-High interaction. It would be likely that there is some counterexample to the bilinear estimate in $X^{-\frac{3}{4},\frac{1}{2},1}$, perhaps of the same type as that for $X^{-\frac{3}{4},\frac{1}{2}}$ in [16]. On the other hand, if we set $X = X^{-\frac{3}{4},\frac{1}{2}}$ in low frequency and $X = X^{-\frac{3}{4},\frac{1}{2},1}$ in high frequency, then this structure would be so weak that we cannot possibly control the High-Low interaction. This difficulty can be seen in the proof of our bilinear estimate (see Section 3, proof of Proposition 3.4, (iv) and (v)).

This problem can be overcome by setting $X = X^{-\frac{3}{4},\frac{1}{2}}$ in just a part of the low frequency region. We find that the High-High interaction is hard to control, not in the whole set $\{\, |\xi| \le 1 \,\}$, but only in the region

$$D := \big\{\, (\tau,\xi) \in \mathbb{R}^2 \,:\, |\xi| \le 1,\ |\tau| \ge |\xi|^{-3} \,\big\}. \tag{2.3}$$

We finally define the space X by the norm

$$\big\|u\big\|_X := \big\|\mathcal{F}^{-1}\big[\mathbf{1}_{\mathbb{R}^2\setminus D}\hat{u}\big]\big\|_{X^{-\frac{3}{4},\frac{1}{2},1}} + \big\|\mathcal{F}^{-1}\big[\mathbf{1}_D\hat{u}\big]\big\|_{X^{-\frac{3}{4},\frac{1}{2}}}. \tag{2.4}$$

This type of modification, which gives a different structure to a specific part of the frequency space (depending also on τ), has appeared in the author's previous papers on nonlinear Schrödinger equations ([12], [13]).

The space X is not embedded into $C_t^0(\mathbb{R}; H^{-\frac{3}{4}})$, so we will make the iteration argument in $X \cap C_t^0(\mathbb{R}; H^{-\frac{3}{4}})$ and need the bilinear estimate given in Proposition 1.1, which will be established in the next section. Local-in-time solutions are then obtained in the time localized space $X_T \cap C_t^0([-T,T]; H^{-\frac{3}{4}})$, where X_T consists of the restriction of functions in X to the time interval $[-T,T]$, equipped with the norm

$$\big\|u\big\|_{X_T} := \inf\big\{\, \big\|v\big\|_X \,:\, v \in X,\ u(t) = v(t) \text{ for } -T \le t \le T \,\big\}. \tag{2.5}$$

Remark 2.3. The function space in the work of Guo [8] is identical with our space in high frequency. The only difference is in low frequency $\{\, |\xi| \le 1 \,\}$; the space in [8] has the maximal function norm $\big\|u\big\|_{L_x^2 L_t^\infty}$, while our space is defined by

$$\big\|\mathcal{F}^{-1}\big[\mathbf{1}_D\hat{u}\big]\big\|_{X^{0,\frac{1}{2}}} + \big\|\mathcal{F}^{-1}\big[\mathbf{1}_{A_0\setminus D}\hat{u}\big]\big\|_{X^{0,\frac{1}{2},1}}\ \big(\, + \big\|u\big\|_{L_t^\infty L_x^2}\,\big).$$

These structures share some common properties; for instance, both are weaker than $X^{-\frac{3}{4},\frac{1}{2},1}$ for the high frequency part and stronger than $C_t^0(H^{-\frac{3}{4}})$. On the other hand, in contrast to the space in [8] defined on the physical space $\mathbb{R}^2_{t,x}$ in low frequency, we define our space X totally on the Fourier space $\mathbb{R}^2_{\tau,\xi}$ similarly to the standard $X^{s,b}$ (note that we can complete the iteration in X as well as in $X \cap C_t^0(H^{-\frac{3}{4}})$), which allows us to define the space for the estimate of nonlinearity simply as $\langle \partial_t + \partial_x^3 \rangle X$ and completely separate the estimate for the Duhamel term of the integral equation ((4.3) below) into the linear Duhamel estimate (Lemma 4.1, (4.2)) and the bilinear estimate (Proposition 1.1). The same reduction would be nontrivial for function spaces including the norm on the physical space, and in [8] a certain part of the estimate for the nonlinear term was actually proved in the Duhamel integral form (see [8], Proposition 3.9). We also remark that our space X has the monotonicity in frequency, namely $|\hat{u}| \leq |\hat{v}|$ implies $\|u\|_X \leq \|v\|_X$, which does not hold in the space of [8]. We will actually use this property in the proof of (4.2). Such structure is also compatible with the I-method and admits the identical proof for GWP as the previous one ([6]) working on the standard $X^{s,b}$.

3. Proof for the bilinear estimate

For the bilinear estimate we work on the frequency space, so it is convenient to define the space $\hat{X}$ (similarly $\hat{X}^{s,b}$ and $\hat{X}^{s,b,1}$) as the Fourier transform of functions in X. $\hat{X}$ is the space of functions in (τ,ξ) with the norm $\|f\|_{\hat{X}} := \|\mathcal{F}^{-1}f\|_X$. Then, Proposition 1.1 is rewritten as

$$\big\|\langle\tau-\xi^3\rangle^{-1}\xi\ f*g\big\|_{\hat{X}} + \big\|\langle\xi\rangle^{-\frac{3}{4}}\langle\tau-\xi^3\rangle^{-1}\xi\ f*g\big\|_{L^2_\xi L^1_\tau} \lesssim \|f\|_{\hat{X}}\|g\|_{\hat{X}}, \quad (3.1)$$

where $*$ denotes the space-time convolution. We also use the following notation for simplicity:

$$A_{<j_1} := \bigcup_{j<j_1} A_j, \quad B_{[k_1,k_2)} := \bigcup_{k_1\leq k<k_2} B_k, \quad \text{etc.}$$

Lemma 3.1. *The space $\hat{X}$ has the following properties.*

(i) *For any $b > \frac{1}{2}$ there exists $C > 0$ such that*

$$\|f\|_{\hat{X}} \leq C\|f\|_{\hat{X}^{-\frac{3}{4},b}}.$$

(ii) *For $1 < p \leq 2$ there exists $C > 0$ such that*

$$\big\|\langle\xi\rangle^{-\frac{3}{4}}f\big\|_{L^2_\xi L^p_\tau} \leq C\|f\|_{\hat{X}}.$$

Moreover, if f is supported outside the set D defined in (2.3), *then*

$$\big\|\langle\xi\rangle^{-\frac{3}{4}}f\big\|_{L^2_\xi L^1_\tau} \leq C\|f\|_{\hat{X}}.$$

Proof. (i) is a consequence of the embedding $\hat{X}^{s,b} \hookrightarrow \hat{X}^{s,\frac{1}{2},1}$, which is easily verified with the Cauchy-Schwarz inequality. (ii) can be showed by decomposing f into dyadic sets $\{A_j\}$, $\{B_k\}$ and using the Hölder inequality in τ. □

The next two lemmas are the essential part of the bilinear estimate.

Lemma 3.2. *Assume that each f and g is supported on a single A_j. Then, we have*

$$\big\||\xi|^{\frac{1}{4}}\ f*g\big\|_{L^2(\mathbb{R}^2)} \lesssim \|f\|_{\hat{X}^{0,\frac{1}{2},1}}\|g\|_{\hat{X}^{0,\frac{1}{2},1}}. \quad (3.2)$$

Moreover, if

$$K := \inf \left\{ |\xi_1 - \xi_2| \,:\, \exists \tau_1, \tau_2 \text{ s.t. } (\tau_1, \xi_1) \in \operatorname{supp} f,\, (\tau_2, \xi_2) \in \operatorname{supp} g \right\} > 0,$$

then

$$\big\| |\xi|^{\frac{1}{2}} \; f * g \big\|_{L^2(\mathbb{R}^2)} \lesssim K^{-\frac{1}{2}} \big\| f \big\|_{\hat{X}^{0,\frac{1}{2},1}} \big\| g \big\|_{\hat{X}^{0,\frac{1}{2},1}}. \tag{3.3}$$

Proof. If we restrict f to an A_j, we have

$$\sum_{l \in \mathbb{Z}} \big\| \mathbf{1}_{\{ l \leq \tau - \xi^3 < l+1 \}} f \big\|_{L^2(\mathbb{R}^2)} \lesssim \big\| f \big\|_{\hat{X}^{0,\frac{1}{2},1}}$$

by applying the Cauchy-Schwarz inequality, so it suffices to show

$$\big\| |\xi|^{\frac{1}{4}} \; f * g \big\|_{L^2(\mathbb{R}^2)} \lesssim \big\| f \big\|_{L^2(\mathbb{R}^2)} \big\| g \big\|_{L^2(\mathbb{R}^2)}$$

and

$$\big\| |\xi|^{\frac{1}{2}} \; f * g \big\|_{L^2(\mathbb{R}^2)} \lesssim K^{-\frac{1}{2}} \big\| f \big\|_{L^2(\mathbb{R}^2)} \big\| g \big\|_{L^2(\mathbb{R}^2)}$$

under the assumption that f is restricted to the set $\left\{ (\tau, \xi) \,:\, l \leq \tau - \xi^3 < l + 1 \right\}$ for some $l \in \mathbb{Z}$ and similarly for g.

The Cauchy-Schwarz inequality and the Fubini theorem imply

$$\big\| |\xi|^{\alpha} \; f * g \big\|_{L^2(\mathbb{R}^2)} \leq \big\| f \big\|_{L^2(\mathbb{R}^2)} \big\| g \big\|_{L^2(\mathbb{R}^2)} \sup_{\tau, \xi} \Big(|\xi|^{\alpha} m^{\frac{1}{2}}(\tau, \xi) \Big),$$

where $m(\tau, \xi) := \left| \left\{ (\tau', \xi') \in \operatorname{supp} f \,:\, (\tau - \tau', \xi - \xi') \in \operatorname{supp} g \right\} \right|$, and we estimate m. Fix τ and $\xi \neq 0$, and assume $(\tau', \xi') \in \operatorname{supp} f$, $(\tau - \tau', \xi - \xi') \in \operatorname{supp} g$. We use the identity

$$(\tau - \frac{\xi^3}{4}) - (\tau' - \xi'^3) - (\tau - \tau' - (\xi - \xi')^3) = \frac{3}{4} \xi \left| \xi' - (\xi - \xi') \right|^2$$

to verify

$$\max\{ K^2, \, \frac{M - 10}{|\xi|} \} \leq \left| \xi' - (\xi - \xi') \right|^2 \leq \frac{M + 10}{|\xi|}$$

with a positive constant M depending on τ, ξ and the supports of f, g. With this we can confine ξ' to a set of size $\lesssim |\xi|^{-1} (K^2 + |\xi|^{-1})^{-\frac{1}{2}}$, and then obtain the bound $m(\tau, \xi) \lesssim \min\{ |\xi|^{-\frac{1}{2}}, \, |\xi|^{-1} K^{-1} \}$, which is the desired one. □

Lemma 3.3. *Assume that f is supported on a single A_j and g is an arbitrary test function. Then, we have*

$$\big\| f * g \big\|_{L^2(B_k)} \lesssim 2^{\frac{k}{4}} \big\| f \big\|_{\hat{X}^{0,\frac{1}{2},1}} \big\| |\xi|^{-\frac{1}{4}} g \big\|_{L^2(\mathbb{R}^2)} \tag{3.4}$$

for $k \geq 0$. *Moreover, if a non-empty set* $\Omega \in \mathbb{R}^2$ *satisfies*

$$K := \inf \left\{ |\xi + \xi_1| \,:\, \exists \tau, \tau_1 \ s.t.\ (\tau,\xi) \in \Omega,\ (\tau_1,\xi_1) \in \text{ supp } f \right\} > 0,$$

then

$$\left\| f * g \right\|_{L^2(\Omega \cap B_k)} \lesssim 2^{\frac{k}{2}} K^{-\frac{1}{2}} \left\| f \right\|_{\hat{X}^{0,\frac{1}{2},1}} \left\| |\xi|^{-\frac{1}{2}} g \right\|_{L^2(\mathbb{R}^2)}. \tag{3.5}$$

Proof. Similarly to Lemma 3.2, we restrict f to the set $\left\{ (\tau,\xi) \,:\, l \leq \tau - \xi^3 < l+1 \right\}$ for each $l \in \mathbb{Z}$, and need to show by duality

$$\left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(\tau - \tau', \xi - \xi') g(\tau', \xi')\, d\tau' d\xi'\, h(\tau,\xi)\, d\tau d\xi \right| \lesssim 2^{\frac{k}{4}} \left\| f \right\|_{L^2} \left\| |\xi|^{-\frac{1}{4}} g \right\|_{L^2} \left\| h \right\|_{L^2}$$

for $h \in L^2(B_k)$ and

$$\left| \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} f(\tau - \tau', \xi - \xi') g(\tau', \xi')\, d\tau' d\xi'\, h(\tau,\xi)\, d\tau d\xi \right| \lesssim 2^{\frac{k}{2}} K^{-\frac{1}{2}} \left\| f \right\|_{L^2} \left\| |\xi|^{-\frac{1}{2}} g \right\|_{L^2} \left\| h \right\|_{L^2}$$

for $h \in L^2(\Omega \cap B_k)$. We apply the Cauchy-Schwarz inequality first to the integral with respect to (τ,ξ), and next to that of (τ',ξ'), to see that the left-hand side is bounded by $\left\| f \right\|_{L^2} \left\| m^{\frac{1}{2}} g \right\|_{L^2} \left\| h \right\|_{L^2}$, where

$$m(\tau',\xi') := \left| \left\{ (\tau,\xi) \in \text{ supp } h \,:\, (\tau - \tau', \xi - \xi') \in \text{ supp } f \right\} \right|.$$

The desired bound for m,

$$m(\tau',\xi') \lesssim \min\{ 2^{\frac{k}{2}} |\xi'|^{-\frac{1}{2}},\ 2^k |\xi'|^{-1} K^{-1} \},$$

is then obtained by following the proof of Lemma 3.2 with the identity

$$(\tau' - \frac{\xi'^3}{4}) - (\tau - \xi^3) + (\tau - \tau' - (\xi - \xi')^3) = \frac{3}{4} \xi' \left| \xi + (\xi - \xi') \right|^2,$$

which implies

$$\max\{ K^2,\ \frac{M - 10 \cdot 2^k}{|\xi'|} \} \leq \left| \xi + (\xi - \xi') \right|^2 \leq \frac{M + 10 \cdot 2^k}{|\xi'|}. \qquad \square$$

We now prove the key bilinear estimate (3.1).

Proposition 3.4. *Suppose* f *and* g *are restricted to* A_{j_1} *and* A_{j_2}, *respectively. Then we have the estimates*

$$\left\| \mathbf{1}_{A_j} \langle \tau - \xi^3 \rangle^{-1} \xi \ f * g \right\|_{\hat{X}} \lesssim \left\| f \right\|_{\hat{X}} \left\| g \right\|_{\hat{X}}, \tag{3.6}$$

$$\big\| \mathbf{1}_{A_j} \langle \xi \rangle^{-\frac{3}{4}} \langle \tau - \xi^3 \rangle^{-1} \xi \ f * g \big\|_{L^2_\xi L^1_\tau} \lesssim \big\| f \big\|_{\hat{X}} \big\| g \big\|_{\hat{X}} \tag{3.7}$$

for $j \geq 0$ in the following cases.

(i) *At least two of j, j_1, j_2 are less than* 20.
(ii) $j_1, j_2 \geq 20$ *and* $0 < j < j_1 - 10$, *with the additional factor* $2^{-\delta j}$ *for some* $\delta > 0$.
(iii) $j, j_1 \geq 20$, $|j - j_1| \leq 10$ *and* $j_2 > 0$, *with the additional factor* $2^{-\delta j_2} + 2^{-\delta(j - j_2)}$ *for some* $\delta > 0$.
(iv) $j_1, j_2 \geq 20$ *and* $j = 0$.
(v) $j, j_1 \geq 20$ *and* $j_2 = 0$.

We remark that the cases (iii), (v) are also true with j_1 and j_2 exchanged, because of symmetry. We can then verify the bilinear estimate (3.1) from the above proposition by summing up in j, j_1 and j_2, using the L^2_ξ property of $\hat{X}$: $\big\| f \big\|^2_{\hat{X}} \sim \sum_j \big\| \mathbf{1}_{A_j} f \big\|^2_{\hat{X}}$.

Proof of Proposition 3.4. We see from Lemma 3.1 (ii) that in the case $j \neq 0$ (the cases (ii), (iii) and (v)) it suffices to prove only (3.6).

Estimate for (i). In this case we may assume that j, j_1, j_2 are all less than 30. Note that the left-hand sides of both (3.6) and (3.7) are bounded by $\big\| \mathbf{1}_{A_j} \langle \tau - \xi^3 \rangle^{-1} \xi \ f * g \big\|_{\hat{X}^{-\frac{3}{4}, \frac{1}{2}, 1}}$. We first use Lemma 3.1 (i) and ignore all the weight factors, and then apply the Young inequality as

$$\big\| f * g \big\|_{L^2(\mathbb{R}^2)} \leq \big\| f \big\|_{L^2_\xi L^{\frac{4}{3}}_\tau} \big\| g \big\|_{L^1_\xi L^{\frac{4}{3}}_\tau},$$

for example. Lemma 3.1 (ii) then implies the desired estimate (use the Hölder inequality in ξ for the estimate of g).

Estimate for (ii). We may assume $|j_1 - j_2| \leq 1$; otherwise, $\mathbf{1}_{A_j} \ f * g$ vanishes. What needs to be proved is the estimate

$$2^{\frac{j}{4}} \sum_{k \geq 0} 2^{-\frac{k}{2}} \big\| f * g \big\|_{L^2(A_j \cap B_k)} \lesssim 2^{-\delta j} 2^{-\frac{3}{2} j_1} \big\| f \big\|_{\hat{X}^{0, \frac{1}{2}, 1}} \big\| g \big\|_{\hat{X}^{0, \frac{1}{2}, 1}}. \tag{3.8}$$

If we restrict f to B_{k_1} and g to B_{k_2}, the identity

$$(\tau - \xi^3) - (\tau' - \xi'^3) - ((\tau - \tau') - (\xi - \xi')^3) = -3\xi\xi'(\xi - \xi')$$

shows that $\mathbf{1}_{A_j \cap B_k} \ f * g = 0$ unless

$$2^{k_{\max}} := 2^{\max\{ k, k_1, k_2 \}} \gtrsim 2^{j + 2j_1}.$$

When $2^k \sim 2^{j+2j_1}$, we have $2^{\frac{j}{4}}2^{-\frac{k}{2}} \sim 2^{-\frac{3}{4}j}2^{-j_1}\cdot 2^{\frac{j}{2}}$ and (3.8) follows from (3.3) with $K\sim 2^{j_1}$,

$$\text{L.H.S.} \lesssim 2^{-\frac{3}{4}j}2^{-j_1}\sum_{k=j+2j_1+O(1)}\big\||\xi|^{\frac{1}{2}}\, f{*}g\big\|_{L^2} \lesssim 2^{-\frac{3}{4}j}2^{-\frac{3}{2}j_1}\|f\|_{\hat{X}^{0,\frac{1}{2},1}}\|g\|_{\hat{X}^{0,\frac{1}{2},1}}.$$

Note that the high-frequency component has to interact with another high-frequency one in the opposite side to produce the low-frequency one, which allows us to assume $K\sim 2^{j_1}$. In the other cases we use (3.5) with $K\sim 2^{j_1}$; for example, if $k_{max}=k_1$ we multiply by $2^{\frac{3}{8}k_1}2^{-\frac{3}{8}(j+2j_1)}\cdot 2^{\frac{1}{8}(k_1-k)}\gtrsim 1$ to have

$$\begin{aligned}\text{L.H.S.} &\lesssim 2^{-\frac{j}{8}}2^{-\frac{3}{4}j_1}\sum_{k\geq 0}2^{-\frac{5}{8}k}\big\|(\langle\tau-\xi^3\rangle^{\frac{1}{2}}f)*g\big\|_{L^2(A_j\cap B_k)}\\ &\lesssim 2^{-\frac{j}{8}}2^{-\frac{7}{4}j_1}\|f\|_{\hat{X}^{0,\frac{1}{2}}}\|g\|_{\hat{X}^{0,\frac{1}{2},1}}.\end{aligned}$$

Estimate for (iii). We may assume $j_2\leq j+11$, and we need to show the estimate

$$\begin{aligned}&2^j\sum_{k\geq 0}2^{-\frac{k}{2}}\|f*g\|_{L^2(A_j\cap B_k)}\\ &\lesssim\Big(2^{-(\frac{3}{4}+\delta)j_2}+2^{-(\frac{3}{4}-\delta)j_2-\delta j}\Big)\|f\|_{\hat{X}^{0,\frac{1}{2},1}}\|g\|_{\hat{X}^{0,\frac{1}{2},1}}.\end{aligned}\tag{3.9}$$

Note that $2^{k_{max}}\gtrsim 2^{2j+j_2}$ in the present case. If $2^k\sim 2^{2j+j_2}$, we use (3.2) to obtain an acceptable bound $2^{-\frac{j}{4}}2^{-\frac{j_2}{2}}\|f\|_{\hat{X}^{0,\frac{1}{2},1}}\|g\|_{\hat{X}^{0,\frac{1}{2},1}}$. The case $k_{max}=k_1$ is treated with (3.4) and the fact that $2^{\frac{k_1}{2}}2^{-\frac{1}{2}(2j+j_2)}\gtrsim 1$, which imply the same bound. In the case $k_{max}=k_2$, however, the same argument leads to an unacceptable bound with $2^{-\frac{3}{4}j_2}$. We consider two subcases $2^{j_2}\sim 2^j$ and $\ll 2^j$ separately. The bound $2^{-\frac{3}{4}j_2}$ is sufficient in the former subcase, while in the latter we can apply (3.5) with $K\sim 2^j$ and have a suitable estimate.

Estimate for (iv). We only consider the worst case $(|\tau|\sim)2^k\sim 2^{k_{max}}\sim 2^{2j_1}|\xi|$, where $(\tau,\xi)\in A_0\cap B_k$ denotes the argument for $f*g$; otherwise, namely if $2^k\ll 2^{k_{max}}$ or $2^k\sim 2^{k_{max}}\gg 2^{2j_1}|\xi|$, we have $2^{k_1}\sim 2^{k_{max}}$ or $2^{k_2}\sim 2^{k_{max}}$ and verify the estimate in almost the same way as the corresponding cases of (ii). We thus assume $|\xi|\langle\tau-\xi^3\rangle^{-1}\sim 2^{-2j_1}$, or $|\xi|\sim 2^{k-2j_1}$ in B_k. We first see the easier (3.7), which can be verified from the Hölder inequality in ξ and the Young inequality, followed by Lemma 3.1 (ii),

$$\begin{aligned}&\big\|\langle\tau-\xi^3\rangle^{-1}\xi f*g\big\|_{L^2_\xi L^1_\tau(A_0)}\lesssim 2^{-2j_1}\|f*g\|_{L^\infty_\xi L^1_\tau}\\ &\lesssim 2^{-\frac{1}{2}j_1}\big\|\langle\xi\rangle^{-\frac{3}{4}}f\big\|_{L^2_\xi L^1_\tau}\big\|\langle\xi\rangle^{-\frac{3}{4}}g\big\|_{L^2_\xi L^1_\tau}\lesssim 2^{-\frac{1}{2}j_1}\|f\|_{\hat{X}}\|g\|_{\hat{X}}.\end{aligned}$$

Next we prove (3.6). If we measure $f * g$ in the weak space $\hat{X}^{-\frac{3}{4},\frac{1}{2}}$, which is the case when $f * g$ is restricted to D, the estimate immediately follows from (3.3) with $K \sim 2^{j_1}$,

$$\big\|\langle\tau-\xi^3\rangle^{-\frac{1}{2}}\xi\ f*g\big\|_{L^2} \lesssim 2^{-j_1}\cdot 2^{-\frac{j_1}{2}}\big\|f\big\|_{\hat{X}^{0,\frac{1}{2},1}}\big\|g\big\|_{\hat{X}^{0,\frac{1}{2},1}}.$$

However, when we try to estimate $f*g$ in $\hat{X}^{-\frac{3}{4},\frac{1}{2},1}$ with (3.3), the logarithmic divergence will occur and we will be forced to bound with the constant of $O(j_1)$ to sum up the norm in B_k. We adopt a different way with the Hölder inequality and the Young inequality as

$$\begin{aligned}\sum_k 2^{-\frac{k}{2}}\big\|\xi\ f*g\big\|_{L^2(B_k)} &\lesssim \sum_k 2^{-\frac{k}{2}}\big\|\xi\big\|_{L^2(\{\,|\xi|\sim 2^{k-2j_1}\,\})}\big\|f\big\|_{L^2_{\tau,\xi}}\big\|g\big\|_{L^2_\xi L^1_\tau}\\ &\lesssim \big\|f\big\|_{\hat{X}^{0,\frac{1}{2},1}}\big\|g\big\|_{\hat{X}^{0,\frac{1}{2},1}}\sum_k 2^{k-3j_1}.\end{aligned}$$

If we restrict $f*g$ to the outside of D, then $|\xi|^{-3} \geq |\tau| \sim 2^{2j_1}|\xi|$, which implies $|\xi| \lesssim 2^{-\frac{j_1}{2}}$ and $|\tau| \sim 2^{2j_1}|\xi| \lesssim 2^{\frac{3}{2}j_1}$, so the above estimate is appropriate.

Estimate for (v). For $(\tau_2,\xi_2) \in A_0$, the argument for g, we assume $|\xi_2| \geq 2^{-2j}$; otherwise, we can easily estimate $f*g$ with an $L^2_\xi L^1_\tau \cdot L^1_\xi L^2_\tau$ application of the Young inequality. If $2^k \sim 2^{k_{max}}$, we use (3.3) with $K \sim 2^j$, combining Lemma 3.1 (i) and a multiplication by $2^{\frac{k}{4}}\big(2^{2j}|\xi_2|\big)^{-\frac{1}{4}}\cdot 2^{\varepsilon(k-k_2)} \gtrsim 1$ with $0<\varepsilon \ll 1$, to have

$$\begin{aligned}&\big\|\xi\ f*g\big\|_{\hat{X}^{-\frac{3}{4},-\frac{1}{2},1}}\\ &\lesssim 2^{\frac{j}{2}}\big\||\xi|^{\frac{1}{2}}\langle\tau-\xi^3\rangle^{\frac{1}{4}+\varepsilon}\,(\langle\xi\rangle^{-\frac{3}{4}}f)*((2^{2j}|\xi|)^{-\frac{1}{4}}\langle\tau-\xi^3\rangle^{-\varepsilon}g)\big\|_{\hat{X}^{0,-\frac{1}{2}+\varepsilon}}\\ &\lesssim 2^{-\frac{j}{2}}\big\|f\big\|_{\hat{X}^{-\frac{3}{4},\frac{1}{2},1}}\big\|\mathbf{1}_{\{\,|\xi|\geq 2^{-2j}\,\}}|\xi|^{-\frac{1}{4}}\langle\tau-\xi^3\rangle^{-\varepsilon}g\big\|_{\hat{X}^{0,\frac{1}{2},1}}\\ &\lesssim \big\|f\big\|_{\hat{X}^{-\frac{3}{4},\frac{1}{2},1}}\big\|g\big\|_{\hat{X}^{-\frac{3}{4},\frac{1}{2}}}.\end{aligned}$$

In the second case $2^{k_1} \sim 2^{k_{max}} \gtrsim 2^{2j}|\xi_2|$; we will not use (3.5). Note that $|\xi_2| \lesssim 2^{k_1-2j}$ when f is on B_{k_1}. We again use Lemma 3.1 (i) to obtain

$$\begin{aligned}&\big\|\xi\ f*g\big\|_{\hat{X}^{-\frac{3}{4},-\frac{1}{2},1}}\\ &\lesssim 2^j\big\|(\langle\xi\rangle^{-\frac{3}{4}}f)*g\big\|_{\hat{X}^{0,-\frac{1}{2}+\varepsilon}} \lesssim 2^j\big\|(\langle\xi\rangle^{-\frac{3}{4}}f)*g\big\|_{L^2_\xi L^4_\tau} \qquad (3.10)\\ &\lesssim 2^j\sum_{k_1}\big\|\langle\xi\rangle^{-\frac{3}{4}}f\big\|_{L^2_{\tau,\xi}(B_{k_1})}\big\|g\big\|_{L^1_\xi L^{\frac{4}{3}}_\tau(\{\,|\xi|\lesssim 2^{k_1-2j}\,\})}\end{aligned}$$

$$\lesssim 2^j \sum_{k_1} \big\|\langle\xi\rangle^{-\frac{3}{4}} f\big\|_{L^2_{\tau,\xi}(B_{k_1})} 2^{\frac{k_1}{2}-j} \big\|g\big\|_{\hat{X}^{0,\frac{1}{2}}} \sim \big\|f\big\|_{\hat{X}^{-\frac{3}{4},\frac{1}{2},1}} \big\|g\big\|_{\hat{X}^{-\frac{3}{4},\frac{1}{2}}}.$$

Lastly we treat the worst case $(|\tau_2| \sim)2^{k_2} \sim 2^{k_{max}} \sim 2^{2j}|\xi_2|$, where we have $|\xi_2| \sim 2^{k_2-2j}$ if g is on B_{k_2}. As long as g is measured in the strong space $\hat{X}^{-\frac{3}{4},\frac{1}{2},1}$ there is no difficulty:

$$(3.10) \lesssim 2^j \big\|\langle\xi\rangle^{-\frac{3}{4}} f\big\|_{L^2_\tau L^{\frac{4}{3}}_\xi} \sum_{k_2} \big\|g\big\|_{L^1_\xi L^2_\tau(B_{k_2}\cap\{\,|\xi|\lesssim 2^{k_2-2j}\,\})}$$
$$\lesssim \big\|f\big\|_{\hat{X}^{-\frac{3}{4},\frac{1}{2}}} \big\|g\big\|_{\hat{X}^{-\frac{3}{4},\frac{1}{2},1}}.$$

Similarly to (iv), we see that $(\tau_2,\xi_2) \in D$ implies $|\tau_2| \gtrsim 2^{\frac{3}{2}j}$. We also have $|\tau_2| \lesssim 2^{2j}$, so let us assume g to be supported on $A_0 \cap B_{[\frac{3}{2}j,2j]}$ and prove the bound $\big\|f\big\|_{\hat{X}^{-\frac{3}{4},\frac{1}{2},1}} \big\|g\big\|_{\hat{X}^{-\frac{3}{4},\frac{1}{2}}}$. The following two estimates are now obtained by different arguments:

(a) If g is restricted to $B_{[\frac{3}{2}j,\frac{3}{2}j+\gamma]}$ with some $0 \le \gamma \le \frac{j}{2}$, then for $\gamma' \ge 0$

$$\big\|\mathbf{1}_{B_{\ge\gamma'}}\xi\ f*g\big\|_{\hat{X}^{-\frac{3}{4},-\frac{1}{2},1}}$$
$$\lesssim \sum_{k\ge\gamma'} 2^{-\frac{k}{2}} \big\|\langle\xi\rangle^{-\frac{3}{4}} f\big\|_{L^2_\xi L^1_\tau} \big\||\xi|^{-\frac{1}{2}} \langle\tau-\xi^3\rangle^{\frac{1}{2}} g\big\|_{L^1_\xi L^2_\tau(B_{[\frac{3}{2}j,\frac{3}{2}j+\gamma]})}$$
$$\lesssim 2^{-\frac{\gamma'}{2}} \big\|f\big\|_{\hat{X}^{-\frac{3}{4},\frac{1}{2},1}} \big\||\xi|^{-\frac{1}{2}}\big\|_{L^2_\xi(\{\,2^{-\frac{j}{2}}\lesssim|\xi|\lesssim 2^{-\frac{j}{2}+\gamma}\,\})} \big\|g\big\|_{\hat{X}^{0,\frac{1}{2}}}$$
$$\lesssim \sqrt{\langle\gamma\rangle}\, 2^{-\frac{\gamma'}{2}} \big\|f\big\|_{\hat{X}^{-\frac{3}{4},\frac{1}{2},1}} \big\|g\big\|_{\hat{X}^{-\frac{3}{4},\frac{1}{2}}}.$$

(b) If g is restricted to $B_{[\frac{3}{2}j+\gamma',2j]}$ with some $0 \le \gamma' \le \frac{j}{2}$, then for $\gamma \ge 0$ we use (3.5) with $K \sim 2^j$ to have

$$\big\|\mathbf{1}_{B_{\le\gamma}}\xi\ f*g\big\|_{\hat{X}^{-\frac{3}{4},-\frac{1}{2},1}} \lesssim \sum_{k\le\gamma} 2^{-\frac{k}{2}} \big\|(\langle\xi\rangle^{-\frac{3}{4}} f) * (|\xi|^{-\frac{1}{2}} \langle\tau-\xi^3\rangle^{\frac{1}{2}} g)\big\|_{L^2(B_k)}$$
$$\lesssim \langle\gamma\rangle 2^{-\frac{j}{2}} \big\|f\big\|_{\hat{X}^{-\frac{3}{4},\frac{1}{2},1}} \big\||\xi|^{-1} \langle\tau-\xi^3\rangle^{\frac{1}{2}} g\big\|_{L^2_{\tau,\xi}(\{\,|\xi|\gtrsim 2^{-\frac{j}{2}+\gamma'}\,\})}$$
$$\lesssim \langle\gamma\rangle 2^{-\gamma'} \big\|f\big\|_{\hat{X}^{-\frac{3}{4},\frac{1}{2},1}} \big\|g\big\|_{\hat{X}^{-\frac{3}{4},\frac{1}{2}}}.$$

Each of these estimates is, however, not enough by itself because of some logarithmic divergences; in fact if we choose $\gamma = O(j)$ and $\gamma' = 0$ there will remain the term of $O(j)$ or $O(j^{\frac{1}{2}})$ in the resulting estimates. Now, let $\{\,\gamma_n\,\}_{0\le n\le N}$ be the decreasing sequence uniquely defined by

$$\gamma_0 = \tfrac{j}{2}\ (\ge 10), \quad \gamma_{n+1} = 2\log_2\gamma_n, \quad 6 \le \gamma_N < 8.$$

We first apply (a) with $\gamma = \gamma_0$ and $\gamma' = \gamma_1$, next (b) with $\gamma = \gamma_1$ and $\gamma' = \gamma_2$, and thereafter apply (a) and (b) alternately until γ' accords with γ_N, and at the end apply (a) with $\gamma = \gamma_{N-1}$ (< 16) and $\gamma' = 0$, to obtain

$$\big\|\xi\ f * g\big\|_{\hat{X}^{-\frac34,-\frac12,1}} \lesssim (1 + \sum_{n=0}^{N-1} \frac{1}{\sqrt{\gamma_n}})\big\|f\big\|_{\hat{X}^{-\frac34,\frac12,1}}\big\|g\big\|_{\hat{X}^{-\frac34,\frac12}},$$

which shows the claim since $\sum_{n=0}^{N-1} \frac{1}{\sqrt{\gamma_n}}$ is bounded uniformly in j. □

4. Proof of Theorem 1.2

We begin with some linear estimates. Recall the definition of the restricted norm (2.5).

Lemma 4.1. *Let $e^{-t\partial_x^3}$ be the propagator of the linear equation, namely $\widehat{e^{-t\partial_x^3}u_0}(t) = e^{it\xi^3}\hat{u}_0$. Then we have the following estimates:*

$$\big\|e^{-t\partial_x^3}u_0\big\|_{X_T} + \sup_{-T\le t\le T}\big\|e^{-t\partial_x^3}u_0\big\|_{H_x^{-\frac34}(\mathbb{R})} \lesssim \big\|u_0\big\|_{H_x^{-\frac34}(\mathbb{R})}, \tag{4.1}$$

$$\begin{aligned}\Big\|\int_0^t e^{-(t-s)\partial_x^3}F(s)\,ds\Big\|_{X_T} + \sup_{-T\le t\le T}\Big\|\int_0^t e^{-(t-s)\partial_x^3}F(s)\,ds\Big\|_{H_x^{-\frac34}(\mathbb{R})} \\ \lesssim \big\|\mathcal{F}^{-1}\big[\langle\tau-\xi^3\rangle^{-1}F\big]\big\|_X + \big\|\mathcal{F}^{-1}\big[\langle\tau-\xi^3\rangle^{-1}F\big]\big\|_Y\end{aligned} \tag{4.2}$$

for any $0 < T \le 1$. The implicit constants do not depend on T.

Proof. From the definition of the restricted norm we may assume $T = 1$. For (4.1), we use a smooth characteristic function $\psi(t)$ of the interval $[-1, 1]$ to remove the time restriction of each norm; estimates of this type are quite familiar and have been treated in much of the literature. Note that our space X satisfies the embedding $X^{-\frac34,\frac12+\varepsilon} \hookrightarrow X$ for any $\varepsilon > 0$. Then, (4.1) can be verified by the corresponding estimate in $X^{-\frac34,\frac12+\varepsilon}$.

For the estimate of the X norm in (4.2) we refer to [7]. Following Lemma 2.1 in [7] we decompose the norm into three parts as

$$\begin{aligned}&\mathcal{F}_x\Big[\int_0^t e^{-(t-s)\partial_x^3}F(s)\,ds\Big](\xi) = ce^{it\xi^3}\int_{\mathbb{R}} \frac{e^{it(\tau-\xi^3)}-1}{\tau-\xi^3}\hat{F}(\tau,\xi)\,d\tau \\ &= ce^{it\xi^3}\int_{\mathbb{R}} \frac{e^{it(\tau-\xi^3)}-1}{\tau-\xi^3}\,(\mathbf{1}_{B_0}\hat{F})(\tau,\xi)\,d\tau \\ &+ c\int_{\mathbb{R}}\Big(\frac{e^{it\tau}}{\tau-\xi^3} + \frac{-e^{it\xi^3}}{\tau-\xi^3}\Big)\,(\mathbf{1}_{B_{>0}}\hat{F})(\tau,\xi)\,d\tau.\end{aligned}$$

For the first and the third terms we give $\psi(t)$ and replace by the $X^{-\frac{3}{4},\frac{1}{2}+\varepsilon}$ norm, which is estimated by the Y norm in the same way as [7]. The difference from [7] is that we directly estimate the second term without $\psi(t)$ or any cutoff, which is bounded by the same X norm with the additional term $\langle\tau-\xi^3\rangle^{-1}$. From the fact that

$$\Big|\frac{e^{it(\tau-\xi^3)}-1}{\tau-\xi^3}\Big| \lesssim \langle t\rangle\langle\tau-\xi^3\rangle^{-1},$$

the rest of the left-hand side in (4.2) can be easily bounded by the Y norm. □

To obtain a local-in-time solution we replace (1.1) with the integral equation

$$u(t)=e^{-t\partial_x^3}u_0+\int_0^t e^{-(t-s)\partial_x^3}[\partial_x(u(s)^2)]\,ds. \quad (\;=:\Phi(u)(t)\;) \tag{4.3}$$

Then, the above lemma and Proposition 1.1 imply the estimate

$$\big\|\Phi(u)\big\|_{X_1}+\sup_{-1\le t\le 1}\big\|\Phi(u)(t)\big\|_{H^{-\frac{3}{4}}(\mathbb{R})}\lesssim\big\|u_0\big\|_{H^{-\frac{3}{4}}(\mathbb{R})}+\big\|u\big\|_{X_1}^2,$$

which shows that if $\|u_0\|_{H^{-\frac{3}{4}}}$ is sufficiently small Φ will become a contraction mapping on a closed ball in $X_1\cap C_t^0([-1,1];H^{-\frac{3}{4}}(\mathbb{R}))$ and produce a solution on the time interval $[-1,1]$. The Lipschitz dependence of solutions upon the data automatically follows. For large data we take a familiar scaling argument. Note that our regularity $s=-\frac{3}{4}$ is well above the critical regularity for KdV with respect to the scaling ($s=-\frac{3}{2}$).

At the end we prove the uniqueness of solutions following the argument in [15]. We recall the space in [15] and define $B_{2,1}^{(s,b)}$ by

$$\big\|u\big\|_{B_{2,1}^{(s,b)}}:=\Big\|\Big\{\|\langle\xi\rangle^s\langle\tau-\xi^3\rangle^b\hat{u}\|_{L^2_{\tau,\xi}(A_j\cap B_k)}\Big\}_{j,k\ge 0}\Big\|_{\ell^1_{j,k}}.$$

The space $B_{2,1}^{(s,b)}$ is similar to, but slightly stronger than, our space $X^{s,b,1}$.

Note that using the equation (4.3) with Lemma 4.1 and Proposition 1.1 we can easily bound the $C_t^0([-T,T];H^{-\frac{3}{4}})$ norm of any solution $u(t)$ in terms of $\|u(0)\|_{H^{-\frac{3}{4}}}$ and $\|u\|_{X_T}$ (if $T>1$, then repeat the estimate). Therefore, a solution of (1.1) in X_T automatically lies in $C_t^0([-T,T];H^{-\frac{3}{4}})$, and the uniqueness just in $X_T\cap C_t^0([-T,T];H^{-\frac{3}{4}})$ will suffice for our assertion, as we have mentioned in Remark 1.3 (i).

Lemma 4.2. *Let $s \in \mathbb{R}$. Assume that the Banach space of spacetime functions Z is included in $C_t^0(\mathbb{R}; H^s(\mathbb{R}))$, containing $\mathcal{S}(\mathbb{R}^2)$ densely, and satisfies the estimate*

$$\sup_{t\in\mathbb{R}} \|u(t)\|_{H^s} \lesssim \|u\|_Z \lesssim \|u\|_{X^{s,b}}$$

for all $u \in X^{s,b}$ with some $\frac{1}{2} < b < 1$. We define the time restricted norm Z_T similarly to (2.5). Then,

$$\lim_{T\to+0} \|u\|_{Z_T} = 0 \tag{4.4}$$

for all $u \in Z$ satisfying $u(0,x) = 0$.

Proof. (4.4) has been proved for $B_{2,1}^{(s,b)}$ with $\frac{1}{2} \le b < 1$ ([15], Theorem 2.5). Let ε be an arbitrary positive number and assume $u \in Z$, $u(0,x) = 0$. We choose a Schwartz function v satisfying $\|u - v\|_Z < \varepsilon$. It does not necessarily hold that $v(0,x) = 0$, but from the assumption we have

$$\|v(0)\|_{H^s} = \|(u-v)(0)\|_{H^s} \lesssim \|u - v\|_Z < \varepsilon.$$

Combining this with the $X^{s,b}$ version of (4.1), we have

$$\begin{aligned} \|u\|_{Z_T} &\le \|u - v\|_Z + \|v - e^{-t\partial_x^3} v(0)\|_{Z_T} + \|e^{-t\partial_x^3} v(0)\|_{X_T^{s,b}} \\ &\lesssim \varepsilon + \|v - e^{-t\partial_x^3} v(0)\|_{B_{2,1,T}^{(s,b)}}. \end{aligned}$$

Since the second term tends to 0 as $T \to 0$, we have (4.4). □

In the following we consider $Z = X \cap C_t^0(H^{-\frac{3}{4}})$. Let $u_1, u_2 \in Z_{T_0}$ be solutions to (1.1) on $[-T_0, T_0]$ with the same u_0, and set $u := u_1 - u_2$. From the scaling we assume $\|u_0\|_{H^{-\frac{3}{4}}}$ to be sufficiently small. Using the integral equation (4.3), Lemma 4.1 and Proposition 1.1 we have

$$\begin{aligned} \|u\|_{Z_T} &\lesssim \|u\|_{Z_T} \|u_1 + u_2\|_{Z_T} \\ &\lesssim \|u\|_{Z_T} \big(\|u_1 + u_2 - 2e^{-t\partial_x^3} u_0\|_{Z_T} + \|u_0\|_{H^{-\frac{3}{4}}} \big). \end{aligned}$$

By Lemma 4.2 we can choose $T > 0$ small so that $\|u\|_{Z_T} \le \frac{1}{2}\|u\|_{Z_T}$, which implies $u(t) = 0$ for $-T \le t \le T$. This procedure is repeated to obtain $u(t) = 0$ in the whole existence interval $[-T_0, T_0]$ and thus we have the uniqueness as desired.

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